Foundation of Mathematics I

# *Chapter 4   Functions*

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# *Chapter Four*

# *Functions*

**Definition 4.1.** A **function** or a **mapping** from $A$ to $B$, denoted by $f: A \to B$ is a relation $f$ from $A$ to $B$ in which every element from $A$ appears exactly once as the first component of an ordered pair in the relation. That is, each $a \in A$ the relation $f$ contains exactly one ordered pair of form $(a, b)$.

**Equivalent statements to the function definition.**

**(i)** A relation $f$ from $A$ to $B$ is function iff

$$\forall x \in A \ \exists! \, y \in B \text{ such that } (x, y) \in f$$

**(ii)** A relation $f$ from $A$ to $B$ is function iff

$$\forall x \in A \ \forall y, z \in B, \text{ if } (x, y) \in f \wedge (x, z) \in f, \text{ then } y = z.$$

**(iii)** A relation $f$ from $A$ to $B$ is function iff

$$(x_1, y_1) \text{ and } (x_2, y_2) \in f \text{ such that if } x_1 = x_2, \text{ then } y_1 = y_2.$$

This property called **the well-defined relation**.

**Example 4.2.**

**(i)** Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5\}$.

**(1)** $R_1 = \{(1, 2), (2, 4), (3, 4), (4, 5)\}$ function from $A$ to $B$.

**(2)** $R_2 = \{(1, 2), (2, 4), (2, 5), (4, 5)\}$ not a function.

**(3)** $R_3 = \{(1, 2), (2, 4), (4, 5)\}$ function from $\{1, 2, 4\}$ to $B$.

**(4)** $R_4 = A \times B$ not a function.

**(ii)** Consider the relations described below.

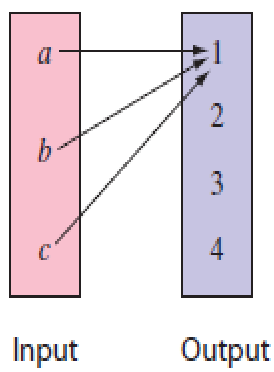
| Relation | Orderd pairs | Sample Relation |
|---|---|---|
| 1 | (person, month) | {(A, May), (B, Dec), (C, Oct), . . } |
| 2 | (hours, pay) | {(12, 84), (4, 28), (6, 42), (15, 105), . . . } |
| 3 | (instructor, course) | {(A, MATH001), (A, MATH002), . . . } |
| 4 | (time, temperature) | {(8, 70°), (10, 78°), (12, 78°), . . . } |

**The first** relation is a function because each person has only one birth month.
**The second** relation is a function because the number of hours worked at a particular job can yield only one paycheck amount.
**The third** relation is not a function because an instructor can teach more than one course.
**The fourth** relation is a function. Note that the ordered pairs (10, 78°), (12, 78°) do not violate the definition of a function.

**(iii)** Decide whether each relation represents a function.

**a.** Input: $a, b, c$
Output: 2, 3, 4
$\{(a, 2), (b, 3), (c, 4)\}$

**b.**




**c.**

| Input $x$ | Output $y$ | $(x, y)$ |
|---|---|---|
| 3 | 1 | (3, 1) |
| 4 | 3 | (4, 3) |
| 5 | 4 | (5, 4) |
| 3 | 2 | (3, 2) |

**Solution.**
**a.** This set of ordered pairs does represent a function. No first component has two different second components.
**b.** This diagram does represent a function. No first component has two different second components.
**c.** This table does not represent a function. The first component 3 is paired with two different second components, 1 and 2.

**Notation 4.3.** We write $f(a) = b$ when $(a, b) \in f$ where $f$ is a function. We say that $b$ is the **image** of $a$ under $f$, and $a$ is a **preimage** of $b$.

**Definition 4.4.** Let $f: A \rightarrow B$ be a function from $A$ to $B$.

**(i)** The set $A$ is called the **domain** of $f$, ($\boldsymbol{D(f)}$), and the set $B$ is called the **codomain** of $f$.

