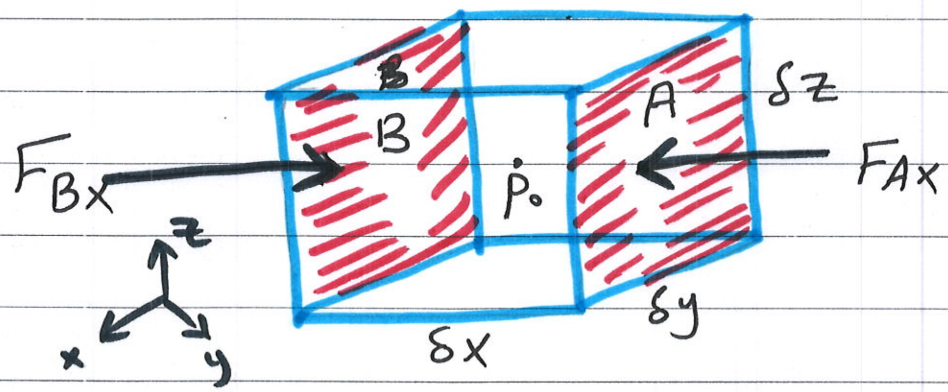
$$\frac{d\vec{V}}{dt} = PGF + \underset{\text{force}}{\text{Gravitional}} + \underset{\text{"friction"}}{\text{viscous}} - 2\vec{\Omega} \times \vec{V} + \Omega^2 \vec{R}$$

The forces in the Equation of motion:

1/ The pressure gradient force.

The pressure force is due to the variation of the atmospheric pressure from point to point. The pressure at a given point and a given time is independent of direction.

Consider the air parcel like the box in the fig. the pressure at the center of box is $P_o$.




The volume of the box is $\delta V = \delta x \, \delta y \, \delta z$

The pressure at the surface A is

$$P_{Ax} = P_o + \frac{\partial P}{\partial x}\left(\frac{\delta x}{2}\right).$$

The Force at the surface A is:

$$\vec{F}_{Ax} = -\left[p_0 + \frac{\partial p}{\partial x}\left(\frac{\delta x}{2}\right)\right]\delta y\,\delta z\, i$$

The pressure at the surface B is:

$$p_{Bx} = p_0 - \frac{\partial p}{\partial x}\left(\frac{\delta x}{2}\right)$$

The force at the surface B is

$$\vec{F}_{Bx} = \left[p_0 - \frac{\partial p}{\partial x}\left(\frac{\delta x}{2}\right)\right]\delta y\,\delta z\, i$$

the net pressure force in the x-direction is

$$\vec{F}_x = \vec{F}_{Bx} - \vec{F}_{Ax}$$

$$\vec{F}_x = \left[p_0 - \frac{\partial p}{\partial x}\left(\frac{\delta x}{2}\right)\right]\delta y\,\delta z\, i - \left[p_0 + \frac{\partial p}{\partial x}\left(\frac{\delta x}{2}\right)\right]\delta y\,\delta z\, i$$

$$\vec{F}_x = -2\frac{\partial p}{\partial x}\left(\frac{\delta x}{2}\right)\delta y\,\delta z\, i$$

$$\vec{F}_x = -\frac{\partial p}{\partial x}\,\delta v\; i$$

$$\delta v = \delta x\,\delta y\,\delta z$$

since $\delta V = \frac{m}{\rho}$ where m is mass
and $\rho$ is density.

$$\vec{F_x} = -\frac{\partial p}{\partial x}\,\delta V i = -\frac{m}{\rho}\frac{\partial p}{\partial x}\, i \qquad \div m$$

$$\vec{f_x} = \frac{\vec{F_x}}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\, i \qquad x\text{-direction.}$$

Similarly for y and z directions

$$\vec{f_y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}\, j. \qquad y\text{-direction}$$

$$\vec{f_z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}\, k. \qquad z\text{-direction.}$$

