FOUNDATION OF MATHEMATICS I

CHAPTER THREE (CROSS PRODUCT AND RELATIONS)

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3.2 Relations

Definition 3.2.1. Any subset " R" of $A \times B$ is called a **relation between A and B** and denoted by $R(A,B)$. Any subset of $A \times A$ is called a **relation on A**.

In other words, if A is a set, any set of ordered pairs with components in A is a relation on A. Since a relation R on A is a subset of $A \times A$, it is an element of the power set of $A \times A$; that is, $R \subseteq P(A \times A)$.

If R is a relation on A and $(x,y) \in R$, then we write $\boldsymbol{xRy}$, read as "x is in R-relation to y", or simply, x is in relation to y, if R is understood.

Example 3.2.2.

(i) Let $A = \{2,4,6,8\}$, and define the relation R on A by $(x,y) \in R$ iff x divides y. Then, $R = \{(2,2),(2,4),(2,6),(2,8),(4,4),(4,8),(6,6),(8,8)\}$.

(ii) Let $A = \mathbb{N}$, and define $R \subseteq A \times A$ by xRy iff x and y have the same remainder when divided 3.

Since A is infinite, we cannot explicitly list all elements of R; but, for example $(1,4),(1,7),(1,10),\ldots,(2,5),(2,8),\ldots,(0,0),(1,1),\ldots \in R$. Observe, that xRx for $x \in N$ and, whenever xRy then also yRx.

(iii) Let $A = \mathbb{R}$, and define the relation R on $\mathbb{R}$ by xRy iff $y = x^2$. Then R consists of all points on the parabola $y = x^2$.

(iv) Let $A = \mathbb{R}$, and define R on $\mathbb{R}$ by xRy iff $x \cdot y = 1$. Then R consists of all pairs $(x, \frac{1}{x})$, where x is non-zero real number.

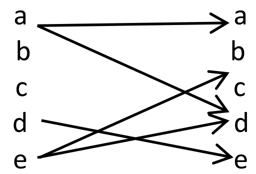
(v) Let $A = \{1,2,3\}$, and define R on A by xRy iff $x + y = 7$. Since the sum of two elements of A is at most 6, we see that xRy for no two elements of A; hence, $R = \emptyset$.

For small sets we can use a pictorial representation of a relation R on A: Sketch two copies of A and, if xRy then draw an arrow from the x in the left sketch to the y in the right sketch.

(vi) Let $A = \{a,b,c,d,e\}$, and consider the relation

$$R = \{(a,a),(a,c),(c,a),(d,b),(d,c)\}.$$

An arrow representation of R is given in Fi



(vii) Let A be any set. Then the relation $R = \{(x,x): x \in A\} = I_A$ on A is called the **identity relation on A**. Thus, in an identity relation, every element is related to itself only.

Definition 3.2.3. Let R be a relation on A. Then

(i) $\text{Dom}(R) = \{x \in A :$ There exists some $y \in A$ such that $(x,y) \in R\}$ is called the **domain of R**.

(ii) $\text{Ran}(R) = \{y \in A$: There exists some $x \in A$ such that $(x,y) \in R\}$ is called the **range of R**.

Observe that $\text{Dom}(R)$ and $\text{Ran}(R)$ are both subsets of A.

Example 3.2.4.

(i) Let A and R be as in Example 3.2.2.(vi). Then
$\text{Dom}(R) = \{a, c, d\}$, $\text{Ran}(R) = \{a, b, c, d\}$.

(ii) Let $A = R$, and define R by xRy iff $y = x^2$. Then
$\text{Dom}(R) = R$, $\text{Ran}(R) = \{y \in R : y \geq 0\}$.

(iii) Let $A = \{1,2,3,4,5,6\}$, and define R by xRy iff $x \lneqq y$ and x divides y; $R = \{(1,2),(1,3),\dots,(1,6),(2,4),(2,6),(3,6)\}$, and $\text{Dom}(R) = \{1,2,3\}$, $\text{Ran}(R) = \{2,3,4,5,6\}$.

(iv) Let $A = \mathbb{R}$, and R be defined as $(x,y) \in R$ iff $x^2 + y^2 = 1$. Then $(x,y) \in R$ iff (x,y) is on the unit circle with centre at the origin. So,

$$\text{Dom}(R) = \text{Ran}(R) = \{z \in \mathbb{R}: -1 \leq z \leq 1\}.$$

Definition 3.2.5. (Reflexive, Symmetric and Transitive Relations)

Let R be a relation on a nonempty set A.

(i) R is **reflexive** if $(x,x) \in R$ for all $x \in A$.

(ii) R is **antisymmetric** if for all $x, y \in A$, $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$.

(iii) R is **transitive** if for all $x, y, z \in A$, $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$.

(iv) R is **symmetric** if whenever $(x,y) \in R$ then $(y,x) \in R$.