**(ii)**The set $f(A) = \{f(x) \mid x \in A\}$ is called the **range** of $f$, ($\boldsymbol{R(f)}$).

**Remark 4.5.**

**(i)** Think of the domain as the set of possible **"input values"** for $f$.

**(ii)** Think of the range as the set of all possible **"output values"** for $f$.

**Example 4.6.**

**(i)** Let $A = \{p, q, r, s\}$ and $B = \{0,1,2\}$ and

$$f = \{(p, 0), (q, 1), (r, 2), (s, 2)\} \subseteq A \times B.$$

This is a function $f: A \to B$ because each element of $A$ occurs exactly once as a first coordinate of an ordered pair in $f$.

We have $f(p) = 0, f(q) = 1, f(r) = 2$ and $f(s) = 2$. The domain of $f$ is $A$, and the codomain and range are both $B$.

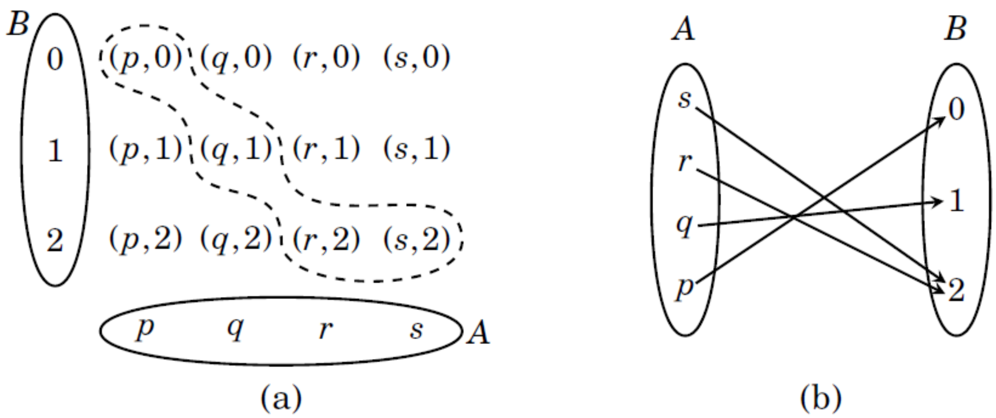



**Figure.** Two ways of drawing the function $f = \{(p,0),(q,1),(r,2),(s,2)\}$

**(ii)** Say a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f(m, n) = 6m - 9n$.

Note that as a set, this function is

$$f = \{((m, n), 6m - 9n): (m, n) \in \mathbb{Z} \times \mathbb{Z}\} \subseteq (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z}.$$

**What is the range of ?**

To answer this, first observe that for any $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, the value

