Mustansiriyah University

College of Engineering

Department of Computer Engineering

Fundamentals of Electric Circuits

For

First Stage

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1st Stage

Electrical Circuits | code: CSE103 | Hours/Units: 4/4	
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Chapter One

Basic Concepts

1.1. Introduction

Electric circuit theory and electromagnetic theory are the two fundamental theories upon which all branches of computer and electrical engineering are built. Many branches of electrical engineering, such as power, electric machines, control, electronics, communications, and instrumentation, are based on electric circuit theory. Therefore, the basic electric circuit theory course is the most important course for a computer and electrical engineering student, and always an excellent starting point for a beginning student in computer and electrical engineering education. Circuit theory is also valuable to students specializing in other branches of the physical sciences because circuits are a good model for the study of energy systems in general, and because of the applied mathematics, physics, and topology involved.

In computer and electrical engineering, we are often interested in communicating or transferring energy from one point to another. Doing this requires an interconnection of electrical devices. Such interconnection is referred to as an ***electric circuit***, and each component of the circuit is known as an ***element***.

- An *electric circuit* is an interconnection of electrical elements

A simple electric circuit is shown on **Fig. 1.1**. It consists of three basic elements: a battery, a lamp, and connecting wires. Such a simple circuit can exist by itself; it has several applications, such as a flashlight, a search light, and so forth. Electric circuits are used in numerous electrical systems to accomplish different tasks. Our objective in this chapter is not to study various uses and applications of circuits. Rather our major concern is the analysis of the circuits. By the analysis of a circuit, we mean a study of the behavior of the circuit: How does it respond to a given input? How do the interconnected elements and devices in the circuit interact? We commence our study by defining some basic concepts. These concepts include charge, current, voltage, circuit elements, power, and energy. Before defining these concepts, we must first establish a system of units that we will use throughout the text.

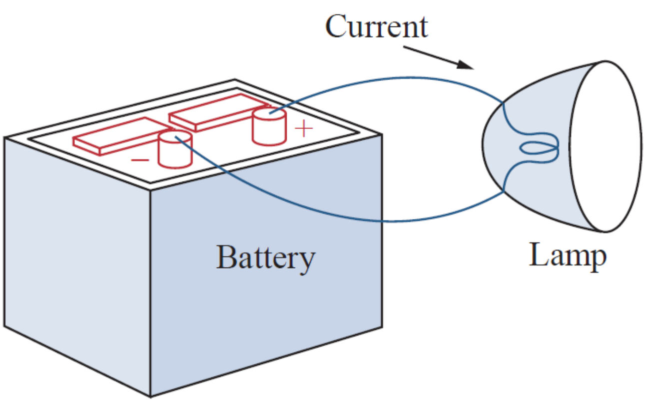


Figure 1.1 A simple Electric Circuit.

1.2. Systems of Units

As electrical engineers, we deal with measurable quantities. Our measurement, however, must be communicated in a standard language that virtually all professionals can understand, irrespective of the country where the measurement is conducted. Such an international measurement language is the ***International System of Units*** (SI), adopted by the General Conference on Weights and Measures in 1960. In this system, there are six principal units from which the units of all other physical quantities can be derived. **Table 1.1** shows the six units, their symbols, and the physical quantities they represent. The SI units are used throughout this text. One great advantage of the SI unit is that it uses prefixes based on the power of 10 to relate larger and smaller units to the basic unit. **Table 1.2** shows the SI prefixes and their symbols. For example, the following are expressions of the same distance in meters (m):

600,000,000 mm 600,000 m 600 km

TABLE 1.1

Six basic SI units and one derived unit relevant to this text.

Quantity	Basic unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Thermodynamic temperature	kelvin	K
Luminous intensity	candela	cd
Charge	coulomb	C

TABLE 1.2

The SI prefixes.

Multiplier	Prefix	Symbol
10^{18}	exa	E
10^{15}	peta	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

1.3. Charge and Current

The concept of electric charge is the underlying principle for explaining all electrical phenomena. Also, the most basic quantity in an electric circuit is the ***electric charge***.

The following points should be noted about the electric charge:

- The coulomb is a large unit for charges. In 1 C of charge, there are $1/(1.602 \times 10^{-19}) = 6.24 \times 10^{18}$ electrons. Thus, realistic or laboratory values of charges are on the order of pC, nC, or μC.
- According to experimental observations, the only charges that occur in nature are integral multiples of the electronic charge $e = -1.602 \times 10^{-19}$ C.
- The ***law of conversation of charge*** states that charge can neither be created nor destroyed, only transferred. Thus, the algebraic sum of electric charges in a system does not change.

- **Electric Charge (q)**

The ***electric charge is an electrical property of the atomic particles of which matter consists, measured in coulombs (C).*** The charge e on an electron is negative and equal in magnitude to (**1.602×10−19 C**), while a proton carries a positive charge of the same magnitude as the electron.

$$q = \int_{t_1}^{t_2} i\, dt \qquad \text{Eq. (1.1)}$$

- **Electric Current (i)**

The electric current is the time rate of change of charge, measured in amperes (A).

$$i = \frac{dq}{dt} \qquad \text{Eq. (1.2)}$$

The way we define **current** as i in Eq. (1.1) suggests that current need not be a constant-valued function. There can be several types of current; that is, charge can vary with time in several ways. If the current does not change with time, but remains constant, we call it a ***direct current*** (dc).

A **direct current** (dc) is a current that remains constant with time. Where current is measured in **amperes** (**A**), an

1 ampere = 1 coulomb/second

By convention the symbol I is used to represent such a constant current. A time-varying current is represented by the symbol i. A common form of time-varying current is the sinusoidal current or ***alternating current*** (ac).

An **alternating current** (ac) is a current that varies sinusoidally with time.

Such current is used in your household, to run the air conditioner, refrigerator, washing machine, and other electric appliances. **Figure 1.2**.

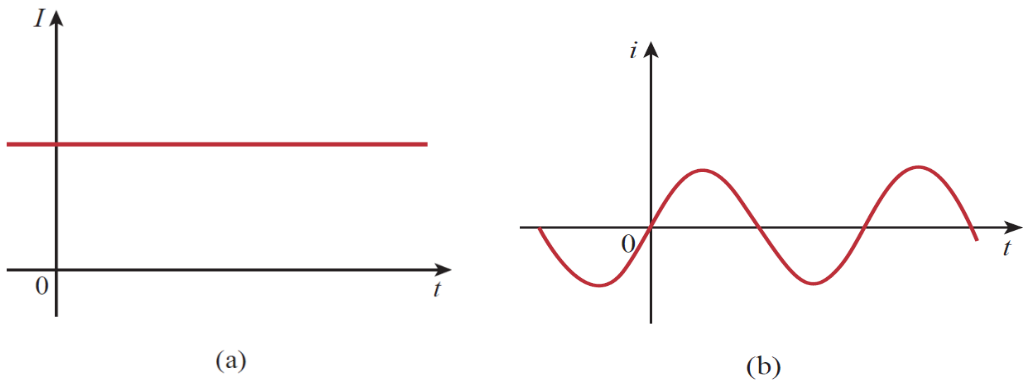


Figure 1.2. Two common types of current: (a) **direct current** (dc), (b) **alternating current** (ac).

Example 1.1:

How much is the charge represented by 4.600 electrons?

Solution:

Each electron has -1.602×10^{-19} C. Hence 4.600 electrons will have:

-1.602×10^{-19} C/electron × 4.600 electrons = -7.369×10^{-16} C.

Example 1.2:

The total charge entering a terminal is given by $q = 5t\sin 4\pi t$ mC. Calculate the current at $t = 0.5$ s.

Solution:

$i = \frac{dq}{dt} = \frac{d}{dt}(5t\sin 4\pi t)$ mC/s = $(5t\sin 4\pi t + 20\pi t\cos 4\pi t)$mA

At $t = 0.5$,

$i = 5\sin 2\pi + 10\pi\cos 2\pi = 0 + 10\pi = 31.42$ mA.

Practice Example:

If previous Example has $q = (10 - e^{-2t})$ mC, find the current at $t = 0.5$ s.

Answer:

7.36 mA.

Example 1.3: Determine the total charge for entering a terminal between $t = 1$ s and $t = 2$ s, if the current passing the terminal is $i = (3t^2 - t)$ A.

Solution:

$$q = \int_{t=1}^{2} i\,dt = \int_{t=1}^{2} (3t^2 - t)dt$$

$$= \left(t^3 - \frac{t^2}{2}\right)\Big|_1^2 = (8 - 2) - \left(1 - \frac{1}{2}\right) = 5.5 \text{ C.}$$

Practice Example:

The current flowing through an element is

$$q = \begin{cases} 2\ A, & 0 < t < 1 \\ 2t^2\ A, & t > 1 \end{cases}$$

Calculate the charge entering the element from $t = 0$ to $t = 2$ s.

Answer:

6.667 C.

1.4. Voltage

As explained briefly in the previous section, ***to move the electron in a conductor in a particular direction requires some work or energy transfer***. This work is performed by an ***external electromotive force (emf)***, typically represented by the battery in **Fig. 1.3**. This emf is also known as ***voltage*** or ***potential difference***. The voltage v_{ab} between two points *a* and *b* in an electric circuit is the energy (or work) needed to move a unit charge from *a* to *b*; mathematically:

$$v_{ab} = \frac{dw}{dq} \qquad \text{Eq. (1.3)}$$

where w is energy in **joules** (**J**) and q is charge in **coulombs** (**C**). The voltage v_{ab} or simply v is measured in **volts** (**V**).

1 volt = 1 joule/coulomb =1 newton-meter/coulomb

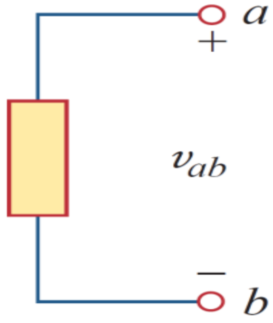


Figure 1.3. Polarity of voltage v_{ab}.

Thus, **Voltage** (or **potential difference**) is the energy required to move a unit charge through an element, measured in **volts (V)**.

Figure 1.3 shows the voltage across an element (represented by a rectangular block) connected to points a and b. The plus (+) and minus (-) signs are used to define reference direction or voltage polarity. The v_{ab} can be interpreted in two ways: (1) point a is at a potential of v_{ab} volts higher than point b, or (2) the potential at point a with respect to point b is v_{ab}. It follows logically that in general:

$$v_{ab} = -v_{ba} \qquad \text{Eq. (1.4)}$$

Current and voltage are the two basic variables in electric circuits. Like electric current, a constant voltage is called a ***dc voltage*** and is represented by **V**, whereas a sinusoidally time-varying voltage is called an ***ac voltage*** and is represented by ***v***. A dc voltage is commonly produced by a battery; ac voltage is produced by an electric generator.

