Lecture 3: Full Derivation of the Rotating Coordinates

Q By Using the total derivative of a vector in a rotating system relationship, derive the equation of momentum in the dynamic meteorology.

Sol

$$\underbrace{\frac{d_a\vec{A}}{dt}}_{\text{absolute system}} = \underbrace{\frac{d\vec{A}}{dt}}_{\text{rotational system}} + \vec{\Omega}\times\vec{A} \quad \ldots(1)$$ Total Der. of a vector in a Rot. sys.

$$\frac{d_a\vec{V}_a}{dt} = \sum\vec{F} \quad \ldots(2)$$ Newton's 2nd law of motion in an absolute reference

Now, Apply eqn 1 to a position vector $(\vec{r})$

$$\frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r} \quad \ldots(3)$$

$$\therefore \quad \vec{V}_a = \underbrace{\vec{V}}_{\text{relative to coord.}} + \underbrace{\vec{\Omega}\times\vec{r}}_{\text{velocity of the coord. itself}} \quad \ldots(4)$$

Now Apply eqn 1 to the velocity vector $\vec{V}_a$

$$\left(\frac{d_a\vec{V}_a}{dt}\right) = \left(\frac{d\vec{V}_a}{dt}\right) + \vec{\Omega}\times\vec{V}_a \quad \ldots(5)$$

substitute eqn 4 in eqn 5,

$$\left(\frac{d_a\vec{V}_a}{dt}\right) = \frac{d}{dt}\left[\vec{V}+\vec{\Omega}\times\vec{r}\right] + \vec{\Omega}\times\left[\vec{V}+\vec{\Omega}\times\vec{r}\right]$$

$$= \left(\frac{d\vec{V}}{dt}\right) + \frac{d}{dt}(\vec{\Omega}\times\vec{r}) + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$= \left(\frac{d\vec{V}}{dt}\right) + \underbrace{\left(\frac{d\vec{\Omega}}{dt}\times\vec{r}\right)}_{=0 \text{ (why?)}} + \left(\vec{\Omega}\times\frac{d\vec{r}}{dt}\right) + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\therefore \left(\frac{d_a\vec{V}_a}{dt}\right) = \left(\frac{d\vec{V}}{dt}\right) + \left(\vec{\Omega}\times\frac{d\vec{r}}{dt}\right) + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

Thus

$$= \left(\frac{d\vec{V}}{dt}\right) + 2(\vec{\Omega}\times\vec{V}) + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

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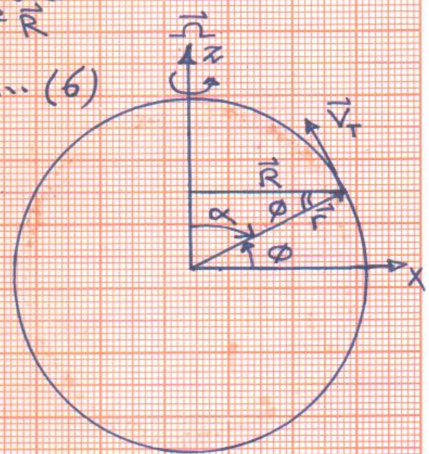
To be continued...

By using a vector triple product :

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

Thus $$\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = \underbrace{(\vec{\Omega}\cdot\vec{r})\vec{\Omega} - (\vec{\Omega}\cdot\vec{\Omega})\vec{r}}$$
$$= -\Omega^2\vec{R}$$

$$\therefore \left(\frac{d_a\vec{V}_a}{dt}\right) = \left(\frac{d\vec{V}}{dt}\right) + 2(\vec{\Omega}\times\vec{V}) - \Omega^2\vec{R} \quad \cdots (6)$$



Using eqn 2

$$\therefore \frac{d\vec{V}_a}{dt} = \frac{d\vec{V}}{dt} + 2(\vec{\Omega}\times\vec{V}) - \Omega^2\vec{R} = \sum\vec{F}$$

$|\vec{R}| = |\vec{r}|\cos\phi$

Tangetial velocity $\vec{V}_T = \vec{\Omega}\times\vec{r}$

$$|\vec{V}_T| = |\vec{\Omega}|R$$
$$= |\vec{\Omega}||\vec{r}|\sin\alpha$$
$$= |\vec{\Omega}\times\vec{r}|$$

If the only forces acting on the atmosphere are :

1. pgf
2. gravitation
3. Friction

we rewrite Newton's 2nd law with the aid of eqn 6 .

$$\frac{d\vec{V}}{dt} = \underbrace{-2\vec{\Omega}\times\vec{V}}_{\text{Coriolis force}} \underbrace{-\frac{1}{\rho}\nabla p}_{\text{pressure gradient force (pgf)}} + \underbrace{\vec{g}}_{\text{gravity}} + \underbrace{\vec{F}_r}_{\text{Frictional Force}} \quad \cdots (7)$$

gravity
- centrifugal force
- gravitation

This form of the momentum equation is the basic to most work in dynamic meteorlogy.

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