3.2 Relations

Definition 3.2.1. Any subset " R" of $A \times B$ is called a **relation between A and B** and denoted by $R(A, B)$. Any subset of $A \times A$ is called a **relation on A**.

In other words, if A is a set, any set of ordered pairs with components in A is a relation on A. Since a relation R on A is a subset of $A \times A$, it is an element of the powerset of $A \times A$; that is, $R \subseteq P(A \times A)$.

If R is a relation on A and $(x, y) \in R$, then we write $\boldsymbol{xRy}$, read as "x is in R-relation to y", or simply, x is in relation to y, if R is understood.

Example 3.2.2.

(i) Let $A = \{2,4,6,8\}$, and define the relation R on A by $(x, y) \in R$ iff x divides y. Then, $R = \{(2,2),(2,4),(2,6),(2,8),(4,4),(4,8),(6,6),(8,8)\}$.

(ii) Let $A = \mathbb{N}$, and define $R \subseteq A \times A$ by xRy iff x and y have the same remainder when divided 3.

Since A is infinite, we cannot explicitly list all elements of R; but,

for example $(1,4),(1,7),(1,10),\ldots,(2,5),(2,8),\ldots,(0,0),(1,1),\ldots \in R$. Observe, that xRx for $x \in N$ and, whenever xRy then also yRx.

(iii) Let $A = \mathbb{R}$, and define the relation R on $\mathbb{R}$ by xRy iff $y = x^2$. Then R consists of all points on the parabola $y = x^2$.

(iv) Let $A = \mathbb{R}$, and define R on $\mathbb{R}$ by xRy iff $x \cdot y = 1$. Then R consists of all pairs $(x, \frac{1}{x})$, where x is non-zero real number.

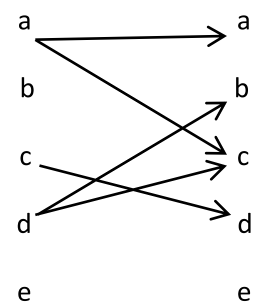
(v) Let $A = \{1,2,3\}$, and define R on A by xRy iff $x + y = 7$. Since the sum of two elements of A is at most 6, we see that xRy for no two elements of A; hence, $R = \emptyset$.

For small sets we can use a pictorial representation of a relation R on A: Sketch two copies of A and, if xRy then draw an arrow from the x in the left sketch to the y in the right sketch.

(vi) Let $A = \{a, b, c, d, e\}$, and consider the relation

$$R = \{(a,a),(a,c),(c,d),(d,b),(d,c)\}.$$

An arrow representation of R is given in Fig.



We observe that $\boldsymbol{e}$ does not appear at all in the elements of R, and that, for example, $\boldsymbol{b}$ is not the first component of any pair in R. In order to give names to the sets of those elements of A which are involved in R, we make the following.

(vii) Let A be any set. Then the relation $R = \{(x,x): x \in A\} = i_A$ on A is called the **identity relation on $\boldsymbol{A}$**. Thus, in an identity relation, every element is related to itself only.

Definition 3.2.3. Let R be a relation on A. Then

(i) dom $R = \{x \in A :$ There exists some $y \in A$ such that $(x,y) \in R\}$.
dom R is called the **domain of $\boldsymbol{R}$**.

(ii) ran $R = \{y \in A$: There exists some $x \in A$ such that $(x,y) \in R\}$

is called the **range of $\boldsymbol{R}$**.

(iii) Finally, $fld\ R = dom\ R \cup ran\ R$ is called the **field of $\boldsymbol{R}$**.

Observe that $dom\ R$, $ran\ R$, and $fld\ R$ are all subsets of A.

Example 3.2.4.

(i) Let A and R be as in Example 3.2.2.(vi); then $dom\ R = \{a,c,d\}$, ran $= \{a,b,c,d\}$, $fld\ R = \{a,b,c,d\}$.

(ii) Let $A = \mathbb{R}$, and define R by xRy iff $y = x^2$; then, $dom\ R = \mathbb{R}$,

$ran\ R = \{y \in \mathbb{R} : y \geq 0\}, fld\ R = \mathbb{R}$.

(iii) Let $A = \{1, 2, 3, 4, 5, 6\}$, and define R by xRy iff $x \lneqq y$ and x divides y; $R = \{(1, 2), (1, 3), \ldots, (1, 6), (2, 4), (2, 6), (3, 6)\}$, and dom $R = \{1, 2, 3\}$, $ran\ R = \{2, 3, 4, 5, 6\}$, $fld\ R = A$.

(iv) Let $A = R$, and R be defined as $(x, y) \in R$ iff $x^2 + y^2 = 1$. Then $(x, y) \in R$ iff (x, y) is on the unit circle with centre at the origin. So,

$$dom\ R = ran\ R = fld\ R = \{z \in \mathbb{R}: -1 \leq z \leq 1\}.$$

Definition 3.2.5. Reflexive, Symmetric and Transitive Relations
Let R be a relation on a nonempty set A.

- **(i)** R is **reflexive** if $(x, x) \in R$ for all $x \in A$.
- **(ii)** R is **antisymmetric** if for all $x, y \in A$, $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$.
- **(iii)** R is **transitive** if for all $x, y, z \in A$, $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.
- **(iv)** R is **symmetric** if whenever $(x, y) \in R$ then $(x, y) \in R$.