Action of Group on The Projective Plane Over Finite Fields

❖ Introduction

1. $GF(q)$ denote the Galois field of q elements.
2. $V(3,q) = \{(a_1, a_2, a_3) | a_i \in GF(q)\}$ be the respective vector space of row vectors of length three with entries in $GF(q)$.
3. $PG(2,q)$ be the projective plane over the field $GF(q)$.

The number of points.
The number of lines in $PG(2,q)$ is $q^2 + q + 1$.
There are $q + 1$ points on every line.
There are $q + 1$ lines passes through a point.
Companion Matrix

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix}$$

The points are $P(i) = [1,0,0]T^{i-1}$ and the lines are $\ell_i = \ell_1 T^{i-1}$, $i = 1, \dots, q^2 + q + 1$ where $\ell_1 = V(X_2)$ be the line passing through points $P(X_0, X_1, X_2)$ with $X_2 = 0$

Definition An n-arc K or arc of degree 2 in $PG(2,q)$ with $n \geq 3$ is a set of n points with property that every lines meets K in at most two points and there is some lines meeting K in exactly two points.

Definition A line ℓ of $PG(2,q)$ is an i-secant of an n-arc K if $|\ell \cap K| = i$. A 2-secant is called a *bisecant*, a 1-secant a *unisecant* and a 0-secant is an *external line*.

A_n = Alternating group of degree n.

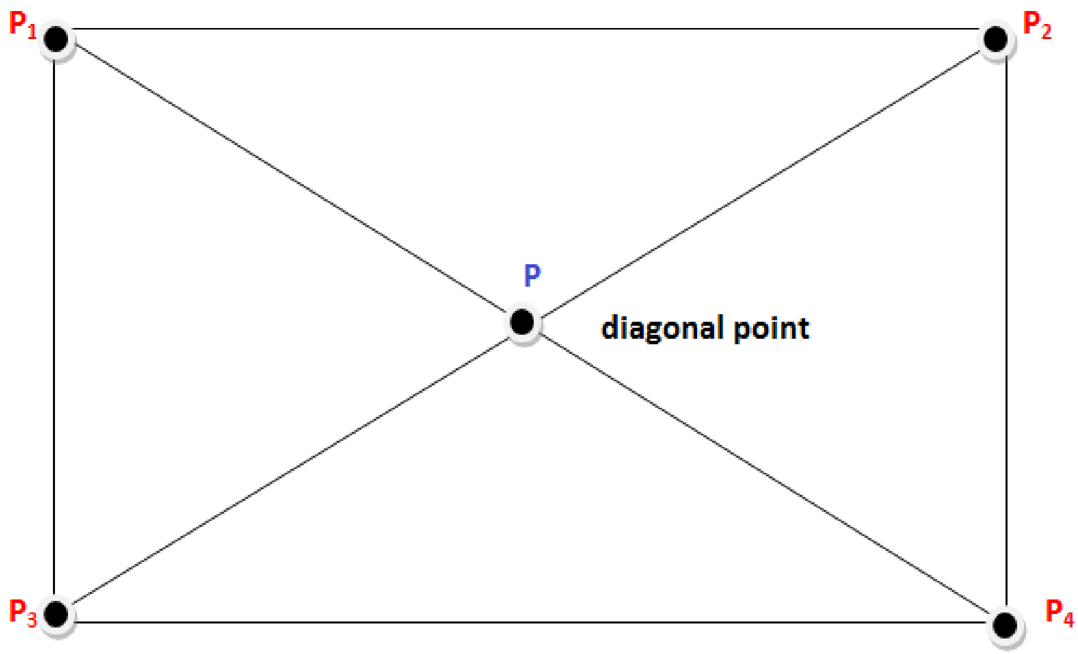
D_n = Dihedral group of order $2n = \langle r, s | r^n = s^2 = (rs)^2 = 1 \rangle$.

For details and full descriptions about above groups of order less than 32 see [3].