The net total of force per unit mass:

$$\vec{f} = \vec{f_x} + \vec{f_y} + \vec{f_z}$$

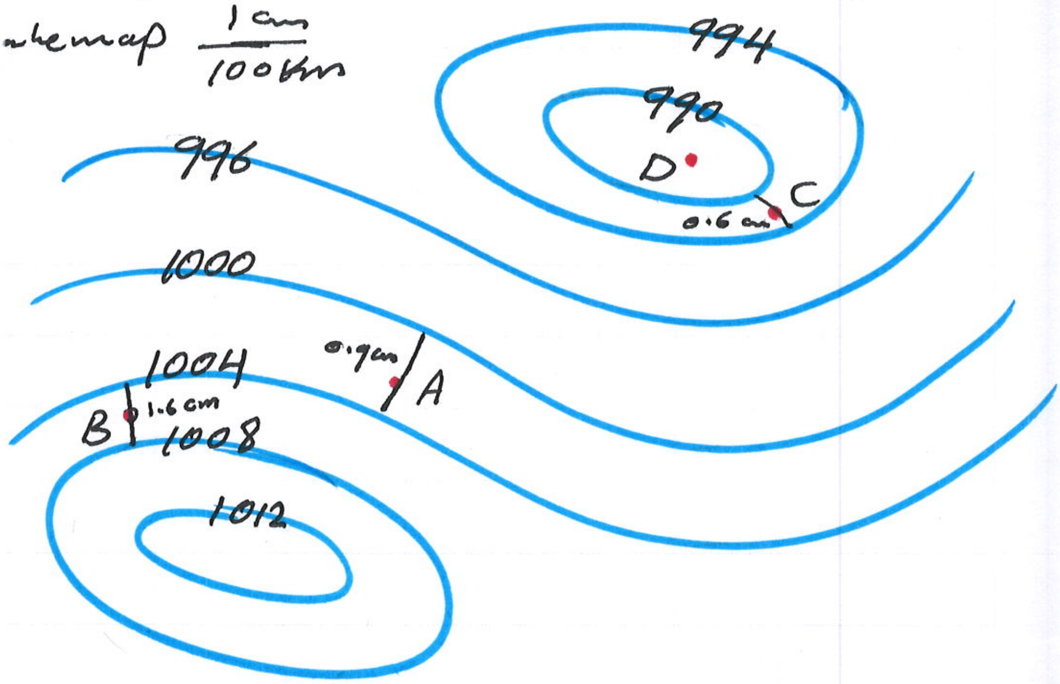
$$\vec{f} = -\frac{1}{\rho}\left[\frac{\partial p}{\partial x} i + \frac{\partial p}{\partial y} j + \frac{\partial p}{\partial z} k\right].$$

$$\boxed{\vec{f} = -\frac{1}{\rho}\vec{\nabla} p = -\frac{1}{\rho}\cdot \mathrm{grad}\, p.}$$

The direction of the pressure gradient force $\vec{f}$ is in the opposite direction of the pressure gradient $\vec{\nabla} p$, and $\nabla p \approx \frac{\Delta p}{\Delta n}$ where $\Delta p$ is the contour interval for the isobars and $\Delta n$ is the horizontal distance between the isobars.

At the four points shown in the picture below, estimate the magnitude of the acceleration due to the pressure gradient force. Assume a density of 1.23 Kg/m³. The isobars are labeled in mb.

Scale map $\frac{1\text{ cm}}{100\text{ km}}$




$$\vec{f} = -\frac{1}{\rho}\vec{\nabla}P$$

$$|\vec{f}| = \frac{1}{\rho}\nabla P = \frac{1}{\rho}\frac{\Delta P}{\Delta n}$$

المسافة بين خطوط
تساوي الضغط

point B :

$$\Delta P = 1008 - 1004 = 4\text{ mb} = 4\text{ hpa}$$

$$= 4\times10^2\text{ pa} = 4\times10^2\text{ N/m}^2$$

$$\Delta n = 1.6\text{ cm}\times\frac{100\text{ Km}}{1\text{ cm}} = 160\text{ km}$$

$$\Delta n = 160\times10^3\text{ m}$$

$$|\vec{f}| = \frac{1}{1.23}\,\frac{4\times10^2}{160\times10^3} = 0.002\text{ m/sec}^2$$

point A :