1.5. Power and Energy

Although current and voltage are the two basic variables in an electric circuit, they are not sufficient by themselves. For practical purposes, we need to know how much power an electric device can handle. **Power** ***is the time rate of expending or absorbing energy***, measured in **watts (W)**,

$$p = \frac{dw}{dt} \qquad \text{Eq. (1.5)}$$

where ***p*** is power in **watts (W)**, ***w*** is energy in **joules (J),** and ***t*** is time in **seconds (s).** From Eqs. (1.1), (1.3), and (1.5), it follows that:

$$p = \frac{dw}{dt} = \frac{dw}{dq} \cdot \frac{dq}{dt} = vi \qquad \text{Eq. (1.6)}$$

Or

$$p = vi \qquad \text{Eq. (1.7)}$$

Thus, the power absorbed or supplied by an element is the product of the voltage across the element and the current through it. If the power has a + sign, power is being delivered to or absorbed by the element. If, on the other hand, the power has a − sign, power is being supplied by the element

Current direction and voltage polarity play a major role in determining the sign of power. It is therefore important that we pay attention to the relationship between current i and voltage v in **Fig. 1.4 (a)**. The voltage polarity and current direction must conform with those shown in **Fig. 1.4 (a)** in order for the power to have a positive sign. This is known as the passive sign convention.

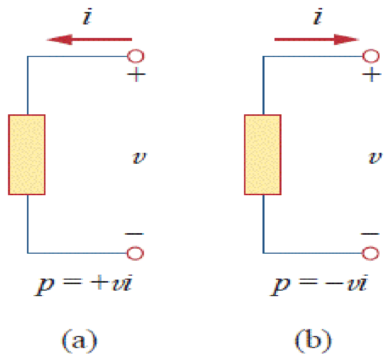


Figure 1.4. Reference polarities for power using the passive sign convention: (a) absorbing power, (b) supplying power.

Passive sign convention is satisfied when the current enters through the positive terminal of an element and $\boldsymbol{p = +vi}$. If the current enters through the negative terminal, $\boldsymbol{p = -vi}$.

Unless otherwise stated, we will follow the passive sign convention throughout this text. For example, the element in both circuits of **Fig. 1.5** has an absorbing power of W **because a positive current enters the positive terminal in both cases**.

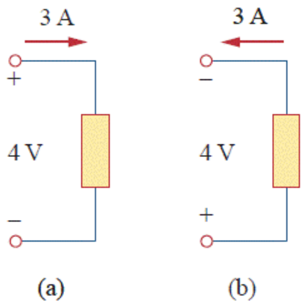


Figure 1.5. Two cases of an element with an absorbing power of 12 W:

(a) $p = 4 \times 3 = 12$ W, (b) $p = 4 \times 3 = 12$ 12W

In **Fig. 1.6**, however, the element is supplying power of $+12$ W because a positive current enters the negative terminal. Of course, an absorbing power of -12 W is equivalent to a supplying power of $+12$W. In general,

$$+\text{Power absorbed} = -\text{Power supplied}$$

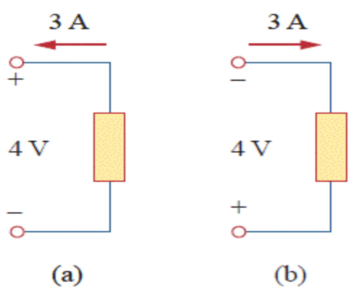


Figure 1.6 Two cases of an element with a supplying power of 12 W:

(a) $p = -4 \times 3 = -12$ W, (b) $p = -4 \times 3 = -12$ W.

Energy is the capacity to do work, measured in **joules (J)**. The energy absorbed or supplied by an element from time $\boldsymbol{t_o}$ to time $\boldsymbol{t}$ is:

$$\boldsymbol{w = \int_{t_o}^{t} p\, dt = \int_{t_o}^{t} vi\, dt} \qquad \text{Eq. (1.8)}$$

The electric power utility companies measure energy in watt-hours (Wh), where

1 Wh = 3.600 J

Example 1.4: An energy sources a constant current of 2 A for 10 s to flow through a lightbulb. If 2.3 kJ is given off in the form of light and heat energy, calculate the voltage drop across the bulb.

Solution:

The total charge is: $\Delta q = i\,\Delta \boldsymbol{t} = 2 \times 10 = 20$ C.

The voltage drop is: $v = \frac{\Delta w}{\Delta q} = \frac{2.3\times10^3}{20} = 115$ V.

Example 1.5: Find the power delivered to an element at $t = 3$ ms if the current entering its positive terminal is $i = 5\cos 60\,\pi t$ A, and the voltage is:

(a) $v = 3i$, (b) $v = 3\, di/dt$.

Solution:

(a) The voltage is $v = 3i = 15\cos 60\,\pi t$; hence, the power is

$p = vi = 75\cos^2 60\pi t$ W.

At $t = 3$ ms,

$p = 75\cos^2(60\pi \times 3 \times 10^{-3}) = 75\cos^2 0.18\pi = 53.48$ W.

(b) We find the voltage and the power as:

$v = 3\frac{di}{dt} = 3(-60\pi)5\sin 60\,\pi t = -900\pi \sin 60\,\pi t$ V.

$p = vi = -4500\pi \sin 60\,\pi t \cos 60\,\pi t$ W.

At $t = 3$ ms,

$p = -4500\pi \sin 0.18\,\pi \cos 0.18\,\pi$ W

$= -14137.167\ \sin 32.4^\circ \cos 32.4^\circ = -6.396$ kW.

Note: Converting radian to degree:

1 radian = 180/ $\boldsymbol{\pi}$ degree. Thus, 0.18×180 = 32.4.

Practice Example: Find the power delivered to the element in Example 1.5 at $t = 5$ ms, if the current remains the same but the voltage is:

(a) $v = 2i$ V, (b) $v = \left(10 + 5\int_0^t i\, dt\right)$ V.

Answer: (a) 17.27 W, (b) 29.7 W.

1.6. Circuit Elements

An electric circuit is simply an interconnection of the elements. **Circuit analysis** is the process of determining voltages across (or the currents through) the elements of the circuit. There are two types of elements found in electric circuits which are:

- **Active Elements**: These elements are capable of generating energy such as generators, batteries, and operational amplifiers
- **Passive Elements**: These elements are incapable of generating energy such as resistors, capacitors, and inductors.

The most important active elements are voltage or current sources that generally deliver power to the circuit connected to them. There are two kinds of sources:

- **Independent Source**: is an active element that provides a specified voltage or current that is completely independent of other circuit elements.
- **Dependent Source or controlled source**: is an active element in which the source quantity is controlled by another voltage or current.

Figure 1.7 shows the symbols for independent voltage sources. Notice that both symbols in **Fig. 1.7** (**a**) and (**b**) can be used to represent a dc voltage source, but only the symbol in **Fig. 1.7** (**a**) can be used for a time-varying voltage source. Similarly, an ideal independent current source is an active element that provides a specified current completely independent of the voltage across the source.

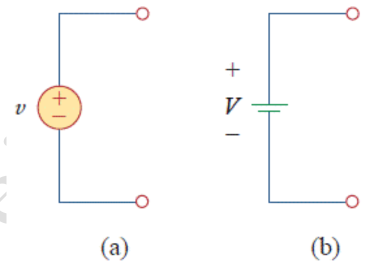


Figure 1.7. Symbols for independent voltage sources: (**a**) used for constant or time-varying voltage, (**b**) used for constant voltage (dc).

Dependent sources are usually designated by diamond-shaped symbols, as shown in **Fig. 1.8**. Since the control of the dependent source is achieved by a voltage or current of some other element in the circuit, and the source can be voltage or current

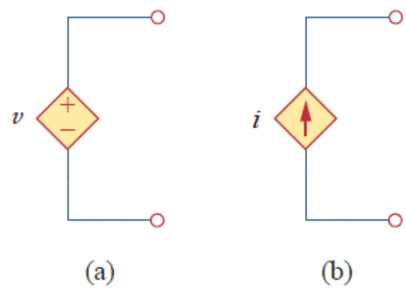
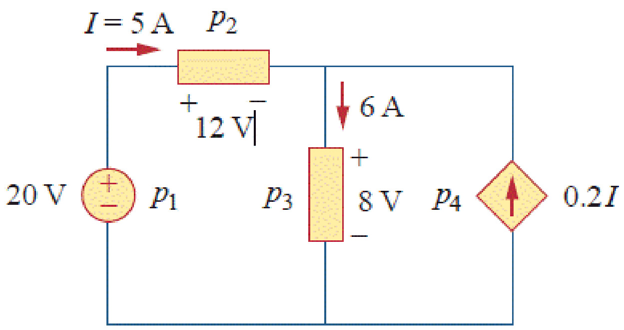


Figure 1.8. Symbols for: (a) dependent voltage source, (b) dependent current source.

Example 1.6: Calculate the power supplied or absorbed by each element in **Figure**.



Solution: We apply the sign convention for power. For p_1, the 5-A current is out of the positive terminal (or into the negative terminal). Thus,

$p_1 = 20(-5) = -100$ W   Supplied power

For p_2 and p_3, the current flows into positive terminal of the element in each case.

$p_2 = 12(5) = 60$ W   Absorbed power

$p_3 = 8(6) = 48$ W   Absorbed power

For p_4, we should note that the voltage is 8 V, the same as the voltage for p_3, since both the passive element and the dependent source are connected to the same terminals. (Remember that voltage is always measured across an element in a circuit.) Since the current flows out of the positive terminal,

$p_4 = 8(-0.2I) = 8(-0.2 \times 5) = -8$ W   Supplied power

We should observe that the 20-V independent voltage source and dependent current source are supplying power to the rest of the network, while the two passive elements are absorbing power.

$$p_1 + p_2 + p_3 + p_4 = -100 + 60 + 48 - 8 = 0$$

The total power supplied equals the total power absorbed.

Chapter Two

Basic Laws

2. Ohm's Law

2.1. Ohm's Law

Materials in general have a characteristic behavior of resisting the flow of electric charge. This physical property, or ability to resist current, is known as **resistance** and is represented by the **symbol *R***. The resistance of any material with a uniform cross-sectional **area *A*** depends on *A* and its **length *ℓ***, as shown in **Fig. 2.1**(**a**). We can represent resistance (as measured in the laboratory), in mathematical form,

$$R = \rho \frac{\ell}{A} \qquad \text{Eq. (2.1)}$$

where ρ is known as the **resistivity** of the material in ohm-meters. Good conductors, such as copper and aluminum, have low resistivities, while insulators, such as mica and paper, have high resistivities.

