Coding Theory

Sheet 7 Solutions

Spring 2014

1. Let C' be the set of words of even weight in the binary linear code C. Then, by Sheet 6, Exercise 5, the sum of two words of even weight also has even weight, and so C' is a linear code. Let x be any word of odd weight in C, if it exists. Then, if y is any other word of odd weight, $x+y$ has even weight and so is in C'; that is, $y \in x + C'$. Hence
$$C = C' \cup (x + C'),$$
in which case $|C| = 2|C'|$. So, either $C' = C$ or $|C'| = \frac{1}{2}|C|$.

2. Let C be a binary $[n,k]$ code. Since
$$W_{C^\perp}(T) = 2^{-k}\,(1+T)^n W_C\left(\frac{1-T}{1+T}\right),$$
so replacing C by $C^\perp$ gives
$$W_C(T) = 2^{-(n-k)}\,(1+T)^n W_{C^\perp}\left(\frac{1-T}{1+T}\right).$$

3. (a) Since
$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$
so
$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix},$$
(b) The elements of C and their weights are as follows:
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & & 3 \\ 0 & 1 & 0 & 0 & 1 & 1 & & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 & & 3 \\ 1 & 1 & 0 & 1 & 0 & 1 & & 4 \\ 0 & 1 & 1 & 1 & 0 & 1 & & 4 \\ 1 & 0 & 1 & 0 & 1 & 1 & & 4 \\ 1 & 1 & 1 & 0 & 0 & 0 & & 3 \end{bmatrix}.$$

So $W_C(T) = 1 + 4T^3 + 3T^4$.

(c) Applying the MacWilliams theorem gives

$$\begin{aligned} W_C(T) &= 2^{-3}\,(1+T)^6 W_{C^\perp}\left(\frac{1-T}{1+T}\right) \\ &= \tfrac{1}{8}(1+T)^6\left\{1 + 4\left(\tfrac{1-T}{1+T}\right)^3 + 3\left(\tfrac{1-T}{1+T}\right)^4\right\} \\ &= \tfrac{1}{8}\left\{(1+T)^6 + 4(1+T)^3(1-T)^3 + 3(1+T)^2(1-T)^4\right\}. \end{aligned}$$

Now, this can be evaluated in various ways. Write the coefficients of the various terms:

$(1+T)^6$	1	6	15	20	15	6	1

$4(1+T)^3(1-T)^3 = 4(1-T^2)^3$

$(1-T^2)^3$	1	0	-3	0	3	0	-1
$4(1-T^2)^3$	4	0	-12	0	12	0	-4

Similarly,

$3(1+T)^2(1-T)^4$	1	-4	6	-4	1		
		2	-8	12	-8	2	
			1	-4	6	-4	1
	1	-2	-1	4	-1	-2	1
$\times 3$	3	-6	-3	12	-3	-6	3
	4	0	-12	0	12	0	-4
	1	6	15	20	15	6	1
	8	0	0	32	24	0	0
$\div 8$	1	0	0	4	3	0	0

Hence

$$W_C^\perp(T) \quad = \quad 1 + 4T^3 + 3T^4.$$

As a check, $W_{C\perp}(1) = 8 = 2^3$.

(d) The elements of $C^\perp$ and their weights are as follows:

0	0	0	0	0	0	0
1	0	1	1	0	0	3
1	1	0	0	1	0	3
0	1	1	0	0	1	3
0	1	1	1	1	0	4
1	1	0	1	0	1	4
1	0	1	0	1	1	4
0	0	0	1	1	1	3

So $W_C^\perp(T) = 1 + 4T^3 + 3T^4$, in agreement with the previous calculation.

(e) Thus $W_C^{\perp}(T) = W_C(T)$. However, $C^{\perp} \neq C$; but $C^{\perp}$ is equivalent to C as the columns of H are a permutation of the columns of G.

