Lecture 11

Three Additional Equations

11.1 Closing the System of Equations

So far, we have taken three equations of the seven closed system of equations that govern the atmospheric dynamics. These three equations are the momentum equations in x-, y-, and z-direction. The four other equations are:

1. The gas equation
2. The thermodynamic equation
3. The continuity equation
4. The conservation law of water vapor substance.

11.2 The Gas Equation

A perfect gas (ideal gas) obeys the physical laws of Boyle and Charles. The gas equation (or equation of state) is:

$$P = \rho\, R\, T$$

where P is pressure ; ρ is density ; $R = 287\, J\, kg^{-1} K^{-1}$; T is temperature

Boyle's law:

$$P_1 V_1 = P_2 V_2 \qquad at\ constant\ \ T$$

Charles'law:

$$\frac{P_1}{T_1} = \frac{P_2}{T_2} \qquad \text{at constant}\ \ V$$

where V is volume. Combining the two laws we get:

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} = C$$

Where C is constant depends on the mass of gas and equal to $287\, J\, kg^{-1} K^{-1}$ (specific gas constant).

$$\frac{P\alpha}{T} = R \qquad where \quad \alpha = \frac{1}{\rho} \quad \text{is the specific volume}$$

Thus, $\quad P = \rho R T$

Note: *the specific volume of a substance is the ratio of the substance's volume to its mass and equal to the reciprocal of the density.*

Question: what is the difference between ideal gas and the real gas?

11.3 The Thermodynamic Equation

The thermodynamic energy equation comes from the first law of thermodynamics (conservation of energy):

$$dH = du + dw$$

where: dH is the amount of heat added to system per unit mass.

du is the change in energy per unit mass; $du = C_v\, dT$

dw is the work done by unit mass on a system ; $dw = p\, d\alpha$

$$dH = C_v\, dT + p\, d\alpha \qquad (11.1)$$

Differentiation of the equation of state ($P\alpha = RT$) gives:

$$P\, d\alpha + \alpha\, dP = R\, dT$$

$$P\, d\alpha = R\, dT - \alpha\, dP \qquad (11.2)$$

Substitute (11.2) in (11.1) we get:

$$dH = C_v\, dT + R\, dT - \alpha\, dP$$

$$dH = (C_v + R)\, dT - \alpha\, dP$$

Recall that $R = C_p - C_v \quad \Rightarrow \quad C_p = R - C_v$

where $C_p = 1004\, Jkg^{-1}K^{-1}$ and $C_v = 717\, Jkg^{-1}K^{-1}$ are the specific heat at constants pressure and volume, respectively.

Hence,

$$dH = C_p\, dT - \alpha\, dP \qquad (11.3)$$

Both of equations (11.1) and (11.3) represent the first law of thermodynamics.

11.4 Adiabatic Assumption

It is assumed that *dH=0* for most air parcel movements. This assumption can be made whenever the motion is fast so that the heat exchange between the parcel and the surroundings is negligible. (*Why?*)

For adiabatic motion, the equations of first law of thermodynamics becomes:

$$C_v\, dT + p\, d\alpha = 0 \qquad (1)$$

$$C_p\, dT - \alpha\, dP = 0 \qquad (2)$$

Solving for P in equation (1) from the equation of state ($P = \frac{1}{\alpha} R\, T$)

$$C_v\, dT + \frac{1}{\alpha} R\, T\, d\alpha = 0$$

$$C_v\, dT = -R\, T \frac{d\alpha}{\alpha}$$

$$C_v\, \frac{dT}{T} = -R\, \frac{d\alpha}{\alpha}$$

$$\int_{T_1}^{T} \frac{dT}{T} = -\frac{R}{C_v} \int_{\alpha_1}^{\alpha} \frac{d\alpha}{\alpha}$$

$$\ln\, (T - T_1) = -\frac{R}{C_v} (\ln(\alpha - \alpha_1))$$

Now, by taking the exponential (e) for the two sides:

$$\frac{T}{T_1} = (\frac{\alpha}{\alpha_1})^{-\frac{R}{C_v}} \qquad (4)$$

If T increases, α will decreases and vice versa.

From equation (2)

$$C_p\, dT - \alpha\, dP = 0$$

From state equation $P = \frac{1}{\alpha} R\, T$

or

$$\alpha = \frac{R\,T}{P} \qquad (5)$$

Substitute (5) in (2) we get:

$$C_p\,dT - \frac{R\,T}{P}\,dP$$

$$\int_{T_1}^{T} \frac{dT}{T} = -\frac{R}{C_v}\int_{P_1}^{P} \frac{dP}{P}$$

$$\ln\frac{T}{T_1} = \frac{R}{C_P}\ln\frac{P}{P_1}$$

$$\frac{T}{T_1} = (\frac{P}{P_1})^{\frac{R}{C_P}} \qquad (6)$$

If T increases P will increases and vice versa.

From equation (4) an Equation (6) we get:

$$(\frac{\alpha}{\alpha_1})^{-\frac{R}{C_v}} = (\frac{P}{P_1})^{\frac{R}{C_P}}$$

$$\frac{\alpha}{\alpha_1} = (\frac{P}{P_1})^{-\frac{C_v}{C_P}}$$

An increase in P corresponds to a decrease in α and vice versa