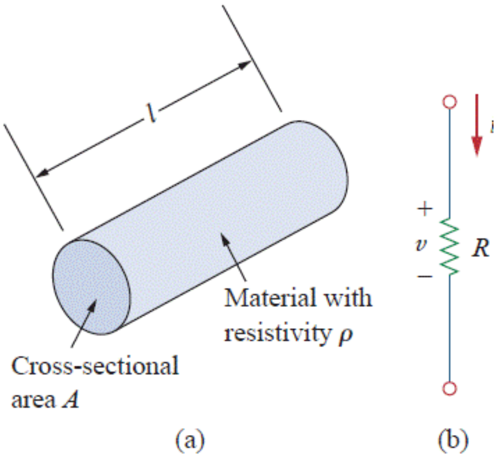


Figure 2.1 (a) Resistor, (b) Circuit symbol for resistance.

Ohm's law states that the voltage v across a resistor is directly proportional to the current i flowing through the resistor.

$$v \propto i \qquad \text{Eq. (2.2)}$$

Ohm defined the constant of proportionality for a resistor to be the **resistance**, ***R***. (The resistance is a material property which can change if the internal or external conditions of the element are altered, e.g., if there are changes in the temperature.). The Eq. (2.2) becomes:

$$v = iR \qquad \text{Eq. (2.3)}$$

The resistance R of an element denotes its ability to resist the flow of electric current; it is measured in **ohms** (***Ω***).

$$R = V/i \qquad \text{Eq. (2.4)}$$

So that

$$1\Omega = 1V/A \qquad \text{Eq. (2.5)}$$

Since the value of R can range from ***zero*** to ***infinity***, it is important that we consider the two extreme possible values of R. An element with $R = 0$ is called a **short circuit**, as shown in **Figure 2.2** (**a**). For a short circuit,

$$v = iR = 0 \qquad \text{Eq. (2.6)}$$

showing that the voltage is zero but the current could be anything. In practice, a short circuit is usually a connecting wire assumed to be a perfect conductor. Thus,

A **short circuit** is a circuit element with resistance approaching **zero**.

Similarly, an element with is known as an ***open circuit***, as shown in **Fig. 2.2**(**b**). For an open circuit,

$$i = \lim_{R\to\infty} \frac{v}{R} = 0 \qquad \text{Eq. (2.7)}$$

indicating that the current is zero though the voltage could be anything. Thus,

An **open circuit** is a circuit element with resistance approaching **infinity**.

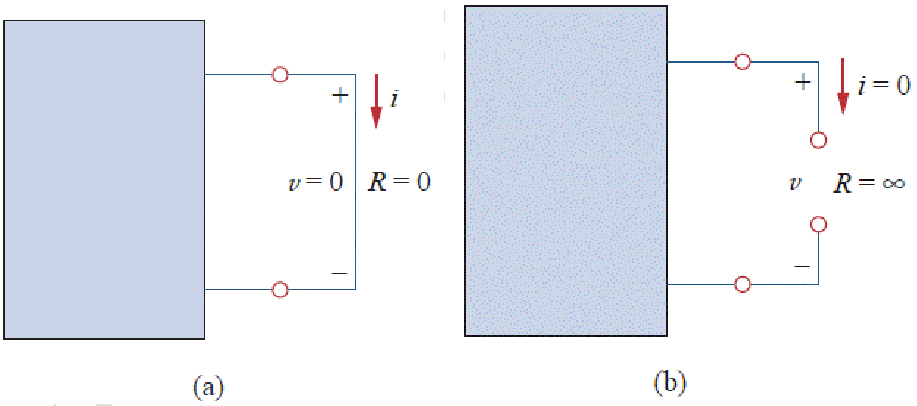


Figure 2.2: (**a**) Short circuit ($R \to 0$), (**b**) Open circuit ($R \to \infty$).

A useful quantity in circuit analysis is the reciprocal of resistance R, known as ***conductance*** and denoted by G:

$$G = \frac{1}{R} = \frac{i}{v} \qquad \text{Eq. (2.8)}$$

The circuit symbol for the resistor is shown in **Fig. 2.1**(**b**), where R stands for the resistance of the resistor.

Conductance (G): is the ability of an element to conduct electric current; it is measured in ***mho*** (℧) or **Siemens** (**S**).

The same resistance can be expressed in ohms or siemens. For example, 10 Ω is the same as 0.1 S. Thus,

$$\boldsymbol{i = Gv} \qquad \text{Eq. (2.9)}$$

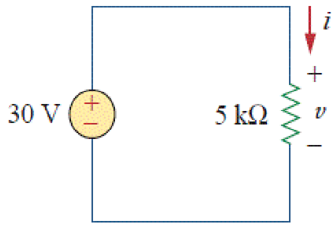
The power dissipated by a resistor can be expressed in terms of R, as follows:

$$\boldsymbol{p = vi = i^2R = \frac{v^2}{R}} \qquad \text{Eq. (2.10)}$$

The power dissipated by a resistor may also be expressed in terms of G, as follows:

$$\boldsymbol{p = vi = v^2G = \frac{i^2}{G}} \qquad \text{Eq. (2.11)}$$

Example 2.1: In the circuit shown below, calculate the current i, the conductance G, and the power p.



Solution: The voltage across the resistor is the same as the source voltage (30 V) because the resistor and the voltage source are connected to the same pair of terminals. Hence, the current is

$$i = \frac{v}{R} = \frac{30}{5\times10^3} = 6\text{mA}$$

The conductance is

$$G = \frac{1}{R} = \frac{1}{5\times10^3} = 0.2 \text{ mS}$$

The power can be calculated in various ways:

$$p = vi = 30(6 \times 10^{-3}) = 180 \text{ mW}$$

Or $$p = i^2R = (6 \times 10^{-3})^2 5 \times 10^3 = 180 \text{ mW}$$

Or $$p = v^2G = (30)^2 0.2 \times 10^{-3} = 180 \text{ mW}$$

Example 2.2: A voltage source of $20 \sin \pi t$ V is connected across a 5-kΩ resistor. Find the current through the resistor and the power dissipated.

Solution:

$$i = \frac{v}{R} = \frac{20 \sin \pi t}{5\times10^3} = 4 \sin \pi t \text{ mA}$$

Hence, $$p = vi = 80 \sin^2 \pi t \text{ mW}$$

Practice Example: A resistor absorbs an instantaneous power of $20 \cos^2 t$ mW when connected to a voltage source $v = 10 \cos t$ V. Find i and R.

Answer: $2 \cos t$ mA, 5 kΩ.

2.2. Nodes, Branches, and Loops

We study the properties relating to the placement of elements in the network and the geometric configuration of the network. Such elements include branches, nodes, and loops.

- A **branch** represents a single element such as a voltage source or a resistor.
- A **node** is the point of connection between two or more branches.
- A **loop** is any closed path in a circuit.

The circuit in **Fig. 2.3** has five branches, namely, the 10-V voltage source, the 2-A current source, and the three resistors.

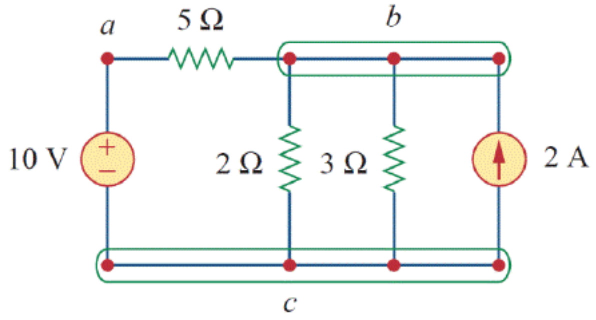


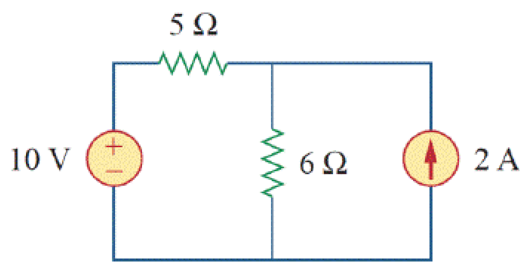
Figure 2.3: Nodes, branches, and loops.

A loop is a closed path formed by starting at a node, passing through a set of nodes, and returning to the starting node. A network with $\boldsymbol{b}$ branches, $\boldsymbol{n}$ nodes, and $\boldsymbol{l}$ independent loops will satisfy the fundamental theorem of network topology:

$$\boldsymbol{b = l + n - 1} \quad \text{Eq. (2.12)}$$

- Two or more elements are in **series** if they exclusively share a **single node** and consequently carry the **same current**.
- Two or more elements are in **parallel** if they are connected to the **same two nodes** and consequently have the **same voltage** across them.

Example 2.3: Determine the number of branches and nodes in the circuit shown in Figure below. Identify which elements are in series and which are in parallel

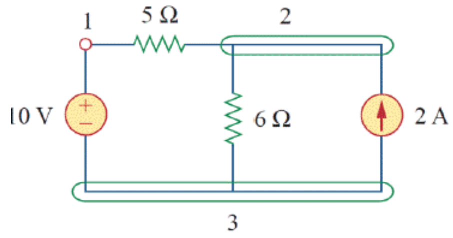


Solution: Since there are four elements in the circuit, the circuit has four branches: 10 V, 5 Ω, 6 Ω, and 2 A. The circuit has three nodes as identified in Figure below. The 5 Ω resistor is in series with the 10-V voltage source because the same current would flow in both. The 6-Ω resistor is in parallel with the 2-A current source because both are connected to the same nodes 2 and 3.

$l = 2, b = 4, n = 3$

$\boldsymbol{b = l + n - 1}$

$4 = 2 + 3 - 1 = 4$



2.3. Kirchhoff's Laws

Ohm's law by itself is not sufficient to analyze circuits. However, when it is coupled with Kirchhoff's two laws, we have a sufficient, powerful set of tools for analyzing a large variety of electric circuits. Kirchhoff's laws were first introduced in 1847 by the German physicist Gustav Robert Kirchhoff (1824–1887). These laws are formally known as **Kirchhoff's current law** (**KCL**) and **Kirchhoff's voltage law** (**KVL**).