❖ Pentastigm with Collinearities of its Diagonal Points

Definition An n-stigm K in $PG(2,q)$ is a set of n points, no three of which are collinear, together with the $\frac{1}{2}n(n-1)$ lines that are joins of pairs of the points. The points and lines are called vertices and sides of K.

The intersection points of two sides of K which do not pass through the same vertex is called *diagonal points*.



- ❖ Since the vertices of K form an n-arc, so, to construct a 5-stigm, started with unique projectively 4-arc, $\Gamma_{41} = \{U_0, U_1, U_2, U\}$ (*standard frame*) in the projective plane which has stabilizers group isomorphic to S_4, where
  $U_0 = [1,0,0],\ U_1 = [0,1,0],\ U_2 = [0,0,1], U = [1,1,1]$.

The condition to existence a pentastigm with five diagonal points are collinear in $PG(2,q)$ is that $x^2 - x - 1 = 0$ has solution in F_q.

1. If $q = 19$, the equation $x^2 - x - 1 = 0$ has two solutions $5, -4$.
2. If $q = 29$, the equation $x^2 - x - 1 = 0$ has two solutions $6, -5$.
3. If $q = 31$, the equation $x^2 - x - 1 = 0$ has two solutions $13, -12$.
4. If $q = 41$, the equation $x^2 - x - 1 = 0$ has two solutions α^{21}, α^{39}

Theorem In $PG(2,q)$, the pentastigm which has the 5-arc $\mathcal{A}_i$

1. $|\mathcal{A}_{19} = \Gamma_{41} \cup \{P(-5\,,-4\,,1)\}$,
2. $\mathcal{A}_{29} = \Gamma_{41} \cup \{P(v^{22}\,,v^{16}\,,1)\}$,
3. $\mathcal{A}_{31} = \Gamma_{41} \cup \{P(w^{4}\,,w^{27}\,,1)\}$,
4. $\mathcal{A}_{41} = \Gamma_{41} \cup \{P(\alpha^{39}\,,\alpha\,,1)\}$,

as vertices has five diagonal points which are collinear on the line

1. If $q = 19$, $\ell = V(-X_0 + 5X_1 + X_2)$,
2. If $q = 41$, $\ell = V(X_0 - X_1 - 5X_2)$,
3. If $q = 41$, $\ell = V(X_0 + 19X_1 + 12X_2)$,
4. If $q = 41$, $\ell = V(\alpha^{21}X_0 - X_1 + X_2)$.

❖ Action of D_5 on $PG(2,41)$

$$C_{\mathcal{A}_{41}} = \quad V(X_0X_1 + \alpha^{20}X_0X_2 - \alpha^{20}X_1X_2)$$

The Dihedral group D_5 generated by

$$r = \begin{bmatrix} \alpha & 0 & 0 \\ 1 & 1 & 1 \\ \alpha^{19} & \alpha^{12} & \alpha^{20} \end{bmatrix}, s = \begin{bmatrix} 0 & \alpha^{22} & 0 \\ \alpha^{19} & \alpha^{21} & \alpha^{20} \\ 1 & 1 & 1 \end{bmatrix}$$

which stabilized the 5-arc $\mathcal{A}_{41}$ has the following effects on the points of $PG(2,41)$.

1- Fixes the conic $C_{\mathcal{A}_{41}}$.

2- Acts transitively on $\mathcal{A}_{41}$ since

$$(U_0, rs) \mapsto U_1,$$

$$(U_0, rs^3) \mapsto U_2,$$

$$(U_0, rs^4) \mapsto U,$$

$$(U_0, rs^2) \mapsto \; P(\alpha^{39}, \alpha, 1) = 112.$$

3- The elements of D_5 divided into two classes according to fixing points of $PG(2,41)$ by sending each point to itself as illustrated bellow.