4. Since C is a $[10, 7]$ code, so $C^{\perp}$ is a $[10, 3]$ code. Its elements with their weights are as follows:

	1	0	0	1	0	1	1	1	1	1	7
	0	1	0	1	1	1	1	0	1	1	7
	0	0	1	0	1	1	0	1	1	0	5
$C^{\perp}$	1	1	0	0	1	0	0	1	0	0	4
	1	0	1	1	1	0	1	0	0	1	6
	0	1	1	1	0	0	1	1	0	1	6
	1	1	1	0	0	1	0	0	1	0	5
	0	0	0	0	0	0	0	0	0	0	0

Hence

$$W_{C^{\perp}}(T) = 1 + T^4 + 2T^5 + 2T^6 + 2T^7.$$

Applying the MacWilliams theorem gives

$$\begin{aligned} W_C(T) &= 2^{-3}\,(1+T)^{10} W_{C^{\perp}}\left(\frac{1-T}{1+T}\right) \\ &= \tfrac{1}{8}(1+T)^3\left\{(1+T)^7 + (1+T)^3(1-T)^4 + 2(1+T)^2(1-T)^5 \right. \\ &\qquad \left. +2(1+T)(1-T)^6 + 2(1-T)^7\right\} \end{aligned}$$

The last four terms sum to

$$\begin{aligned} & (1-T)^4\{(1+T)^3 + 2(1+T)^2(1-T) + 2(1+T)(1-T)^2 + 2(1-T)^3\} \\ = \; & (1-T)^4(7 - 3T + 5T^2 - T^3) \end{aligned}$$

Now, just writing the coefficients gives

	1	−4	6	−4	1			
	7	−28	42	−28	7			
		−3	12	−18	12	−3		
			5	−20	30	−20	5	
				−1	4	−6	4	−1
	7	−31	59	−67	53	−29	9	−1
Putting in the term $(1+T)^7$:								
	1	7	21	35	35	21	7	1
	8	−24	80	−32	88	−8	16	0
Dividing by 8:								
	1	−3	10	−4	11	−1	2	0

Multiply this by $(1+T)^3$:

1	−3	10	−4	11	−1	2	0	0	0	0
	3	−9	30	−12	33	−3	6	0	0	
		3	−9	30	−12	33	−3	6	0	
			1	−3	10	−4	11	−1	2	0
1	0	4	18	26	30	28	14	5	2	0

Hence

$$W_C(T) = 1+4T^2+18T^3+26T^4+30T^5+28T^6+14T^7+5T^8+2T^9.$$

As a check, $W_C(1)=128=2^7$.

The weight distribution of C is $(1,0,4,18,26,30,28,14,5,2,0)$.

5. (a) From Sheet 6, Exercise 7, all non-zero elements of to $C^\perp$ have weight $2^{r-1}=(n+1)/2$. As $C^\perp$ is a $[2^r-1,r]$ code, so

$$W_{C^\perp}(T) = 1+(2^r-1)T^{2^{r-1}} = 1+nT^{(n+1)/2}.$$

(b)

$$\begin{aligned} W_C(T) &= 2^{-r}\,(1+T)^n W_{C^\perp}\left(\frac{1-T}{1+T}\right) \\ &= \frac{1}{n+1}(1+T)^n\left\{1+n\left(\frac{1-T}{1+T}\right)^{(n+1)/2}\right\} \\ &= \frac{1}{n+1}\{(1+T)^n + n(1+T)^{(n-1)/2}(1-T)^{(n+1)/2}\} \end{aligned}$$

(c) $C=\mathrm{Ham}(r,q)$ is an

$$\left[n=\frac{q^r-1}{q-1},\ k=n-r,\ 3\right]$$

code.

