Lecture 9

The Components of Motion Equation

9.1 The Primary Equation

The equation of motion is:

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla \mathrm{p} + \vec{g}_* + \vec{\mathrm{F}}_r - 2\vec{\Omega} \times \vec{V} + \Omega^2\vec{R}$$

Each of $\vec{\mathrm{g}}_*$ *and* $\Omega^2\vec{R}$ are the only forces which depend solely on position, hence the gravity is:

$$\vec{g} = \vec{g}_* + \Omega^2\vec{R}$$

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla \mathrm{p} + \vec{g} + \vec{\mathrm{F}}_r - 2\vec{\Omega} \times \vec{V} \qquad (9.1)$$

Now we are going to find the x, y and z components of each term in Equation (9.1):

A. The left side

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}i + \frac{dv}{dt}j + \frac{dw}{dt}k \qquad (A)$$

B. The pressure gradient force

$$-\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\frac{\partial p}{\partial x}i - \frac{1}{\rho}\frac{\partial p}{\partial y}j - \frac{1}{\rho}\frac{\partial p}{\partial z}k \qquad (B)$$

C. Gravity force

$$\vec{g} = -gk \qquad (C)$$

D. Viscosity force

$$\vec{\mathrm{F}}_r = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)i - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)j - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)k \qquad (D)$$

E. Coriolis Force

$$-2\vec{\Omega} \times \vec{V} = -2\Omega \begin{vmatrix} i & j & k \\ 0 & \cos\phi & \sin\phi \\ u & v & w \end{vmatrix}$$

$$-2\vec{\Omega} \times \vec{V} = -(2\Omega w \cos\phi - 2\Omega v \sin\phi)i - 2\Omega u \sin\phi\, j + 2\Omega u \cos\phi\, k \qquad (E)$$

Now put each of the equations A, B, C, D, and E in equation (9.1):

$$\frac{du}{dt}i + \frac{dv}{dt}j + \frac{dw}{dt}k$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial x}i - \frac{1}{\rho}\frac{\partial p}{\partial y}j - \frac{1}{\rho}\frac{\partial p}{\partial z}k - gk - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)i - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)j$$
$$-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)k - (2\Omega w \cos\phi - 2\Omega v \sin\phi)i - 2\Omega u \sin\phi\, j + 2\Omega u \cos\phi\, k$$

If we split this equation into the three directions, we get:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) + 2\Omega w \cos\phi - 2\Omega v \sin\phi \qquad x - direction$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) - 2\Omega u \sin\phi \qquad y - direction$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) - g + 2\Omega u \cos\phi\, k \qquad z - direction$$

8.1 The Order of Magnitude (or Scale Analysis)

To assess which terms can be neglected, we use "order of magnitude" method to all variables and parameters in the equations.

For synoptic scale, we can apply the values in Table (9.1)

Table (9.1)

Variable or parameter	Symbol	approximated typical value
horizontal velocity	u	10 m/sec
vertical velocity	w	0.01 m/sec
horizontal distance	L	1000 km = 10^6 m
vertical distance	H	10 km = 10^4 m
horizontal pressure	Δp	10 mb = 10 hPa = 1000 Pa
density	ρ	1 kg/m^3
time	$\frac{L}{u}$	1 day (10^5 sec)
viscosity	μ	1.46 ×10^{-5} m^2/sec
angular velocity	Ω	7.3 × 10^{-5} sec $^{-1}$
latitude	ϕ	45^o
Distance from axis of Earth	R	≈ 3×10^6 m

Using these scales and parameters in the momentum equation in x-direction we have the following order of magnitude:

$$\frac{du}{dt}=\frac{u^2}{L}=\frac{10^2}{10^6}=10^{-4}\quad m/sec^2$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\Delta p}{\mathrm{\rho\ L}}=\frac{10^3}{1\times 10^6}=10^{-3}\ m/sec^2$$

Similarly,

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)=10^{-12}\ m/sec^2$$

$$2\Omega v\sin\phi=10^{-3}\ m/sec^2$$

$$2\Omega w\cos\phi=10^{-6}\ \ m/sec^2$$

We made the same analysis for y and z equations.

Many terms are very small compared to others and can therefore be ignored without much loss of accuracy. We can therefore ignore the viscous terms and Coriolis term that involves the vertical velocity.

Thus the components of equation of motion are:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega v \sin\phi$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi$$

$$\frac{\partial p}{\partial z} = -\rho g$$

- The first two are the semi-geostrophic equations; the last one is the hydrostatic equation.
- The hydrostatic equation expresses the balance between gravity acting downward and the vertical component of the pressure force acting upwards.
- This system of equations is quite often used in analysis of dynamical problems.