Week	Sunday & Tuesday Class (Text Sec.)	
1	Concepts of differential Equations, General form of solution	
2	First-degree first-order equations, Separable-variable and Exact equations	
3	Inexact, integrating factors and Linear equations	
4	Homogeneous, Isobaric and Bernoulli's equations	
5	Problems and applications	
6	Higher-degree first-order equations and their applications	
7	Higher-order ordinary differential equations	
8	Linear equations with constant coefficients	
9	Solution from complementary function and particular integral	
10	Laplace transform method	
11	Linear equations with variable coefficients	
12	Legendre and Euler linear equations and Green's functions	
13	Problems and applications	
14	Series solutions of differential equations	
15	Regular singular point and Frobenius method	
	First Exam In (??-??-2016)	**Return and Discussions of first exam Results**
	Final Exams	

Textbooks [الكتاب المنهجى]

1. "Mathematical Methods for Physics and Engineering", Riley K F, Hobson M P and Bence S J, 3rd ed., CUP 2006.

Suggested references [المراجع المساعدة للمنهج]

1. "Essential Mathematical Methods for the Physical Sciences", Riley K F, Hobson M P, 1st ed., CUP 2011.
2. "Mathematical Methods in the Physical Sciences", Boas M L, 3rd ed., Wiley 2006.

Marking [توزيع الدرجات]

First Exam	10 marks	Second Exam	10 marks
Activity	10 marks	Final Exam	70 marks

A. A review of the fundamental mathematical concepts for solving differential equations: **(استعراض المفاهيم الرياضية الأساسية لحل المعادلات التفاضلية).**

(Reference: Introduction to Differential Equations, Lecture notes for MATH 2351/2352, Jeffrey R. Chasnov)

A bas ic understanding of calculus is required to undertake a study of differential equation فهم اساسيات حسبان التفاضل والتكامل ضروري لحل المعادلات التفاضليه

The trigonometric functions **الدوال المثلثيه**

The Pythagorean trigonometric identity is متطابقة فيثاغورس المثلثيه هي

$$sin^2x + cos^2x = 1$$

And the addition theorems are نظريات اضافيه مهمه هي

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

The following symmetry are also useful: المتامثله التاليه ايضا ممكن استخدامها

$$\sin(\pi/2 - x) = \cos x \quad , \quad \cos(\pi/2 - x) = \sin x;$$

And

$$\sin(-x) = -\sin(x) \quad \text{(Odd function)} \quad , \quad \cos(-x) = \cos(x) \quad \text{(Even function)}$$

Also, the values of sin x in the first quadrant can be remembered by the rule of quarters, with 0^o=0, 3^0=$\pi/6$, 45^o= $\pi/4$, 60^o= $\pi/3$, 90^o= $\pi/2$:

كذلك قيم دالة الجيب للزوايا في الربع الاول ممكن تذكرها من خلال قانون الارباع

$$\text{Sin}0^o=\sqrt{\tfrac{0}{4}}, \quad \sin30^o=\sqrt{\tfrac{1}{4}}, \quad \sin45^o=\sqrt{\tfrac{2}{4}}, \quad \sin60^o=\sqrt{\tfrac{3}{4}}, \quad \sin90^o=\sqrt{\tfrac{4}{4}}.$$

1. The exponential function and the natural logarithm **الداله الاسيه واللوغارتيم الطبيعي**

The exponential function exp(x)=e^x and natural logarithm ln x are inverse functions satisfying الداله الاسيه ودالة اللوغارتيم الطبيعي هي دوال انعكاسيه لبعضها

$$e^{\ln x} = x, \quad \ln e^x = x$$

The usual rules of exponents apply: القوانين الاعتياديه التي تطبق على الداله الاسيه

$$e^x e^y = e^{x+y}, \quad e^x/e^y = e^{x-y}, \quad (e^x)^p = e^{px}$$

The corresponding rules for the logarithmic function are: القوانين المماثله للدالة اللوغارتيم هي

$$\ln(xy) = lnx + lny, \quad \ln\left({}^{x}\!/_{y}\right) = lnx - lny, \qquad lnx^{p} = plnx$$

2. Definition of the derivative تعريف المشتقه

The derivate of the function y=*f (x)*, denoted as $f'(x)$ or *dy/dx*, is defined as the slope of the tangent line to the curve *y=f (x)*, at the point (*x*,*y*). This slope is obtained by a limit, and is defined as:

اشتقاق الداله يعرف بانه ميل الخط المستقيم المماس للمنحني الذي يمثل الداله عند نقطة معرفة. قيمة الميل ممكن ايجاده باستخدام الحد ويعرف :

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

3. Differentiating a combination of functions الاشتقاق لمجموعة دوال

3.1 The sum or difference rule قانون الجمع اوالطرح

The derivative of the sum *f*(*x*) and *g*(*x*) is اشتقاق جمع دالتين هو

$$(f+g)' = f' + g'$$

Similarly, the derivative of the difference is بصورة مشابها اشتقاق حصل طرح دالتين هو

$$(f-g)' = f' - g'$$

3.2 The product rule قانون الضرب

The derivative of the product of *f*(*x*) and *g*(*x*) is مشتقة حاصل ضرب دالتين هو

$$(fg)' = f'g + fg'$$

And should be memorized as “the derivative of the first times the second plus the first times the derivative of the second”.

ويجب تذكر بانها حاصل ضرب مشتقة الداله الاولى في الداله الثانيه مضاف الى حاصل ضرب مشتقة الداله

الثانيه في الداله الاولى.

3.3 The Quotient rule قانون حاصل القسمه

The derivative of the quotient of *f* (*x*) and *g* (*x*) is مشتقة حاصل قسمة دالتين هو

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

And should be memorized as “the derivative of the top times the bottom minus the top times the derivative of the second”.

ويجب تذكر بانها مشتقة داله البسط في دالة المقام مطروح من حاصل ضرب مشتقة المقام في دالة البسط والمقدار الناتج يقسم على مربع قيمة دالة البسط.

3.4 The chain rule قانون السلسله

The derivative of the composition of $f(x)$ and $g(x)$ is مشتقة الدالة المركبه هو

$$\left(f\left(g(x)\right)\right)^2 = f'\left(g(x)\right) \cdot g'(x)$$

4. Differentiating elementary functions مشتقة الدوال الاوليه

5.1 The power rule قانون القوى

The derivative of a power of x is given by مشتقة دالة القوى تعطى ب

$$\frac{d}{dx} x^p = \mathrm{p} x^{p-1}$$

5.2 Trigonometric functions الدوال المثلثيه

The derivative of sin x and cos x are مشتقة دالة الجيب والجيب تمام هي

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x.$$

We thus say that “the derivative of sine is cosine,” and “the derivative of cosine is minus sine”. Notice that the second derivatives satisfy

ممكن ان نقول ان "مشتقة دالة الجيب هي جيب تمام ومشتقة الجيب تمام هي جيب ". لاحظ ان المشتقة الثانية تحقق

$$(\sin x)'' = -\sin x, \quad (\cos x)'' = -\cos x.$$

5.3 Exponential and natural logarithm functions الدوال الاسيه واللوغارتميه

The derivative of e^x and ln x are مشتقة الداله الاسيه والداله اللوغارتميه هي

$$(e^x)' = e^x, \quad (\ln x)' = \frac{1}{x}$$

5. Definition of the integral تعريف التكامل

The definite integral of a function $f(x) > 0$ from $x = a$ to b ($b > a$) is defined as the area bounded by the vertical lines $x = a$, $x = b$, the x-axis and the curve $y = f(x)$. This “area under the curve” is obtained by a limit. First, the area is approximated by a sum of rectangle areas. Second, the integral is defined to be the limit of the rectangle areas as the width of each individual rectangle goes to zero and the number of rectangles goes to infinity. This resulting infinite sum is called a Riemann Sum, and we define.

التكامل المحدد للداله من النقطة a الى النقطة b وممكن تعريفه بانه المساحه المحدده بين منحني الدالة والمحور x وكل من الخطيين العموديين عند النقطتين a و b . هذه المساحه تحت المنحني ممكن ايجادها بواسطة الحدود. اولا,هذه المساحه تقرب لتكون بشكل جمع مساحات مستطيلات متعدده. ثانيا, التكامل يعرف ليكون الحدود للمساحة لكل مستطيل منفرد من الصفر الى عرض المستطيل وعدد المستطيلات ممكن ان يكون غير منتهي. نتيجه الجمع غير المنتهي تسمى جمع رايمان وتعرف.

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{n=1}^{N} f(a+(n-1)h).\,h_1$$

where $N = (b-a)/h$ is the number of terms in the sum. The symbols on the left-hand-side of Eq. are read as “the integral from a to b of f of x dee x.” The Riemann Sum definition is extended to all values of a and b and for all values of $f(x)$ (positive and negative). Accordingly,

حيث عدد المستطيلات يمكن ايجاده من الفرق بين حدود التكامل مقسم على مساحة كل مستقيم .الرموز في الجهه اليسرى السفلى من المعادله يقرأ كتكامل من النقطه a الى b للداله المعينه لقيم x. تعريف جمع رايمان يوسع لكل قيم a و b ولكل قيم الداله (الموجبه والسالبه), وفقا

$$\int_b^a f(x)\,dx = -\int_a^b f(x)dx \text{ and } \int_a^b \big(-f(x)\big)\,dx = -\int_a^b f(x)\,dx$$

Also, if $a < b < c$, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)dx + \int_b^c f(x)\,dx$$

Which states (when $f(x) > 0$) that the total area equals the sum of its parts.

وبالتي تنص ان لكل دالة قيمتها اكبر من صفر ,فان المساحه تساوي مجموع اجزائها

7.The Fundamental theorem of calculus النظريه الاساسيه لحسبان التفاضل والتكامل

Using the definition of the derivative, we differentiate the following integral:

باستخدام تعريف المشتقة، ونحن نشتق صيغة التكامل التالي:

$$\frac{d}{dx}\int_a^x f(s)ds = \lim_{h\to 0}\frac{\int_a^{x+h} f(s)ds - \int_a^x f(s)ds}{h}$$

$$= \lim_{h\to 0}\frac{\int_a^{x+h} f(s)ds}{h}$$

$$= \lim_{h\to 0}\frac{hf(x)}{h} = f(x)$$

This result is called the fundamental theorem of calculus, and provides a connection between differentiation and integration.

النتيجه اعلاه تسمى النظريه الاساسيه للحسبان التفاضل والتكامل, وتوفر صيغة تربط المشتقات مع التكامل

The fundamental theorem teaches us how to integrate function. Let $F(x)$ be a function such that $F'(x) = f(x)$. We say that $F(x)$ is antiderivative of $f(x)$. Then from the fundamental theorem and the fact that the derivative of a constant equals zero,

النظريه الاساسيه تعلمنا كيفة تكامل الداله. تعرف الداله الناتجه من التكامل بانها معكوس مشتقة الداله. اذن من النظريه الاساسيه وحقيقة ان مشتقة الثابت تساوي صفر.

$$F(x) = \int_a^x f(s)\,ds + c$$

Unfortunately, finding antiderivatives is much harder than finding derivatives, and indeed, most complicated functions cannot be integrated analytically.

للأسف، إيجاد معكوس المشتقات هو أصعب بكثير من العثور على المشتقات، بالواقع، دوال معقده جدا لا يمكن ايجاد تكاملها بالطرق التحليلية.

We can also derive the very important results directly from the definition of the derivative and the definite integral. We will see it is convenient to choose the same h in both limits. With $F'(x)= f(x)$, we have

ممكن ايضا ان نشتق نتيجه مهمه جدا مباشرة من تعريف المشتقه و التكامل المحدد.سوف نرى انه من المناسب اختيار نفس القيمه ل h لحديين.

$$\int_a^b f(s)ds = \int_a^b F'(s)ds$$

$$= \lim_{h\to 0} \sum_{n=1}^{N} F'\,(a+(n-1)h)\cdot h$$

$$= \lim_{h\to 0} \sum_{n=1}^{N} \frac{F(a+nh)-(a+(n-1)h)}{h}\cdot h$$

$$= \lim_{h\to 0} \sum_{n=1}^{N} F(a+nh)-(a+(n-1)h)$$

The last expression has an interesting structure. All the values of *F*(*x*) evaluated at the points lying between the endpoints *a* and *b* cancel each other in consecutive terms. Only the value -*F*(*a*) survives when *n*=1, and the value +*F*(*b*) when *n*=*N*, yielding again

التعبير الأخير لديه بنية مثيرة للاهتمام. كل قيم الداله المحسوبه في النقاط الواقعة بين النهاية *a* و *b*, يلغي كل منهما الآخر في الفترات المتتالية. بالتالي فقط قيمة الداله عند النقطه *a* ممكن حسابها عندما تكون قيمة n=1 ، وقيمة الداله عند النقطة *b* ممكن حسابها عندما *n*=*N*، لنحصل بالنهايه على صيغة مماثلة للتكامل المحدد المذكوره اعلاه.

8. Definite and indefinite integrals التكامل المحدد وغير المحدد

The Riemann sum definition of an integral is called a definite integral. It is convenient to also define an indefinite integral by

$$\int f(x)dx = F(x),$$

Where F(x) is the antiderivative of *f* (*x*) حيث الداله الناتجه هي معكوس المشتقه

9. Indefinite integrals of elementary functions

From our known derivatives of elementary functions, we can determine some simple indefinite integrals. The power rule gives us

من خلال معرفتنا للمشتقات بعض الدوال الاوليه, نستطيع تحديد بعض من تكاملاتها المحدده البسيطه . قانون القوى يعطي

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq 1$$

When *n*=-1, and *x* is positive, we have

عندما تكون *n*=-1 وقيمه *x* موجبة, يكون لدينا

$$\int \frac{1}{x} dx = \ln x + c$$

If *x* is negative, using the chain rule we have

اذا كانت x سالبه, باستخدام قانون السلسله يكون لدينا

$$\frac{d}{dx}\ln(-x) = \frac{1}{x}$$

Therefore, since لذلك, بما ان

$$|x| = \begin{cases} -x & if\ x < 0; \\ x & if\ x > 0; \end{cases}$$

We can generalize our indefinite integral to strictly positive or strictly negative x:

يمكننا تعميم تكاملنا غير المحدد لقيم x الموجبه او السالبه

$$\int \frac{1}{x}dx = \ln|x| + c$$

Trigonometric functions can also be integrated:

الدوال المثلثيه ممكن ايجاد تكاملها ايضا:

$$\int cosx\ dx = sinx + c, \int sinx\ dx = -cosx + c.$$

Easily proved identities are an addition rule

ممكن بسهوله اثبت متطابقات كقواعد اضافيه

$$\int \big(f(x) + g(x)\big) = \int f(x)\,dx + \int g(x)\,dx\,;$$

And application by a constant: والمضروبه بثابت

$$\int Af(x)dx = A\int f(x)dx.$$

This permits integration of functions such as هذا القاعده تسمح لتكامل دوال مثل

$$\int (x^2 + 7x + 2)dx = \frac{x^3}{3} + \frac{7x^2}{2} + 2x + c,$$

And

$$\int (5\ cosx + sinx)dx = 5sinx - cosx + c,$$

10.Substituation الاستبدال

More complicated functions can be integrated using the chain rule, Since

دوال اكثر تعقيدا ممكن ايجاد تكاملها بواسطة استخدام قانون السلسله, حيث ان

$$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x),$$

We have

$$\int f'\,\big(g(x)\big) \cdot g'(x) dx = f\big(g(x)\big) + c.$$

This integration formula is usually implemented by letting *y*=*g*(*x*). Then one write *dy* = *g*\`(*x*) *dx* to obtain

صيغة التكامل ممكن بصورة اعتياديه استخدامها بترك *y*=*g*(*x*). ثم ممكن كتابة *dy* = *g*\`(*x*) *dx* لايجاد

$$\int f'\,\big(g(x)\big) g'(x) dx = \int f'(y) dy \ = f(y) + c \ = f\big(g(x)\big) + c.$$

11. Integration by parts **التكامل بالتجزئه**

Another integration technique makes use of the product rule for differentiation.

تقنية تكامل اخرى يستثمر استغلال قانون السلسله للمشتقات

Since حيث

$$(fg)' = f'g + fg'$$

We have

$$f'g = (fg)' - fg'$$

Therefore,

$$\int f'\,(x) g\,(x) = f(x) g(x) - \int f(x)\, g'(x) dx$$

Commonly, the above integral is done by writing

عادة, التكامل اعلاه ينفذ بكتابة

$$u = g(x) \qquad\qquad dv = f'(x) dx$$
$$du = g'(x) dx \qquad\qquad v = f(x)$$

Then, the formula to be memorized is ثم, الصيغة النهائيه التي يجب تذكرها هي

$$\int u dv = uv - \int v du$$

12. Tylor series سلسله تايلر

A Tylor series of a function *f* (*x*) about a point *x* =*a* is a power series representation of *f* (*x*) developed so that all the derivatives of *f* (*x*) at *a* match all the derivatives of the power series. Without worrying about convergence here,

سلسلة تايلر لداله حول نقطه معينه هي عباره عن اعادة تمثيل الداله المطوره بواسطة سلسلة القوى الاسيه ,بحيث جميع مشتقاتها عند تلك النقطه تطابق مشتقات سلسلة القوى. بدون القلق حول التقارب هنا

We have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Notice that the first term in the power series matches $f(a)$, all other terms vanishing, the second term matches $f'(a)$, all other terms vanishing, etc … Commonly, the Tylor series is developed with a=0. We also make use of the Tylor series in a slightly different form, with $x=x_*+\varepsilon$ and $a=x_*$:

لاحظ ان الحد الاول من سلسلة القوى يطابق $f(a)$, وكل الحدود الاخرى تهمل, الحد الثاني يطابق $f'(a)$, والحدود الاخرى تهمل وهكذا لبقية الحدود. بشكل عام فان سلسه تايلر تم تطويرها مع a=0. ممكن استخدام متسلسلة تايلر مع تحوير بسيط

$$f(x_* + \epsilon) = f(x_*) + f'(x_*)\epsilon + \frac{f''(x_*)}{2!}\epsilon^2 + \frac{f'''(x_*)}{3!}\epsilon^3 + \cdots$$

Another way to view this series is that of $g(x) = f(x_* + \epsilon)$ expand about $\epsilon = 0$.