Let $H=[h_1,\dots,h_r]^{\mathrm{T}}$ be a parity-check matrix of C with rows $h_1,\dots,h_r$, and let $h=\sum\lambda_i h_i$ be an element of $C^\perp$. If $(x_1,\dots,x_r)^{\mathrm{T}}$ is the j-th column of H, then the j-th coordinate of h is zero if $\sum\lambda_i x_i=0$. However, the number of columns $(x_1,\dots,x_r)$ that are solutions of $\sum\lambda_i x_i=0$ is the number N of points of $PG(r-1,q)$ in a subspace of dimension $r-2$. Hence $N=\frac{q^{r-1}-1}{q-1}$. So

$$\begin{aligned} w(h)=n-N &= \frac{q^r-1}{q-1}-\frac{q^{r-1}-1}{q-1} \\ &= \frac{q^r-q^{r-1}}{q-1} \\ &= q^{r-1}. \end{aligned}$$

So

$$\overline{W}_{C^\perp}(X,Y) = X^{\frac{q^r-1}{q-1}} + (q^r-1)X^{\frac{q^{r-1}-1}{q-1}}Y^{q^{r-1}},$$

and

$$\begin{aligned}\overline{W}_C(X,Y) &= q^{-r}\overline{W}_{C^\perp}(X+(q-1)Y, X-Y) \\ &= q^{-r}\left\{[X+(q-1)Y]^{\frac{q^r-1}{q-1}} + (q^r-1)[X+(q-1)Y]^{\frac{q^{r-1}-1}{q-1}}(X-Y)^{q^{r-1}}\right\}.\end{aligned}$$

6. (a) The eight codewords y, $x+ty$ of C for $t \in \mathbf{F}_7$ are as follows:

1	0	4	2	3	6	x	5
0	1	4	6	5	2	y	5
1	1	1	1	1	1	$x+y$	6
1	2	5	0	6	3	$x+2y$	5
1	3	2	6	4	5	$x+3y$	6
1	4	6	5	2	0	$x+4y$	5
1	5	3	4	0	2	$x+5y$	5
1	6	0	3	5	4	$x+6y$	5

(b) Every non-zero word in C is $\lambda x + \mu y = \lambda[x + (\mu/\lambda)y]$, when $\lambda \neq 0$, or μy when $\lambda = 0$. Hence every non-zero word in C is a multiple of one of the words in (a). So

$$A_0 = 1, \quad A_5 = 6 \times 6 = 36, \quad A_6 = 6 \times 2 = 12.$$

(c) From (b), $\overline{W}_C(X,Y) = X^6 + 36XY^5 + 12Y^6$.

(d) By the MacWilliams formula,

$$\begin{aligned}\overline{W}_{C^\perp}(X,Y) &= \frac{1}{49}\overline{W}_C(X+6Y, X-Y) \\ &= \frac{1}{49}\left[(X+6Y)^6 + 36(X+6Y)(X-Y)^5 + 12(X-Y)^6\right] \\ &= X^6 + 120X^3Y^3 + 360X^2Y^4 + 972XY^5 + 948Y^6.\end{aligned}$$

Note that $\overline{W}_{C^\perp}(1,1) = 2401 = 7^4$.
In more detail,

$$6^2 = 36, \quad 6^2 = 36, \quad 6^3 = 216, \quad 6^4 = 1296, \quad 6^5 = 7776, \quad 6^6 = 46656.$$

$(X+Y)^6$	1	6	15	20	15	6	1
$(X+6Y)^6$	1	36	540	4320	19440	46656	46656
$(X-Y)^5$	1	-5	10	-10	5	-1	
$(X+6Y)(X-Y)^5$	1	-5	10	-10	5	-1	0
		6	-30	60	-60	30	-6
	1	1	-20	50	-55	29	-6
$\times 36$	36	36	-720	1800	-1980	1044	-216
$12(X-Y)^6$	12	-72	180	-240	180	-72	12
$(X+6Y)^6$	1	36	540	4320	19440	46656	46656
	49	0	0	5880	18140	47628	46452
$\div 49$	1	0	0	120	360	972	948