Definition 3.2.6.
(i) R is an **equivalence relation** A, if R is reflexive, symmetric, and transitive.
The set

$$[x] = \{y \in A: xRy\}$$

is called **equivalence class**. The set of all different equivalence classes A/R is called the **quotient set**.
(ii) R is a **partial order** on A(an **order** on A, or an **ordering** of A), if R is reflexive, antisymmetric, and transitive. We usually write $\leq$ for R; that is,

$$\boxed{x \leq y \text{ iff } xRy}.$$

(iii) If R is a **partial order** on A, then the element $a \in A$ is called **least element of *A* with respect to *R*** if and only if aRx for all $x \in A$.
(iv) If R is a **partial order** on A, then the element $a \in A$ is called **greatest element of *A* with respect to *R*** if and only if xRa for all $x \in A$.
(v) If R is a **partial order** on A, then the element $a \in A$ is called **minimal element of *A* with respect to *R*** if and only if xRa then $a = x$ for all $x \in A$.
(vi) If R is a **partial order** on A, then the element $a \in A$ is called **maximal element of *A* with respect to *R*** if and only if aRx then $a = x$ for all $x \in A$.

Example 3.2.7.

(i) The relation on the set of integers $\mathbb{Z}$ defined by

$$\boxed{(x,y) \in R \text{ if } x - y = 2k, \qquad \text{for some } k \in \mathbb{Z}}$$

is an equivalence relation, and partitions the set integers into two equivalence classes, i.e., the even and odd integers.
If $y = 0$, then $[x] = \mathbb{Z}_e$. If $y = 1$, then $[x] = \mathbb{Z}_o$. $\mathbb{Z} = \mathbb{Z}_e \cup \mathbb{Z}_o$, $\mathbb{Z}/R$={$\mathbb{Z}_e, \mathbb{Z}_o$}.

(ii) The inclusion relation $\subseteq$ is a partial order on power set $P(X)$ of a set X.
(iii) Let $A = \{3,6,7\}$, and

$$R_1 = \{(x,y) \in A \times A: x \leq y\}, R_2 = \{(x,y) \in A \times A: x \geq y\}$$
$$R_3 = \{(x,y) \in A \times A: y \text{ divisible by } x\}$$

are relations defined on A.

$$R_1 = \{(3,3), (3,6), (3,7), (6,6), (6,7), (7,7)\},$$

$$R_2 = \{(3,3),(6,3),(6,6),(7,3),(7,6),(7,7)\}.$$
$$R_3 = \{(3,3),(3,6),(6,6),(7,7)\}.$$

R_1,R_2 and R_3 are partial orders on A.

(1) The least element of A with respect to R_1 is --------------.
(2) The least element of A with respect to R_2 is --------------.
(3) The greatest element of A with respect to R_1 is --------------.
(4) The greatest element of A with respect to R_2 is --------------.
(5)A has no least and greatest element with respect to R_3 since, --------------.
(6) The maximal element of A with respect to R_3 is --------------.
(7) The minimal element of A with respect to R_3 is --------------.

(iv) Let $X = \{1,2,4,7\}$, $K = \{\{1,2\},\{4,7\},\{1,2,4\},X\}$ and

$$R_1 = \{(A,B) \in K \times K : A \subseteq B\},$$
$$R_2 = \{(A,B) \in K \times K : A \supseteq B\},$$

are relations defined on K.

$$R_1 = \begin{matrix} (\{1,2\},\{1,2\}), & (\{1,2\},\{1,2,4\}), & (\{1,2\},X), \\ (\{4,7\},\{4,7\}), & (\{4,7\},X), & \\ (\{1,2,4\},\{1,2,4\}), & (\{1,2,4\},X), & \\ (X,X) & & \end{matrix}$$

$$R_2 = \begin{matrix} (\{1,2\},\{1,2\}), & & & \\ (\{4,7\},\{4,7\}), & & & \\ (\{1,2,4\},\{1,2\}), & (\{1,2,4\},\{1,2,4\}), & & \\ (X,\{1,2\}), & (X,\{4,7\}), & (X,\{1,2,4\}), & (X,X) \end{matrix}$$

R_1 and R_2 are partial orders on K.

(1)K has no least element with respect to R_1 since, --------------.
(2) The greatest element of K with respect to R_1 is --------------.
(3) The least element of K with respect to to R_2 is --------------.
(4)K has no greatest element with respect to R_2 since, --------------.
(5) The minimal elements of K with respect to R_1 are --------------.

(6) The maximal element of K with respect to R_1 is --------------.
(7) The minimal element of K with respect to R_2 is --------------.
(8) The maximal element of K with respect to R_2 is --------------.

Remark 3.2.8.
(i) Every greatest (least) element is maximal (minimal). The converse is not true.
(ii) The greatest (least) element if exist, it is unique.
(iii) every finite partially ordered set has maximal (minimal) element.
Properties of equivalence classes
(iv) For all $a \in X$, $a \in [a]$.
(v) $aRb \Leftrightarrow [a] = [b]$.
(vi) $[a] = [b] \Leftrightarrow (a, b) \in R \Leftrightarrow aRb$.
(vii) $[a] \cap [b] \neq \emptyset \Leftrightarrow [a] = [b]$.
(viii) $[a] \cap [b] = \emptyset \Leftrightarrow [a] \neq [b]$.
(ix) For all $a \in X$, $[a] \in X/R$ but $[a] \subseteq X$.