هناك طريقة اخرى لاظهار تلك السلسلة وهي بان $g(x) = f(x_* + \epsilon)$ توسع حول $\epsilon = 0$

Taylor series that are commonly used include سلسلة تايلر المستخدمه بصوره شائعه تتضمن

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots, \quad for\ |x| < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad for\ |x| < 1$$

A Taylor series of a function of several variables can also be developed. Here all partial derivatives of *f* (*x*, *y*) at (*a*, *b*) match all the partial derivatives of the power series. With the notation

سلسله تايلر لداله تمتلك عدة متغيرات ممكن ان تطور ايضا. حيث جميع المشتقات الجزئيه ل $f(x, y)$ عند (a, b) تطابق المشتقات الجزئيه لسلسلة القوى . مع التنويه

$$f_x = \frac{\partial f}{\partial x}, \qquad f_y = \frac{\partial f}{\partial y}, \quad f_{xx} = \frac{\partial^2}{\partial x^2}, \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \qquad f_{yy} = \frac{\partial^2 f}{\partial x^2}, \quad etc,$$

We have

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+ \frac{1}{2!}\left(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b)\right.$$
$$\left. + f_{yy}(a,b)(y-b)^2\right) + \cdots$$

13. Complex numbers

We define the imaginary number i to be one of the two numbers that satisfies the rule $(i)^2 = -1$, Formally, we write $i = \sqrt{-1}$. A complex number z is written as

ممكن تعريف العدد المركب i ليكون واحد من عددين التي تحقق القاعدة $(i)^2 = -1$. الصيغة الرسميه تكتب $i = \sqrt{-1}$. العدد المركب يكتب

$$z = x + iy$$

where x and y are real numbers. We call x the real part of z and y the imaginary part and write.

حيث x و y هي اعداد حقيقيه. و تسمى x الجزء الحقيقي من z و y الجزء الخيالي .

$$x = Re\, z, \qquad y = \mathrm{Im} z.$$

Two complex numbers are equal if and only if their real and imaginary parts are equal. The complex conjugate of z = x + iy, denoted as $\bar{z}$, is defined as

عددين مركبين يكونان متساويان فقط اذا كان الاجزاء الحقيقة والمعقده متساويه. المرافق للعدد المركب z = x + iy يشار له $\bar{z}$ ويعرف

$$\bar{z} = x - iy$$

Using z and $\bar{z}$, we have

$$\mathrm{Re}\, z = \frac{1}{2}(z + \bar{z}), \qquad \mathrm{Im}\, z = \frac{1}{2i}(z + \bar{z})$$

Furthermore,

$$z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2 + y^2$$

and we define the absolute value of z, also called the modulus of z, by

معامل العدد المركب وممكن تعريف القيمه المطلقه للعدد المركب والتي تسمى المعامل

$$|z| = (z\bar{z})^{1/2}$$
$$= \sqrt{x^2+y^2}$$

We can add, subtract, multiply and divide complex numbers to get new complex numbers. With $z = x + iy$ and $w = s + it$, and x, y, s, t real numbers, we have

ممكن ان نجري عمليات الجمع الطرح الضرب والقسمه على الاعداد المركبه لايجاد عدد مركب جديد

$$z + w = (x + s) + i(y + t); \quad z - w = (x - s) + i(y - t);$$
$$zw = (x + iy)(s + it) = (xs - yt) + i(xt + ys);$$
$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}}$$
$$= \frac{(x + iy)(s - it)}{s^2 + t^2}$$
$$= \frac{(xs+yt)}{x^2+y^2} + i\,\frac{(ys-xt)}{s^2+t^2}$$

Furthermore

$$|zw| = \sqrt{(xs - yt)^2 + (xt + ys)^2}$$
$$= \sqrt{(x^2 + y^2)(s^2 + t^2)}$$
$$= |z||w|$$

And

$$\overline{zw} = (xs - yt) - i(xt + ys)$$
$$= (x - iy)(s - it) = \bar{z}\bar{w}$$

Similarly

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}, \overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\overline{w}}$$

Also, $\overline{z + w} = \bar{z} + \overline{w}$, However, $|z + w| < |z| + |w|$, a theorem as the triangle inequality.

هذه نظريه عدم تساوي المثلث

It is especially interesting and useful to consider the exponential function of an imaginary argument. Using the Taylor series expansion of an exponential function, we have

انه بشكل خاص مثير للاهتمام ومفيد اعتمد الداله الاسيه للمناقشة الجزء التخيلي. باستخدام توزيع سلسلة تايلر للداله الاسيه , يكون لدينا

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$
$$= cos\theta + isin\theta$$

Therefore, we have

$$cos\theta = \text{Re}e^{i\theta}, \qquad sin\theta = \text{Im}e^{i\theta}$$

Since $cos\pi = -1$ and $sin\pi = 0$, we derive the celebrated= Euler's identity

وفقا لشروط المذكوره اعلاه لدالة الجيب تمام والجيب, نستطيع اشتقاق متطابقة اويلر الشهيره

$$e^{-i\theta} + 1 = 0$$

That links five fundamental numbers, 0, 1, i, e and π, using three basic mathematical operations, addition, multiplication and exponential, only once.

هذا العلاقه تربط خمس اعداد اساسيه , وتستخدام ثلاث عمليات رياضيه الجمع الضرب و الاس مرة واحدة.

Using the even property $\cos(-\theta) = cos\theta$ and the odd property $\sin(-\theta) = -sin\theta$, we also have

وباستخدام الخاصية الزوجيه للدالة الجيب تمام والخاصيه الفردية لدالة الجيب نحصل على

$$e^{-i\theta} = cos\theta - isin\theta$$

And the identities for $e^{i\theta}$ and $e^{-i\theta}$ results in the frequently used expressions,

والمتطابقات للدوال الاسيه ينتج عنه التعبير الشائع الاستخدام

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

The complex number *z* can be represented in the complex plane with Re *z* as the *x*-axis and Im *z* as the *y*-axis. This leads to the polar representation of *z*=*x*+i*y*

العدد المركب ممكن تمثيله بواسطة المستوى المعقد مع الاشارة للجزء الحقيقي ليكون على محور السينات والجزء التخيلي على محور الصادات. وهذا يقود الى التمثيل بواسطة المحاور القطبيه

$$z = re^{i\theta},$$

Where $r = |z|$ and $\tan\theta = {}^{y}/_{x}$. We define $\arg z = \theta$. Note that θ is not unique, through it is conventional to choose the value such that $-\pi \le \theta \le \pi$, and $\theta = 0$ where $r = 0$.

انه من المناسب اختيار قيم الزوايا وقيمة طول المحور كما مذكور اعلاه.

Differential Equations (DE's):

Differential Equation are the language in which the laws of nature are expressed. Understanding properties of solutions of differential equations is the Fundamental to much of contemporary science and engineering. (Reference: MIT)

المعادلات التفاضليه هي اللغه التي يتم استخدامها للتعبير عن قوانين الطبيعه. فهم خصائص حل المعادلات التفاضليه امر اساسي لكثير من تطبيقات العلوم والهندسه المعاصره.

A differential equation is an equation involving an unknown function and its derivatives. (Reference: DE SCHAUMS)

المعادله التفاضليه هي عباره عن معادله تحتوي على داله مجهوله ومشتقاتها.

The differential equations can be classified into two kinds; An ordinary differential equation (ODE), and a partial differential equation (PDE).

المعادلات التفاضليه ممكن ان تقسم الى نوعين: المعادلات التفاضليه الاعتياديه و المعادلات التفاضليه الجزئيه.

1. An ordinary differential equation (ODE) is a differential equation for a function of a single variable, (i.e. unknown function depends on only one independent variable) e.g. *x(t)*.

المعادله التفاضليه الاعتياديه هي معادله تفاضليه للداله ذات متغير واحدا (اي داله مجهوله تعتمد على فقط متغير مستقل واحد) .

2. A partial differential equation (PDE) is a differential equation for a function of several variables, (i.e. unknown function depends on two or more independent variables) e.g. *v(x,y,z,t)*.

المعادله التفاضليه الجزئيه هي معادله تفاضليه للداله ذات عدة متغيرات (اي داله مجهوله تعتمد على متغيريين مستقلين او اكثر) .

Several formulas of the derivatives can be used to express the differential equations.

عدة صيغ للمشتقات ممكن ان تستخدام للتعبير عن المعادلات التفاضليه

$$y_x, \dot{y}, y', y'', Dy, \quad or \quad \frac{dy}{dx}, \frac{\Delta y}{\Delta x}, \frac{\partial y}{\partial x}, \frac{\partial f}{\partial x \partial y \partial z}, f_{xyz}$$

Examples of several formula of differential equations involving the unknown function y.

امثلة لعدة صيغ من المعادلات التفاضليه المتضمنه داله مجهوله y.

$$\frac{dy}{dx} = 5x + 3 \qquad \text{(ODE)}$$

$$e^y \frac{d^2y}{dx^2} + 2\left\{\frac{dy}{dx}\right\}^2 = 1 \qquad \text{(ODE)}$$

$$4\frac{d^3y}{dx^3} + (sinx)\frac{d^2y}{dx^2} + 5xy = 0 \qquad \text{(ODE)}$$

$$\left\{\frac{d^2y}{dx^2}\right\}^3 + 3y\left\{\frac{dy}{dx}\right\}^7 + y^3\left\{\frac{dy}{dx}\right\}^2 = 5x \qquad \text{(ODE)}$$

$$\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0 \qquad \text{(PDE)}$$

In this course we will be concerned solely with ordinary differential equations

خلال هذا الكورس سوف نتناول دراسة فقط المعادلات التفاضليه الاعتياديه

Differential equations are often classified with respect to order. The order of a differential equation is the order of the highest order derivative present in the equation.

المعادلات التفاضليه غالبا تصنف بالنسبه الى الرتبة. رتبة المعادله التفاضليه هي رتبة اعلى مشتقه موجودة في المعادله.

$$\frac{d^3y}{dx^3} + 4x\left(\frac{dy}{dx}\right)^2 = y\frac{d^2y}{dx^2} + e^x \qquad 3^{rd} \text{ order DE}$$

The expressions $y', y'', y''', y^{(4)}, \dots, y^{(n)}$ are often used to represent, respectively, the first, second, third, fourth, . . ., nth derivatives of y with respect to the independent variable under consideration.

التعبيرات اعلاه غالبا تستخدام للتعبير على التوالي عن المشتقات الاول, الثاني, الثالث, الرابع , ...,n مشتقة للدالة y مع المتغير المعتمد.

The degree of a differential equation is the power of the highest order derivative in the equation.

درجة المعادله التفاضليه هي القوى الاسيه لاعلى مشتقه في المعادلة التفاضليه.

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = \sin x \qquad 2^{nd} \text{ order , } 3^{rd} \text{ degree DE}$$

$$\frac{d^2y}{dx^2} + 5\left\{\left(\frac{dy}{dx}\right)^2 + y\right\}^{1/3} = 0 \qquad 2^{nd} \text{ order , } 3^{rd} \text{ degree DE}$$

General solution for DE

A general solution of a differential equation in the unknown function y and the independent variable x on the interval $\wp$ is a function $y(x)$ that satisfies the differential equation identically for all x in $\wp$.

الحل العام للمعادلة التفاضليه لداله غير معروفه وتعتمد على متغير مستقل واحد و ضمن فترة معينة هي عبارة عن داله تعتمد على ذلك المتغير وتحقق المعادلة التفاضليه ضمن تلك الفتره.

Example: Is $y(x) = c_1\, sin2x + c_2\, cos2x$, where c_1 and c_2 are arbitrary constants, a solution of y''+4y=0

Differentiating y, we found $y' = 2c_1\, cos2x - 2c_2\, sin2x$ and $y'' = -4c_1\, sin2x - 4c_2\, cos2x$. Here

$$y' + 4\text{y} = (-4c_1\, sin2x - 4c_2\, cos2x) + 4(c_1\, sin2x + c_2\, cos2x)$$

$$= (-4c_1 + 4c_1)\, sin2x + (-4c_2 + 4c_2)\, cos2x$$

The $y(x) = c_1\, sin2x + c_2\, cos2x$ satisfies the differential equation for all values of *x* and is a solution on the interval (-∞, ∞).

الداله اعلاه تحقق المعادله التفاضليه لكل قيم *x* وتعتبر حل للترة غير محدده.

Example:

Determine whether $y = x^2 - 1$ is a solution of $(y')^4 + y^2$= -1.

Note that the left side of the differential equation must be nonnegative for every real function *y*(*x*) and any *x*, since it is the sum of terms raised to the second and fourth powers, while the right side of the equation is negative. Since no function *y*(*x*) will satisfy this equation, the given differential equation has no solutions. We see that some differential equations have *infinitely many solutions*, whereas other differential equations have *no solutions*. It is also possible that a differential equation has *exactly one solution.*

ملاحظه: الجانب الايسر من المعادله التفاضليه اعلاه يجب ان يكون غير سالب لكل الدوال الحقيقيه ولكل قيم *x*. وذلك لان للجزء الاول والثاني من المعادله التفاضليه مرفوع الى اس 2 و 4 . وبما ان الجزء الايمن من المعادله هو سالب.

لذلك لايمكن ايجاد اي داله ممكن ان تكون حل لهذه المعادلة التفاضليه. بالنتيجه ممكن ان نستنتج ان المعادلات التفاضليه ممكن ان يكون لايوجد حل لها . في حين بعض الدوال التفاضليه يمكن ان يوجد لها عدد غير محدد من الحلول, وكذلك ممكن ان يكون لها حل وحيد فقط.

Every particular solution of the differential equation has this general form. A few particular solutions are: (a) $y(x) = 5\, sin2x - 3cos2x$ (choose $c_1 = 5$ and $c_2 = -3$), (b) $y(X) = sin2x$ (choose $c_1 = 1$ and $c_2 = 0$), and (c) $y(x) = 0$ (choose $c_1 = 0$ and $c_2 = 0$).

The general solution of a differential equation cannot always be expressed by a single formula. As an example consider the differential equation y'+4y=0, which has two particular solutions and $y(x) = {}^{1}/_{x}$ and y=0.

الحل العام للمعادله التفاضليه لا يمكن ان ياخذ تعبير رياضي واحد بل من الممكن ان يكون هناك عدة تعابير.

Initial-conditions and Boundary conditions **الشروط الاوليه والشروط الحديه**

في بعض مسائل المعادلات التفاضليه الاعتياديه ممكن تعريف بعض الشروط الاوليه التي تتحققبحل تلك المعادلات.
هذا الشروط تساعد على تحديد قيم الثوابت الاختياريه المعرفه ضمن الحل العام للمعادله التفاضليه.

Example: Find the solution of DE $y' = 2x$, that satisfy the condition $y(2) = 3$.

$$y = x^2 + c$$

$$\therefore 3 = 4 + c, \longrightarrow c = -1$$

The final solution is

$$y = x^2 - 1$$

First- order ordinary differential equations المعادلات التفاضليه الاعتياديه من الرتبه الاولى

Standard form for a first-order differential equation in the unknown function y(x) is:

الصيغه القياسيه للمعادله التفاضليه الاعتياديه من الدرجه الاولى هي:

$$y' = \frac{dy}{dx} = f(x, y)$$

Or alternative form which is

$$M(X, Y)dx + N(x, y)dy = 0$$

Separation of Variables **فصل المتغيرات**

A separable-variable equation is one which may be written in the conventional form

المعادله ذات المتغيرات المفصوله ممكن ان تكتب بالصيغة التاليه

$$\frac{dy}{dx} = f(x)g(y) \quad or \quad f(x)dx + g(y)dy = 0$$

where $f(x)$ and $g(y)$ are functions of x and y respectively, including cases in which $f(x)$ or $g(y)$ is simply a constant. Rearranging this equation so that the terms depending on x and on y appear on opposite sides (i.e. are separated), and integrating, we obtain

حيث $f(x)$ و $g(y)$ هي دوال ل x و y على التوالي . والني ممكن ان تكون احدى هاتين الدالنين ثابت . اعادة ترتيب المعادله التفاضليه وجعل كل حد يعتمد على متغير واحد فقط وعلى طرفي المعدله ومن ثم ايجاد التكامل لكل حد هو اساس الحل بطريقة فصل المتغيرات.

$$\int f(x)dx + \int g(y)dy = c$$

Example1: Find the solution for a differential equation following:

$$\frac{dy}{dx} = x + xy$$

Sol.

$$\int \frac{dy}{1+y} = \int x dx = \ln(1+y) = \frac{x^2}{2} + c,$$

$$1+y = \exp(\frac{x^2}{2} + c) = A\exp(\frac{x^2}{2}),$$

$Where\ A = exp$(c) and both c and A are an arbitrary constant.

Example2: Find the solution for a differential equation following:

$$e^x \cos y\, dx + (1 + e^x) \sin y\, dy = 0$$

Sol.

$$\int \frac{e^x}{1 + e^x}\, dx + \int \frac{\sin y}{\cos y}\, dy = 0$$

$$\therefore\ \ln(1 + e^x) - \ln(\cos y) = \ln c$$

$$\ln \frac{(1+e^x)}{\cos y} = \ln c, \quad 1 + e^x = c\,(\cos y)$$

Example3: Find the solution for a differential equation following:

$$\frac{dy}{dx} = \frac{2\cos 2x}{3 + 2y}$$

Sol.

$$\int (3 + 2y) dy = \int 2\cos 2x\, dx$$

$$3y + y^2 = \sin 2x + c$$

The particular solution it the condition y (0) = -1 is:

$$\int_{-1}^{y} (3 + 2y) dy = \int_{0}^{x} 2\cos 2x\, dx$$

$$3y + y^2\,]_{-1}^{y} = \sin 2x\,]_{0}^{x}$$

$$y^2 + 3y + 2 - \sin 2x = 0$$

$$y_{\pm} = \frac{1}{2}\left[-3 \pm \sqrt{1 + 4\sin 2x}\right]$$

H.w.

Q.1 Find the general solution of $y' + e^x y = e^x y^2$

Q.2 Find the general solution of $y' = 3x^2 e^{-y}$ and the particular solution that satisfies the condition y (0)=1.

Q.3 Find the solution of $y' = e^{2x+y}$ that has y = 0 when x =0.

Q.4 Find the general solution of $x \ sin^2 y y' = (x+1)^2$.

Q.5 Solve $y' = -2\, x\, tany$ subject to the condition $y = \frac{\pi}{2}$ when x=0.