$$f(m, n) = 6m - 9n = 3(2m - 3n)$$

is a multiple of 3. Thus every number in the range is a multiple of 3, so

$$R(f) \subseteq \{3k: k \in \mathbb{Z}\}. \qquad \cdots (1)$$

On the other hand if $b = 3k$ is a multiple of 3 we have

$$f(-k,-k) = 6(-k) - 9(-k) = -6k + 9k = 3k,$$

which means any multiple of 3 is in the range of $f$, so

$$\{3k : k \in \mathbb{Z}\} \subseteq R(f). \qquad \cdots (2)$$

Therefore, from (1) and (2) we get

$$R(f) = \{3k : k \in \mathbb{Z}\}.$$

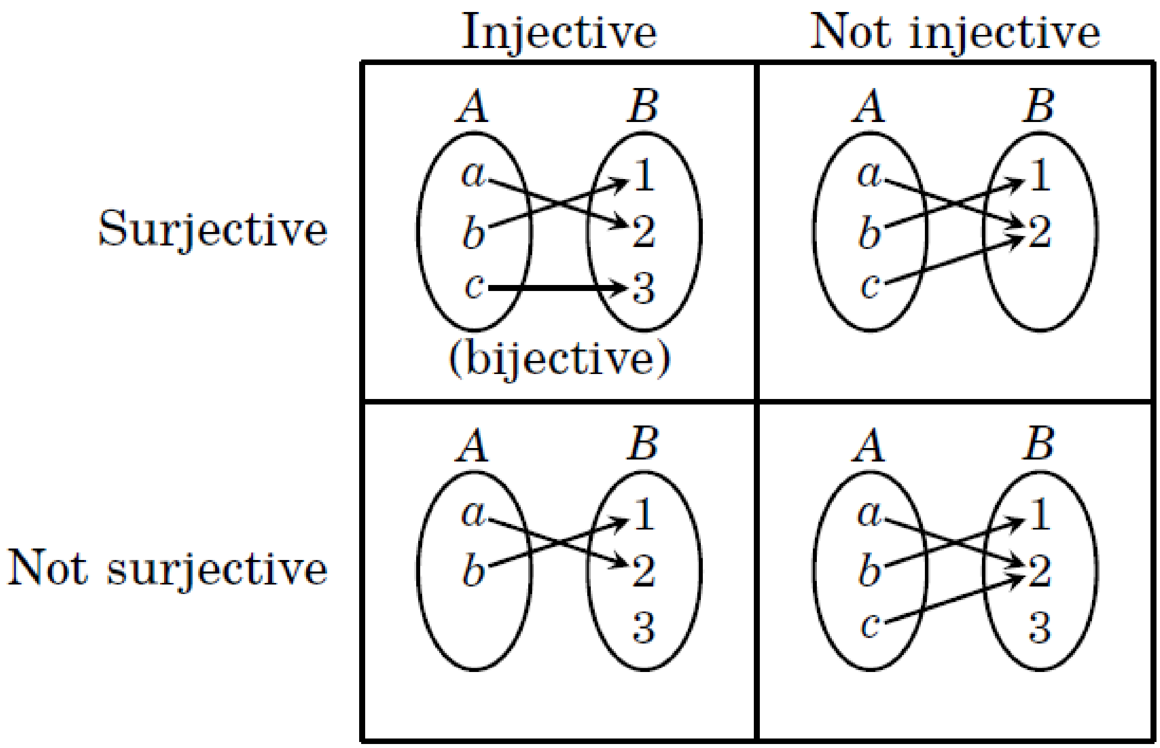
**Definition 4.7.** Two functions $f: A \to B$ and $g: C \to D$ are **equal** if $A = C, B = D$ and $f(x) = g(x)$ for every $x \in A$.

**Example 4.8.**
**(i)** Suppose that $A = \{1,2,3\}$ and $B = \{a, b\}$. The two functions $f = \{(1, a), (2, a), (3, b)\}$ and $g = \{(3, b), (2, a), (1, a)\}$ from $A$ to $B$ are equal because the sets $f$ and $g$ are equal. Observe that the equality $f = g$ means $f(x) = g(x)$ for every $x \in A$.
**(ii)** Let $f(x) = (x^2 - 1)/(x - 1)$ and $g(x) = x + 1$, where $x \in \mathbb{R}$.
$f(x) = (x - 1)(x + 1)/(x - 1) = (x + 1)$.
$D(f) = \mathbb{R} - \{1\}$, $R(f) = \mathbb{R} - \{2\}$.
$D(g) = \mathbb{R}$, $R(f) = \mathbb{R}$.
$f \neq g$.

**Definition 4.9.**
**(i)** A function $f: A \to B$ is **one-to-one** or **injective** if each element of $B$ appears at most once as the image of an element of $A$. That is, a function $f: A \to B$ is injective if $\forall x, y \in A,\ f(x) = f(y) \Rightarrow x = y$ **or** $\forall x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$.

**(ii)** A function $f: A \to B$ is **onto** or **surjective** if $f(A) = B$, that is, each element of $B$ appears at least once as the image of an element of $A$. That is, a function $f: A \to B$ is surjective if $\forall y \in B\ \exists x \in A$ such that $f(x) = y$.

**(iii)** A function $f : A \to B$ is **bijective** iff it is one-to-one and onto.




**Example 4.10.** Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined as $f(x) = 3x + 7$.

$$f = \{\dots, (-3, -2), (-2, 1), (-1, 4), (0, 7), (1, 10), (2, 13), \dots\}.$$

**(i)** $f$ is injective. Suppose otherwise; that is,

$$f(x) = f(y) \Rightarrow 3x + 7 = 3y + 7 \Rightarrow 3x = 3y \Rightarrow x = y$$

**(ii)** $f$ is not surjective. For $b = 2$ there is no $a$ such that $f(a) = b$; that is, $2 = 3a + 7$ holds for $a = -\frac{5}{3}$ which is not in $\mathbb{Z} = D(f)$.