$$|\vec{f}| = \frac{1}{\rho} \frac{\Delta p}{\Delta n}$$

$$\Delta p = 1004 - 1000 = 4\,mb = 4 \times 10^{2}\ N/m^{2}$$

$$\Delta n = 0.9\ cm \times \frac{100\,Km}{1\,cm} = 90\ Km$$

$$|\vec{f}| = \frac{1}{1.23} \times \frac{4 \times 10^{2}}{90 \times 10^{3}} = 0.0036\ m/sec^{2}$$

point C :

$$\Delta p = 994 - 990 = 4\,mb = 4 \times 10^{2}\ N/m^{2}$$

$$\Delta n = 0.6\ cm \times \frac{100\,Km}{1} = 60\ Km$$

$$|\vec{f}| = \frac{1}{1.23} \times \frac{4 \times 10^{2}}{60 \times 10^{3}} = 0.0054\ m/se^{2}$$

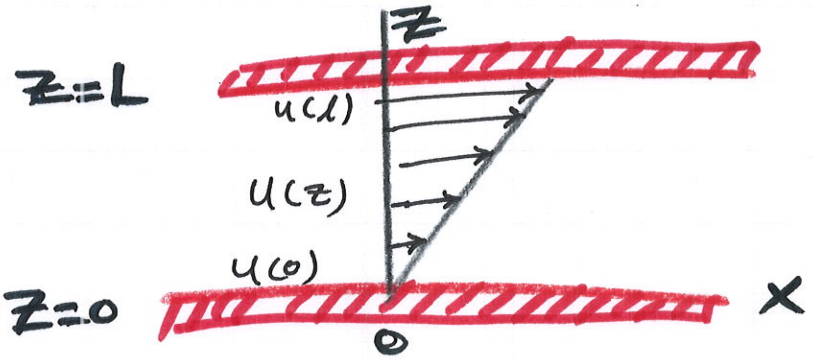
point D

$$|\vec{f}| = \text{Zero}.$$

because $\Delta p = \text{Zero}$.

## 2- The viscous Force.

viscous force is due to friction caused by interactions of molecules of a fluid.
IF we consider a layer of incompressible fluid between two horizontal plates separated by a distance (L) as shown in the fig.




The layer in contact with the upper and lower plates will moved as the following:

at $z=L$ the fluid moves at speed $u(L)=u_0$ and at $z=0$ the fluid is motionless $u(0)=0$.

$$F \propto \frac{A u_0}{L}$$

where $u_0$ is peak speed, $A$ is area and $L$ is depth.

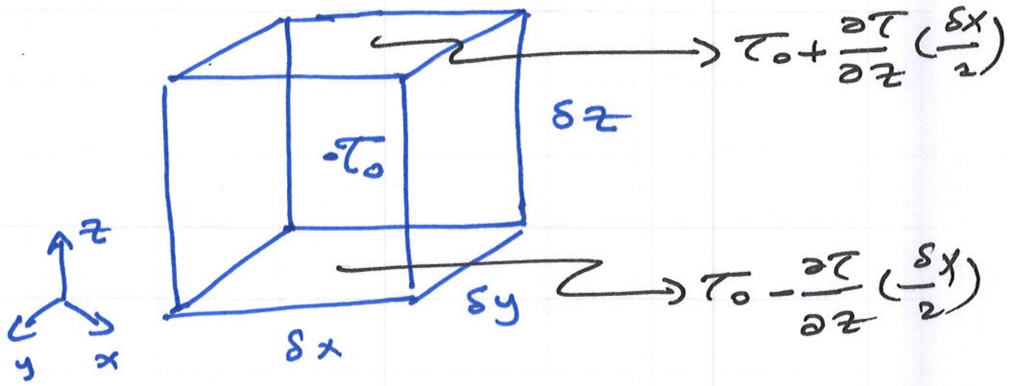
$$\therefore F = \mu \frac{A u_0}{L}$$

where $\mu$ dynamical viscosity coefficient.