Q.6 Find the general solution of $\frac{1}{y} y' = \frac{x}{x^2+1}$.

Q.7 Find the general solution of $cosec^3 x y' = cos^2 y$.

Q.8 Find the general solution of $(1-x^2)y' = x(y-a) = 0$ where a is a constant.

Exact equations: **المعادلات التامه**

The formula of exact differential equation is:

صيغة المعادله التفاضليه التامه هي:

$$df\ (x,y) = \frac{\partial f}{\partial x}\partial x + \frac{\partial f}{\partial y}\partial y = 0$$

$$M(x,y)dx + N(x,y)dy = 0$$

Where $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$

Test for exactness: If M (x, y) and N (x, y) are continuous functions and have continuous first partial derivatives on some rectangle of the xy-plane, then exact equation is exact if and only if

اختبار وجود المعادله التامه: اذا كانت M (x, y) و N (x, y) هي دوال مستمره ولديها استمراريه ايضا للمشتقة الجزئيه الاولى داخل بعض المستطيلات في المستوي-xy . اذا الداله التامه موجوده فقط واذا فقط يتحقق الشرط التالي:

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

The solution of exact differential equation subject to the following rule:

$$F(x,y) = \int M(x,y)dx + G(y)$$

The function G(y) can be found from $\frac{\partial f}{\partial y} = N(x,y)$ by differentiating the equation above with respect to y and equating to N (x, y).

الداله $G(y)$ ممكن ايجاده من $N(x,y) = \frac{\partial f}{\partial y}$ بواسطة اشتقاق المعادله اعلاه بالنسبه ل y ومساوتها مع N (x, y).

Example: Given $F(x, y) = x^3 siny + y^2 x$ then its partial derivatives are:

$$M(x, y) = \frac{\partial F}{\partial x} = 3x^2 siny + y^2 \quad \text{and} \quad N(x, y) = \frac{\partial F}{\partial y} = x^3 cosy + 2yx$$

Therefore d F(x, y) = $(3x^2 siny + y^2)\, dx + (x^3 cosy + 2yx)\, dy$

Test the condition of exactness:

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

$$3x^2 cosy + 2y = 3x^2 cosy + 2y$$

Example: Solve the differential equation $\frac{1}{x} dy - \frac{y}{x^2} dx = 0$

Sol.

Test the exactness

$$\frac{\partial N(x, y)}{\partial x} = \frac{\partial M(x, y)}{\partial y}$$

$$\frac{-1}{x^2} = \frac{-1}{x^2}$$

$$F(x, y) = \int \frac{1}{x} dx + G(y) = c_1$$

$$F(x, y) = \frac{y}{x} + G(y) = c_1$$

$$\frac{d}{dy} F(x, y) = \frac{d}{dy}\left(\frac{y}{x} + G(y) = c_1\right) = N(x, y)$$

$$\frac{1}{x} + \frac{d\, G(y)}{dy} = \frac{1}{x} \implies \frac{d\, G(y)}{dy} = 0 \implies G(y) = c_2$$

$$\therefore \frac{y}{x} + c_2 = c_1 \implies y = cx \quad where \ c = c_1 - c_2$$

Example: Solve the differential equation $(e^{4x} + 2xy^2)dx + (cosy + 2x^2 y)dy = 0$

Sol.

Test the exactness

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

$$4xy = 4xy$$

$$F(x, y) = \int (e^{4x} + 2xy^2)dx + G(y) = c_1$$

$$F(x,y) = \frac{1}{4}e^{4x} + x^2y^2 + G(y) = c_1$$

$$\frac{d}{dy}F(x,y) = \frac{d}{dy}\left(\frac{1}{4}e^{4x} + x^2y^2 + G(y) = c_1\right) = N(x,y)$$

$$2x^2y + \frac{d\,G(y)}{dy} = cosy + 2x^2y$$

$$\therefore\ \frac{d\,G(y)}{dy} = cosy$$

$$G(y) = \sin y + c_2$$

$$\frac{1}{4}e^{4x} + x^2y^2 + \sin y + c_2 = c_1 \Rightarrow \frac{1}{4}e^{4x} + x^2y^2 + \sin y = c, \qquad \text{where } c = c_1 - c_2$$

H.W.

Q.1 Find the general solution of $2(y+1)e^x dx + 2(e^x - 2y)dy = 0$

Q.2 Find the general solution of $(2xy + 6x)dx + (x^2 + 4y^3)dy = 0$

Q.3 Find the general solution of $(3x^2 + ycosx)dx + (sinx - 4y^3)dy = 0$

Q.4 Find the general solution of $xtan^{-1}ydx + \frac{x^2}{2(1+y^2)}dy = 0$

Q.5 Find the general solution of $(2x^3 - 3x^2y + y^3)\frac{dy}{dx} = 2x^3 - 6x^2y + 3xy^2$

Q.6 Find the general solution of $(y^2cosx - sinx)dx + (2ysinx + 2)dy = 0$

Inexact equations: integrating factors **المعادله غير التامه: معامل التكامل**

Equations that may be written in the form الداله ممكن ان تكتب بالصيغة

$$M(x,y)dx + N(x,y)dy = 0$$

But for which $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$

Are known as inexact equations. However, the differential $Mdx + Ndy$ can always be made exact by multiplying by an integrating factor I(x, y), which obeys

تعرف بمعادله غير تامه. على كل حال, هذه الداله ممكن ان تحول الى تامه بواسطة ضربها بعامل تكامل والذي يتبع

$$\frac{\partial IM(x,y)}{\partial y} = \frac{\partial IN(x,y)}{\partial x}$$

$$I[M(x,y)dx + N(x,y)dy] = 0$$

For an integrating factor that is a function of both *x* and y, i.e. I= I (x, y), there exists no general method for finding it; in such cases it may sometimes be found by inspection. If, however, an integrating factor exists that is a function of either *x* or *y* alone then equation above can be solved to find it. For example, if we assume that the integrating factor is a function of x alone, i.e. I= I(x), then equation reads

للعامل التكامل الذي يكون داله لكل من x و y . لاتوجد طريقه عامه لايجاده. وفي هذه الحاله ممكن ايجاده بواسطة بالتوقع. اذا على كل حال, اذا كان يوجد عامل تكامل كداله لمتغير واحد فقط ,اذن المعادله اعلاه ممكن حلها لايجاد هذا العامل. كمثال, اذا افترضنا ان عامل التكامل كداله ل x. اذا لمعادلة تقرا

$$I\frac{\partial M(x,y)}{\partial y} = I\frac{\partial N(x,y)}{\partial x} + N(x,y)\frac{dI}{dx}$$

Rearranging this expression, we find *باعادة ترتيب التعبير, نجد*

$$\frac{1}{I}\frac{dI}{dx} = \frac{1}{N}\left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right)\text{dx} = f(x)\ \text{d}x$$

where we require $f(x)$ also to be a function of x only; indeed, this provides a general method of determining whether the integrating factor I is a function of x alone. This integrating factor is then given by

حيث من الضروري f (x) ان تكون داله ل x فقط. وهذا يكون كافي لتوفير طريقه عامه لتحيد عامل التكامل , *والذي يعطى بواسطة*

$$I = \exp\{\int f(x)dx\}$$

Where $$f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial X}\right)$$

Similarly, if I = I(y) then

$$I = \exp\{\int g(y)dy\}$$

Where $$g(y) = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Example: Solve the differential equation $\frac{dy}{dx} = -\frac{2}{y} - \frac{3y}{2x}$

Sol.

Rearranging into the following form:

$$(4x + 3y^2)dx + 2xydy = 0$$

Test the exactness

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

$$6y \neq 2y$$

So, the ODE is not exact in its present form.

However, we see that

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{2}{x} \ ,$$

a function of *x* alone.

Therefore, an integrating factor exists that is also a function of *x* and, ignoring arbitrary constant, is given by

$$I(x) = exp\left\{\int \frac{2}{x}\,dx\right\} = exp(2lnx) = x^2$$

Multiplying the ODE above by this integrating factor, we obtain

$$(4x^3 + 3x^2y^2)dx + 2x^3ydy = 0$$

Test the exactness again

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

$$6x^2y \ = \ 6x^2y$$

Know the ODE is exact with this form.

$$F(x,y) = \int (4x^3 + 3x^2y^2)dx + G(y) = c_1$$

$$F(x,y) = x^4 + x^3y^2 + G(y) = c_1$$

$$\frac{dF(x,y)}{dy} = 2x^3y \ + \frac{dG(y)}{dy} = N(x,y)$$

$$\frac{dF(x,y)}{dy} = 2x^3y \ + \frac{dG(y)}{dy} = 2x^3y$$

$$\therefore \ \frac{dG(y)}{dy} = 0 \ \Longrightarrow \ G(y) = c_2$$

By inspection this integrates immediately to give the solution $x^4 + y^2x^3 = c$,

where $c = c_1 + c_2$.

Example: Solve the differential equation $(5xe^{-y} + 2cos3x)y' + (5e^{-y} - 3sin3x) = 0$

Rearranging into the following form:

$$(5xe^{-y} + 2cos3x)dy + (5e^{-y} - 3sin3x)dx = 0$$

Test the exactness

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

$$-5e^{-y} \neq 5e^{-y} - 6sin3x$$

So, the ODE is not exact in its present form.

However, we see that

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{y}\right)=\frac{5e^{-y}-6sin3x-(-5e^{-y})}{5e^{-y}-3sin3x}=\frac{10e^{-y}-6sin3x}{5e^{-y}-3sin3x}=2\,,$$

a function of *y* alone.

Therefore, an integrating factor exists that is also a function of *y* and, ignoring arbitrary constant, is given by

$$I(y)=exp\left\{\int 2\,dy\right\}=exp(2y)$$

Multiplying the ODE above by this integrating factor, we obtain

$$(5xe^{y}+2cos3xe^{2y})dy+(5e^{y}-3sin3xe^{2y})dx=0$$

Test the exactness again

$$\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$$

$$5e^{y}-6sin3xe^{2y}=\ 5e^{y}-6sin3xe^{2y}$$

Know the ODE is exact with this form.

$$F(x,y)=\int(5e^{y}-3sin3xe^{2y})dx+G(y)=c_1$$

$$F(x,y)=5xe^{y}+cos3xe^{2y}+G(y)=c_1$$

$$\frac{dF(x,y)}{dy}=5xe^{y}+2cos3xe^{2y}+\frac{dG(y)}{dy}=N(x,y)$$

$$\frac{dF(x,y)}{dy}=5xe^{y}+2cos3xe^{2y}+\frac{dG(y)}{dy}=5xe^{y}+2cos3xe^{2y}$$

$$\therefore\ \frac{dG(y)}{dy}=0\ \Longrightarrow\ G(y)=c_2$$

$$\therefore 5xe^{y}+cos3xe^{2y}=c\quad \text{where}\quad c=c_1-c_2$$

Linear Equations

Linear first-order ODEs are a special case of inexact ODEs and can be written in the conventional form

المعادلات التفاضليه الاعتياديه الخطيه من الدرجه الاولى هي حاله خاصة من المعادلات غير التامه وممكن ان تكتب بالصيغه التقليديه

$$\frac{dy}{dx}+P(x)y=Q(x)$$

Such equations can be made exact by multiplying through by an appropriate integrating factor which is always a function of x alone. An integrating factor I(x) must be such that

هذه المعادلات ممكن ان تصبح تامه من خلال ضربها بعامل تكامل مناسب والذي يكون دائما داله ل x فقط. عامل التكامل يجب I(x) ان يكون كما

$$I(x)\frac{dy}{dx}+I(x)P(x)y=\frac{d}{dx}[I(x)y]=I(x)Q(x)$$

which may then be integrated directly to give

$$I(x)y=\int I(x)Q(x)dx$$

The required integrating factor I(x) is determined by the first equality of a pervious equation for above equation

عامل التكامل المطلوب يحدد بواسطة المساواة للحد الاول للمعادله السابقه للمعادله اعلاه.

which gives the simple relation والتي تعطى الصيغه البسيطه

$$\frac{dI}{dx}=I(x)P(x)=I(x)=e^{\{\int P(x)dx\}}$$

Similarly, if I = I(y) then

$$\frac{dx}{dy}+H(y)x=K(y)$$

And

$$\mathrm{I(y)}=e^{\{\int H(y)dy\}}$$

Example: Solve $\frac{dy}{dx}+2xy=4x$

Answer: The integrating factor is given by

$$\mathrm{I}(x)=e^{\{\int P(x)dx\}}=e^{\{\int 2xdx\}}=e^{x^2}$$

Multiplying through the ODE by I(x) $=e^{x^2}$, and integrating, we have

$$e^{x^2}y=4\int x\,e^{x^2}dx=2e^{x^2}+c$$

The solution to the ODE is therefore given by

$$y=2+ce^{-x^2}$$

Example: Solve the differential equation $\frac{dy}{dx}+3x^2y=6x^2$

Answer: The integrating factor is given by

$$\mathrm{I}(x)=e^{\{\int P(x)dx\}}=e^{\{\int 3x^2dx\}}=e^{x^3}$$

Multiplying both sides of the differential equation by e^{x^3} , we get

$$e^{x^3}\frac{dy}{dx}+3x^2e^{x^3}y=6x^2e^{x^3}$$

Or

$$\frac{d\left(e^{x^3}y\right)}{dx}=6x^2e^{x^3}$$

Integrating both sides, we have

$$e^{x^3}y=\int 6x^2e^{x^3}dx$$

$$e^{x^3}y=2e^{x^3}+c \Leftrightarrow y=2+ce^{-x^3}$$

Example: Find the solution of the initial-value problem

$$x^2y'+xy=1 \qquad x>0 \quad y(1)=2$$

Answer: We must first divide both sides by the coefficient of x^2 to put the differential equation into standard form:

$$y'+\frac{1}{x}y=\frac{1}{x^2}$$

The integrating factor is

$$\mathrm{I}(x)=e^{\{\int P(x)dx\}}=e^{\left\{\int\frac{1}{x}dx\right\}}=e^{lnx}=x$$

Multiplying both sides of the differential equation by x , we get

$$xy'+y=\frac{1}{x} \; or \; (xy)'=\frac{1}{x}$$

Then $\qquad xy=\int\frac{1}{x}dx$=ln$x$+c

And also $\qquad y=\frac{lnx+c}{x}$

Since $y(1)=2,$ we have $2=\frac{ln1+c}{1}=c$

Therefore, the solution to the initial-value problem is

$$y=\frac{lnx+2}{x}$$

H.W.

Q.1 Solve the differential equation y'+2y=2e^x

Q.2 Solve the differential equation xy'+y=$\sqrt{x}$

Q.3 Solve the differential equation x^2y'+2xy=cos^2x

Q.4 Solve the differential equation $\frac{dy}{dx} = xsin2x + ytanx, \quad -\pi/2 < \ x < \pi/2$

Q.5 Solve the initial-value problem $\frac{dv}{dt} - 2tv = 3t^2 e^{t^2}, \ v(0) = 5$

Q.6 Solve the initial-value problem $xy' = y + x^2 sinx, \qquad y(\pi) = 0$

Homogeneous Equations المعادلة التفاضليه المتجانسه

Homogeneous equations are ODEs that may be written in the form

المعادلات المتجانسه هي معادله تفاضليه اعتياديه التي ممكن ان تكتب بالشكل

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} = F\left(\frac{y}{x}\right)$$

where $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree. A function $f(x,y)$ is homogeneous of degree n if, for any λ, it obeys

حيث $A(x,y)$ و $B(x,y)$ كلاهما دوال متجانسه بنفس الدرجه. الداله $f(x,y)$ هي متجانسه من الدرجه n. اذا لاي λ تكون تطيع

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

For example, if $A = x^2y - xy^2$ and $B = x^3 + y^3$ then we see that A and B are both homogeneous functions of degree 3

The RHS of a homogeneous ODE can be written as a function of y/x. The equation can then be solved by making the substitution $y = vx$ so that.

الجانب الايمن من المعادله التفاضليه الاعتياديه المتجانسه ممكن ان تكتب كداله y/x . المعادله اذن ممكن ان تحل من خلال تعويض $y = vx$ بحيث

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = F(v)$$

This is now a separable equation and can be integrated to give

هذه الان هي معادله مفصوله و ممكن ان تكامل لتعطي

$$\int \frac{v}{F(v) - v} = \int \frac{dx}{x}$$

Example: Solve $$\frac{dy}{dx} = \frac{y}{x} + tan\left(\frac{y}{x}\right)$$

Answer:

Substituting $y = vx$, we obtain

$$v + x\frac{dv}{dx} = v + tanv$$

Cancelling v on both sides, rearranging and integrating gives

$$\int cotv\, dv = \int \frac{dx}{x} = lnx + c_1$$

But

$$\int cotv\, dv = \int \frac{cosv}{sinv} = \int \frac{dx}{x} = ln(sinv) + c_2$$

so the solution to the ODE is $y = x\sin^{-1} Ax$, where A is a constant

Example: Solve the following differential equations

$$(x^2 - 3y^2)dx + 2xydy = 0$$

The coefficients of the differential equations are homogeneous, since for any a ≠ 0

$$\frac{a^2x^2 - 3a^2y^2}{2(ax)(ay)} = \frac{a^2x^2 - 3a^2y^2}{2a^2xy} = \frac{x^2 - 3y^2}{2xy}$$

Substituting $y = vx,$ we obtain

$$(x^2 - 3v^2x^2)dx + 2vx^3dv + 2v^2x^2dx = 0$$

$$(x^2 - 3v^2x^2 + 2v^2x^2)dx + 2vx^3dv = 0$$

$$(x^2 - v^2x^2)dx + 2vx^3dv = 0$$

$$x^2(1 - v^2)dx + 2vx^3dv = 0$$

separating variables

$$\frac{1}{x}\, dx + \frac{2v}{1 - v^2}dv = 0$$

$$\frac{2v}{v^2 - 1}dv = \frac{1}{x}\, dx$$

Integrating

$$\int \frac{2v}{v^2 - 1}dv = \int \frac{1}{x}\, dx$$

$$ln(v^2 - 1) = lnx + lnc$$

$$ln(v^2 - 1) = lncx$$

$$|v^2 - 1| = |cx|$$

replacing v = y/x,

$$\left|\left(\frac{y}{x}\right)^2 - 1\right| = |cx| \;\; or \;\; |y^2 - x^2| = |cx|\, x^2$$

Example: Solve the following differential equations

$$\left(x\sin\frac{y}{x} - y\cos\frac{y}{x}\right)dx + x\cos\frac{y}{x}dy = 0$$

It is readily seen that the differential equation is homogeneous. Putting *y* = *xv* we obtain

$$(x\sin v - xv\cos v)dx + x\cos v(vdx + xdv) = 0$$

$$x\sin v\ dx + x^2\cos v\ dv = 0 \quad or \quad \sin v\ dx + x\cos v\ dv = 0$$

separating variables

$$\frac{1}{x}\ dx + \frac{\cos v}{\sin v}\ dv = 0$$

By integrating,

$$\ln x + \ln v = \ln c \quad or\ \ln cx = \ln \sin v$$

$$\therefore \sin v = cx$$

replacing v = y/x,

$$\sin\frac{y}{x} = cx$$

$$y = x\sin^{-1}(cx)$$

H.W.