- ***Kirchhoff's current law (KCL)*** states that the algebraic sum of currents entering a node (or a closed boundary) is zero. Mathematically, KCL is expressed as:

$$\sum_{n=1}^{N} i_n = 0 \qquad \text{Eq. (2.12)}$$

where $\boldsymbol{N}$ is the number of branches connected to the node and $\boldsymbol{i_n}$ is the nth current entering (or leaving) the node. By this law, currents **entering** a **node** may be regarded as **positive**, while currents **leaving** the **node** may be taken as **negative** or vice versa as shown in **Fig. 2.4**:

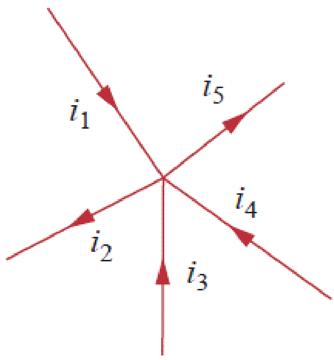


Figure 2.4: Currents at a node illustrating KCL.

$$i_1 + (-i_2) + i_3 + i_4 + (-i_5) = 0$$

By rearranging the terms, we get:

$$i_1 + i_3 + i_4 = i_2 + i_5$$

The sum of the currents entering a node is equal to the sum of the currents leaving the node.

- ***Kirchhoff's voltage law (KVL)*** states that the algebraic sum of all voltages around a closed path (or loop) is zero.

Expressed mathematically, KVL states that:

$$\sum_{m=1}^{M} v_m = 0 \qquad \text{Eq. (2.13)}$$

where $\boldsymbol{M}$ is the number of voltages in the loop (or the number of branches in the loop) and $\boldsymbol{v_m}$ is the mth voltage.

To illustrate KVL, consider the circuit in **Fig. 2.5**. The sign on each voltage is the polarity of the terminal encountered first as we travel around the loop. We can start with any branch and go around the loop either clockwise or counterclockwise. Suppose we start with the voltage source and go clockwise around the loop as shown; then voltages would be $-v_1$, $+v_2$, $+v_3$, $-v_4$, and $+v_5$, in that order. For example, as we reach branch 3, the positive terminal is met first; hence we have $+v_3$. For branch 4, we reach the negative terminal first; hence, $-v_4$. Thus, KVL yields

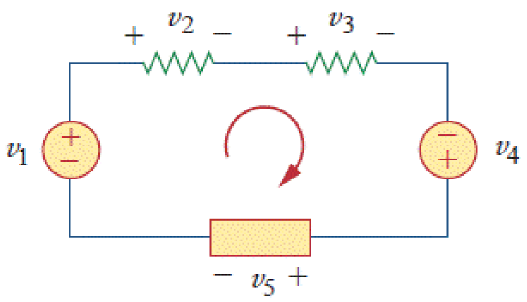


Figure 2.5. A single-loop circuit illustrating KVL

$$-\boldsymbol{v_1} + \boldsymbol{v_2} + \boldsymbol{v_3} - \boldsymbol{v_4} + \boldsymbol{v_5} = 0 \qquad \text{Eq. (2.14)}$$

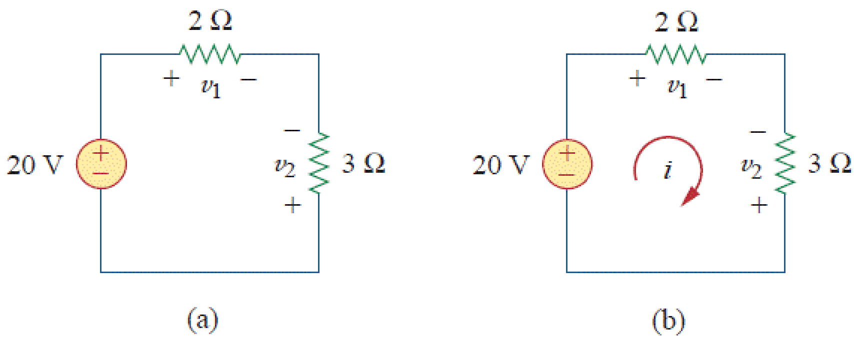
Rearranging terms gives:

$$\boldsymbol{v_2} + \boldsymbol{v_3} + \boldsymbol{v_5} = \boldsymbol{v_1} + \boldsymbol{v_4} \qquad \text{Eq. (2.15)}$$

which may be interpreted as:

Sum of voltage drops = Sum of voltage rise Eq. (2.16)

Example 2.4: For the circuit in Figure below, find v_1 and v_2.



Solution: To find v_1 and v_2, we apply Ohm's law and Kirchhoff's voltage law.

$v_1 = 2i, \qquad v_2 = -3i$

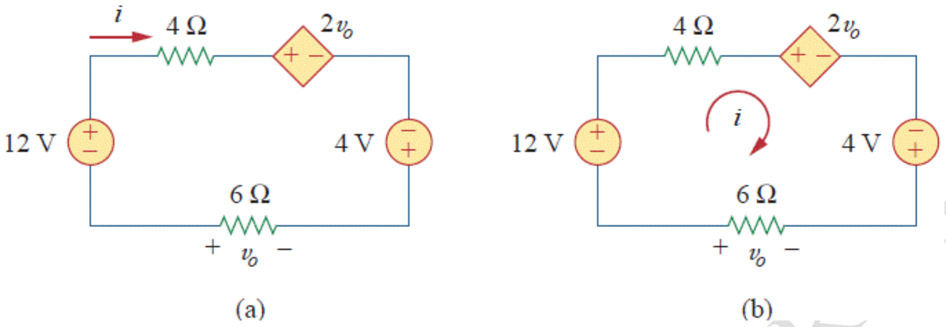
Applying KVL around the loop gives:

$-20 + v_1 - v_2 = 0$

$-20 + 2i + 3i = 0$ or $5i = 20 \Rightarrow i = 4$ A

$v_1 = 8$ V, $v_2 = -12$ V

Example 2.5: Determine v_o and i in the circuit shown below.



Solution: We apply KVL around the loop as shown in Figure. The result is

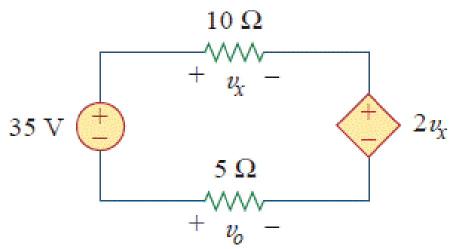
$-12 + 4i + 2v_o - 4 + 6i = 0$

Applying Ohm's law to the 6 Ω resistor gives:

$v_o = -6i$

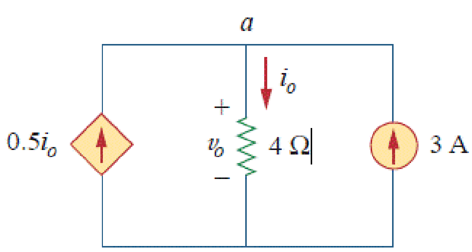
$-16 + 10i - 12i = 0 \Rightarrow i = -8$ A, and $v_o = 48$ V.

Practice Example: Find v_x and v_o in the circuit of figure below.



Answer: 10 V, -5 V.

Example 2.6: Determine i_o and v_o in the circuit shown below.



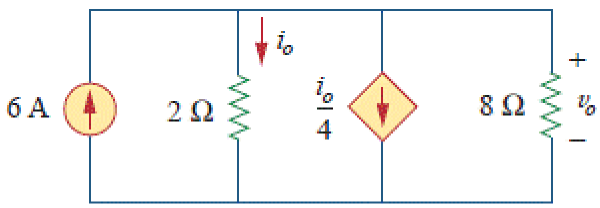
Solution: We apply KVL to node a, we obtain

$3 + 0.5i_o = i_o \Rightarrow i_o = 6$ A

For the 4 Ω resistor, Ohm's law gives:

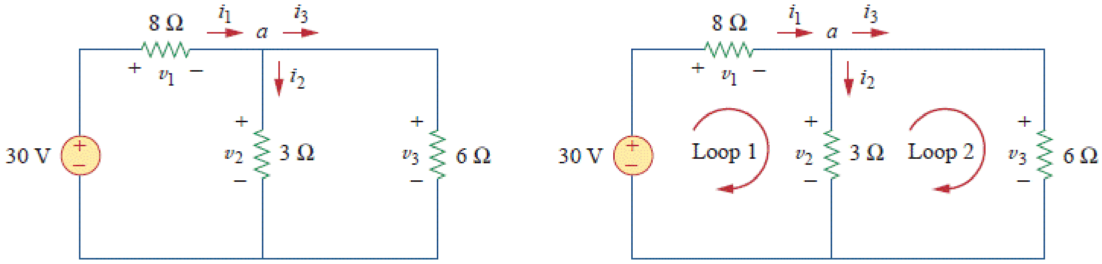
$v_0 = 4i_o = 24$ V

Practice Example: Find v_o and i_o in the circuit shown below.



Answer: 8 V, 4 A

Example 2.7: Find currents and voltages in the circuit shown below:



(a) (b)

Solution: We apply Ohm's law and KVL's law. By Ohm's law,

$v_1 = 8i_1, \qquad v_2 = 3i_2, \qquad v_3 = 6i_3,$

Since the voltage and current of each resistor are related by Ohm's law as shown, we are really looking for three things: (v_1, v_2, v_3) or (i_1, i_2, i_3). At node a, KCL gives:

$i_1 - i_2 - i_3 = 0$

Applying KVL to loop 1,

$-30 + v_1 + v_2 = 0 \quad \Rightarrow -30 + 8i_1 + 3i_2 = 0 \;\Rightarrow i_1 = \frac{(30-3i_2)}{8}$

Applying KVL to loop 2,

$-v_2 + v_3 = 0 \quad \Rightarrow v_2 = v_3$

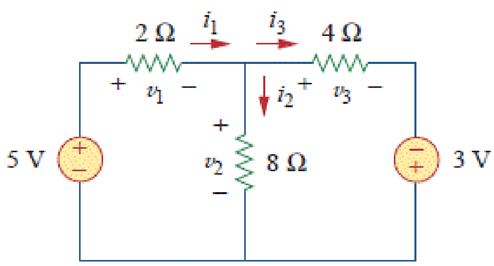
since the two resistors are in parallel. We express v_1 and v_2 in terms of i_1 and i_2,

$6i_3 = 3i_2 \;\Rightarrow\; i_3 = \frac{i_2}{2}$, Substituting in the first equation,

$\frac{(30-3i_2)}{8} - i_2 - \frac{i_2}{2} = 0$ multiply by 8, $\Rightarrow\; 30 - 3i_2 - 8i_2 - 4i_2 = 0 \;\Rightarrow\; i_2 = 2$

$i_1 = 3$ A, $i_3 = 1$ A, $v_1 = 24$ V, $v_2 = 6$ V, $v_3 = 6$ V

Practice Example: Find currents and voltages in the circuit shown below:



Answer: $v_1 = 3$ V, $v_2 = 2$ V, $v_3 = 5$ V,

$i_1 = 1.5$ A, $i_2 = 0.25$ A, $i_3 = 1.25$ A,

2.4. Series Resistors and Voltages Division

The need to combine resistors in series or in parallel occurs so frequently that it warrants special attention. The process of combining the resistors is facilitated by combining two of them at a time. Consider the single-loop circuit of **Fig. 2.6** The two resistors are in series, since the same current i flows in both. Apply both to each of the resistors, we obtain,

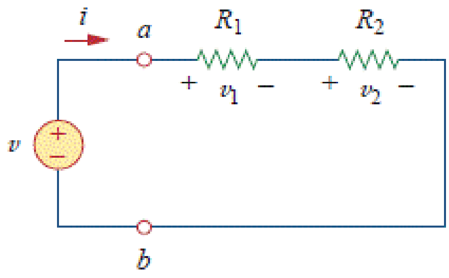


Figure 2.6. A single-loop circuit with two resistors in series.

$$v_1 = iR_1, \qquad v_2 = iR_1 \qquad \text{Eq. (2.17)}$$

If we apply KVL to the loop (moving in the clockwise direction), we have:

$$-v + v_1 + v_2 = 0 \qquad \text{Eq. (2.18)}$$

Combining the two previous equations, we get:

$$v = v_1 + v_2 = i(R_1 + R_2) \qquad \text{Eq. (2.19)}$$

Or

$$i = \frac{v}{R_1 + R_2} \qquad \text{Eq. (2.20)}$$

Notice that Eq. (2.20) can be written as follows:

$$v = iR_{eq} \qquad \text{Eq. (2.21)}$$

the two resistors can be replaced by an equivalent resistor R_{eq} as shown in **Fig.2.7.**

$$R_{eq} = R_1 + R_2 \qquad \text{Eq. (2.22)}$$

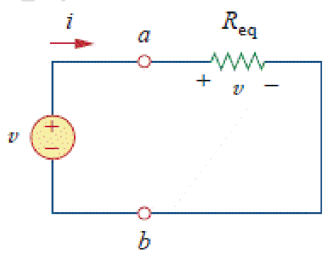


Figure 2.7. Equivalent circuit of **Fig.2.6** circuit.

The equivalent resistance of any number of **resistors connected in series** is the **sum of the individual resistances**.

For *N* resistors in series then,

$$R_{eq} = R_1 + R_2 + \cdots + R_N = \sum_{n=1}^{N} R_n \qquad \text{Eq. (2.23)}$$

To determine the voltage across each resistor in **Fig. 2.7**, we substitute Eq. (2.20) into Eq. (2.17) and obtain the following:

$$v_1 = \frac{R_1}{R_1+R_2} v, \qquad v_2 = \frac{R_2}{R_1+R_2} v \qquad \text{Eq. (2.24)}$$

Notice that the source voltage v is divided among the resistors in direct proportion to their resistances; the larger the resistance, the larger the voltage drop. This is called the ***principle of voltage division***, and the circuit in **Fig. 2.6** is called a ***voltage divider***. In general, if a voltage divider has N resistors $(R_1, R_2, \dots, R_N)$ in series with the source voltage v, the nth resistor (R_n) will have a voltage drop of

$$v_n = \frac{R_n}{R_1+R_{2,\dots,N}} v \qquad \text{Eq. (2.25)}$$

2.5. Parallel Resistors and Current Division

Consider the circuit in **Fig. 2.8**, where two resistors are connected in parallel and therefore have the same voltage across them. From Ohm's law,

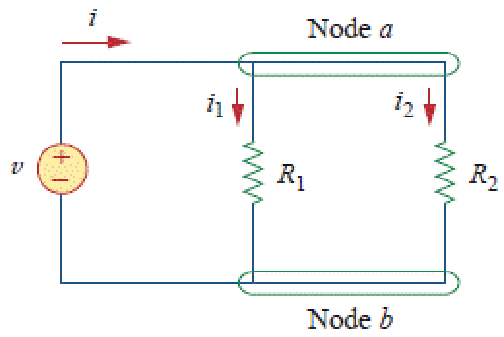


Figure 2.8. Two resistors in parallel.

$$v = i_1 R_1 = i_2 R_2 \quad \text{Or} \quad i_1 = \frac{v}{R_1},\ i_2 = \frac{v}{R_2} \qquad \text{Eq. (2.26)}$$

Applying KCL at node a gives the total current i as:

$$i = i_1 + i_2 \qquad \text{Eq. (2.27)}$$

Substituting Eq. (2.26) into Eq. (2.27), we get:

$$i = \frac{v}{R_1} + \frac{v}{R_2} = v\left(\frac{1}{R_1} + \frac{1}{R_1}\right) = \frac{v}{R_{eq}} \qquad \text{Eq. (2.28)}$$

where R_{eq} is the equivalent resistance of the resistors in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{Or} \quad \frac{1}{R_{eq}} = \frac{R_1+R_2}{R_1 R_2} \qquad \text{Eq. (2.29)}$$

$$R_{eq} = \frac{R_1 R_2}{R_1+R_2} \qquad \text{Eq. (2.30)}$$

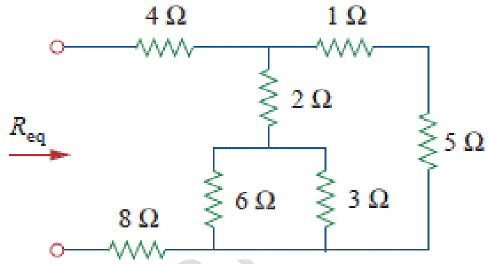
The ***equivalent resistance*** of **two parallel resistors** is equal to the **product of their resistances divided by their sum**.

$$v = iR_{eq} = \frac{iR_1 R_2}{R_1+R_2} \qquad \text{Eq. (2.31)}$$

Substituting in Eq. (2.26) $\quad i_1 = \frac{iR_2}{R_1+R_2}, \; i_2 = \frac{iR_1}{R_1+R_2}$ Eq. (2.32)

which shows that the total current i is shared by the resistors in inverse proportion to their resistances. This is known as the ***principle of current division***, and the circuit in **Fig. 2.8** is known as a ***current divider***. Notice that the larger current flows through the smaller resistance.

Example 2.8: Find R_{eq} for the circuit given below:



Solution: To get R_{eq}, we combine resistors in series and in parallel. The 6Ω and 3Ω resistors are in parallel, so their equivalent resistance is

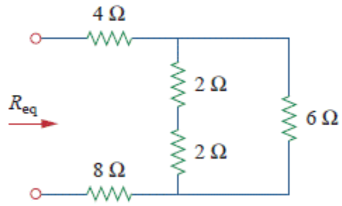
$6\Omega||3\Omega = \frac{6\times3}{6+3} = 2\Omega$

Also, the 1Ω and 5Ω are in series ⇒ $1\Omega + 5\Omega = 6\Omega$

According to right figure, the circuit is reduced, we notice

That the two 2Ω resistors are in series, so the equivalent

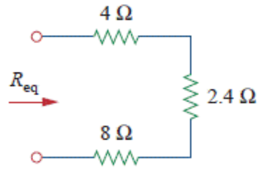
Resistance is,



(a)

$2\Omega + 2\Omega = 4\Omega$. Now, the 4Ω is parallel with 6Ω, thus their equivalent resistance is

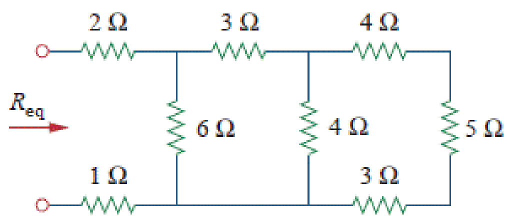
$4\Omega||6\Omega = \frac{4\times6}{4+6} = 2.4\Omega$. In the right figure (b), there are three resistors are in series. The equivalent resistance for the circuit is



(b)

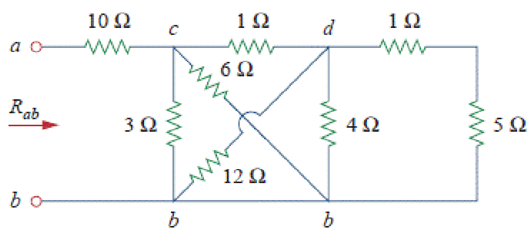
$R_{eq} = 4\Omega + 2.4\Omega + 8\Omega = 14.4\Omega$

Practice Example: Find R_{eq} for the circuit given below:



Answer: 6Ω.

Example 2.9: Find R_{ab} for the circuit given below:



Solution:

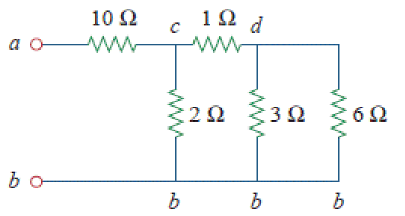
$1 + 5 = 6\Omega, \frac{4\times12}{4+12} = 3\Omega, \frac{3\times6}{3+6} = 2\Omega,$

In figure (a), we have:

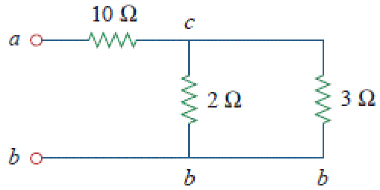
$\frac{3\times6}{3+6} = 2\Omega$, and $2\Omega + 1\Omega = 3\Omega$

$\frac{3\times2}{3+2} = 1.2\Omega$,

Thus, the $R_{ab} = 10 + 1.2 = 11.2\ \Omega$.

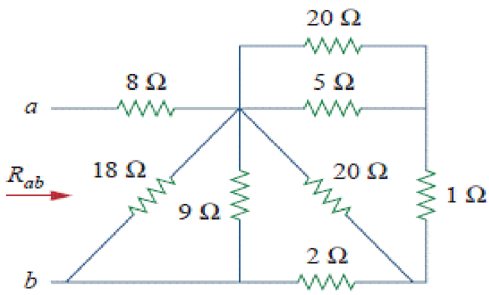


(a)



(b)

Practice Example: Find R_{eq} for the circuit given below:

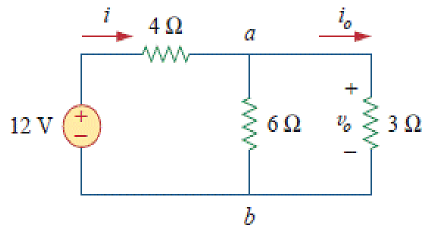


Answer: 11Ω.

Example 2.10: Find i_o and v_o in the circuit given below. Calculate the power dissipated in the 3 Ω resistor.