The shearing stress $\tau = \frac{F}{A}$

Thus $\tau_x = \frac{F_x}{A} = \mu \frac{u_o}{L} = \mu \frac{\partial u}{\partial z}$

From the fig. along the top of the cube the force in x-direction is: $\tau_o + \frac{\partial \tau}{\partial z}\left(\frac{\delta x}{2}\right)$




on the bottom face of the cube, the force in the x-direction is: $\tau_o - \frac{\partial \tau}{\partial z}\left(\frac{\delta x}{2}\right)$

The net force is: $\vec{F}_x = \left[\sum \vec{\tau}_x\right] A$

$$\vec{F}_x = \left\{\left[\tau_o - \frac{\partial \tau}{\partial z}\left(\frac{\delta x}{2}\right)\right] - \left[\tau_o + \frac{\partial \tau}{\partial z}\left(\frac{\delta x}{2}\right)\right]\right\} \delta y \delta z i$$

$$\vec{F}_x = -\frac{\partial \tau}{\partial z} \delta x \delta y \delta z$$

$$\vec{F_x} = -\frac{\partial \tau}{\partial z} \delta v \, i$$

$$\delta v = \frac{m}{\rho} \Rightarrow \vec{F_x} = -\frac{\partial \tau}{\partial z} \frac{m}{\rho} i$$

$$\therefore \frac{\vec{F_x}}{m} = \vec{f_x} = -\frac{1}{\rho} \frac{\partial \tau}{\partial z} i$$

$$\vec{f_x} = -\frac{1}{\rho} \frac{\partial}{\partial z} \left(\mu \frac{\partial u}{\partial z}\right) i \quad \text{x-direction}$$

H.w $$\vec{f_y} = -\frac{1}{\rho} \frac{\partial}{\partial z} \left(\mu \frac{\partial v}{\partial z}\right) j \quad \text{y-direction.}$$

H.w $$\vec{f_z} = -\frac{1}{\rho} \frac{\partial}{\partial z} \left(\mu \frac{\partial w}{\partial z}\right) k \quad \text{z-direction}$$

The total force per unit mass:
$$\vec{f} = \vec{f_x} + \vec{f_y} + \vec{f_z}$$

$$\vec{f} = -\frac{1}{\rho} \left[\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) i + \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) j + \frac{\partial}{\partial z}\left(\mu \frac{\partial w}{\partial z}\right) k\right]$$

$$\vec{f} = -\frac{\mu}{\rho} \left[\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) i + \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right) j + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}\right) k\right].$$

$$\vec{f} = -\frac{\mu}{\rho} \frac{\partial}{\partial z} \left[\left(\frac{\partial u}{\partial z}\right) i + \left(\frac{\partial v}{\partial z}\right) j + \frac{\partial w}{\partial z} k\right].$$

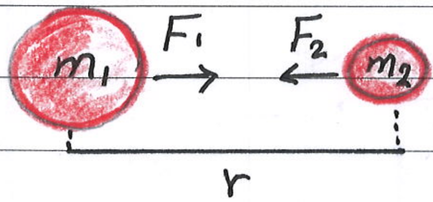
$$\vec{f} = -\frac{\mu}{\rho} \frac{\partial}{\partial z} \left[\frac{\partial \vec{V}}{\partial z}\right]. \quad \text{where } \vec{V} = ui + vj + wk.$$

The acceleration due to the viscous force can be written as:
$$\boxed{\vec{f} = -\frac{1}{\rho} \frac{\partial \tau}{\partial z}}$$

The viscosity of the atmosphere is small, so we can ignore it.

3) The Gravitional force.

Newton's Law of universal gravitation states that "any two element of mass in the universe attracts each other with a force proportional to their masses and inversely proportional to the square of the distance between them".




Newton's Law can be written as a vector equation:

$$\vec{g} = -G \frac{m_1 m_2}{r^3} \vec{r}$$

where $\vec{g}$ attraction of $m_1$ on $m_2$ (force of gravition)

$\vec{r}$ is vector from $m_1$ to $m_2$

$G$ is universal gravitional constant $= 6.66 \times 10^{-11}\ N m^2 Kg^{-2}$

If we assume $m_2 = 1\ Kg$

$$\therefore \vec{g} = -G \frac{m_1}{r^3} \vec{r}$$

IF $m_1 = M$ 'total mass of earth is equal to $5.988 \times 10^{14}$ Kg.

