1.6. Method To Construct DNF

To construct DNF of a logical proposition we use the following way.

Construct a truth table for the proposition.

(i) Use the rows of the truth table where the proposition is True to construct minterms

- If the variable is true, use the propositional variable in the minterm.
- If a variable is false, use the negation of the variable in the minterm.

(ii) Connect the minterms with ∨'s.

Example 1.6.1. Find the disjunctive normal form for the following logical proposition
(i) $p \to q$.
(ii) $(p \to q) \land \sim r$.
Solution. (i) Construct a truth table for $p \to q$:

p	q	p → q	
T	T	T	←
T	F	F	
F	T	T	←
F	F	T	←

$p \to q$ is true when either
p is true and q is true, or
p is false and q is true, or
p is false and q is false.
The disjunctive normal form is then

$$(p \land q) \lor (\sim p \land q) \lor (\sim p \land \sim q).$$

(ii) Write out the truth table for $(p \to q) \land \sim r$

p	q	r	p → q	~ r	(p → q) ∧ ~ r
T	T	T	T	F	F

T	T	F	T	T	T	←
T	F	T	F	F	F	
T	F	F	F	T	F	
F	T	T	T	F	F	
F	T	F	T	T	T	←
F	F	T	F	F	F	
F	F	F	T	T	T	←

The disjunctive normal form for $(p \to q) \wedge \sim r$ is

$$(p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r).$$

Remark 1.6.2. If we want to get the conjunctive normal form of a logical proposition, construct

(1) the disjunctive normal form of its negation,

(2) negate again and apply De Morgan's Law.

Example 1.6.3. Find the conjunctive normal form of the logical proposition

$$(p \wedge \sim q) \vee r.$$

Solution.

(1) Negate: $\sim[(p \wedge \sim q) \vee r] \equiv (\sim p \vee q) \wedge \sim r$.

(2) Find the disjunctive normal form of $(\sim p \vee q) \wedge \sim r$.

p	q	r	$\sim p$	$\sim r$	$\sim p \vee q$	$(\sim p \vee q) \wedge \sim r$	
T	T	T	F	F	T	F	
T	T	F	F	T	T	T	←
T	F	T	F	F	F	F	
T	F	F	F	T	F	F	
F	T	T	T	F	T	F	

F	T	F	T	T	T	T	←
F	F	T	T	F	T	F	
F	F	F	T	T	T	T	←

The disjunctive normal form for (∼ p ∨ q) ∧ ∼ r is

$$(p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r).$$

(3) The conjunctive normal form for (p ∧ ∼ q) ∨ r is then the negation of this last expression, which, by De Morgan's Laws, is

$$(\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee r).$$

Remark 1.6.4.

(1) p∨q can be written in terms of ∧ and ∼.

(2) We can write every compound logical proposition in terms of ∧ and ∼.

1.7. Logical Implication

Definition 1.7.1. (Logical implication)
We say the logical proposition r implies the logical proposition s (or s logically deduced from r) and write r ⟹ s if r → s is a tautology.
Example 1.7.2. Show that (p → t) ∧ (t →q) ⟹ p → q.

Solution. Let P: the proposition (p → t) ∧ (t →q)

Q: the proposition p → q

p	t	q	p → t	t → **q**	P	Q	P → Q
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T

Remark 1.7.3.

T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

(i) We use r ⟹ s to imply that the statement r ⟶ s is true, while the statement r ⟶ s alone does not imply any particular truth value. The symbol is often used in proofs as shorthand for "implies".

(ii) If r ⟹ s and s ⟹ r, then we denote that by r ⟺s.

Example 1.7.4. Show that
(i) r ⟹ s if and only if ~r ∨ s is tautology.
(ii) r ⟺s if and only if r ≡ s.
Solution.
(i) r ⟹ s if and only if r ⟶ s is a tautology (by def.)
But ~r ∨ s ≡ r ⟶ s is a tautology.
Then, r ⟹ s if and only if ~r ∨ s is tautology.

(ii) r ⟹ s if and only if r → s is tautology (by def.)
s ⟹ r if and only if s → r is tautology (by def.)
Then, r → s ∧ s → r is tautology. Therefore, r ≡ s.
Definition 1.7.5.
The statement q ⟶p is called the **converse** of the statement p ⟶q and the statement ~p ⟶~q is called the **inverse**.

Generally, the statement and its converse not necessary equivalent. Therefore, p ⟹ q does not mean that q ⟹ p.

Example 1.7.6. The statement "the triangle which has equal sides, has two equal legs" equivalent to the statement " the triangle which has not two equal legs has no equal sides".

1.8. Quantifiers

Recall that a formula is a statement whose truth value may depend on the values of some variables. For example,

"$x \leq 5 \wedge x > 3$" is true for $x = 4$ and false for $x = 6$.

Compare this with the statement

"For every x, $x \leq 5 \wedge x > 3$," which is definitely false and the statement

"There exists an x such that $x \leq 5 \wedge x > 3$," which is definitely true.

Definition 1.8.1.