Q.1 Find the general solution of $\frac{dy}{dx} = \frac{xy+y^2}{x^2}$

Q.2 Find the general solution of $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

Q.3 Find the general solution of $x\frac{dy}{dx} = y + xe^{\frac{y}{x}}$

Q.4 Solve $2xy\frac{dy}{dx} = x^2 + y^2$ given that *y* = 0 at *x* = 1

Q.5 Solve $\frac{dy}{dx} = \frac{x+y}{x}$ and find the particular solution when *y* (1) = 1

Isobaric equations **المعادلات الايزوباريك**

An isobaric ODE is a generalization of the homogeneous ODE and is of the form

المعادله التفاضليه الاعتياديه الايزوباريه هي صيغه عامه للمعادلات المتجانسه وتكون بالصيغه

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$$

where the RHS is dimensionally consistent if *y* and *dy* are each given a weight m relative to *x* and *dx*, i.e. if the substitution $y = vx^m$ makes the equation separable.

حيث الجانب الايمن يكون متسق الابعاد اذا *y* و *dy* كلاهما تعطى كوزن *m* بالنسبه الى *x* و *dx*, اي اذا عوضنا $y = vx^m$ تصبح الداله مفصوله.

Example: Solve $\frac{dy}{dx} = \frac{-1}{2xy}\left(y^2 + \frac{2}{x}\right)$

Answer: Rearranging we have

$$\left(y^2 + \frac{2}{x}\right)dx + 2xydy = 0$$

Giving *y* and *dy* the weight *m* and *x* and *dx* the weight 1, the sums of the powers in each term on the LHS are $2m + 1$, 0 and $2m + 1$ respectively. These are equal if $2m + 1 = 0$, i.e. if $m = -\frac{1}{2}$. Substituting $y = vx^m = vx^{-1/2}$, with the result that $dy = x^{-1/2}\, dv - \frac{1}{2}vx^{-3/2}\, dx$, we obtain

$$vdv + \frac{dx}{x} = 0$$

which is separable and integrated to give

Replacing *v* by $y\sqrt{x}$, we obtain the solution $\frac{1}{2}y^2\, x + lnx = c$.

Example: Solve $2x^3y' = 1 + \sqrt{1 + 4x^2y}$

Answer: The weights of each term are 3 +(m-1),0,$\frac{1}{2}(0{,}2 + m)$, if m =-2, every term has the same weight. Substituting $y = vx^m = vx^{-2}$, with the result that $y' = x^{-2}v' - 2vx^{-3}$, we obtain

$$2x^3\,(x^{-2}v' - 2vx^{-3}) = 1 + \sqrt{1 + 4v}$$

$$2xv' - 4v = 1 + \sqrt{1 + 4v}$$

$$2xv' = 1 + 4v + \sqrt{1 + 4v}$$

$$2x\frac{dv}{dx} = 1 + 4v + \sqrt{1 + 4v}$$

$$\frac{dv}{1 + 4v + \sqrt{1 + 4v}} = \frac{dx}{2x}$$

$$\frac{dv}{\sqrt{1 + 4v}\left(\sqrt{1 + 4v} + 1\right)} = \frac{dx}{2x}$$

$$\frac{d\left(\sqrt{1 + 4v} + 1\right)}{2\left(\sqrt{1 + 4v} + 1\right)} = \frac{dx}{2x}$$

$$ln\left(\sqrt{1 + 4v} + 1\right) = lnx + c$$

$$\frac{\sqrt{1 + 4v} + 1}{x} = c$$

$$\frac{\sqrt{1+4x^2y}+1}{x}=c$$

H.W.

Q.1 Solve $y^2+(1+xy)y'=0$

Q.2 Solve $x^3y'-x^2y+y^2=0$

Q.3 Solve $2x^2y'-x^2y^2+2xy+1=0$

Q.4 Solve $x^3y'+4x^2y+1=0$

Q.5 Solve $(x+2x^2y)y'+2y+3xy^2=0$

Bernoulli's equations

A Bernoulli differential equation is an equation of the form

معادلة برنولي التفاضليه هي معادلة تكون بالصيغه

$$\frac{dy}{dx}+P(x)y=Q(x)y^n$$

where n denotes a real number. When $n = 1$ or $n = 0$, a Bernoulli equation reduces to a linear equation.

حيث n تشير الى عدد حقيقي. وعندما n تساوي واحد او صفر فان معادلة برنولي تتحول الى معادله خطيه.

To find the solution, change the dependent variable from y to z, where $z=y^{1-n}$ This gives a differential equation in x and z that is **linear**, and can be solved using the integrating factor method

لايجاد الحل , نغير المتغير المعتمد من y ال z بحيث $z=y^{1-n}$. هذا يعطي معادله تفاضليه بدلالة x و z والتي تكون خطيه, وممكن حاها باستخدام طريقة عامل التكامل.

Note: Dividing the above standard form by y^n gives:

$$\frac{1}{y^n}\frac{dy}{dx}+P(x)y^{1-n}=Q(x)$$

$$i.e.\quad \frac{1}{(1-n)}\frac{dz}{dx}+P(x)z=Q(x)$$

(where we have used $\frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$) .

Example: Solve $y'+xy=xy^2$

We make the substitution, namely $z = y^{1-2} = y^{-1}$, from which follow

نقوم بعملية التعويض, ونحصل على

$$y=\frac{1}{z}\ \text{ and }\ y'=-\frac{z'}{z^2}$$

Substituting these equations into the differential equation, we obtain

نعوض هذه المعادلات في المعادله التفاضليه فنحصل

$$-\frac{z'}{z^2}+\frac{x}{z}=\frac{x}{z^2} \quad or \;\; z'-xz=-x$$

This last equation is linear for the unknown function z(x).

المعادله الاخيره هي خطيه لداله مجهوله z(x).

The integrating factor is

$$I(x)=e^{\int(-x)dx}=e^{-x^2/2}$$

Multiplying the differential equation by I(x), we obtain

$$e^{-x^2/2}\,z'-xe^{-x^2/2}z=-xe^{-x^2/2}$$

Or

$$\frac{d}{dx}\left(ze^{-x^2/2}\right)=-xe^{-x^2/2}$$

Upon integrating both sides of this last equation, we have

$$ze^{-x^2/2}=e^{-x^2/2}+c$$

Whereupon

$$z(x)=ce^{x^2/2}+1$$

The solution of the original differential equation is then

$$y=\frac{1}{z}=\frac{1}{ce^{x^2/2}+1}$$

Example: Solve $x\frac{dy}{dx}+y=xy^3$

Answer: Rearranging we have

$$\frac{dy}{dx}+\frac{1}{x}y=y^3$$

We make the substitution, namely $z = y^{1-3} = y^{-2}$, from which follow

نقوم بعملية التعويض, ونحصل على

$$y=\frac{1}{z^{\frac{1}{2}}} \text{ and } y'=-\frac{1}{2}\frac{z'}{z^{\frac{3}{2}}}$$

Substituting these equations into the differential equation, we obtain

نعوض هذه المعادلات في المعادله التفاضليه فنحصل

$$-\frac{1}{2}\frac{z'}{z^{\frac{3}{2}}}+\frac{1}{xz^{\frac{1}{2}}}=\frac{1}{z^{\frac{3}{2}}} \quad or\; z'-2\frac{z}{x}=-2$$

This last equation is linear for the unknown function z(x).

المعادله الاخيره هي خطيه لداله مجهوله z(x).

The integrating factor is

$$I(x)=e^{\int\left(-\frac{2}{x}\right)dx}=e^{-2lnx}=e^{lnx^{-2}}=\frac{1}{x^2}$$

Multiplying the differential equation by I(x), we obtain

$$\frac{z'}{x^2}-2\frac{z}{x^3}=\frac{-2}{x^2}$$

Or

$$\frac{d}{dx}\left(\frac{z}{x^2}\right) = \frac{-2}{x^2}$$

Upon integrating both sides of this last equation, we have

$$\frac{z}{x^2} = \frac{2}{x} + c$$

Whereupon

$$z(x) = 2x + cx^2$$

The solution of the original differential equation is then

$$y = \frac{1}{z^{\frac{1}{2}}} = \frac{1}{\sqrt{2x + cx^2}}$$

H.W.

Q.1 Solve $\frac{dy}{dx} + \frac{1}{3}y = e^x y^4$

Q.2 Solve $x\frac{dy}{dx} + y = xy^3$

Q.3 Solve $\frac{dy}{dx} + \frac{2}{x}y = -x^2 cosxy^2$

Q.4 Solve $2\frac{dy}{dx} + tanxy = \frac{(4x+5)^2}{cosx}y^3$

Q.5 Solve $x\frac{dy}{dx} + y = y^2x^2 lnx$

Problems and application

Differential equations play a prominent role in many disciplines including engineering, physics, economics and biology.

المعادلات التفاضلية تلعب دورا بارزا في العديد من التخصصات بما في ذلك الهندسة والفيزياء والاقتصاد وعلم الأحياء.

Differential equations are physics, almost all differential equations are derived for physics to model physical problems.

المعادلات التفاضليه هي الفيزياء, اغلب المعادلات التفاضليه هي مشتقه لتطبيقات الفيزياء لمحاكاة مشاكل فيزيائيه.

Example 1: Exponential growth and decay **النمو الاسي والاضمحلال**

The rate at which new organisms are produced (dx/dt) is proportional to the number that are already there, with constant of proportionality α. So the differential equation is:

ان معدل نمو او انتاج كائنات جديده (dx/dt) يتناسب مع عدد الكائنات الموجوده x مضروب بثابت تناسب معين α.

بالنتيجه المعادله التفاضليه لهذا النمو هي:

$$\frac{dx}{dt} = \propto x$$

This differential equation can be solved by separable the variables.

$$\frac{dx}{dx} = \propto dt$$

$$\int \frac{dx}{dx} = \int \propto dt$$

$$lnx = \alpha t + c$$

The constant(s) of integration are usually found from the boundary conditions: which in this case means from knowledge of x at some value of t. For this example, suppose we know that, at time t = 0, $x = x_0$. Substitution gives

ثابت التكامل يتم ايجاده عادة من خلال تطبيق الشروط الحدوديه: ولهذه الحاله من خلال معرفة قيمة x لقيمه معينه ل t. على سبيل المثال, نفترض اننا نعرف عند زمن t = 0 ان قيمة $x = x_0$.التعويض يعطي

$$lnx_o = \alpha * 0 + c$$
$$\therefore c = lnx_o$$

The final equation is

$$lnx = \alpha t + lnx_o \Longrightarrow lnx - lnx_o = \alpha t \Longrightarrow ln\left(\frac{x}{x_o}\right) = \alpha t$$
$$\therefore x = x_o e^{\alpha t}$$

Example2: Terminal velocity **سرعة المنتهى**

Using Newton's law, we model a mass m free falling under gravity but with air resistance. We assume that the force of air resistance is proportional to the speed of the mass and opposes the direction of motion.

باستخدام قانون نيوتن, نحن نحاكي سقوط كتله بصورة حره تحت تاثير الجاذبيه ولكن مع وجود مقاوة هواء. نحن نفترض أن قوة مقاومة الهواء تتناسب طرديا مع سرعة الكتلة وبصوره معاكسه لاتجاه الحركة.

Near the surface of the Earth, the force of gravity is approximately constant and is given by −mg, with $g = 9.8\ m/s^2$ the usual gravitational acceleration. The force of air resistance is modeled by −kv, where v is the vertical velocity of the mass and k is a positive constant. When the mass is falling, $v < 0$ and the force of air resistance is positive, pointing upward and opposing the motion. The total force on the mass is therefore given by F = −mg−kv. With F = ma and a = dv/dt, we obtain the differential equation

قرب سطح الارض. قوة الجاذبيه تكون تقريبا ثابته وتعطى بواسطة $-mg$ مع $g = 9.8\ m/s^2$ التي تمثل تعجيل الجاذبيه. القوة الناتجه بواسطة مقاومة الهواء تحاكى بواسطة −kv, حيث v تمثل السرعه العموديه للكتله و k هو ثابت موجب . عندما الكتله تسقط $v < 0$ و قوة مقاومة الهواء هي موجبه وتشير الى الاعلى وبصوره معاكسه للاتجاه الحركه. القوة الكليه المسلطه على الكتله ممكن ان تعبر بالمعادله F = −mg−kv مع F = ma و a = dv/dt. وبالتالي المعادله التفاضليه

$$m\frac{dv}{dt} = -mg - kv$$

The terminal velocity v_∞ of the mass is defined as the asymptotic velocity after air resistance balances the gravitational force. When the mass is at terminal velocity, $\frac{dv}{dt} = 0$ so that

سرعة المنتهى v_∞ للكتله تعرف كسرعة تقريبيه بعد مقاومة الهواء تتعادل مع قوة الجاذبيه. وعندما الكتله تصل الى سرعة المنتهى $\frac{dv}{dt} = 0$ يكون

$$v_\infty = -\frac{mg}{k}$$

The approach to the terminal velocity of a mass initially at rest is obtained by solving the differential equation of mass falling with initial condition $v(0) = 0$.

التقريب للسرعة المنتهى للكتله ابتدا عند السكون يتم الحصول بحل المعادله التفاضليه للكتله الساقطه مع شرط حدودي $v(0) = 0$.

The equation is both linear and separable, and I solve by separating variables.

المعادله هي خطيه ومفصوله . وممكن ان تحل بفصل المتغيرات.

$$m\int_0^v \frac{dv}{mg+kv} = -\int_0^t dt$$

$$\frac{m}{k} ln\left(\frac{mg+kv}{mg}\right) = -t$$

$$1+\frac{kv}{mg} = e^{-\frac{kt}{m}}$$

$$v = -\frac{mg}{k}\left(1-e^{-\frac{kt}{m}}\right)$$

Therefore, $v = v_\infty(1-e^{-kt/m})$, and v approaches v_∞ as the exponential term decays to zero.

لذلك, $v = v_\infty(1-e^{-kt/m})$ و v_∞ تقترب عندما مقدار الكميه الاسيه تنحل لتصل الى الصفر.

As an example, a skydiver of mass $m = 100$ kg with his parachute closed may have a terminal velocity of 200 km/hr. With

كمثال, القفز بمظله لكتله $m = 100\ kg$ مع مظله مغلقة ممكن ان تمتلك سرعة منتهى 200 km/hr مع تعجيل ارضي

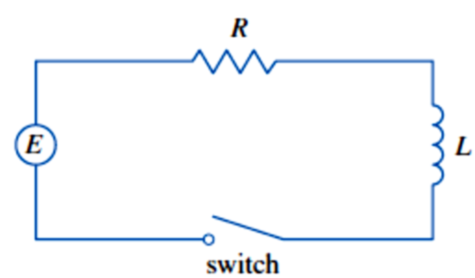
$$g = (9.8\text{m/s}^2)\,(10^{-3}\text{ km/m})\,(60\text{s/min})^2\,(60\text{ min/hr})^2 = 127{,}008\text{ km/hr}^2,$$

one obtains from the terminal velocity equation, $k = 63{,}504$ kg/hr. One-half of the terminal velocity for free-fall (100 km/hr) is therefore attained when $(1-e^{-kt/m}) = 1/2$, or $t = m \ln 2/k \approx 4$ sec. Approximately 95% of the terminal velocity (190 km/hr) is attained after 17 sec.

ممكن الحصول من معادلة سرعة المنتهى على قيمة الثابت k. نصف قيمة سرعة المنتهى للسقوط الحر (100 km/hr) ممكن بلوغها عندما $(1-e^{-kt/m}) = 1/2$ او $t = m \ln 2/k \approx 4$ sec. *تقريبا 95% من سرعة المنتهى ممكن بلوغها بعد 17 ثانية.*

Example 3: Application to Electric Circuits

we considered the simple electric circuit shown in Figure: نفترض دائرة كهربائيه بسيطه كما موضح بالشكل



An electromotive force (usually a battery or generator) produces a voltage $E(t)$ of volts (V) and a current $I(t)$ of amperes (A) at time. The circuit also contains a resistor with a resistance R of ohms (Ω) and an inductor with an inductance L of henries (H).

قوة دافعه كهربائيه (عادة بطاريه او مولد) تنتج فرق جهد وتيار بزمن معين. هذه الدائره تحتوي ايضا مقومه وملف حث.

Ohm's Law gives the drop in voltage due to the resistor as *RI*. The voltage drop due to the inductor is *L*(*dI/dt*). One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage *E* (*t*). Thus, we have

قانون اوم يعطي الانحدار بالجهد نتيجه المقاومة كا *RT*. انحدار الجهد بسب ملف الحث هو *L*(*dI/dt*). احد قوانين كيرشوف ينص على ان مجموع انحدار الجهد يساوي الفولتية المجهزه. كما في المعادله

$$L\frac{dI}{dt} + RI = E(t)$$

which is a first-order linear differential equation. The solution gives the current at time

والتي هي معادله تفاضليه خطيه من الدرجه الاولى. الحل لهذه المعادله يعطي قيمة التيار عند زمن محدد.

Suppose that in the simple circuit of Figure above the resistance is 12 Ω and the inductance is 4 H. If a battery gives a constant voltage of 60 V and the switch is closed when *t*=0 so the current starts with *I* (0) =0, find (a) *I* (*t*), (b) the current after 1 s, and (c) the limiting value of the current.