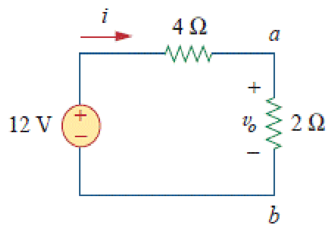
Solution:

$6\Omega \,\|\, 3\Omega = \frac{3\times6}{3+6} = 2\Omega$



(a)

In Fig. (b), notice that it is not affected by the combination of the resistors because the resistors are in parallel and therefore, have the same voltage. From Fig.(b), we can obtain in two ways. One way is to apply Ohm's law to get,



(b)

$i = \frac{12}{4+2} = 2\text{ A},\ v_o = 2i = 2\times2 = 4\text{ V}.$

Another way is to apply voltage division,

$v_o = \frac{2}{2+4}(12) = 4\text{ V}.$

Similarly, i_o can be obtained in two ways. One approach is to apply Ohm's law to the 3 Ω resistor in Fig. (a),

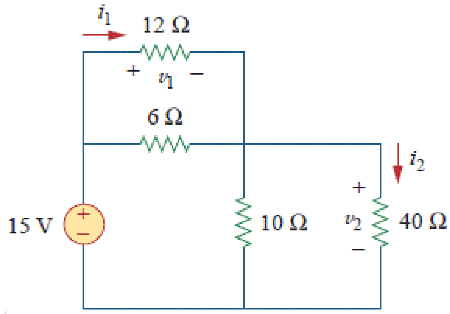
$v_o = 3i_o = 4 \Rightarrow i_o = \frac{4}{3}\text{ A}$

Another approach is to apply current division to the circuit in Fig. (a),

$$i_o = \frac{6}{6+3} i = \frac{2}{3}(2\text{ A}) = \frac{4}{3}\text{ A.}$$

The power dissipated in the 3 Ω is $p_o = i_o v_o = 4\left(\frac{4}{3}\right) = 5.333$ W.

Practice Example: Find v_1 and v_2 in the circuit shown below in figure. Also calculate i_1 and i_2 and the power dissipated in the 12 Ω and 40 Ω resistors.



Answer: $v_1 = 5$ V, $i_1 = 416.7$ mA, $p_1 = 2.083$ W,

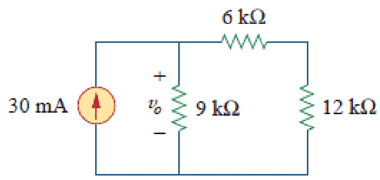
$v_2 = 10$ V, $i_2 = 250$ mA, $p_2 = 2.5$ W.

Example 2.11: For the circuit shown below, find: (a) the voltage v_0, (b) the power supplied by the current source, (c) the power absorbed by each resistor.

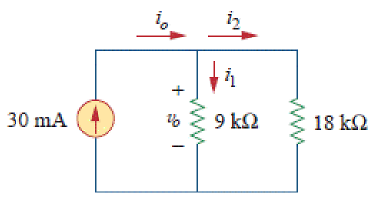
Solution:

(a) The 6 Ω and 12 Ω resistors are in series, so 6+12=18 Ω

We apply current division technique to find i_1 and i_2:



(a)



(b)

$$i_1 = \frac{18000}{9000+18000}(30\text{ mA}) = 20\text{ mA}$$

$$i_2 = \frac{9000}{9000+18000}(30\text{ mA}) = 10\text{ mA}$$

$$v_o = 9000 i_1 = 18000 i_2 = 180\text{ V.}$$

(b) Power supplied by the source is

$$p_o = v_o i_o = 180(30)\text{ mW} = 5.4\text{ W.}$$

(c) Power absorbed by 12 kΩ resistor is

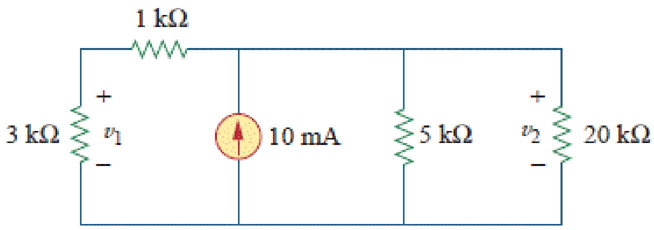
$$p = iv = i_2(i_2 R) = {i_2}^2 R = (10 \times 10^{-3})^2(12000) = 1.2\text{ W}$$

Power absorbed by 6 kΩ resistor is

$$p = iv = {i_2}^2 R = (10 \times 10^{-3})^2(6000) = 0.6\text{ W}$$

Power absorbed by 9 kΩ resistor is $p = \frac{v_o^2}{R} = \frac{180^2}{9000} = 3.6$ W

Practice Example: For the circuit shown below, find: (a) v_1 and v_2, (b) the power dissipated in the 3 Ω and 20 Ω resistors, and (c) the power supplied by the current sources.



Answer:

(a) 15 V, 20 V, (b) 75 mW, 20 mW,

(c) 200 mW.

2.6. Wye-Delta Transformations

Situations often arise in circuit analysis when the resistors are neither in parallel nor in series. For example, consider the bridge circuit in **Fig. 2.9**.

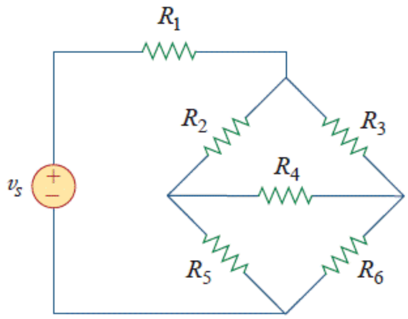
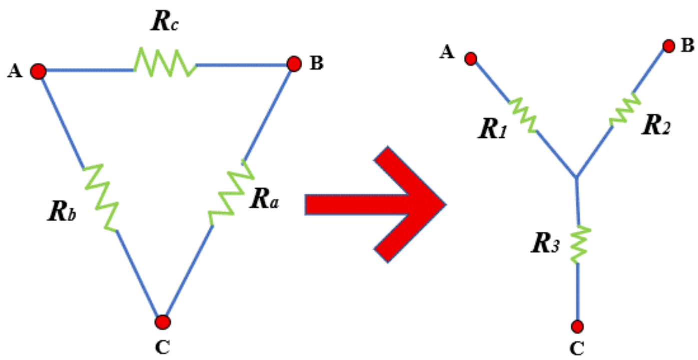


Figure 2.9. The bridge network.

2.6.1. Delta to Wye conversion

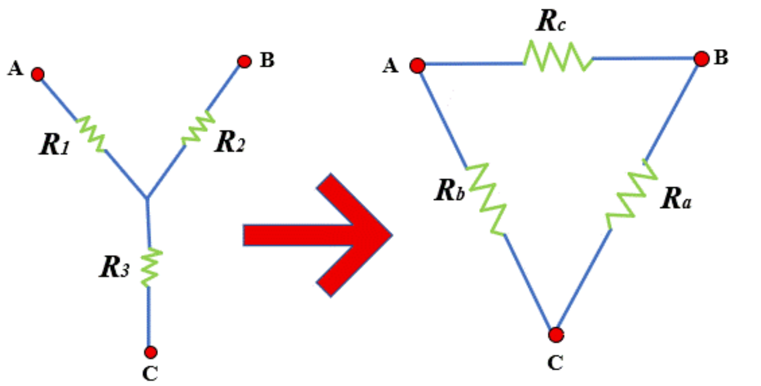
Each resistor in the Y network is the product of the resistors in the two adjacent Δ branches, divided by the sum of the three Δ resistors.



$$R_1 = \frac{R_b R_c}{R_a+R_b+R_c}, \qquad R_2 = \frac{R_a R_c}{R_a+R_b+R_c}, \qquad R_3 = \frac{R_a R_b}{R_a+R_b+R_c}, \qquad \text{Eq. (2.33)}$$

2.6.2. Wye to Delta conversion

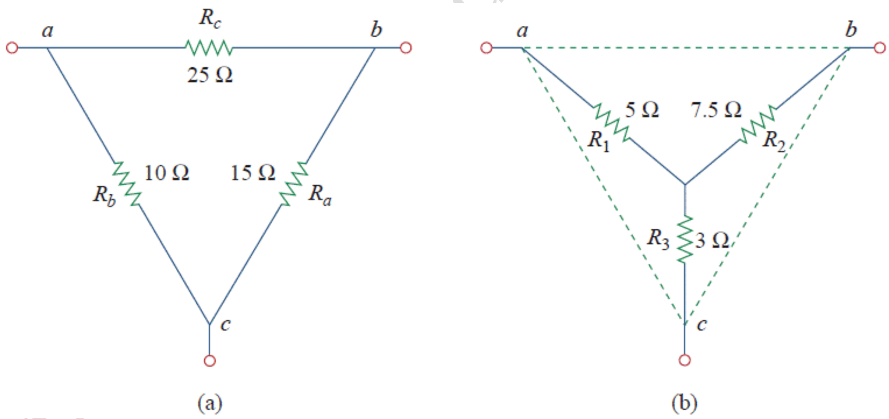
Each resistor in the Δ network is the sum of all possible products of Y resistors taken two at a time, divided by the opposite Y resistor.



$$R_a = \frac{R_1R_2+R_2R_3+R_3R_1}{R_1}, \quad R_b = \frac{R_1R_2+R_2R_3+R_3R_1}{R_2}, \quad R_c = \frac{R_1R_2+R_2R_3+R_3R_1}{R_3} \quad \text{Eq. (2.34)}$$

Example 2.12: Convert the Δ network in the below figure to an equivalent Y network.

Solution:



$$R_1 = \frac{R_bR_c}{R_a+R_b+R_c},$$

$$R_1 = \frac{10\times25}{15+10+25} = 5\ \Omega$$

$$R_2 = \frac{R_aR_c}{R_a+R_b+R_c}$$

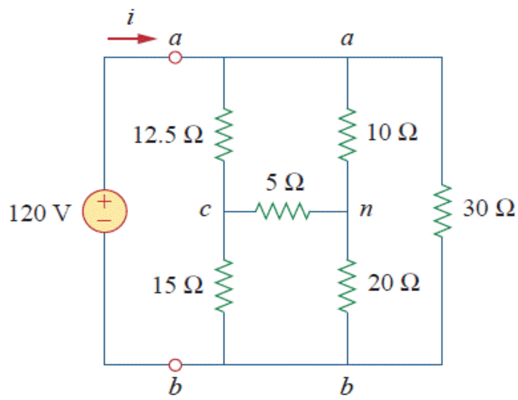
$$R_2 = \frac{25\times15}{15+10+25} = 7.5\ \Omega$$

$$R_3 = \frac{R_aR_b}{R_a+R_b+R_c}$$

$$R_3 = \frac{15\times10}{15+10+25} = 3\ \Omega$$

Example 2.13: Obtain the equivalent resistance R_{ab} for the below figure and use it to find i.