(i) The phrase **"for all x"** (**"for every x"**, **"for each x"**) is called a **universal quantifier** and is denoted by $\forall\, x$.

(ii) The phrase **"for some x"** (**"there exists an x"**) is called an **existential quantifier** and is denoted by $\exists\, x$.

(iii) A formula that contains variables is not simply true or false unless each of these variables is **bound** by a quantifier.

(iv) If a variable is not bound the truth of the formula is contingent on the value assigned to the variable from the universe of discourse.

Definition 1.8.2. (The Universal Quantifier)

Let $f(x)$ be a logical proposition which depend only on x. A sentence $\forall\, x\, f(x)$ is true if and only if $f(x)$ is true no matter what value (from the universe of discourse) is substituted for x.

Example 1.8.3.

$\forall\, x : (x^2 \geq 0)$, i.e., "the square of any number is not negative."
$\forall\, x$ and $\forall\, y, (x + y = y + x)$, i.e., the commutative law of addition.
$\forall\, x, \forall\, y$ and $\forall\, z, ((x + y) + z = x + (y + z))$, i.e. the associative law of addition.

Remark .1.8.4. The *"***all***"* form, the universal quantifier, is frequently encountered in the following context: $\forall\, x\, (f(x) \Rightarrow Q(x))$, which may be read, "For all x satisfying $f(x)$ also satisfy $Q(x)$. " Parentheses are crucial here; be sure you understand the difference between the "all" form and $\forall\, x\, f(x) \Rightarrow \forall\, x\, Q(x)$ and $(\forall\, x\, f(x)) \Rightarrow Q(x)$.

Definition 1.8.5. (The Existential Quantifier)

A sentence $\exists\, x\, f(x)$ is true if and only if there is at least one value of x (from the universe discourse of) that makes $f(x)$ is true.

Example 1.8.6.

$\exists\, x$: $(x \geq x^2)$ is true since $x = 0$ is a solution. There are many others.

$\exists\, x\, \exists\, y$: $(x^2 + y^2 = 2xy)$ is true since $x = y = 1$ is one of many solutions

Negation Rules 1.8.7. When we negate a quantified statement, we negate all the quantifiers first, from left to right (keeping the same order), then we negative the statement.

Definition 1.8.8.

(i) $\forall x\, f(x) = \sim \exists\, x \sim f(x)$.

(ii) $\exists\, x\, f(x) = \sim \forall\, x \sim f(x)$.

Example 1.8.9. Express each of the following sentences in symbolic form and then give its negation.

(i) r: The square of every real number is non-negative.

Solution. Symbolically, r can be expressed as $\forall x \in \mathbb{R}, x^2 \geq 0$.

~r: $\sim (\forall x \in \mathbb{R}, x^2 \geq 0) \equiv \exists x \in \mathbb{R}, \sim (x^2 \geq 0) \equiv \exists x \in \mathbb{R}, x^2 < 0$.

In words, this is "~r: There exists a real number whose square is negative".

(ii) r: For all x, there exists y such that $xy = 1$.

Solution.

r: $\forall\, x, \exists\, y$ such that $xy = 1$.

~r: $\sim (\forall\, x, \exists\, y$ such that $xy = 1) \equiv \exists x, \forall\, y$ such that $\sim (xy = 1) \equiv \exists x, \forall\, y$ such that $xy \neq 1$.

In words, this is "~r: There exists x for all y such that $xy \neq 1$".

(iii) p: student who is intelligent will succeed.

Solution.

Let r: student who is intelligent.

s: succeed.

p: r → s
∼p: ∼ (r → s) ≡∼ (∼ r ∨ s) Implication Low.

≡ r ∧∼ s. De Morgan's Law
∼p: student who is intelligent will not succeed.

There are six ways in which the quantifiers can be combined when two variables are present:

(1) $\forall x \forall y f(x,y)$= $\forall y \forall x f(x,y)$=For every x, for every y $f(x,y)$.

(2) $\forall x \exists y f(x,y)$= For every x, there exists a y such that $f(x,y)$.

(3) $\forall y \exists x f(x,y)$= For every y, there exists an x such that $f(x,y)$.

(4) $\exists x \forall y f(x,y)$= There exists an x such that for every y $f(x,y)$.

(5) $\exists y \forall x f(x,y)$= There exists a y such that for every y $f(x,y)$.

(6) $\exists x \exists y f(x,y)$= $\exists y \exists x f(x,y)$= There exists an x such that there exists a y $f(x,y)$.

Example 1.8.10. Show that the following are equivalents.

(i) $\sim[\forall x \forall y f(x,y)] \equiv \exists x \ \exists y \sim f(x,y)$.

(ii) $\sim[\exists x \forall \exists f(x,y)] \equiv \forall x \ \forall y \sim f(x,y)$.

(iii) $\sim[\forall x \exists y f(x,y)] \equiv \exists x \ \forall y \sim f(x,y)$.

(iii) $\sim[\exists x \forall y f(x,y)] \equiv \forall x \ \exists y \sim f(x,y)$.

Solution. Exercise.