نفترض دائرة كهربائيه بسيطه كما فالشكل اعلاه مع مقاومه 12 Ω وحث ملف 4 H. اذا كانت البطاريه تعطي فرق جهد ثابت مقداره 60 V والدائرة تغلق عند زمن *t*=0 لذلك التيار يبدا مع *I* (0) =0. جد *I* (*t*) , التيار بعد ثانيه واحده, والقيمه المحدده للتيار.

a) If we put L= 4H, R=12, and E(t)= 60 in the Kirchhoff's laws that defined in the differential equation above, we obtain the initial-value problem

اذا عوضنا قيم الحث و المقاومه وفرق الجهد في قانون كيرشوف المعرف بالمعادله التفاضليه اعلاه , نحصل على

$$4\frac{dI}{dt} + 12I = 60 \qquad I(0) = 0$$

Or

$$\frac{dI}{dt} + 3I = 15 \qquad I(0) = 0$$

Multiplying by the integrating factor $e^{\int 3dt} = e^{3t}$, we get

$$e^{3t}\frac{dI}{dt} + 3Ie^{3t} = 15e^{3t}$$

$$d(Ie^{3t}) = 15e^{3t}$$

$$Ie^{3t} = \int 15e^{3t}\ dt = 5e^{3t} + C$$

$$I(t) = 5 + Ce^{-3t}$$

Since *I* (0)=0 , we have 5+C=0 , so C=-5 and

$$I(t) = 5(1 - 5e^{-3t})$$

(b) After 1 second the current is

$$I(1) = 5(1 - 5e^{-3}) \approx 4.75\ A$$

(c)

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} 5(1 - 5e^{-3t})$$

$$= 5 - 5\lim_{t\to\infty} e^{-3t}$$

$$= 5 - 0 = 5$$

Higher-degree first-order equations

The differential equation of first degree can write as a formula:

المعادله التفاضليه من الدرجه الاولى تاخذ الصيغة التاليه:

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

Or

$$F(x, y, p) = 0, \quad where\ p = \frac{dy}{dx}$$

Higher-degree first-order equations can be written as *F*(*x*,*y*,*dy/dx*) = 0. The most general standard form is

المعادله التفاضليه من الرتبه الاولى والدرجات العليا ممكن ان تكتب . الشكل القياسي الأعم هو

$$p^n + a_{n-1}(x, y)p^{n-2} + \cdots + a_1(x, y)p + a_o(x, y) = 0$$

1. **Equations soluble for *p***

Sometime the LHS of Equation above can be factorized into

بعض الاحيان الجانب الايسر من المعادله اعلاه ممكن ان تحلل الى

$$(p - F_1)(p - F_2) \ldots .(p - F_n) = 0$$

where $F_i = F_i(x,y)$. We are then left with solving the n first-degree equations $p = F_i(x,y)$. Writing the solutions to these first-degree equations as $G_i(x, y) = 0$, the general solution to Equation above is given by the product

نحن نبقى مع حل للمعادله تفاضليه من الرتبه الاولى درجة n . ويكون الحل للهذه المعادله ياخذ الصيغ العامه التاليه:

$$G_1(x, y)G_2(x, y) \ldots .G_n(x, y) = 0$$

Example1: Solve $(y')^3 - (y')^2 - 2y' = 0$

Sol:

Let $p = y'$, Then equation rewrite as

$$p^3 - p^2 - 2p = 0$$
$$p(p - 2)(p + 1) = 0$$
$$\therefore p = 0 \longrightarrow y = c_1$$
$$p = 2 \longrightarrow y = 2x + c_2$$
$$p = -1 \longrightarrow y = -x + c_3$$

So the general solation as

$$(y - c_1)(y - 2x - c_2)(y + x - c_3) = 0$$

Since, the differential equation is from 1st order, so the general solution must have only one arbitrary constant.

بما ان المعادله التفاضليه هي من الرتبه الاولى, لذلك يجب ان يكون حلها العام لديه ثابت اختياري واحد فقط.

$$(y - c)(y - 2x - c)(y + x - c) = 0$$

Example2: Solve $(x^3 + x^2 + x + 1)p^2 - (3x^2 + 2x + 1)yp - 2xy^2 = 0$

Sol.

The equation may be factorized to give

$$[(x+1)p - y][(x^2+1)p - 2xy] = 0$$

Turn each bracket in turn we have

$$(x+1)\frac{dy}{dx} - y = 0$$

$$(x^2+1)\frac{dy}{dx} - 2xy = 0$$

Which can give the solution

$$y - c(x+1) = 0 \ \& \ y - c(x^2+1) = 0$$

So, the general solution is

$$[y - c(x+1)][y - c(x^2+1)] = 0$$

2. Equations soluble for *x*

Equations that can be solved for *x*, i.e. such that they may be written in the form

المعادلات التي يمكن حلها ل x ، مما يعني بحيث أنها قد تكون مكتوبة بالصيغه

$$x = F(y,p)$$

can be reduced to first-degree first-order equations in *p* by differentiating both sides with respect to y, so that

ممكن ان تختصر الى معادله من الرتبه الاولى في p من خلال الاشتقاق الطرفين بالنسبه y بحيث

$$\frac{dx}{dy} = \frac{1}{p} = \frac{dF}{dy} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y}$$

This results in an equation of the form $G(y,p) = 0$, which can be used together with $x = F(y,p)$ to eliminate *p* and give the general solution. Note that often a singular solution to the equation will be found at the same time.

النتيجه هي داله بالصيغه $G(y,p) = 0$, والتي ممكن ان تستخدام معا $x = F(y,p)$ لحذف p وتعطي الحل العام. لاحظا انه غالبا ممكن ايجاد حل خاص للمعادله بنفس الوقت.

Example1: Solve $6y^2p^2 + 3xp - y = 0$

Sol.

This equation can be solved for *x* explicitly to give $3x = (y/p) - 6y^2p$. Differentiating both sides with respect to *y*, we find

هذه المعادله ممكن ان تحل لx وتفسر ل0تعطي $3x = (y/p) - 6y^2p$. باشتقاق كلا الطرفين بالنسبه y , ونحصل على

$$3\frac{dx}{dy} = \frac{3}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 6y^2\frac{dp}{dy} - 12py$$

which factorizes to give التي تحلل لتعطي

$$(1 + 6yp^2)\left(2p + y\frac{dp}{dy}\right) = 0$$

Setting the factor containing *dp*/*dy* equal to zero gives a first-degree first-order equation in *p*, which may be solved to give $py^2 = c$. Substituting for *p* in the differential equation given then yields the general solution of this equation

بوضع الحد الذي يحتاوي *dp/dy* مساوي الى صفر, يعطي معادله من الدرجه الاولى ل *p* والتي ممكن ان تحل لتعطي $py^2 = c$. وبتعويض قيمة في المعادله التفاضليه تعطي حل عام لهذه المعادله:

$$y^3 = 3cx + 6c^2$$

If we now consider the first factor in the primary solution of the differential equation after factories, we find $6p^2y = -1$ as a possible solution. Substituting for *p* in the differential equation we find the singular solution

اذا اخذنا بالاعتبار العامل الاول في الحل الابتدائي للمعادله التفاضليه بعد التحليل, نحن نجد $6p^2y = -1$ كحل محتمل. وبتعويض *p* في المعادله التفاضليه نجد الحل المنفرد

$$8y^3 + 3x^2 = 0$$

Note that the singular solution contains no arbitrary constants and cannot be found from the general solution the differential equation by any choice of the constant c.

لاحظ ان الحل المنفرد لايحتوي ثابت اختياري ولايمكن ايجاده من الحل العام للمعادله التفاضليه باختيار اي قيمه للثابت.

Solution method. *Write the equation in the form $x = F(y, p)$ and differentiate both sides with respect to y. Rearrange the resulting equation into the form G(y,p)=0 , which can be used together with the original ODE to eliminate p and so give the general solution. If G(y,p) can be factorized then the factor containing dp/dy should be used to eliminate p and give the general solution. Using the other factors in this fashion will instead lead to singular solutions.*

طريقة الحل: اكتب المعادله بالصيغة $x = F(y, p)$ واشتق الطرفين بالنسبه ل. اعد ترتيب الداله الناتجه بالصيغه *G(y,p)=0* . والتي ممكن ان تستخدام مع المعادله التفاضليه الاعتيادية لاستبعاد *p* وهكذا الحصول على الحل العام.
اذا G(y,p) ممكن ان تحلل , ثم الحد الذي يحتوي dp/dy يجب ان يستخدام لاستبعاد p واعطاء الحل العام. باستخدام الحد الثاني بنفس الطريقة سيؤدي بدلا من ذلك إلى حلول منفرده.

1. **Equations soluble for *y***

Equations that can be solved for *y*, i.e. such that they may be written in the form

المعادلات التي يمكن حلها ل x ، مما يعني بحيث أنها قد تكون مكتوبة بالصيغه

$$y = F(x, p)$$

can be reduced to first-degree first-order equations in *p* by differentiating both sides with respect to *x*, so that

ممكن ان تختصر الى معادله من الرتبه الاولى في *p* من خلال الاشتقاق الطرفين بالنسبه *x* بحيث

$$\frac{dy}{dx} = \frac{1}{p} = \frac{dF}{dx} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x}$$

This results in an equation of the form $G(x, p) = 0$, which can be used together with $y = F(x, p)$ to eliminate *p* and give the general solution. Note that often a singular solution to the equation will be found at the same time.

النتيجه هي داله بالصيغه $G(x,p) = 0$, والتي ممكن ان تستخدام معا $y = F(x,p)$ لحذف p وتعطي الحل العام. لاحظا انه غالبا ممكن ايجاد حل خاص للمعادله بنفس الوقت.

Example1: Solve $xp^2 + 2xp - y = 0$

Sol.

This equation can be solved for *x* explicitly to give $y = xp^2 + 2xp$. Differentiating both sides with respect to *y*, we find

هذه المعادله ممكن ان تحل لx وتفسر لتعطي $y = xp^2 + 2xp$. باشتقاق كلا الطرفين بالنسبه x , ونحصل على

$$\frac{dy}{dx} = p = 2xp\frac{dp}{dx} + p^2 + 2x\frac{dp}{dx} + 2p$$

which factorizes to give

التي تحلل لتعطي

$$(p+1)\left(p + 2x\frac{dp}{dx}\right) = 0$$

To obtain the general solution of the differential equation, we first consider the factor containing *dp/dx*. This first-degree first-order equation in p has the solution $xp^2 = c$, which we then use to eliminate *p* from the differential equation. We therefore find that the general solution to the differential equation is

لايجاد حل عام للمعادله التفاضليه, نحن نعتمد اولا العامل الذي يحتوي *dp/dx*. المعادله التفاضليه من الدرجه الاولى المرتبه الاولى ل p تمتلك الحل $xp^2 = c$, والذي سوف نستخدامه لاستبعاد p من المعادله التفاضليه. نحن لذلك نجد ان الحل العام للمعادله التفاضليه هو

$$(y-c)^2 = 4cx.$$

If we now consider the first factor in the equation above, we find this has the simple solution $p = -1$. Substituting this into the differential equation then gives

اذا نحن الان نعتمد في الحل العامل الاول بالمعادله اعلاه. نحن نجد انها تمتلك حل بسيط $p = -1$. يعوض بالمعادله التفاضليه ليعطي.

$$x + y = 0$$

which is a singular solution to the differential equation.

والذي هو حل منفرد للمعادله التفاضليه.

Clairaut's equation ***معادلة كليرو***

The Clairaut's equation has the form

معادلة كليرو لديها الصيغه

$$y = px + F(y)$$

and is therefore a special case of equations soluble for *y*,

وهي بذلك حالة خاصة من المعادله القابله للحل ل y

Differentiating Equation above with respect to *x*, we find

اشتقاق المعادله اعلاه بالنسبه لx , نجد

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{dF}{dx}\frac{dp}{dx} \Rightarrow \frac{dp}{dx}\left(\frac{dF}{dx} + x\right) = 0$$

Considering first the factor containing *dp/dx,* we find

بالاخذ بالاعتبار الحد الاول الذي يحتوي *dp/dx*, نجد

$$\frac{dp}{dx} = \frac{d^2y}{dx^2} = 0 \Longrightarrow y = c_1x + c_2$$

Since $p = dy/dx = c_1$, if we substitute Equation above into Clairaut's equation, we find

$$c_1x + c_2 = c_1x + F(c_1)$$

Therefore, the constant c_2 is given by $F(c_1)$, and the general solution to Clairaut's equation

$$y = c_1x + F(c_1)$$

i.e. the general solution to Clairaut's equation can be obtained by replacing p in the ODE by the arbitrary constant c_1. Now considering the second factor in the derivative Equation of Clairaut's equation with respect to x, also have

$$\frac{dF}{dx} + x = 0$$

which has the form $G(x,p) = 0$. This relation may be used to eliminate p from the Clairaut's equation to give a singular solution.

Example: Solve

$$y = px + p^2$$

Sol.

According to the clarification above, the general solution is $y = cx+c^2$.

But from the second part of solution that explain above, we also have

$$2p+x = 0 \Rightarrow p = -x/2.$$

Substituting this into Equation of the question, we find the singular solution

$$x^2 + 4y = 0.$$

Higher- order ordinary differential equations **المعادلات التفاضليه الاعتياديه من الرتب العليا**

Higher-order ordinary differential equations are expressions that involve derivatives other than the first and, as you might expect, their properties are different to those of first-order ODEs.

المعادلات التفاضليه الاعتياديه من الرتب العليا هي التعبيرات التي تحتوي مشتقات غير الاولى, كما نتوقع , فان خواصها تختلف عن المعادلات التفاضليه الاعتياديه من الدرجه الاولى.

A linear ODE of general order n has the form

المعادلات التفاضليه الاعتياديه الخطيه من الدرجات العامه n تمتلك الصيغة التاليه

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_o(x)y = f(x)$$

If $f(x) = 0$ then the equation is called homogeneous; otherwise it is inhomogeneous. The general solution to Equation above will contain n arbitrary constants

في حالة $f(x) = 0$ اذن المعادله تدعى متجانسه: وبغيره تكون غير متجانسه. الحل العام للمعادله اعلاه سوف تحتوي n من الثوابت الاختياريه.

For an n-th order homogeneous linear equation with constant coefficients:

لمعادله خطيه متجانسه من الرتب n مع عوامل ثابته.

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_oy = 0, \quad a_n \neq 0$$

It has a general solution of the form يكون لديها حل عام بالصيغه

$$y = c_1y_1 + c_2y_2 + \cdots + c_{n-1}y_{n-1} + c_ny_n$$

where $y_1, y_2, \ldots, y_{n-1}, y_n$ are any n linearly independent solutions of the equation. (Thus, they form a set of fundamental solutions of the differential equation.) The linear independence of those solutions can be determined by their Wronskian.

i.e., $W(y_1, y_2, \ldots, y_{n-1}, y_n)(t) \neq 0.$

حيث الدوال الخطيه المستقله تعتبرحل للمعادله.(لذلك, فانها تشكل مجموعه من الحلول الاساسيه للمعادله التفاضليه). الدوال الخطيه المستقله لهذه الحلول ممكن تحيدها بواسطة الرونسكيان.

Such a set of linearly independent solutions, and therefore, a general solution of the equation, can be found by first solving the differential equations.

مثل هذه المجموعه من الحلول الخطيه المستقله, ولذلك , الحل العام لهذه المعادله ممكن ايجادها بواسطة اولا حل المعادله التفاضليه المميزه.

Note 1: In order to determine the n unknown coefficients C_i, each n-th order equation requires a set of n initial conditions in an initial value problem: $y(t_0) = y_0$, $y'(t_0) = y'_0$, $y''(t_0)= y''_0$, and $y^{(n-1)}(t_0) = y^{(n-1)}{}_0$.

ملاحظه 1: من اجل ايجاد n من العوامل المجهوله C_i, كل معادله من الرتبه تتطلب مجموعه من الشروط الاوليه في القيمه الاوليه للمشكله.

Note 2: The Wronskian $W(y_1, y_2, \ldots, y_{n-1}, y_n)(t)$ is defined to be the determinant of the following n × n matrix

ملاحظه 2: الرونسكيان يعرف ليكون المحدد للمصفوفه n × n التاليه.

$$W(y_1, y_2, \ldots, y_n) = \begin{bmatrix} y_1 & y_2 & \cdots & \cdots & y_n \\ y_1' & y_2' & \cdots & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & \cdots & y_n'' \\ \vdots & \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & \cdots & y_n^{(n-1)} \end{bmatrix}$$

تعريف المؤثر التفاضلي D:

$$D = \frac{d}{dx}, \qquad D^2 = \frac{d^2}{dx^2}, \qquad D^3 = \frac{d^3}{dx^3}, \quad D^n = \frac{d^n}{dx^n},$$

Ex:

$$De^{3x} = \frac{d}{dx}e^{3x} = 3e^{3x}, \qquad D^2e^{3x} = \frac{d^2}{dx^2}e^{3x} = 9e^{3x}$$

خواص حلول المعادله التفاضليه الخطيه المتجانسه من الرتبه الثانيه

نفترض ان المعادله التفاضليه الخطيه المتجانسه من الرتبه الثانيه تكون بالشكل التالي:

$$y'' + a_1y' + a_2y = 0$$

اذا كان كل من y_1 , y_2 حل خاص للمعادله اعلاه فان $y = c_1y_1 + c_2y_2$ حل ايضا للمعادله , حيث c_1 , c_2 ثابتان.

1. الحلان y_1 y_2 للمعادله اعلاه مستقلان خطيا اذا كان $\frac{y_2}{y_1} \neq c$, واذا وفقط اذا $W(y_1, y_2) \neq 0$

2. الحلان y_1 y_2 للمعادله اعلاه مرتبطان خطيا اذا كان $\frac{y_2}{y_1}=c$, اي انا $y_2=cy_1$,واذا وفقط اذا $W(y_1,y_2)=0$

تعريف الحل العام

اذا كانy_1 , y_2 حلين مستقلين للمعادله فان $y=c_1y_1+c_2y_2$ يمثل الحل العام للمعادله , حيثc_1 , c_2 ثابتان اختيارين.

نفترض ان المعادله هي y'', a_1y', a_2y

حيثa_1 , a_2 ثابتان.

للحصول على العام لتلك المعادله, نحاول ايجاد حلين خاصين مستقلين خطيا.

نحاول استخدام $y=e^{\lambda x}$ حل للمعادله اعلاه حيث λ ثابت.