Class 1: The five elements r, rs, rs^2, rs^3, rs^4 of order two fixes 43 points if acts on $PG(2,41)$ which is exactly line plus the diagonal point of $\mathcal{A}_{41}$

	$\ell_i \cup P_j$
r	$\ell_{320} \cup P(1, \alpha^2, 1)$
rs	$\ell_3 \cup P(\alpha^{19}, 1, 0)$
rs^2	$\ell_{375} \cup P(\alpha, \alpha, 1)$
rs^3	$\ell_{807} \cup P(\alpha^{39}, 0, 1)$
rs^4	$\ell_{292} \cup P(0, 1, 1)$

Class 2: Each of the four element s, s^2, s^3, s^4 of order five fixes three points one of the points is $P(\alpha^{38}, \alpha^{39}, 1)$ which is intersection point of the five lines ℓ_i, $i = 3,292,320,375,807$.

4- The lines ℓ_i, $i = 3,292,320,375,807$ have the property that unisecant to $\mathcal{A}_{41}$ and bisecant to C_{41}.

- The unique 6-arc K with stabilizer group A_5 is just $\mathcal{A}_{41}$ union the intersection point of the lines ℓ_i, $i = 3, 292, 320, 375, 807$. The arc K in numeral form is

$$K = \{1,2,3,323,112,443\},$$

❖ Conclusion

1- There is an arc of degree five $\xi=\{P_1,P_2,P_3,P_4,P_5\}$ which has stabilizer group $G(\xi)$ of type D_5.

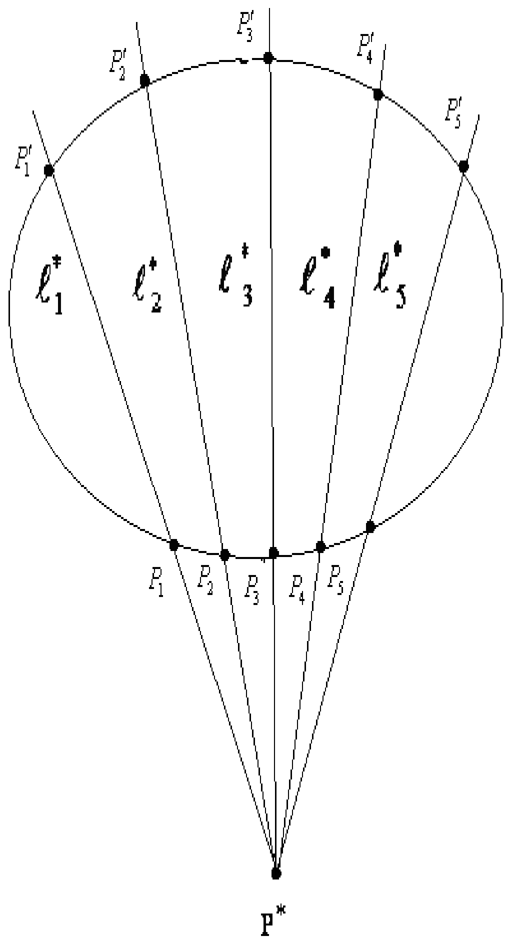
2- The pentastigm which has ξ as a vertex has collinear diagonal points.

3- The effect of the group Dihedral group $G(\xi)$ on points of $PG(2,q), q = 19,29,31,41$ depends on the order of its elements. Let G^2 be the set of five elements of $G(\xi)$ of order two and G^5 be the set of four elements of $G(\xi)$ of order five.

(i) Each elements of G^2 fixes five a subset of the plane of length $q+2$ by sending it to itself. Each of this set, is a line ℓ_i^* with extra point P_i^*, $i=1,2,3,4,5$. The five extra points P_i' are exactly the diagonal points of ξ. Also, these lines are the bisecant to the conic C_ξ which passes through ξ and unisecants to ξ.

(ii) Each elements of G^5 fixes a point $\mathbf{P}^*$ which is the intersection point of the five lines $\ell_i^*, i=1,2,3,4,5$.

4- The unique six arc with stabilizer group of type A_5 is constructed by adding the point $\mathbf{P}^*$ to ξ. So, the following figure is fixed by the group $G(\xi)$.


P_1'
P_2'
P_3'
P_4'
P_5'
ℓ_1^*
ℓ_2^*
ℓ_3^*
ℓ_4^*
ℓ_5^*
P_1
P_2
P_3
P_4
P_5
P^*