The acceleration due to the gravitational force at the surface of earth ( $r = a = 6378$ Km ) is

$$\vec{g}_* = -G \frac{M}{a^2} \vec{r}$$

At some altitude Z above the surface of the earth, the acceleration due to the gravitational force is:

$$\vec{g}_* = -G \frac{M}{(a+Z)^2} \vec{r}$$

* $\vec{r}$ is the vector from the center of earth to partical in the atmosphere.

* $\vec{g}_*$ is directed toward the center of earth.

4/ Centrifugal force.

The earth is no inertial system, but a rotating system; therfore Centrifugal and Coriolis force occur.

The term $\Omega^2 \vec{R}$ represents the centrifugal acceleration due to the earth's rotation, the centrifugal force is always directed away from the axis of rotation. The centrifugal force is combined with the gravitational force to define a new force called GRAVITY.

The acceleration due to gravity force is defined as:

$$\vec{g} = \vec{g}_* + \Omega^2 \vec{R}$$

when ~~are~~ you see gravity in an equation of motion, Keep in mind that it is a combination of the gravitational acceleration plus centrifugal accele.

The gravity in an equation of motion has a plus sign, but Keep in mind that if written in component from it lies solely in the negative K-direc.

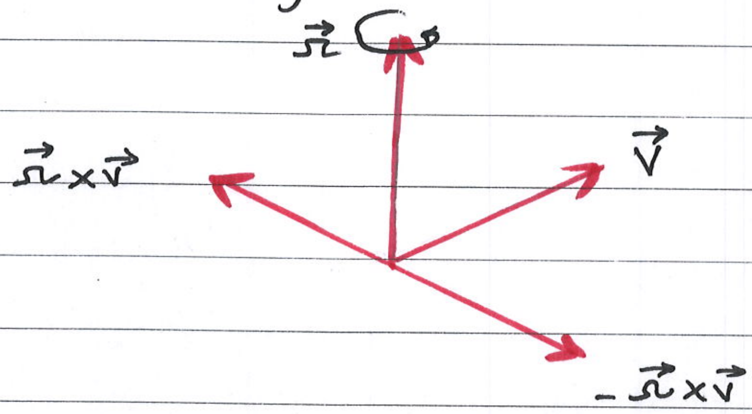
$$\vec{g} = -gK.$$

## 5/ The Coriolis force.

Coriolis force an air parcel with velocity in a coordinate systems with angular velocity $\Omega$ is:

$$\text{Coriolis force} = -2\vec{\Omega} \times \vec{V}$$

The minus sign is important, just like in advection. The coriolis force acts perpendicular to the direction of velocity vector:

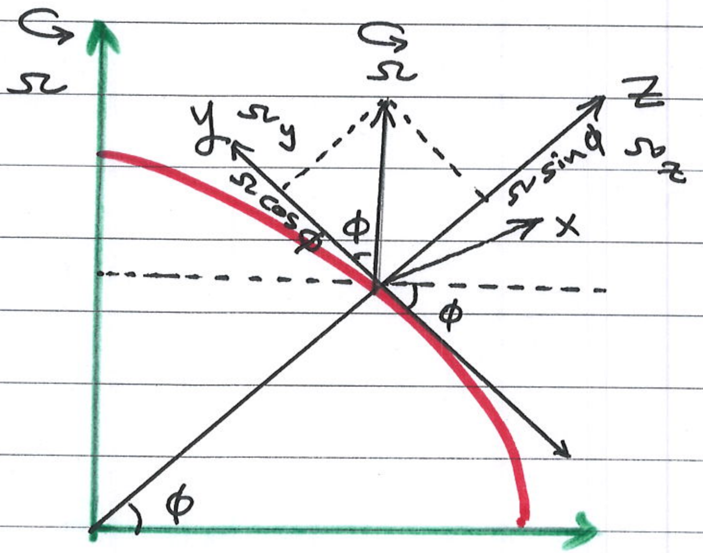



in the northern hemisphere, where $\Omega$ is counterclockwise, the Coriolis force acts to the right of the velocity vector, it tends to deflect air parcel to the right of their direction of motion.

The coriolis force changes only the direction of particle's motion, not it's speed, because the force is always perpendicular to the direction of motion.