نضع المعادله بالصوره $(D^2+a_1D+a_2)y=0$

ثم نعوض بالحل المفروض

$$Dy=De^{\lambda x}=\lambda e^{\lambda x}, \qquad D^2y=De^{\lambda x}=\lambda^2e^{\lambda x}$$

نحصل على المعادله المساعده التاليه

$$(\lambda^2+a_1\lambda+a_2)e^{\lambda x}=0$$

حيث ان $e^{\lambda x}\neq 0$, اذا الطرف الثاني من المعادله اعلاه يجب ان يساوي صفر, فيكون

$$(\lambda^2+a_1\lambda+a_2)=0$$

وتسمى هذه المعادله بالمعادله المميزه المساعده (auxiliary equation). ويمكن الحصول عليها مباشرة من المعادله الاصليه بدلالة المؤثر D, وذلك بوضع λ بدلا من D.

وهذه المعادله عباره عن معادله تربيعيه (من الدرجه الثانيه في λ) وبالتالي لها جذران λ_1 , λ_2. حيث

$$\lambda_1,\lambda_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

وهذان الجذران لهما ثلاث حالات3

1. حقيقيان مختلفان $\lambda_1\neq\lambda_2$.
2. حقيقيان متساويان $\lambda_1=\lambda_2$.
3. مركبان.

1. الحالة الاولى : جذران المعادلة المميزه حقيقيان مختلفان.

اي ان $\lambda_1\neq\lambda_2$, نجد ان $y_1=e^{\lambda_1x}, y_2=e^{\lambda_2x}$ يعتبران حلان خاصان للمعادله ومستقلان خطيا.

وبالتالي فان الحل العام يكون بالصوره

$$y=c_1e^{\lambda_1x}+c_2e^{\lambda_1x}$$

حيثc_1 , c_2 ثابتان اختياريان.

Example1: Find the general solution for the following equation

$$y''+3y'-4y=0$$

Sol.

Rewrite the equation by using the operator D

$$(D^2+3D-4)y=0$$

We suppose that $y=e^{\lambda x}$ is a solution for the equation.

So, the auxiliary equation will be

$$(\lambda^2 + 3\lambda - 4) = 0$$
$$(\lambda + 4)(\lambda - 1) = 0$$

So, $\lambda = -4$, $\lambda = 1$ are the roots of the auxiliary equation.
The general solution will be

$$y = c_1 e^{-4x} + c_2 e^{x}$$

Example2: Find the general solution for the following equation

$$2y'' - 3y' = 0$$

Sol.
Rewrite the equation by using the operator *D*

$$(2D^2 - 3D)y = 0$$

We suppose that $y = e^{\lambda x}$ is a solution for the equation.
So, the auxiliary equation will be

$$(2\lambda^2 - 3\lambda) = 0$$
$$\lambda(2\lambda - 3) = 0$$

So, $\lambda = 0$, $\lambda = 3/2$ are the roots of the auxiliary equation.
The general solution will be

$$y = c_1 e^{0} + c_2 e^{\frac{3}{2}x} = c_1 + c_2 e^{\frac{3}{2}x}$$

2. الحالة الثانيه : جذران المعادلة المميزه حقيقيان متساويان.

اي ان $\lambda_1 = \lambda_2$, في هذه الحاله يكون $y_1 = e^{\lambda_1 x}$ الحل الاول مرتبطا بالحل $y_2 = e^{\lambda_2 x}$, لذا نبحث عن حل اخر y_2 غير مرتبط بالحل $y_1 = e^{\lambda_1 x}$ وقد ثبت ان $y_{21} = xe^{\lambda_2 x}$ يمثل حلا للمعادله و غير مرتبط بالحل الاول y_1 .

على ذلك يكون الحل العام للمعادله بالصيغه التاليه

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

او

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_1 x}$$

حيث $\lambda_1 = \lambda_2 = \lambda$

Example1: Find the general solution for the following equation

$$y'' - 4y' + 4y = 0$$

Sol.
Rewrite the equation by using the operator *D*

$$(D^2 - 4D + 4)y = 0$$

We suppose that $y = e^{\lambda x}$ is a solution for the equation.
So, the auxiliary equation will be

$$(\lambda^2 - 4\lambda + 4) = 0$$
$$(\lambda - 2)^2 = 0$$

So, $\lambda = 2, 2$ are the roots of the auxiliary equation.
The general solution will be

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

Example2: Find the general solution for the following equation

$$y'' - 2y' + y = 0$$

Sol.

Rewrite the equation by using the operator D

$$(D^2 - 2D + 1)y = 0$$

We suppose that $y = e^{\lambda x}$ is a solution for the equation.

So, the auxiliary equation will be

$$(\lambda^2 - 2\lambda + 1) = 0$$

$$(\lambda - 1)^2 = 0$$

So, $\lambda = 1, 1$ are the roots of the auxiliary equation.

The general solution will be

$$y = c_1 e^x + c_2 x e^x$$

3. الحاله الثالثه: جذرا المعادله المميزه مركبان.

اذا كان احد جذري المعادله عدد مركب $\lambda_1 = \alpha + i\beta$ حيث $i = \sqrt{-1}$ فان الجذر الاخر λ_2 يكون على صورة $\lambda_2 = \alpha - i\beta$ (الجذر المرافق) حيث $\beta \neq 0$.

من ذلك فان $\lambda_1 \neq \lambda_2$ ويكون الحل العم هو

$$y = A_1 e^{(\alpha + i\beta)x} + A_2 e^{(\alpha - i\beta)x}$$

حيث A_2, A_1 ثابتان اختياريان.

ويمكن ان يكتب الحل العام بالصيغه.

$$y = e^{\alpha x}[c_1 cos\beta x + c_2 sin\beta x]$$

حيث $c_1 = A_1 + A_2$ و $c_2 = i(A_1 - A_2)$.

Example1: Find the general solution for the following equation

$$y'' + 2y' + 5y = 0$$

Sol.

Rewrite the equation by using the operator D

$$(D^2 + 2D + 5)y = 0$$

We suppose that $y = e^{\lambda x}$ is a solution for the equation.

So, the auxiliary equation will be

$$(\lambda^2 + 2\lambda + 5) = 0$$

$$\therefore \lambda = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 + 2i$$

The general solution will be

$$y = e^{-x}[c_1 cos 2x + c_2 sin 2x]$$

Example.2: Find the general solution for the following equation

$$y'' + 9y = 0$$

Sol.

Rewrite the equation by using the operator D

$$(D^2+9)y=0$$

We suppose that $y=e^{\lambda x}$ is a solution for the equation.
So, the auxiliary equation will be

$$(\lambda^2+9)=0$$

So, $\lambda =\neq 3i$ are the roots of the auxiliary equation.
The general solution will be

$$y=c_1cos3x+c_2sin3x$$

Homogeneous linear differential equations of order *n* with constant coefficients.

المعادلات التفاضليه الخطيه المتجانسه من الرتبه n ذات المعاملات الثابته.

يمكن تعميم الحالات السابقة الخاصه بحل معادلات من الرتبه الثانيه على المعادلات من الرتبه n

$$a_oy^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-2}y''+a_{n-1}y'+a_ny=0$$

نفترض ان $y=e^{\lambda x}$ حلا للمعادله المعطاة. فتكون المعادله المساعده

$$a_o\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-2}y''+a_{n-1}y'+a_ny=0$$

التي منها نحصل على الجذور $\lambda_1,\lambda_2,\dots,\lambda_n$

ونحصل على الحلول المختلفه حسب العلاقه بين تلك الجذور.

1. اذا كانت $\lambda_1\neq\lambda_2\neq\cdots\neq\lambda_n$ (اعداد حقيقيه).
   فان الحل العام.

$$y=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}+\cdots+c_ne^{\lambda_nx}$$

2. اذا كانت جميع الجذور حقيقيه واحد الجذور مكرر k من المرات
   $\lambda_1=\lambda_2=\cdots=\lambda_{k,}\quad \lambda_{k-1}\neq\cdots\neq\lambda_n$ فان الحل العام يكون

$$y=\left(c_1+c_2x+\cdots+c_kx^{k-1}\right)e^{\lambda_1x}+c_{k+1}e^{\lambda_{k+1}x}+\cdots+c_ne^{\lambda_nx}$$

3. اذا كانت الجذور اعداد مركبه

$$\lambda_1=\lambda_2=\lambda_3=\alpha+i\beta$$

فانه يوجد

$$\lambda_4=\lambda_5=\lambda_6=\alpha-i\beta$$

ويكون الحل العام المناظر لتلك الجذور

$$y=e^{\alpha x}[(c_1+c_2x+c_3x^2)cos\beta x+(c_4+c_5x+c_6x^2)sin\beta x]$$

Example1: Find the solution of the differential equation

$$\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}-3\frac{dy}{dx}=0$$

that satisfy the condition; $y(0)=4, y'(0)=8, y''(0)=-4.$

Sol.

Rewrite the equation by using the operator D

$$(D^3+2D^2-3D)y=0$$

We suppose that $y=e^{\lambda x}$ is a solution for the equation.
So, the auxiliary equation will be

$$(\lambda^3 + 2\lambda^2 - 3\lambda)y = 0$$

So, $\lambda = 0, 1, -3$ are the roots of the auxiliary equation.

The general solution will be

$$y = c_1 + c_2e^x + c_3e^{-3x}$$

ولايجاد الحل الذي يحقق الشروط الابتدائيه.

$$y' = c_2e^x - 3c_3e^{-3x}$$
$$y'' = c_2e^x + 9c_3e^{-3x}$$

وبالتعويض من الشروط الابتدائيه في المعادلات اعلاه.

$$\therefore 4 = c_1 + c_2 + c_3, \qquad y(0) = 4$$
$$8 = c_2 - 3c_3, \qquad y'(0) = 8$$
$$-4 = c_2 + 9c_3, \qquad y''(0) = -4$$

بحل المعادلات اعلاه يمكن ايجاد قيم الثوابت $c_1 = 0, c_2 = 5, c_3 = -1$.

ويكون الحل اللمعادله التفاضليه هو

$$y = 5e^x - e^{-3x}$$

Example2: Find the general solution of the differential equation

$$y''' + 2y'' - y' - 2y = 0$$

Rewrite the equation by using the operator *D*

$$(D^3 + 2D^2 - D - 2)y = 0$$

We suppose that $y = e^{\lambda x}$ is a solution for the equation.

So, the auxiliary equation will be

$$(\lambda^3 + 2\lambda^2 + \lambda - 2)y = 0$$
$$\lambda^2(\lambda + 2) - (\lambda + 2) = 0$$
$$(\lambda + 2)(\lambda + 1)(\lambda - 1) = 0$$

So, $\lambda = 1, -1, -2$ are the roots of the auxiliary equation.

The general solution will be

$$y = c_1e^x + c_2e^{-x} + c_3e^{-2x}$$

Example3: Find the general solution of the differential equation

$$y^4 - 2y^3 + y^2 = 0$$

Rewrite the equation by using the operator *D*

$$(D^4 - 2D^3 + D^2)y = 0$$

We suppose that $y = e^{\lambda x}$ is a solution for the equation.

So, the auxiliary equation will be

$$\lambda^2(\lambda^2 - 2\lambda + 1) = 0$$
$$\lambda^2(\lambda - 1)^2 = 0$$

So, $\lambda = 0, 0,1,1$ are the roots of the auxiliary equation.
The general solution will be

$$y = c_1 + c_2x + (c_3 + c_4x)e^x$$

Example4: Find the general solution of the differential equation

$$y^4 + 2y^3 + 2y^2 = 0$$

Rewrite the equation by using the operator D

$$(D^4 + D^3 + 2D^2)y = 0$$

We suppose that $y = e^{\lambda x}$ is a solution for the equation.
So, the auxiliary equation will be

$$(\lambda^4 + \lambda^3 + 2\lambda^2)y = 0$$

$$\lambda^2(\lambda^2 + \lambda + 2) = 0$$

$$(\lambda + 2)(\lambda + 1)(\lambda - 1) = 0$$

So, $\lambda = 1, -1, -2$ are the roots of the auxiliary equation.
The general solution will be

$$y = c_1e^x + c_2e^{-x} + c_3e^{-2x}$$

H.W.

Group A: Find the general solution of each equation.

Q.1: $y^4 - y = 0$

Q.2: $y^6 - y = 0$

Q.3: $y^3 + 27y = 0$

Q.4: $y^3 + 25y' = 0$

Q.5: $y^3 - 3y^2 - 9 - y' + 13y = 0$

Q.6: $y^4 - 3y^2 - 4y = 0$

Q.7: $y^4 - 18y^2 + 81y = 0$

Q.8: $y^4 - 2y^3 + 2y^2 - 2y' + y = 0$

Q.9: $y^5 - 3y^4 + 3y^3 - y^2 = 0$

Q.10: $y^5 + 5y^4 + 10y^3 + 10y^2 + 5y' + y = 0$

Q.11: $y^5 + 2y^4 + 5y^3 = 0$

Q.12: $y^5 + 5y^4 + 10y^3 + 10y^2 + 5y' + y = 0$

Group B: Solve each initial value problem.

Q.1: $y^3 + 4y^2 - 5y' = 0 \quad y(0) = 4, y'(0) = 7, y''(0) = 23.$

Q.2: $y^3 + 3y^2 + 3y' + y = 0 \quad y(0) = 7, y'(0) = -7, y''(0) = 11.$

Q.3: $y^4 - 10y^2 + 9y = 0 \quad y(0) = 5, y'(0) = -1, y''(0) = 21, y^3(0) = -49$

Q.4: $y^4 + 13y^2 + 36y = 0 \quad y(0) = 0, y'(0) = -3, y''(0) = 5, y^3(0) = -3$

Nonhomogeneous linear differential equations of order *n* with constant coefficients.

المعادلات التفاضليه الخطيه غيرالمتجانسه من الرتبه n ذات المعاملات الثابته.

$$a_o y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-2} y'' + a_{n-1} y' + a_n y = f(x)$$

If, however, the equation has $f(x) \neq 0$ (i.e. it is inhomogeneous). Then, the general solution of this type of equation is:

اذا, المعادله تمتلك $f(x) \neq 0$ مما يعني انها غير متجانسه. اذا الحل العام لهذه النوع من المعادله هو:

$$y = y_c(x) + y_p(x)$$

Where $y_c(x)$ or can be write as $y_h(x)$(henceforth called the homogeneous or complementary solution) represent the general solution of the associated homogeneous equation. $y_p(x)$ denote any particular solution of Equation above.

حيث $y_c(x)$ او ممكن ان تكتب كا $y_h(x)$ (من الان وصاعدا تسمى متجانسه او حل مكمل) تمثل الحل العام للمعادله المتجانسه المرتبطه بهذه المعادله. $y_p(x)$ تشير لاي حل معين للمعادله اعلاه.

The general solution of nonhomogeneous linear differential equation is the sum of the homogeneous and particular solution.

الحل العام لمعادله التفاضليه الخطيه غير المتجانسه هو مجموع الحل لمعادله متجانسه و الحل المعين.

Finding the particular solution or particular integral $y_p(x)$

ايجاد الحل المعين او التكامل المعين.

There is no generally applicable method for finding the particular integral $y_p(x)$. but, for linear ODEs with constant coefficients, $y_p(x)$ can often be found by inspection or by assuming a parameterized form similar to $f(x)$.

لاتوجد طريقه عامه قابله للتطبيق لايجاد التكامل المعين$y_p(x)$. ولكن لمعادله تفاضليه اعتياديه خطيه ذات المعاملات الثابته. $y_p(x)$ غالبا ممكن ايجادها بواسطة التفتيش او بواسطة افتراض صيغة معاملات مشابهه للداله $f(x)$.

The method used to solve non homogeneous linear ODE*,* sometimes called the method of undetermined coefficients $f(x)$. If contains only polynomial, exponential, or sine and cosine terms then, by assuming a trial function for $y_p(x)$ of similar form but one which contains a number of undetermined parameters and substituting this trial function into homogeneous linear ODE part, the parameters can be found and $y_p(x)$ deduced. Standard trial functions are as follows.

الطريقة المستخدمه لحل المعادلات التفاضليه الخطيه غير المتجانسه بعض الاحيان تدعى طريقة العوامل غير المحدده. اذا تحتوي فقط الدالة متعددة حدود, الدالة اسية, او دالة الجيب والجيب تمام, اذا بواسطة افترض داله تجريبيه ل $y_p(x)$ وبنفس الصيغة ولكن فقط التي تحتوي عدد من المؤثرات غير المحدده و بتعويض هذه الداله التجريبيه في جزء المعادله التفاضليه الخطيه المتجانسه, فالمؤثرات ممكن ايجادها و $y_p(x)$ تستنتج.

Standard trial functions are as follows: الدوال القياسيه التجريبيه هي التاليه

i. If $f(x) = ce^{bx}$ then try

$$y_p(x) = Ae^{bx}$$

ii. If $f(x) = c_1 sinbx + c_2 cosbx$ (c_1 *or* c_2 may be zero) then try

$$y_p(x) = A_1 sinbx + A_2 cosbx$$

iii. If $f(x) = c_o + c_1 x + c_2\, x^2 + \cdots + c_n\, x^n$ (some A_m may be zero)then try

$$y_p(x) = A_o + A_1 x + A_2\, x^2 + \cdots + A_N\, x^N$$

iv. If $f(x)$ is the sum or the product of any of the above, then try $y_p(x)$ as the sum or product of the corresponding individual trial functions.

اذا $f(x)$ هي عباره عن مجموع او ضرب لاي من الحالات اعلاه. اذا نحاول $y_p(x)$ كمجموع او ضرب للدوال التجريبيه المستقله المقابله.

