1.4. Rules of Proof

(i) Rule of Replacement.

Any term in a logical formula may be replaced be an equivalent term.

For instance, if $q \equiv r$, then $p \wedge q \equiv p \wedge r$ Rep(q:r).

(ii) Rule of Substitution.

A sentence which is obtained by substituting logical propositions for the terms of a theorem is itself a theorem.

For instance, $(p \to q) \vee w \equiv w \vee (p \to q)$ Sub(p: $p \to q$), Theorem $p \vee w \equiv w \vee p$.

(iii) Rule of Inference.

$\begin{array}{c} p \\ p \to q \\ \hline \therefore q \end{array}$	$\begin{array}{c} \sim q \\ p \to q \\ \hline \therefore \sim p \end{array}$
$\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$	$\begin{array}{c} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$
$\begin{array}{c} p \\ \hline \therefore p \vee R \end{array}$	$\begin{array}{c} p \wedge q \\ \hline \therefore p \end{array}$
$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$	$\begin{array}{c} p \vee q \\ \sim p \vee r \\ \hline \therefore q \vee r \end{array}$

Example 1.4.1. Given
(1) “If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on”
(2) “If the sailing race is held, then the trophy will be awarded”
(3) “The trophy was not awarded”
Does this imply that: “It rained”?
Solution.
p: rain
q: foggy
r: the sailing race will be held
s: the lifesaving demonstration will go on
t: then the trophy will be awarded

Symbolically, the proposition is

$$\begin{array}{ll} (1) & \sim p \vee \sim q \;\rightarrow\; r\wedge s \\ (2) & s \rightarrow t \\ (3) & \sim t \\ \hline & p \end{array}$$

1. $\sim t$	3rd hypothesis
2. $s \rightarrow t$	2nd hypothesis
3. $\sim t \rightarrow \sim s$	Contrapositive of 2
4. $\sim s$	inf (1),(3)
5. $\sim p \vee \sim q \rightarrow r\wedge s$	1st hypothesis
6. $\sim(r\wedge s) \rightarrow \sim(\sim p \vee \sim q)$	Contrapositive of 5
7. $\sim r \vee \sim s \rightarrow (p\wedge q)$	De Morgan’s law and double negation law from 5
8. $\sim r \vee \sim s$	inf (4)
9. $p \wedge q$	inf (7),(8)
10. p	inf (9)

Example 1.4.2. Use the logical equivalences to show that
(i) $\sim(p \rightarrow q) \equiv p \wedge \sim q$,
(ii) $\sim(p \vee \sim(p \wedge q))$ is a contradiction,
(iii) $\sim(p\vee(\sim p\wedge q)) \equiv (\sim p \wedge \sim q)$,

(iv) $p\vee(p\wedge q) \equiv p$ (Absorption Law).

Solution.

(i) $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q)$ Implication Law
$\equiv \sim(\sim p) \wedge \sim q.$ De Morgan's Law
$\equiv p \wedge \sim q$ Double Negation Law

(ii) $\sim(p \vee \sim(p \wedge q))$
$\equiv \sim p \wedge \sim(\sim(p \wedge q))$ De Morgan's Law
$\equiv \sim p \wedge (p \wedge q)$ Double Negation Law
$\equiv (\sim p \wedge p) \wedge q$ Associative Law
$\equiv F \wedge q$ Contradiction Law
$\equiv F$ Domination Law and Commutative Law.

(iii) $\sim (p \vee (\sim p \wedge q))$

$\equiv \sim p \wedge \sim (\sim p \wedge q)$ De Morgan's Law

$\equiv \sim p \wedge (\sim\sim p \vee \sim q)$ De Morgan's Law

$\equiv \sim p \wedge (p \vee \sim q)$ Double Negation Law

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ Distribution Law

$\equiv (p \wedge \sim p) \vee (\sim p \wedge \sim q)$ Commutative Law

$\equiv F \vee (\sim p \wedge \sim q)$ Contradiction Law

$\equiv (\sim p \wedge \sim q) \vee F$ Commutative Law

$\equiv (\sim p \wedge \sim q)$ Identity Law

(iv) $p \vee (p \wedge q)$
$\equiv (p \wedge T) \vee (p \wedge q)$ Identity (in reverse)
$\equiv p \wedge (T \vee q)$ Distributive (in reverse)
$\equiv p \wedge T$ Domination
$\equiv p$ Identity

Example 1.4.3. Find a simple form for the negation of the proposition "If the sun is shining, then I am going to the ball game."

Solution.

This proposition is of the form $p \to q$. Since $\sim(p \to q) \equiv \sim(\sim p \vee q) \equiv (p \wedge \sim q)$. This is the proposition "The sun is shining, and I am not going to the ball game."