Solution:

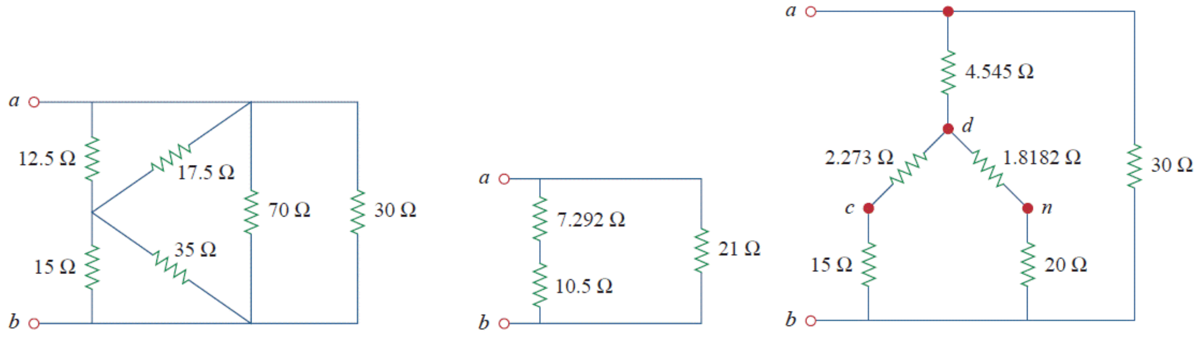


$R_1 = 10\ \Omega,\ R_2 = 20\ \Omega,\ R_3 = 5\ \Omega$

$$R_a = \frac{R_1R_2+R_2R_3+R_3R_1}{R_1} = \frac{10\times20+20\times5+5\times10}{10} = 35\ \Omega$$

$$R_b = \frac{R_1R_2+R_2R_3+R_3R_1}{R_2} = 17.5\ \Omega$$

$$R_c = \frac{R_1R_2+R_2R_3+R_3R_1}{R_3} = 70\ \Omega$$



(a) (b) (c)

$70 \parallel 30 = \frac{70 \times 30}{70+30} = 21\ \Omega$

$12.5 \parallel 17.5 = 7.292\ \Omega$

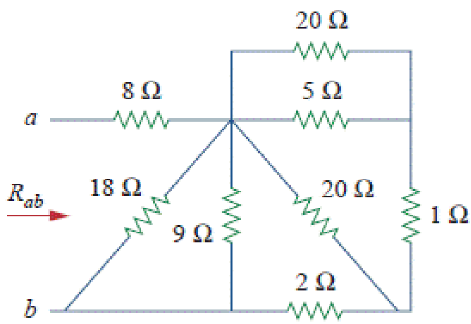
$15 \parallel 35 = 10.5\ \Omega$

$R_{ab} = (7.292 + 10.5) \parallel 21 = 9.632\ \Omega$

$i = \frac{v_s}{R_{ab}} = \frac{120}{9.632} = 12.458$ A

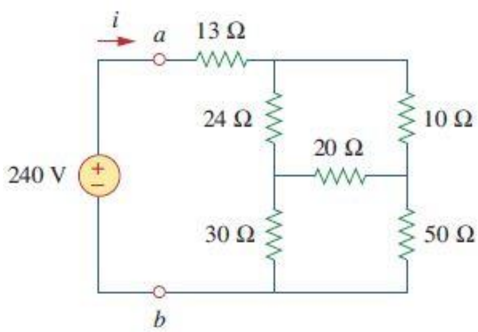
Practice Example: Find R_{ab} for the circuit in the figure below:

Answer: 11 Ω



Practice Example: For the bridge network in Figure below, find R_{ab} and i.

Answer: 40 Ω, 6 A.



Chapter Three

Methods of Analysis

In this chapter, we will prepare to apply these two laws: Ohm's and Kirchhoff's laws, to develop two powerful techniques for circuit analysis: **nodal analysis**, which is based on a systematic application of Kirchhoff's current law (KCL), and **mesh analysis**, which is based on a systematic application of Kirchhoff's voltage law (KVL). These two techniques are so important to be understood.

3.1. Nodal Analysis

Nodal analysis provides a general procedure for analyzing circuits using **node voltages** as the circuit variables. Choosing node voltages instead of element voltages as circuit variables is convenient and reduces the number of equations one must solve simultaneously. To simplify matters, we shall assume in this section that circuits do not contain voltage sources. Circuits that contain voltage sources will be analyzed in the next section.

In ***nodal analysis***, we are interested in finding the node voltages. Given a circuit with n nodes without voltage sources, the nodal analysis of the circuit involves taking the following three steps.

Nodal analysis is also known as the **node-voltage method**.

Steps to Determine Node Voltages:

1. Select a node as the reference node. Assign **voltages** to the remaining nodes. The voltages are referenced with respect to the reference node.
2. Apply **KCL** to each of the nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

We shall now explain and apply these three steps. The first step in nodal analysis is selecting a node as the ***reference*** or ***datum node***. The reference node is commonly called the ***ground*** since it is assumed to have zero potential. A reference node is indicated by any of the three symbols in **Fig.3.1**.

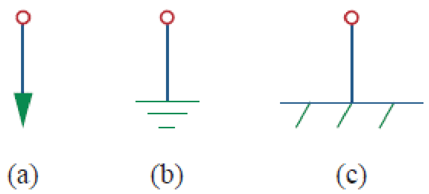
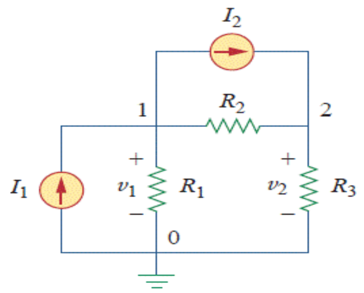


Figure 3.1. Common symbols for indicating a reference node, (a) common ground, (b) ground, (c) chassis ground.

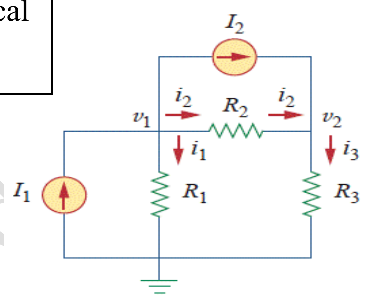
Once we have selected a reference node, we assign voltage designations to nonreference nodes. Consider, for example, the circuit shown. Node 0 is the reference node while nodes **1** and **2** are assigned voltages $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, respectively.



(a)

Each node voltage is the voltage rise from the reference node to the corresponding nonreference node or simply the voltage of that node with respect to the reference node. The **second step**, we apply **KCL** to each nonreference node in the circuit. We now add i_1, i_2, and i_3 as the currents through resistors R_1, R_2, and R_3, respectively.

Figure 3.2. Typical circuit for nodal



(b)

At node **1**, applying **KCL** gives:

$\boldsymbol{I_1 = I_2 + i_1 + i_2}$ Eq. (3.1)

At node 2:

$\boldsymbol{i_3 = i_2 + I_2}$ Eq. (3.2)

We now apply **Ohm's law** to express the unknown currents i_1, i_2, and i_3 in terms of node voltages.

Note: Current flows from a higher potential to a lower potential in a resistor.

We can express this principle as:

$$\boldsymbol{i = \frac{v_{higher} - v_{lower}}{R}} \quad \text{Eq. (3.3)}$$

$$\boldsymbol{i_1 = \frac{v_1 - 0}{R_1}, \quad i_2 = \frac{v_1 - v_2}{R_2}, \quad i_3 = \frac{v_2 - 0}{R_3}} \quad \text{Eq. (3.4)}$$

Substituting Eq. (3.4) in Eqs. (3.1) and (3.2) results, respectively, give:

$$\boldsymbol{I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2}} \quad \text{Eq. (3.5)}$$

$$\boldsymbol{I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}} \quad \text{Eq. (3.6)}$$

The **third step** in nodal analysis is to solve the node voltages. If we apply **KCL** to $\boldsymbol{n-1}$ nonreference nodes, we obtain $\boldsymbol{n-1}$ simultaneous equations such as Eqs. (5) and (6). We solve Eqs. (5) and (6) to obtain the node voltages $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ using any standard method, such as the **substitution method**, the **elimination method**, **Cramer's rule**, or **matrix inversion**. To use either of the last two methods, one must cast the simultaneous equations in **matrix form**. For example, Eqs. (5) and (6) after we simplify them, can be cast in matrix form as:

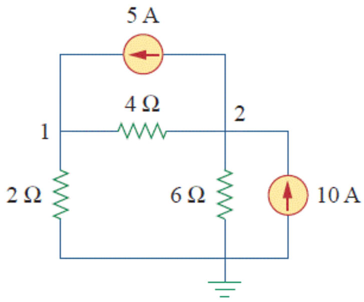
$$\boldsymbol{v_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - v_2\left(\frac{1}{R_2}\right) = I_1 - I_2} \quad \text{Eq. (3.7)}$$

$$\boldsymbol{v_2\left(\frac{1}{R_3} + \frac{1}{R_2}\right) - v_1\left(\frac{1}{R_2}\right) = I_2} \quad \text{Eq. (3.8)}$$

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2+R_3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix} \qquad \text{Eq. (3.9)}$$

Which can be solved to get v_1 and v_2.