If $f(x)$	Use $y_p(x)$
$(c_o + c_1x + c_2\,x^2 + \cdots + c_n\,x^n)e^{bx}$ (a polynomial times an exponential function)	$(A_o + A_1x + A_2\,x^2 + \cdots + A_N\,x^N)e^{bx}$ (another polynomial times an exponential function)
$csin(bx)e^{ax}$ or $ccos(bx)e^{ax}$ (sines or cosines times exponential functions)	$(A_1 sinbx + A_2 cosbx)e^{ax}$ (a product of an exponential function times a linear combination of sine and cosine)
$(c_o + c_1x + c_2\,x^2 + \cdots + c_n\,x^n)sin(bx)$ Or $(c_o + c_1x + c_2\,x^2 + \cdots + c_n\,x^n)cos(bx)$ (a polynomial times sine or cosine)	$(A_o + A_1x + A_2\,x^2 + \cdots + A_N\,x^N)\,cos(bx)$ $+(A_o + A_1x + A_2\,x^2 + \cdots + A_N\,x^N)\,sin(bx)$ (a polynomial times sine and another times cosine)
$(c_o + c_1x + c_2\,x^2 + \cdots + c_n\,x^n)e^{ax}sin(bx)$ Or $(c_o + c_1x + c_2\,x^2 + \cdots + c_n\,x^n)e^{ax}cos(bx)$ (All three together–whoopee!)	$(A_o + A_1x + A_2\,x^2 + \cdots + A_N\,x^N)e^{ax}\,cos(bx)$ $+(A_o + A_1x + A_2\,x^2 + \cdots + A_N\,x^N)e^{ax}sin(bx)$ (what you would expect)

It should be noted that this method fails if any term in the assumed trial function is also contained within the complementary function $y_c(x)$. In such a case the trial function should be multiplied by the smallest integer power of x such that it will then contain no term that already appears in the complementary function. The undetermined coefficients in the trial function can now be found by substitution into nonhomogeneous linear ODE.

يجب ملاحظة ان هذه الطريقة تفشل للاستخدم اذا كان اي حد في الداله التجريبيه المفترضة هو ايضا موجود في دالة الداله المكمله$y_c(x)$. في هذه الحاله الداله التجريبيه يجب ان تضرب بواسطة الداله x التي تمتلك اصغرمعامل اسي صحيح, و هكذا سوف لن تحتوي اي حد ظاهر اصل في الداله المتمه او المكمله. العوامل غير المحددة في الدالة التجريبيه نستطيع الان ايجادها من خلال التعويض بالمعادله التفاضليه الخطيه غير المتجانسه.

Example1: $y'' - 2y' - 3y = e^{2x}$

Sol. The complementary solution is

$$y_c = c_1e^{-x} + c_2e^{3x}$$

Which was found by using the method for homogeneous linear ODE.

Let $y_p = Ae^{2x}$

Then $y_p{}' = 2Ae^{2x}$ & $y_p{}'' = 4Ae^{2x}$

Substitute it's in ODE above

$$4Ae^{2x} - 4Ae^{2x} - 3Ae^{2x} = e^{2x}$$
$$-3Ae^{2x} = e^{2x}$$
$$A = \frac{-1}{3}$$
$$\therefore y_p = \frac{-1}{3}e^{2x}$$

The general solution for the equation is $y = y_c + y_p$

$$y = c_1e^{-x} + c_2e^{3x} - \frac{1}{3}e^{2x}$$

Example2: $y'' - 2y' - 3y = 3x^2 + 4x - 5$

Sol.

The complementary solution is $y_c = c_1e^{-x} + c_2e^{3x}$

Which was found by using the method for homogeneous linear ODE.

Let $y_p = A_1x^2 + A_2x + A_3$

Then $y_p{}' = 2A_1x + A_2$ & $y_p{}'' = 2A_1$

Substitute it's in ODE above

$$2A_1 - 2(2A_1x + A_2) - 3(A_1x^2 + A_2x + A_3) = 3x^2 + 4x - 5$$

$$-3A_1x^2 + (-4A_1 + 3A_2)x + (2A_1 - 2A_2 - 3A_3) = 3x^2 + 4x - 5$$

The corresponding terms on both sides should have the same coefficients, therefore, equating the coefficients of like terms.

$$x^2: \quad 3 = -3A_1 \quad \longrightarrow A_1 = -1$$

$$x: \quad 4 = -4A_1 + 3A_2 \quad \longrightarrow A_2 = 0$$

$$1: \quad -5 = 2A_1 - 2A_2 - 3A_3 \quad \longrightarrow A_3 = 1$$

$$\therefore y_p = -x^2 + 1$$

The general solution for the equation is $y = y_c + y_p$

$$y = c_1e^{-x} + c_2e^{3x} - x^2 + 1$$

Example3: $y'' - 2y' - 3y = 5cos2x$

Sol. The complementary solution is $y_c = c_1e^{-x} + c_2e^{3x}$

Which was found by using the method for homogeneous linear ODE.

Let $y_p = A_1cos2x + A_2cos2x$

Then $y_p{}' = -2A_1sin2x + 2A_2cos2x$ & $y_p{}'' = -4A_1cos2x - 4A_2sin2x$

Substitute it's in ODE above

$$-4A_1cos2x - 4A_2sin2x - 2(-2A_1sin2x + 2A_2cos2x) - 3(A_1cos2x + A_2cos2x) = 5cos2x$$

$$(-4A_1 - 4A_2 - 3A_1)cos2x + (-4A_2 + 4A_1 - 3A_2)sin2x = 5cos2x$$

$$(-7A_1 - 4A_2)cos2x + (4A_1 - 7A_2)sin2x = 5cos2x$$

Compare the coefficients:

$$cos2x: \quad 5 = -7A_1 - 4A_2 \quad \longrightarrow A_1 = -{}^{7}/_{13}$$

$$sin2x: \quad 0 = 4A_1 - 7A_2 \quad \longrightarrow A_2 = -{}^{4}/_{13}$$

$$\therefore y_p = -\frac{7}{13}cos2x - \frac{4}{13}sin2x$$

The general solution for the equation is $y = y_c + y_p$

$$y = c_1e^{-x} + c_2e^{3x} - \frac{7}{13}cos2x - \frac{4}{13}sin2x$$

Note: The method of undetermined coefficients will fail to give us a solution if our proposed particular solution contains elements of the complementary solution.

ملاحظه: طريقة العوامل غير المحدده سوف تفشل لاعطى حل اذا كان الحل المقترح ل y_p يحتوي عنصر اصل موجود في الحل المكمل y_c.

Example: $y'' + 3y' + 2y = 5e^{-2x}$

Sol.

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

Since Ae^{-2x} is one of the homogeneous solutions, we adjust our guess for the specific solution to

$$y_p(x) = Axe^{-2x}$$

Then we have

$$y_p' = Ae^{-2x} - 2Axe^{-2x} = Ae^{-2x}(1-2x)$$

$$y_p'' = -4Ae^{-2x} + 4Axe^{-2x} = 4Ae^{-2x}(x-1)$$

Plugging these in to the differential equation yields

$$y'' + 3y' + 2y = 4Ae^{-2x}(x-1) + 3Ae^{-2x}(1-2x) + 2Axe^{-2x}$$

$$= Ae^{-2x}(4x - 4 + 3 - 6x + 2x)$$

$$= -Ae^{-2x}$$

Setting this equal to 5e−2t, we finally get A = −5, and we have the specific solution

yp(t) = −5te−2t

which gives us the general solution

y(t) = yc(t) + yp(t) = c1e−t + c2e−2t − 5te−2t.

Examples for *f*(*x*) is a sum of several terms

When *f*(*x*) is a sum of several functions: $f(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$, we can break the equation into *n* parts and solve them separately. Given

اذا كانت الداله هي عبارة عن المجموع لعدة دوال. يمكن تجزئة المعادله الى من الاجزاء n وحلها بصورة منفصلة لتعطي.

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x) + \cdots + f_n(x)$$

we change it into

$$y'' + p(x)y' + q(x)y = f_1(x)$$

$$y'' + p(x)y' + q(x)y = f_2(x)$$

$$\vdots$$

$$y'' + p(x)y' + q(x)y = f_n(x)$$

Solve them individually to find respective particular solutions, $y_{p1}, y_{p2}, \ldots, y_{pn}$. Then add up them to get $y_p = y_{p1} + y_{p2} + \cdots + y_{pn}$.

Example: $y'' - 2y' - 3y = e^{2x} + 3x^2 + 4x - 5 + 5cos2x$

Sol:

Solve each of the sub-parts

$y'' - 2y' - 3y = e^{2x}$ $\longrightarrow y_{p1} = \frac{-1}{3}e^{2x}$

$y'' - 2y' - 3y = 3x^2 + 4x - 5$ $\longrightarrow y_{p2} = -x^2 + 1$

$y'' - 2y' - 3y = 5cos2x$ $\longrightarrow y_{p2} = -\frac{7}{13}cos2x - \frac{4}{13}sin2x$

$$y_p = y_{p1} + y_{p2} + y_{p3}.$$

$$y_p = \frac{-1}{3}e^{2x} - x^2 + 1 - \frac{7}{13}cos2x - \frac{4}{13}sin2x$$

The general solution for the equation is $y = y_c + y_p$

$$y = c_1e^{-x} + c_2e^{3x}\frac{-1}{3}e^{2x} - x^2 + 1 - \frac{7}{13}cos2x - \frac{4}{13}sin2x$$

Examples for *f*(*x*) is a product of several functions

If *f*(*x*) is a product of two or more simple functions, e.g. $f(x) = x^2e^{5x}cos(3x)$, then our basic choice (before multiplying by *x*, if necessary) should be a product consist of the corresponding choices of the individual components of $f(x)$. One thing to keep in mind: that there should be only as many undetermined coefficients in y_p as there are distinct terms (after expanding the expression and simplifying algebraically).

اذا كانت الداله هي عباره عن حاصل ضرب دالتسن او اكثر من الدوال البسيطه. كما في المثال, اذا الخيار الاساسي (قبل ضربه ب x اذا ضروري) فيجب ضربها بثابت للخيارات المقابله للمركبات المنفرده للداله $f(x)$. امر اخر يجب اخذه بالاعتبار انه يجب يوجد فقط العديد من العوامل غير المحدده ل y_p كما ان هناك شروط مميزه.(بعد توسيع التعبير والتبسيط الجبري).

Example1: $y'' - 2y' - 3y = x^3e^{5x}cos3x$

Sol:

We have $f(x) = x^2e^{5x}cos3x$. It is a product of a degree 3 polynomial†, an exponential function, and a cosine. Out choice of the form of y_p therefore must be a product of their corresponding choices: a generic degree 3 polynomial, an exponential function, and both cosine and sine. Try

لدينا داله هي عباره عن حاصل ضرب متعددة حدود من الرجه الثالثه مع داله اسيه و دالة جيب تمام. اختيارنا كحل ل y_p لذلك يكون كحاصل ضرب الخيارات المقابله: متعددة حدود عامه من الدرجه الثالثه مع دالة اسيه مع كل من دالة الجيب تمام والجيب.

Correct form

$y_p = (A_1x^3 + A_2x^2 + A_3x + A_4)e^{5x}cos3x + (B_1x^3 + B_2x^2 + B_3x + B_4)e^{5x}sin3x$

Wrong form

$y_p = (A_1x^3 + A_2x^2 + A_3x + A_4)\,Be^{5x}(Ccos3x + Dsin3x)$

Another way (longer, but less prone to mistakes) to come up with the correct form is to do the following.

وهناك طريقة أخرى (أطول، ولكن أقل عرضة للأخطاء) من أجل التوصل إلى الشكل الصحيح هو أن تفعل ما يلي.
with the basic forms of the corresponding functions that are to appear in the product, without assigning any coefficient. In the above example, they are

$$(x^3 + x^2 + x + 1),\ e^{5x} \text{ and } cos3x + sin3x$$

مع الأشكال الأساسية من وظائف المقابلة التي تظهر في الضرب، بدون تحديد أي معامل. في المثال أعلاه، فهي.
Multiply them together to get all the distinct terms in the product:
تضرب معا للحصول على كل الحدود المميزه في الضرب:

$$(x^3 + x^2 + x + 1)e^{5x}(cos3x + sin3x)$$
$$= x^3e^{5x}cos3x + x^2e^{5x}cos3x + xe^{5x}cos3x + e^{5x}cos3x + x^3e^{5x}sin3x + x^2e^{5x}sin3x + xe^{5x}sin3x + e^{5x}sin3x$$

Once we have expanded the product and identified the distinct terms in the product (8, in this example), then we insert the undetermined coefficients into the expression, one for each term:

بعد أن قمنا بايجاد المفكوك للضرب وتحديد حدود واضحه لعملية الضرب (8، في هذا المثال)، نقوم بادراج او ادخال معاملات غير محددة في المعادله، واحد لكل حد:

$$y_p = A_1x^3e^{5x}cos3x + A_2x^2e^{5x}cos3x + A_3xe^{5x}cos3x + A_4e^{5x}cos3x + B_1x^3e^{5x}sin3x + B_2x^2e^{5x}sin3x + B_3xe^{5x}sin3x + B_4e^{5x}sin3x$$

Which is the correct form of y_p seen previously. والتي هي صيغه صحيحه ل كما وضحت سابقا

Example1: $y'' + 25y = 4x^3sin5x - 2e^{3x}cos5x$

Sol.

The complementary solution is $y_c = c_1\ cos5x + c_1\ sin5x$. Let's break up *f*(*x*) into 2 parts and work on them individually.

الحل المتمم هو $y_c = c_1\ cos5x + c_1\ sin5x$. لنفصل *f*(*x*) الى جزئيين ونعمل كل جزاء بصورة مستقله.

$f_1(x) = 4x^3sin5x$ is a product of a degree 3 polynomial and a sine function. Therefore, y_{p1} should be a product of a generic degree 3 polynomial and both cosine and sine.

الداله الاولى هي حاصل ضرب متعددة حدود من الدرجه الثالثه في دالة لجيب. لذلك y_{p1} يجب ان تكون حاصل ضرب متعددة حدود عامه من الدرجه الثالثه في كلا من دالة الجيب والجيب تمام.

$$y_{p1} = (A_1x^3 + A_2x^2 + A_3x + A_4)cos5x + (B_1x^3 + B_2x^2 + B_3x + B_4)sin5x$$

The validity of the above choice of form can be verified by our second (longer) method. Note that the product of a degree 3 polynomial and both cosine and sine: $(x^3 + x^2 + x + 1)\times(cos5x + sin5x)$ contains 8 distinct terms listed below.

صحة الاختيار أعلاه من حيث الشكل يمكن التحقق منه بواسطةالطريقة الثانيه (أطول). لاحظ أنه نتاج متعدد الحدود درجة 3 و كلا من الجيب و الجيب تمام:
$(x^3 + x^2 + x + 1)\times(cos5x + sin5x)$ والتي تحتوي على ثمانية حدود منفصله المدرجه.

$$x^3cos5x \quad x^2cos5x \quad xcos5x \quad cos5x$$
$$x^3sin5x \quad x^2sin5x \quad xsin5x \quad sin5x$$

Now insert 8 independent undetermined coefficients, one for each:
الان ندخل 8 عوامل غير محدده مستقله, زاحد لكل واحد:

$$y_{p1} = A_1x^3cos5x + A_2x^2cos5x + A_3x\ cos5x + A_4cos5x + B_1x^3\ sin5x + B_2x^2sin5x + B_3x\ sin5x + B_4sin5x$$

However, there is still one important detail to check before we could put the above expression down for y_{p1}. Is there anything in the expression that is shared with $y_c = c_1\, cos5x + c_1\, sin5x$? As we can see, there are – both the fourth and the eighth terms. Therefore, we need to multiply everything in this entire expression by x. Hence,

مع ذلك، لا يزال هناك تفصيل واحد مهم للتحقق منه قبل أن نتمكن من وضع التعبير أعلاه إلى أسفل ل y_{p1}. هل هناك أي شيء في التعبير التي يتم مشاركتها مع $y_c = c_1\, cos5x + c_1\, sin5x$. كما يمكننا أن نرى، ان هناك – على حد سواء الحد الرابع الحد الثامن. ولذلك، فإننا بحاجة إلى ضرب كل شيء في هذا التعبير بالكامل بواسطة x. بالتالي

$$y_{p1} = x(A_1x^3 + A_2x^2 + A_3x\ + A_4)cos5x + x(B_1x^3 + B_2x^2 + B_3x\ + B_4)sin5x$$

$$y_{p1} = (A_1x^4 + A_2x^3 + A_3x^2 + A_4x)cos5x + (B_1x^4 + B_2x^3 + B_3x^2\ + B_4x)sin5x$$

The second half of $f(x)$ is $f_1(x) = -2e^{3x}cos5x$. It is a product of an exponential function and cosine. So our choice of form for y_{p2} should be a product of an exponential function with both cosine and sine .

الحد الثاني للداله $f(x)$ هو $f_1(x) = -2e^{3x}cos5x$. انه حاصل ضرب دالة اسيه مع دالة جيب تمام. لذلك اختيارنا لصيغة y_{p2} يجب ان تكون عباره عن حاصل ضرب داله اسيه مع كل من دالة جيب وجيب تمام.

$$y_{p2} = D_1e^{3x}cos5x + D_2e^{3x}sin5x$$

There is no conflict with the complementary solution – even though both $cos5x$ and $sin5x$ are present within both y_c and y_{p2}, they appear alone in y_c, but in products with e^{3x} in y_{p2}, making them parts of completely different functions. Hence this is the correct choice .

لايوجد اي تعارض بين الحل . حتى على الرغم من كل من $cos5x$ و $sin5x$ موجودا في كل من y_c و y_{p2}. نظهر في y_c , ولكن الضرب ب e^{3x} في y_{p2} . مما يجعل اجزاء الوظا ئف مختلفة تماما. وبالتالي هذا هو الخيار الصحيح.

Finally, the complete choice of y_p is the sum of y_{p1} and y_{p2} .

اخيرا, الخيار التام ل y_p هو حاصل جمع y_{p1} و y_{p2} .

$$y_p = y_{p1} + y_{p2}$$

$$y_p = (A_1x^4 + A_2x^3 + A_3x^2 + A_4x)cos5x + (B_1x^4 + B_2x^3 + B_3x^2\ + B_4x)sin5x + D_1e^{3x}cos5x + D_2e^{3x}sin5x$$

Example2: $y'' - 8y' + 12y = x^2e^{6x} - 7xsin2x + 4$

Sol.

Complementary solution: $y_c = c_1\, e^{2x} + c_2e^{6x}$

The form of particular solution is

$$y_p = (A_1x^3 + A_2x^2 + A_3x)e^{6x} + (B_1x + B_2)cos2x + (D_1x + D_2)sin2x + E$$

Example3: $y'' + 10y' + 25y = xe^{-5x} - 7x^2e^{2x}cos4x + 3x^2 - 2$

Sol.