## The components of Coriolis force.

From the Fig.




$\Omega_x = 0$

$\Omega_y = \Omega \cos\phi$

$\Omega_z = \Omega \sin\phi$ , So..

$$\text{Coriolis force} = -2\vec{\Omega}\times\vec{V} = \begin{vmatrix} i & j & k \\ 0 & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix}$$

$$-2\vec{\Omega}\times\vec{V} = -2(\Omega_y w - \Omega_z v)i + (0 - \Omega_z u)j - (0 - \Omega_y u)k$$

$$-2\vec{\Omega}\times\vec{V} = (2\Omega_z v - 2\Omega_y w)i - \Omega_z uj + \Omega_y uk.$$

$$-2\vec{\Omega}\times\vec{V} = (2\Omega \sin\phi\, v - 2\Omega\cos\phi\, w)i - \Omega\sin\phi\, uj + \Omega\cos\phi\, uk$$

let $f = 2\Omega \sin\phi$ and $e = 2\Omega\cos\phi$

Thus $\boxed{-2\vec{\Omega}\times\vec{V} = (fv - ew)i - fuj + euk.}$

The components of the equation of motion:

The Equation of motion is:

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\vec{\nabla}p + \vec{g}_* + \vec{f} - 2\vec{\Omega}\times\vec{V} + \vec{\Omega}^2\vec{R}$$

we not that $\vec{g}_*$ and $\Omega^2\vec{R}$ are the only forces which depend ~~on~~ solely on position.

$$\vec{g} = \vec{g}_* + \Omega^2\vec{R}$$

where $\vec{g}$ is gravity.

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\vec{\nabla}p + \vec{g} + \vec{f} - 2\vec{\Omega}\times\vec{V} \;\ldots\ldots\; (1)$$

Now we find the x, y and z component of each term in equation (1).

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}i + \frac{dv}{dt}j + \frac{dw}{dt}k \;\ldots\ldots\; (A)$$

$$-\frac{1}{\rho}\vec{\nabla}p = -\frac{1}{\rho}\frac{\partial p}{\partial x}i - \frac{1}{\rho}\frac{\partial p}{\partial y}j - \frac{1}{\rho}\frac{\partial p}{\partial z}k \;\ldots\ldots\; (B)$$

$$\vec{g} = -gk \;\ldots\; (C)$$

$$\vec{f} = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)i - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)j - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)k \;\ldots\; (D)$$

$$-2\vec{\Omega}\times\vec{V} = (fv - ew)i - fuj + euk \;\ldots\ldots\; (E)$$

put equations A, B, C, D and E in Equation (1).

$$\frac{du}{dt}i+\frac{dv}{dt}j+\frac{dw}{dt}k=-\frac{1}{\rho}\frac{\partial p}{\partial x}i-\frac{1}{\rho}\frac{\partial p}{\partial y}j-\frac{1}{\rho}\frac{\partial p}{\partial z}k$$

$$-gk+\left(-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)i-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)j-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)k\right)$$

$$+(fv-ew)i-fuj+euk.$$

The components of this equation are:

$$\frac{du}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)+fv-ew \qquad x\text{-Dir.}$$

$$\frac{dv}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)-fu \qquad y\text{-Dir.}$$

$$\frac{dw}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)-g+eu \qquad z\text{-Dir.}$$

To assess which terms can be neglected. we assign an "order of magnitude" to all the variables and parameters in the equations.

For synoptic scales the following order of magnitude are:-

| name | symbol | order of magnitude |
|---|---|---|
| Horizontal velocity | $U$ | $10\ m/sec.$ |
| vertical velocity | $W$ | $0.01\ m/sec.$ |
| Horizontal distance | $L$ | $1000\,Km = 10^6\,m$ |
| vertical distance | $H$ | $10\,Km = 10^4\,m$ |
| Horizontal pressure | $\Delta p$ | $10\,mb$ |
| density | $\rho$ | $1\,kg/m^3$ |
| Time | $\frac{L}{U}$ | $1$ day $(10^5\,sec)$ |
| viscosity | $\mu$ | $1.46\times10^{-5}\ \frac{m^2}{sec}$ |
| omega | $\Omega$ | $7.3\times10^{-5}\ sec^{-1}$ |
| latitude | $\phi$ | $45^\circ$ |
| radius of earth | $R$ | $6.378\times10^6\ m.$ |

Using these scales and parameters, in the x-momentum equation we have the following order of magnitude.