**Example 4.11.**
**(i)** Show that the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined as
$f(x) = (1/x) + 1$ is injective but not surjective.
**Solution.**
We will use the contrapositive approach to show that $f$ is injective.
Suppose $x, y \in \mathbb{R} - \{0\}$ and $f(x) = f(y)$. This means
$\frac{1}{x} + 1 = \frac{1}{y} + 1 \longrightarrow x = y$. Therefore $f$ is injective.
Function $f$ is not surjective because there exists an element $b = 1 \in \mathbb{R}$ for which
$f(x) = (1/x) + 1 \neq 1$ for every $x \in \mathbb{R}$.

**(ii)** Show that the function $f: \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ defined by the formula $f(m,n) = (m+n, m+2n)$, is both injective and surjective.

**Solution.**

**Injective:** Let $(m,n),(r,s) \in \mathbb{Z} \times \mathbb{Z} = D(f)$ such that $f(m,n) = f(r,s)$. To prove $(m,n) = (r,s)$.

1-$f(m,n) = f(r,s) \Longrightarrow (m+n, m+2n) = (r+s, r+2s)$ Hypothesis
2- $m+n = r+s$ Def. of $\times$
3- $m+2n = r+2s$ Def. of $\times$
4- $m = r+2s-2n$ Inf. (3)
5- $n = s$ and $m = r$ Inf. (2),(4)
6- $(m,n) = (r,s)$ Def. of $\times$

**Surjective:** Let $(x,y) = \mathbb{Z} \times \mathbb{Z} = R(f)$. To prove $\exists (m,n) \in \mathbb{Z} \times \mathbb{Z} = D(f) \ni f(m,n) = (x,y)$.

1-$f(m,n) = (m+n, m+2n) = (x,y)$ Def. of $f$
2- $m+n = x$ Def. of $\times$
3- $m+2n = y$ Def. of $\times$
4- $m = x-n$ Inf. (2)
5- $n = y-x$ Inf. (3),(4)
6- $m = -x$ Inf. (2),(5)
7- $(-x, y-x) \in \mathbb{Z} \times \mathbb{Z} = D(f)$, $f(-x, y-x) = (x,y)$

**Definition 4.12.** The **composition** of functions $f: X \to Y$ with $g: Y \to Z$ is the function $g \circ f: X \to Z$ defined by $(g \circ f)(x) = g(f(x))$.

**Remark 4.13.**

**(i)** The composition $g \circ f$ can only be defined if the domain of $g$ includes the range of $f$; that is, $R(f) \subseteq D(g)$, and the existence of $g \circ f$ does not imply that $f \circ g$ even makes sense.

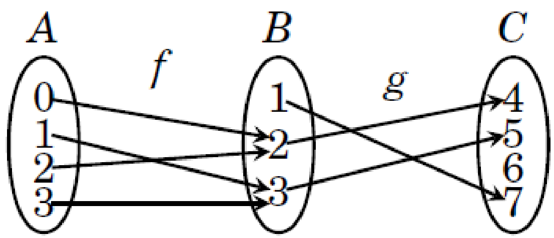
**(ii)** The order of application of the functions in a composition is crucial and is read from from right to left.

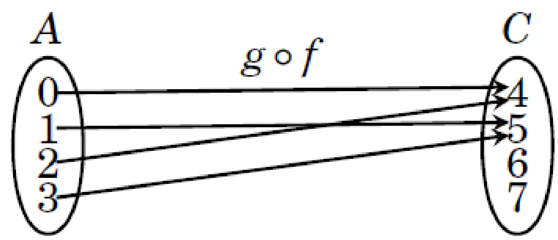
**Example 4.14.**

**(i)** Let $A = \{0,1,2,3\}$, $B = \{1,2,3\}$, $C = \{4,5,6,7\}$. If $f : A \to B$ and $g : B \to C$ are the functions defined as follows.

$f = \{(0,2),(1,3),(2,2),(3,3)\}$, $g = \{(1,7),(2,4),(3,5)\}$.

$g \circ f = \{(0,4), (1,5), (2,4), (3,5)\}$




**(ii)** If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions defined as follows.

$f(x) = x^2$ and $g(x) = \sqrt{x}$. Then $(g \circ f)(x) = g\left(f(x)\right) = g(x^2) = \sqrt{x^2}$.