Complementary solution: $y_c = c_1\, e^{-5x} + c_2xe^{-5x}$

The form of particular solution is

$$y_p = (A_1x^3 + A_2x^2)e^{-5x} + (B_1x^2 + B_2x + B_2)e^{2x}cos4x$$
$$+ (D_1x^2 + D_2x + D_3)e^{2x}sin2x + E_1x^2 + E_2 + E_3$$

H.W.

A. Solve the differential equation or initial-value problem using the method of undetermined coefficients.

Q.1 $y'' + 9y = e^{-4x}$

Q.2 $y'' - 4y' + 5y = e^{-x}$

Q.3 $y'' + 3y' + 2y = sin4x$

Q.4 $y'' + y = cosx$

Q.5 $y'' + 3y' + 2y = x^2$

Q.6 $y'' + y = x^3$

Q.7 $y'' + y = e^x + x^3$ $y(0) = 2,\ y'(0)$=0

Q.8 $y'' - 4y = e^x cosx$ $y(0) = 1,\ y'(0)$=2

Q.9 $y'' + y' - 2y = x + sin2x$ $y(0) = 0,\ y'(0)$=1

B. Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

Q.1 $y'' + 9y = e^{2x} + x^2 sinx$

Q.2 $y'' + 9y = xe^{-x}cos\pi x$

Q.3 $y'' + 9y = 1 + xe^x$

Q.4 $y'' + 3y' - 4y = (x^3 + x)e^x$

Q.5 $y'' + 4 = e^{3x} + xsin2x$

Q.6 $y'' + 2y' + 10y = x^2e^{-x}cos3x$

linear differential equations with variable coefficients.

المعادلات التفاضليه الخطيه غيرالمتجانسه ذات المعاملات المتغيره.

تسمى المعادله التفاضليه التي على الصوره

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_o y = f(x), \quad a_n \neq 0$$

حيث ان كل من $f(x), a_o, a_1, a_2, \dots, a_n$ دوال في المتغير المستقل x بمعادله تفاضليه من الرتبه النونيه غير المتجانسه ذات المعاملات المتغيره. بحيث ان $f(x) \neq 0$ اما اذا كان $f(x) = 0$ فان المعادله التفاضليه تاخذ الصيغه

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_o y = 0$$

تسمى معادله تفاضليه من الرتبه النونيه متجانسه ذات معاملات متغيرة حيث ان كل من $a_1, a_1, a_2, \dots, a_n$ دوال في المتغير المستقل x.

1. معادلة اويلر التفاضليه: **Euler's differential equation**

معادله اويلر التفاضليه من الرتبه الثانيه تاخذ الصيغه

$$a_2 x^2 y'' + a_1 x y' + a_o y = f(x)$$

حيث a_o, a_1, a_2 ثوابت اختياريه.

لحل المعادله اعلاه فاننا نستخدام التعويض

$$x = e^t \ \ or \ t = lnx$$

وهذا التعويض يحول المعادله اعلاه ذات المعاملات المتغيرة الى معادله تفاضليه مناظره ذات معاملات ثابته كالاتي:

نفترض ان $\theta = \frac{d}{dt}, \ D = \frac{d}{dx}$

بذلك نجد ان $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \cdot \frac{dy}{dt}$

اي ان $x \frac{dy}{dx} = \frac{dy}{dt}$

ومنها فان $xD = \theta$

وايضا

$$D^2 = \frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = \frac{d}{dx}\left(\frac{d}{dt}\frac{dt}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{d}{dt} \cdot \frac{1}{x}\right) = \frac{d}{dt} \cdot \frac{-1}{x^2} + \frac{1}{x}\frac{d}{dx}\left(\frac{d}{dt}\right)$$

$$= \frac{-1}{x^2}\frac{d}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{d}{dt}\right) \cdot \frac{dt}{dx}$$

بذلك يكون $D^2 = \frac{-1}{x^2}\theta + \frac{1}{x^2}\theta^2$

اي ان $x^2 D^2 = \theta\,(\theta - 1)$

بنفس الطريقة يمكن بسهوله اثبات ان

$$x^3 D^3 = \theta\,(\theta - 1)(\theta - 2)$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$x^n D^n = \theta\,(\theta - 1)(\theta - 2)(\theta - n + 1)$$

والان بالتعويض المعادلتين اعلاه في معادلة اويلر نجد ان

$$a_2 \theta\,(\theta - 1) y + a_1 \theta y + a_o y = f(e^t)$$

$$(a_2 \theta^2 + (a_1 - a_2)\theta + a_o) y = f(e^t)$$

وهذه معادله تفاضليه من الرتبه الثانيه غير متجانسه ذات معاملات ثابته تحل بنفس الطرق السابقه. وبالتالي يمكن ايجاد حل معادلة اويلر.

Example1: Find the general solution for ODE

$$(x^2D^2 - 2xD + 2)y = 4x^3$$

Sol.

Let $x = e^t$ and $\theta = \frac{d}{dt}$,

so that

$xD = \theta$

$x^2D^2 = \theta(\theta - 1)$

Substitute in the ODE, we get

$$(\theta(\theta - 1) - 2\theta + 2)y = 4e^{3t}$$

$$(\theta^2 - 3\theta + 2)y = 4e^{3t}$$

This is nonhomogeneous ODE of 2nd order with constant variables.

The complementary solution y_c for equation above is

$$(\theta^2 - 3\theta + 2)y = 0$$

Let $y = e^{\lambda t}$, we get the auxiliary equation

$$\lambda^2 - 3\lambda + 2 = 0$$

The roots of this equation are

$\lambda_1 = 1 \quad , \quad \lambda_2 = 2$

$$y_c = c_1e^t + c_2e^{2t} \quad \text{where } c_1, c_2 \text{ are arbitrary constants.}$$

The particular solution y_p for equation above is

$$y_p = Ae^{3t}$$

$$9Ae^{3t} - 9A + 2Ae^{3t} = 4e^{3t}$$

$$\therefore A = 2$$

$$y_p = 2e^{3t}$$

The general solution is

$$y = y_c + y_p = c_1e^t + c_2e^{2t} + 2e^{3t}$$

Since $x = e^t$, so that

$$y = y_c + y_p = c_1x + c_2x^2 + 2x^3$$

Example2: Find the general solution for ODE

$$(x^3D^3 + 2xD - 2)y = 0$$

Sol.

Let $x = e^t$ and $\theta = \frac{d}{dt}$,

so that

$xD = \theta$

$x^2D^2 = \theta(\theta - 1)$

$x^3D^3 = \theta(\theta - 1)(\theta - 2)$

Substitute in the ODE, we get

$$(\theta(\theta - 1)(\theta - 2) + 2\theta + 2)y = 0$$

$$(\theta^3 - 3\theta^2 + 4\theta + 2)y = 0$$

This is homogeneous ODE of 3rd order with constant variables.

The complementary solution y_c for equation above is

$$(\theta^3 - 3\theta^2 + 4\theta + 2)y = 0$$

Let $y = e^{\lambda t}$, we get the auxiliary equation

$$\lambda^3 - 3\lambda^2 + 4\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda^2 - 2\lambda + 2) = 0$$

The roots of this equation are

$\lambda_1 = 1 \quad , \quad \lambda_{2,3} = \frac{-2\mp\sqrt{4-8}}{2} = -1 \mp i$

$$y_c = c_1 e^t + e^{-t}(c_2 cost + c_3 sint) \quad \text{where } c_1, c_2, c_3 \text{ are arbitrary constants.}$$

The general solution is

$$y = y_c + y_p = c_1 e^t + c_2 e^{2t} \quad \text{where } y_p = 0$$

Since $x = e^t$ and $t = lnx$ so that

$$y = y_c + y_p = y_c = c_1 x + \frac{1}{x}\left(c_2 cos(lnx) + c_3 sin(lnx)\right)$$

Example3: Find the general solution for ODE

$$x^2 y'' - 6y = x^2 + \frac{1}{x^2}$$

Sol.

Let $x = e^t$ and $\theta = \frac{d}{dt}$,

so that

$xD = \theta$

$x^2 D^2 = \theta(\theta - 1)$

Substitute in the ODE, we get

$$(\theta(\theta - 1) - 6)y = e^{2t} + e^{-2t}$$

$$(\theta^2 - \theta - 6)y = e^{2t} + e^{-2t}$$

This is nonhomogeneous ODE of 2nd order with constant variables.

The complementary solution y_c for equation above is

$$(\theta^2 - \theta - 6)y = 0$$

Let $y = e^{\lambda t}$, we get the auxiliary equation

$$\lambda^2 - \lambda - 6 = 0$$

he roots of this equation are

$\lambda_1 = -2 \quad , \quad \lambda_2 = 3$

$$y_c = c_1 e^{-2t} + c_2 e^{3t} \quad \text{where } c_1, c_2 \text{ are arbitrary constants.}$$

The particular solution y_p for equation above solve each of the sub-parts:

1. $(\theta^2 - \theta - 6)y = e^{2t}$

   $y_{p1} = \text{A}e^{2t} \quad \Longrightarrow \quad 4Ae^{2t} - 2Ae^{2t} - 6Ae^{3t} = e^{2t} \Longrightarrow A = \frac{-1}{4}$

$$y_{p1} = \frac{-1}{4} e^{2t}$$

2. $(\theta^2 - \theta - 6)y = e^{-2t}$

$y_{p2} = \mathrm{B}te^{-2t}$ since e^{-2t} is already found in complementary solution.

$y_{p2}{}' = -2\mathrm{B}te^{-2t} + \mathrm{B}e^{-2t}$

$y_{p2}{}'' = 4\mathrm{B}te^{-2t} - 2\mathrm{B}e^{-2t} - 2\mathrm{B}e^{-2t} = 4\mathrm{B}te^{-2t} - 4\mathrm{B}e^{-2t}$

$$4\mathrm{B}te^{-2t} - 4\mathrm{B}e^{-2t} + 2\mathrm{B}te^{-2t} - \mathrm{B}e^{-2t} - 6\mathrm{B}te^{-2t} = e^{-2t}$$

$$-5\mathrm{B}e^{-2t} = e^{-2t}$$

$$\therefore \mathrm{B} = \frac{-1}{5}$$

$$y_{p2} = \frac{-1}{5} te^{-2t}$$

$$y_p = y_{p1} + y_{p2} = \frac{-1}{4} e^{2t} - \frac{1}{5} te^{-2t}$$

The general solution is

$$y = y_c + y_p = c_1 e^{-2t} + c_2 e^{3t} - \frac{1}{4} e^{2t} - \frac{1}{5} te^{-2t}$$

Since $x = e^t$ and $t = lnx$ so that

$$y = y_c + y_p = y_c = c_1 x^{-2} + c_2 x^3 - \frac{1}{4} x^2 - \frac{1}{5} lnx \cdot x^{-2}$$

H.W.

Find the general solution of linear ODE.

Q.1 $(x^3 D^3 + 2xD - 2)y = x^2 logx + x$

Q.2 $x^3 y''' - 4x^2 y'' + 8xy' - 8y = 4lnx$

Q.3 $x^2 y'' - xy' - 3y = x^5$

Q.4 $x^2 y'' + 2xy' - 6y = 5x^2 + 6$

Q.5 $x^2 y'' + 6xy' + 6y = lnx$

Lagrangels differential equation **2. معادلة لاكرانج التفاضليه:**

معادله لاكرانج التفاضليه من الرتبه النونية n تاخذ الصيغه

$$a_n(ax+b)^n y^{(n)} + a_{n-1}(ax+b)^{n-1} y^{(n-1)} + \cdots + a_1(ax+b)\, y' + a_o y = f(x)$$

Where a, b, $a_o, a_1, \ldots, a_n$ are constants, and $a_o \neq 0$

واضح انه عندما تاخذ a=1, b=0 فان معادله لاكرانج تتحول الى معادلة اويلر التفاضليه. اي ان معادلة اويلر التفاضليه صوره خاصة من معادله لاكرانج التفاضليه . ولحل معادله لاكرانج التفاضليه فاننا نستخدم التعويض $z = ax + b$, $z = e^t$ فتتحول المعادلة الى معادلة تفاضلية ذات معاملات ثابته تحل بنفس الطريقة المستخدمه لمعادلة اويلر التفاضليه.

Example 1: Find the general solution of linear ODE

$$(3x+2)^2 y'' + 2(3x+2)\, y' - 4y = 3x^2 + 4x + \frac{4}{3}$$

Sol.

Let $z = 3x + 2$, and $\frac{dy}{dx} = \frac{1}{3}\frac{dy}{dz}$, $\frac{d^2y}{dx^2} = \frac{1}{3}\frac{d^2y}{dz^2}$

Substitute in ODE, we get

$$\frac{1}{3}z^2\frac{d^2y}{dz^2} + \frac{2}{3}z\ \frac{dy}{dz} - 4y = 3\left(\frac{z-2}{3}\right)^2 + 4\left(\frac{z-2}{3}\right) + \frac{4}{3}$$

$$= \frac{1}{3}(z^2 - 4z + 4 + 4z - 8 + 3) = \frac{1}{3}z^2$$

$$\left[\frac{1}{3}z^2\frac{d^2y}{dz^2} + \frac{2}{3}z\ \frac{dy}{dz} - 4y = \frac{1}{3}z^2\right] \times 3$$

$$z^2\frac{d^2y}{dz^2} + 2z\ \frac{dy}{dz} - 12y = z^2$$

By using the substitution $z = e^t$ and $\theta = \frac{d}{dt}$

$$(\theta(\theta - 1) + 2\theta - 12)y = e^{2t}$$

$$(\theta^2 + \theta - 12)y = e^{2t}$$

This is linear ODE of 2nd order with constant coefficients.

The complementary solution y_c for equation above is

$$(\theta^2 + \theta - 12)y = 0$$

Let $y = e^{\lambda x}$, we get the auxiliary equation

$$\lambda^2 + \lambda - 12 = 0$$

he roots of this equation are

$\lambda_1 = -3$, $\lambda_2 = 4$

$y_c = c_1e^{-3t} + c_2e^{4t}$ where c_1, c_2 are arbitrary constants.

The particular solution y_p for equation is:

$$y_p = Ae^{2t}$$

$$y_p = \frac{-1}{7}e^{2t}$$

The general solution is

$$y = y_c + y_p = c_1e^{-3t} + c_2e^{4t} - \frac{1}{7}e^{2t}$$

Since $z = e^t$, and $z = 3x + 2$

$$y = y_c + y_p = c_1z^{-3} + c_2z^4 - \frac{1}{7}z^2$$

$$y = y_c + y_p = c_1(3x+2)^{-3} + c_2(3x+2)^4 - \frac{1}{7}(3x+2)^2$$

Example 2: Find the general solution of linear ODE

$$(x+2)^2y'' + (x+2)\,y' - y = x + 2$$

Sol.

Let $z = x + 2$, and $\frac{dy}{dx} = \frac{dy}{dz}$, $\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}$

Substitute in ODE, we get

$$z^2 \frac{d^2y}{dz^2} + z\,\frac{dy}{dz} - y = z$$

By using the substitution $z = e^t$ and $\theta = \frac{d}{dt}$

$$(\theta(\theta - 1) + \theta - 1)y = e^{2t}$$

$$(\theta^2 - 1)y = e^{2t}$$

This is linear ODE of 2^{nd} order with constant coefficients.

The complementary solution y_c for equation above is

$$(\theta^2 - 1)y = 0$$

Let $y = e^{\lambda x}$, we get the auxiliary equation

$$\lambda^2 - 1 = 0$$

he roots of this equation are

$\lambda_1 = -1 \quad , \quad \lambda_2 = 1$

$y_c = c_1 e^{-t} + c_2 e^t$ where c_1, c_2 are arbitrary constants.

The particular solution y_p for equation is:

$$y_p = Ae^{2t}$$

$$y_p = \frac{1}{4}(e^{2t} - 1)$$

The general solution is

$$y = y_c + y_p = c_1 e^{-t} + c_2 e^t + \frac{1}{4}(e^{2t} - 1)$$

Since $z = e^t$, and $z = x + 2$

$$y = y_c + y_p = c_1(x + 2)^{-1} + c_2(3x + 2) + \frac{1}{4}((x + 2)^2 - 1)$$

Example 3: Find the general solution of linear ODE

$$(1 + 2x)^2 y'' - 6(1 + 2x)\, y' + 16y = 8(1 + 2x)^3$$

Sol.

Let $z = 1 + 2x$, and $\frac{dy}{dx} = 2\frac{dy}{dz}$, $\frac{d^2y}{dx^2} = 4\frac{d^2y}{dz^2}$

Substitute in ODE, we get

$$4z^2 \frac{d^2y}{dz^2} - 12z\,\frac{dy}{dz} + 16y = 8z^3$$

By using the substitution $z = e^t$ and $\theta = \frac{d}{dt}$

$$(\theta(\theta - 1) - 3\theta + 4)y = 2e^{3t}$$

$$(\theta^2 - 4\theta + 4)y = 2e^{3t}$$

This is linear ODE of 2^{nd} order with constant coefficients.

The complementary solution y_c for equation above is

$$(\theta^2 - 4\theta + 4)y = 0$$

Let $y = e^{\lambda x}$, we get the auxiliary equation

$$\lambda^2 - 4\lambda + 4 = 0$$

he roots of this equation are

$\lambda_1 = 2 \quad , \quad \lambda_2 = 2$

$$y_c = (c_1 + c_2 t)e^{2t}$$ where c_1, c_2 are arbitrary constants.

The particular solution y_p for equation is:

$$y_p = Ae^{3t}$$

$$y_p = \frac{-1}{3}(2e^{3t} - 4)$$

The general solution is

$$y = y_c + y_p = (c_1 + c_2 t)e^{2t} - \frac{1}{3}(2e^{3t} - 4)$$

Since $z = e^t$, and $z = 1 + 2x$

$$y = y_c + y_p = \left(c_1 + c_2 ln(1 + 2x)\right)(1 + 2x)^2 - \frac{1}{3}(2(1 + 2x)^3 - 4)$$

H.W.

Find the general solution of linear ODE

Q.1 $(3x + 2)^2 y'' + 3(3x + 2)\, y' - 36y = 9$

Q.2 $(x + 1)^2 y'' - (x + 1)\, y' - 3y = x$

Q.3 $(2x + 1)^2 y'' + 2(2x + 1)\, y' - 12y = 6x$

Q.4 $(3x + 2)^2 y'' + 3(3x + 2)\, y' - 36y = 3x^2 + 4x + 1$