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) + fv - ew.$$

$$\frac{\partial u}{\partial t} = \frac{U^2}{L} = \frac{10^2}{10^6} = 10^{-4} \ m/sec^2$$

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{\Delta P}{\rho L} = \frac{10^3}{1 * 10^6} = 10^{-3} \ m/sec^2$$

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) = \mu \frac{\partial^2 u}{\partial z^2} = \frac{\nu U}{H^2} = \frac{1.5 \times 10^{-5} \times 10}{10^8}$$

$$\therefore \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) = 10^{-12} \ m/sec^2$$

$$fv = 2\Omega v \sin\phi = 2\Omega U \sin\phi = 2 * 7.3 \times 10^{-5} * 10 \times \sin 45^\circ$$

$$fv = 10^{-3} \ m/sec^2$$

$$ew = 2\Omega W \cos\phi = 2 \times 7.3 \times 10^{-5} * 10^{-2} * 0.7$$

$$ew = 10^{-6} \ m/sec^2.$$

A similar analysis for y and Z equations.

Many of terms are very small compared to others, and can therfore be ignored without much loss of accuracy. We can therefore ignore the viscous terms and coriolis term that involves the vertical velocity.

Thus the components of equation of motion are:

$$\boxed{\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv}$$

$$\boxed{\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu.}$$

$$\boxed{\frac{\partial p}{\partial z} = -\rho g} \longrightarrow \text{Hydrostatic Equation.}$$

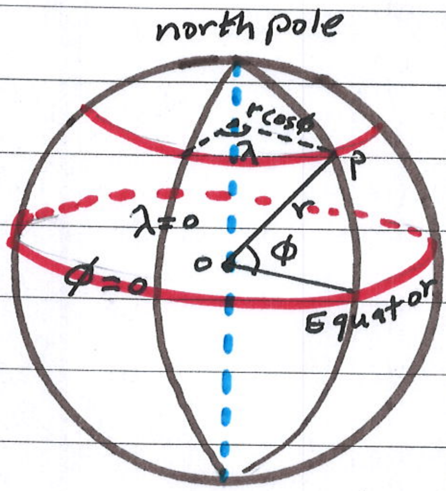
Hydrostatic equation is expressing a balance between gravity acting downwards and the vertical component of the pressure force acting upwards

This system of equation is quite often used in analysis of dynamical problems.

Coordinate system ( spherical system).

It is partical to use aspherical coordinate system with origin at the center of the earth to describe dynamical aspects.

The coordinate system is rotating together with the planet.




$\lambda$ longitude in easterly direction.

$\phi$ latitude in northerly direction.

$Z$ geometric altitude.

$R$ radius of planet $r = R + Z$.

$u = r\cos\phi \dfrac{d\lambda}{dt}$ Zonal component of wind $\vec{V}$ measured towards the east.

$v = r \dfrac{d\phi}{dt}$ meridional component of wind $\vec{V}$ measured towards the north.

$w = \dfrac{dz}{dt}$ Vertical component of $\vec{V}$.

Equation of motion in spherical coordinates.

The whole set in spherical coordinates with out friction.

$$\frac{du}{dt} = \frac{uv}{r}\tan\phi - \frac{uw}{r} + 2\Omega\sin\phi\, v - 2\Omega\cos\phi\, w - \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial \lambda}$$

$$\frac{dv}{dt} = -\frac{u^2}{r}\tan\phi - \frac{vw}{r} - 2\Omega\sin\phi\, u - \frac{1}{\rho r}\frac{\partial p}{\partial \phi}$$

$$\frac{dw}{dt} = \frac{u^2+v^2}{r} + 2\Omega\cos\phi\, u - g - \frac{1}{\rho}\frac{\partial p}{\partial r}.$$

In the Lagrangian frame we have to take care of the total derivative.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \phi} + w\frac{\partial}{\partial z}.$$