Here $R(f) = [0, \infty) \subseteq D(g) = \mathbb{R}$ .

**Theorem 4.15.**

**(i)** Suppose $f : A \to B$ and $g : B \to C$ be functions. If both $f$ and $g$ are injective, then $g \circ f$ is injective. If both $f$ and $g$ are surjective, then $g \circ f$ is surjective.

**(ii)** Composition of functions is associative. That is, if $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $(g \circ f) \circ h = f \circ (g \circ h)$.

**Proof.**

**(i)** To prove $g \circ f$ is 1-1. Let $x, y \in A$ and $(g \circ f)(x) = (g \circ f)(y)$.
To prove $x = y$.

$(g \circ f)(x) = g(f(x)) = g(f(y))$ Def. of $\circ$
$f(x) = f(y)$ Since $g$ is 1-1 and Def. of 1-1on $g$
$x = y$ Since $f$ is 1-1 and Def. of 1-1on $f$
$\therefore g \circ f$ is 1-1.

To prove $g \circ f$ is onto. Let $z \in D$, to prove $\exists\, x \in A$ such that $(g \circ f)(x) = z$.
(1) $\exists y \in B$ such that $g(y) = z$ Since $g$ is onto and Def. of onto on $g$
(2) $\exists x \in A$ such that $f(x) = y$ Since $f$ is onto and Def. of onto on $f$
$g\left(f(x)\right) = z$ Inf. (1), (2)
$(g \circ f)(x) = z$ Def. of $\circ$
$\therefore g \circ f$ is onto.

**(ii) Exercise.**

**Theorem 4.16.** Let $f : X \to Y$ be a function. Then $f$ is bijective iff the inverse relation $f^{-1}$ is a function from $B$ to $A$.

**Proof.**

Suppose $f : X \to Y$ is bijective. To prove $f^{-1}$ is a function from $B$ to $A$.

$(*)$ Let $(y_1, x_1)$ and $(y_2, x_2) \in f^{-1}$ such that $y_1 = y_2$, to prove $x_1 = x_2$.

| | |
|---|---|
| $(x_1, y_1)$ and $(x_2, y_2) \in f$ | Def. of $f^{-1}$ |
| $(x_1, y_1)$ and $(x_2, y_1) \in f$ | By hypothesis $(*)$ |
| $x_1 = x_2$ | Def. of 1-1 on $f$ |

$\therefore f^{-1}$ is a function from $B$ to $A$.

**Conversely,** suppose $f^{-1}$ is a function from $B$ to $A$, to prove $f : X \to Y$ is bijective; that is, 1-1 and onto.

**1-1:** Let $a, b \in X$ and $f(a) = f(b)$. To prove $a = b$.

| | |
|---|---|
| $(a, f(a))$ and $(b, f(b)) \in f$ | Hypothesis ($f$ is function) |
| $(a, f(a))$ and $(b, f(a)) \in f$ | Hypothesis ($f(a) = f(b)$) |
| $(f(a), a)$ and $(f(a), b) \in f^{-1}$ | Def. of inverse relation $f^{-1}$ |
| $a = b$ | Since $f^{-1}$ is function |

$\therefore f$ is 1-1.

**onto:** Let $b \in Y$. To prove $\exists a \in A$ such that $f(a) = b$.

| | |
|---|---|
| $(b, f^{-1}(b)) \in f^{-1}$ | Hypothesis ( $f^{-1}$ is a function from $B$ to $A$) |
| $(f^{-1}(b), b) \in f$ | Def. of inverse relation $f^{-1}$ |

Put $a = f^{-1}(b)$.

| | |
|---|---|
| $a \in A$ and $f(a) = b$ | Hypothesis ( $f$ is function) |

$\therefore f$ is onto.

**Definition 4.17.**

**(i)** A function $I_A : A \to A$ defined by $I_A(x) = x$, for every $x \in A$ is called the **identity** function on $A$. $I_A = \{(x, x): x \in A\}$.

**(ii)** Let $A \subseteq X$. A function $i_A : A \to X$ defined by $i_A(x) = x$, for every $x \in A$ is called the **inclusion** function on $A$.

**Definition 4.18. (Inverse function)**

If $f : X \to Y$ is a bijective function, then its **inverse** is the function $f^{-1} : Y \to X$ such that $f \circ f^{-1} = I_Y$ and $f^{-1} \circ f = I_X$.

