additional equations.

The momentum equation "equations of motion" in component form comprise a system of three equations with 4 unknown quantities (u, v, p and ρ).

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

$$\frac{\partial p}{\partial z} = -\rho g.$$

They are not a closed set, because there are four dependent Variables and only three equations. We need to come up with some more equations in order to close the set.

The three additional equations are:

1- Gas Equation.
2- Thermodynamic Equation.
3- Continuity Equation.

1- Gas Equation.

A perfect gas (ideal gas) obeys the physical laws of Boyle and charles. the general equation of state is:

$$P = \rho R T$$

where:

P: pressure.
ρ: density.
R: gas constant (287 $J Kg^{-1} K^{-1}$
T: Temperature.

Boyle's Law: $P_1 V_1 = P_2 V_2$ where T is constant.

Charles Law: $\dfrac{P_1}{T_1} = \dfrac{P_2}{T_2}$ where V is constant.

Combing Boyles and charle's Law we get:

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} = C$$

where C depends on the mass of gas it is equal to 287 $J Kg^{-1} K^{-1}$ (specific gas constant).

$$\therefore \frac{P\alpha}{T} = R \quad \text{where } \alpha = \frac{1}{\rho}$$

$$\boxed{\therefore P = \rho R T}$$

2- Thermodynamic Equation.

The thermodynamic energy equation comes from 1^{st} law of Thermodynmics (conservation of energy).

The first law of thermodynamic can be written by:

$$dH = du + dw.$$

Where:

dH: is the amount of heat added per unit mass.

du: is the change in energy per unit mass, $du = C_v\, dT$

dw: is the work done by unit mass on system, $dw = p\,d\alpha$.

$$\boxed{\therefore dH = C_v\, dT + p\,d\alpha} \quad \text{——— } (1)$$

Derived equation of state $p\alpha = RT$ we get:

$$p\,d\alpha + \alpha\, dp = R\,dT$$

$$p\,d\alpha = R\,dT - \alpha\, dp \quad \text{——— } (2)$$

put (2) in (1) we get:

$$dH = C_v dT + R\,dT - \alpha\,dp$$

$$dH = (C_v + R)\,dT - \alpha\,dp.$$

Recall that : $R = C_p - C_v$

$$\therefore C_p = R + C_v$$

where $C_p = 1004 \frac{J}{Kg\,K}$

$$C_v = 717 \frac{J}{Kg\,K}$$

C_p and C_v are specific heat at constant p and v.

$$\boxed{\therefore dH = C_p dT - \alpha\,dp}$$

$$\cancel{\frac{dH}{dt} = C_p \frac{dT}{dt} - \alpha \frac{dp}{dt}}$$

dH: is the Heat added per unit mass.

So the equation of first law of thermodyamic are:

$$dH = C_v dT + p\,d\alpha$$

$$dH = C_p dT - \alpha\,dp$$

Addibatic Assmption.

It is assumed that $dH=0$, this assumption can be made whenever the motion is so fast, so that the heat exchange between the particale and the surrounding is negligible.

For addibatic motion the equations of first law of thermodynamics becomes:

$$C_v dT + p d\alpha = 0 \quad \text{—— (1)}$$

$$C_p dT - \alpha dp = 0 \quad \text{—— (2)}$$

The equation of state is $p = \rho R T = \frac{1}{\alpha} RT$ —— (3)

put (3) in (1)

$$C_v dT + \frac{RT}{\alpha} d\alpha = 0$$

$$C_v dT = -RT \frac{d\alpha}{\alpha}$$

$$C_v \frac{dT}{T} = -R \frac{d\alpha}{\alpha} \quad \div C_v$$

$$\int_{T_1}^{T} \frac{dT}{T} = -\frac{R}{C_v} \int_{\alpha_1}^{\alpha} \frac{d\alpha}{\alpha}$$

$$\ln T \Big|_{T_1}^{T} = -\frac{R}{C_v} \ln \alpha \Big|_{\alpha_1}^{\alpha}$$

$$\ln (T - T_1) = -\frac{R}{C_v}\left(\ln(\alpha - \alpha_1)\right)$$

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$$\ln \frac{T}{T_1} = -\frac{R}{C_v} \ln \frac{\alpha}{\alpha}$$

$$\boxed{\therefore \frac{T}{T_1} = \left(\frac{\alpha}{\alpha_1}\right)^{\frac{-R}{C_v}}} \quad \text{—— (4)}$$

if T increases α will decrease and vice versa.

From equation (2)

$$C_p dT - \alpha dp = 0$$

From state equation $p = \frac{1}{\alpha} R T$ $\boxed{\therefore \alpha = \frac{RT}{p}}$ — (5)

put (5) in (2)

$$C_p dT - \frac{RT}{P} dp$$

$$\int_{T_1}^{T} \frac{dT}{T} = \frac{R}{C_p} \int_{P_1}^{P} \frac{dp}{p}$$

$$\ln \frac{T}{T_1} = \frac{R}{C_p} \ln \frac{P}{P_1}$$

$$\boxed{\therefore \frac{T}{T_1} = \left(\frac{P}{P_1}\right)^{\frac{R}{c_p}}} \quad \text{——— } (6)$$

If T increases p will increase and vice versa.

From equation (4) and (6) we get:

$$\left(\frac{\alpha}{\alpha_1}\right)^{-\frac{\cancel{R}}{C_v}} = \left(\frac{P}{P_1}\right)^{\frac{\cancel{R}}{c_p}}$$

$$\boxed{\therefore \frac{\alpha}{\alpha_1} = \left(\frac{P}{P_1}\right)^{-\frac{C_v}{C_p}}}$$

An increase in p correspound to decrease in α and vice versa.