**Example 4.19.** Let $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be a function defined as

$$f(m, n) = (m + n, m + 2n)$$

$f$ is bijective as shown in Example 4.11(ii).

To find the inverse $f^{-1}$ formula, let $f(n, m) = (x, y)$. Then

$(m + n, m + 2n) = (x, y)$. So, the we get the following system

$$\begin{aligned} m + n &= x \ldots.(1) \\ m + 2n &= y \ldots.(2) \end{aligned}$$

$m = x - n \quad \ldots.(3)$ Inf. (1)

$n = y - x \quad \ldots.(4)$ Inf. (2),(3)

$m = 2x - y$ Inf. (4),(3)

Define $f^{-1}$ as follows

$$f^{-1}(x, y) = (2x - y, y - x).$$

We can check our work by confirming that $f \circ f^{-1} = I_Y$.

$$\begin{aligned} (f \circ f^{-1})(x, y) &= f(2x - y, y - x) \\ &= ((2x - y) + (y - x), (2x - y) + 2(y - x)) \\ &= (x, 2x - y + 2y - 2x) = (x, y) = I_Y(x, y) \end{aligned}$$

**Remark 4.20.**

**(i)** If $f : X \to Y$ is oneto-one but not onto, then one can still define an inverse function $f^{-1} : R(f) \to X$ whose domain in the range of $f$.

**(ii)** If $f : X \to Y$ and $g : Y \to Z$ are bijective functions, then

**(a)** $(f^{-1})^{-1} = f$.

**(b)** $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .

**Theorem 4.21.** Let $f : X \to Y$ be a function.

**(i)** $If$ $\{Y_j \subset Y : j \in J\}$ is a collection of subsets of $Y$, then

$f^{-1}(\bigcup_{j\in J} Y_j) = \bigcup_{j\in J} f^{-1}(Y_j)$ and $f^{-1}\left(\bigcap_{j\in J} Y_j\right) = \bigcap_{j\in J} f^{-1}(Y_j)$

**(ii)** If $\{X_i \subset X : i \in I\}$ is a collection of subsets of $X$, then

$f(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} f(X_i)$ and $f(\bigcap_{i\in I} X_i) \subseteq \bigcap_{i\in I} f(X_i)$.

**Proof:** Let $x \in f^{-1}(\bigcup_{j\in J} Y_j)$.

$\exists y \in \bigcup_{j\in J} Y_j$ such that $f(x) = y$ Def. of inverse relation $f^{-1}$

$y \in Y_j$ for some $j \in J$ Def. of $\cup$

$x \in f^{-1}(Y_j)$ Def. of inverse $f^{-1}$

so $x \in \bigcup_{j\in J} f^{-1}(Y_j)$ Def. of $\cup$

It follow that $f^{-1}(\bigcup_{j\in J} Y_j) \subseteq \bigcup_{j\in J} f^{-1}(Y_j)$ Def. of $\subseteq$ …..$(*)$

**Conversely,** If $x \in \bigcup_{j\in J} f^{-1}(Y_j)$, then $x \in f^{-1}(Y_j)$, for some $j \in J$ Def. of $\cup$

So $f(x) \in Y_j$ and $f(x) \in \bigcup_{j\in J} Y_j$ Def. of inverse and $\cup$

$x \in f^{-1}(\bigcup_{j\in J} Y_j)$ Def. of inverse $f^{-1}$

It follow that $\bigcup_{j\in J} f^{-1}(Y_j) \subseteq f^{-1}(\bigcup_{j\in J} Y_j)$ Def. of $\subseteq$ …..$(**)$

$f^{-1}(\bigcup_{j\in J} Y_j) = \bigcup_{j\in J} f^{-1}(Y_j)$ From $(*), (**)$ and Def. of $=$

**Example 4.22.** Let $f: \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function defined as $f(x) = 1, \forall x \in \mathbb{Z}$.

$\mathbb{Z}_e \cap \mathbb{Z}_o = \emptyset$. $f(\mathbb{Z}_e \cap \mathbb{Z}_o) = f(\emptyset) = \emptyset$. But $f(\mathbb{Z}_e) \cap f(\mathbb{Z}_o) = 1$.