Example 3.1: Calculate the node voltages in the circuit shown below:



(a)

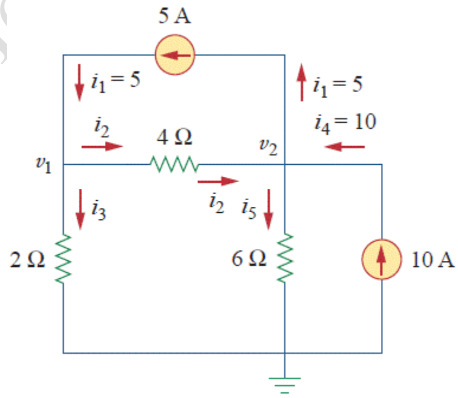
Solution:

At node 1, applying KCL and Ohm's law gives:

$$i_1 = i_2 + i_3 \;\Rightarrow\; 5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2} \;\Rightarrow\; \frac{v_1}{2} + \frac{v_1}{4} - \frac{v_2}{4} = 5$$

$$v_1\left(\frac{1}{2}+\frac{1}{4}\right) - v_2\left(\frac{1}{4}\right) = 5 \;\Rightarrow\; 3v_1 - v_2 = 20 \qquad \text{Eq. (1)}$$

At node 2,



(b)

$$i_2 + i_4 = i_1 + i_5 \;\Rightarrow\; \frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$$

$$\frac{v_2}{4} - \frac{v_1}{4} + \frac{v_2}{6} = 10 - 5 \;\Rightarrow\; v_1\left(-\frac{1}{4}\right) + v_2\left(\frac{1}{4}+\frac{1}{6}\right) = 5$$

$$-3v_1 + 5v_2 = 60 \qquad \text{Eq. (2)}$$

Note that the Eq. (1) and Eq. (2) can be solved in two ways:

- **Elimination technique**,

we add Eq. (1) and Eq. (2)

$$4v_2 = 80 \;\Rightarrow\; v_2 = 20 \text{ V}$$

Substituting v_2 in Eq. (1) gives, $3v_1 - 20 = 20 \;\Rightarrow\; v_1 = 13.333$ V

- **Cramer's rule**

$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$, we now obtain v_1 and v_2 as:

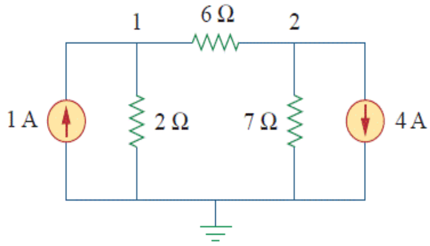
$$v_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{bmatrix} 20 & -1 \\ 60 & 5 \end{bmatrix}}{\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix}} = \frac{20\times5-(-1\times60)}{3\times5-(-1\times-3)} = \frac{160}{12} = 13.333 \text{ V}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{bmatrix} 3 & 20 \\ -3 & 60 \end{bmatrix}}{\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix}} = \frac{3\times60-(20\times-3)}{3\times5-(-1\times-3)} = \frac{180+60}{12} = 20 \text{ V}$$

To compute the values of each current,

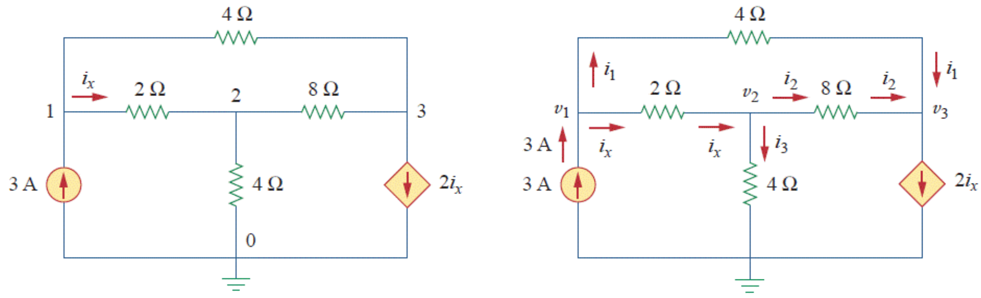
$$i_1 = 5 \text{ A},\; i_2 = \frac{v_1 - v_2}{4} = 1.6668 \text{ A},\; i_3 = \frac{v_1}{2} = 6.666 \text{ A},\; i_4 = 10 \text{ A},\; i_5 = \frac{v_2}{6} = 3.333 \text{ A}$$

Practice Example: Obtain the node voltages in the circuit shown below:



Answer: $v_1 = -2$ V, $v_2 = -14$ V.

Example 3.2: Determine the voltages at the nodes in the figure given below:



(a) (b)

Solution:

At node 1,

$$3 = i_1 + i_x \quad \Rightarrow \quad 3 = \frac{v_1 - v_3}{4} + \frac{v_1 - v_2}{2} \quad \Rightarrow \quad 3 = v_1\left(\frac{1}{4} + \frac{1}{2}\right) - v_2\left(\frac{1}{2}\right) - v_3\left(\frac{1}{4}\right)$$

$$3v_1 - 2v_2 - v_3 = 12 \qquad \text{Eq. (1)}$$

At node 2,

$$i_x = i_2 + i_3 \quad \Rightarrow \quad \frac{v_1 - v_2}{2} = \frac{v_2 - 0}{4} + \frac{v_2 - v_3}{8} \quad \Rightarrow \quad v_1\left(\frac{1}{2}\right) = v_2\left(\frac{1}{4} + \frac{1}{2} + \frac{1}{8}\right) - v_3\left(\frac{1}{8}\right)$$

$$-4v_1 + 7v_2 - v_3 = 0 \qquad \text{Eq. (2)}$$

At node 3,

$$2i_x = i_1 + i_2 \quad \Rightarrow \quad \frac{2(v_1 - v_2)}{2} = \frac{v_1 - v_3}{4} + \frac{v_2 - v_3}{8} \quad \Rightarrow \quad 8(v_1 - v_2) = 2(v_1 - v_3) + (v_2 - v_3)$$

$$6v_1 - 9v_2 + 3v_3 = 0$$

$$2v_1 - 3v_2 + v_3 = 0 \qquad \text{Eq. (3)}$$

- **Elimination technique**, we add Eq. (1) and Eq. (3)

$$v_1 - v_2 = \frac{12}{5} \qquad \text{Eq. (4)}$$

Adding Eq. (2) and Eq. (3)

$-2v_1 + 4v_2 = 0 \;\Rightarrow\; v_1 = 2v_2$ Eq. (5),

substituting Eq. (5) into Eq. (4)

$2v_2 - v_2 = 12/5 \quad \Rightarrow \quad v_2 = 2.4$V, $v_1 = 4.8$ V, $v_3 = -2.4$ V

- **Cramer's rule**

$$\begin{bmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$. we now obtain v_1, v_2, and v_3 as:

$v_1 = \frac{\Delta_1}{\Delta}$, $v_2 = \frac{\Delta_2}{\Delta}$, and $v_3 = \frac{\Delta_3}{\Delta}$, we have **3×3 matrix**, which can be solved by **repeating first two rows and cross multiply**.

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \\ 3 & -2 & -1 \\ -4 & 7 & -1 \end{vmatrix} = 21 - 12 + 4 + 14 - 9 - 8 = 10$$

Similarly, we solve Δ_1, Δ_2, and Δ_3

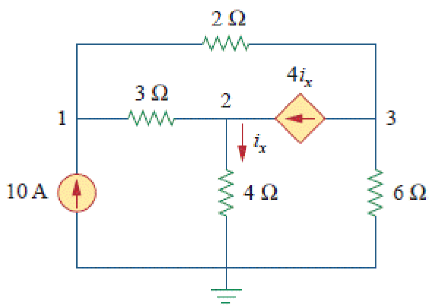
$$\Delta_1 = \begin{vmatrix} 12 & -2 & -1 \\ 0 & 7 & -1 \\ 0 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 12 & -2 & -1 \\ 0 & 7 & -1 \\ 0 & -3 & 1 \\ 12 & -2 & -1 \\ 0 & 7 & -1 \end{vmatrix} = 84 + 0 + 0 - 0 - 36 - 0 = 48$$

$$\Delta_2 = \begin{vmatrix} 3 & 12 & -1 \\ -4 & 0 & -1 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 12 & -1 \\ -4 & 0 & -1 \\ 2 & 0 & 1 \\ 3 & 12 & -1 \\ -4 & 0 & -1 \end{vmatrix} = 0 + 0 - 24 - 0 - 0 + 48 = 24$$

$$\Delta_3 = \begin{vmatrix} 3 & -2 & 12 \\ -4 & 7 & 0 \\ 2 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 12 \\ -4 & 7 & 0 \\ 2 & -3 & 0 \\ 3 & -2 & 12 \\ -4 & 7 & 0 \end{vmatrix} = 0 + 144 + 0 - 168 - 0 - 0 = -24$$

We find $v_1 = \frac{\Delta_1}{\Delta} = \frac{48}{10} = 4.8$ V, $v_2 = \frac{\Delta_2}{\Delta} = \frac{24}{10} = 2.4$ V, and $v_3 = \frac{\Delta_3}{\Delta} = \frac{-24}{10} = -2.4$ V

<u>**Practice Example**</u>: Find the voltages at three nonreference nodes in the circuit shown below:



<u>**Answer**</u>: $v_1 = 80$ V, $v_2 = -64$ V, $v_3 = 156$ V.

3.2. Nodal Analysis with Voltage Sources

We consider how voltage sources affect nodal analysis. We use the circuit in **Fig. 3.3**.

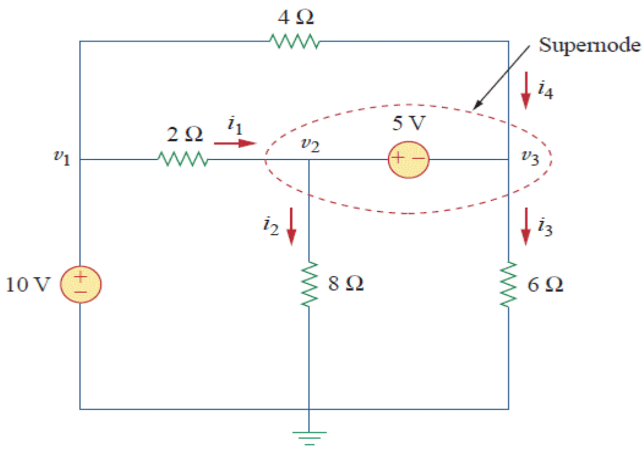


Figure 3.3. A circuit with supernode.

- **Case 1**: If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source. In **Fig. 3.3**, for example,

$$\boldsymbol{v_1 = 10\ \text{V}} \qquad \text{Eq. (3.11a)}$$

- **Case 2**: If the voltage source (dependent or independent) is connected between two nonreference nodes, the two nonreference nodes form ***a generalized node*** or ***supernode***; **we apply both KCL and KVL to determine the node voltages**.

A **supernode** is formed by enclosing a (dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it.

$$\boldsymbol{i_1 + i_4 = i_2 + i_3} \qquad \text{Eq. (3.11a)}$$

Or

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_1 - 0}{8} + \frac{v_3 - 0}{6} \qquad \text{Eq. (3.11b)}$$

To apply Kirchhoff's voltage law to the supernode in **Fig 3.3**. We redraw the circuit as shown in **Fig 3.4**. Going around the loop in the clockwise direction gives,

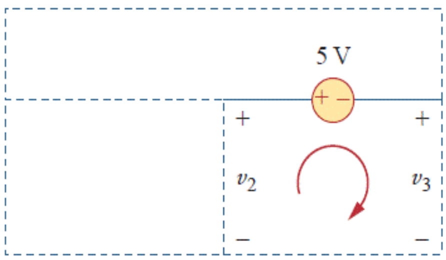


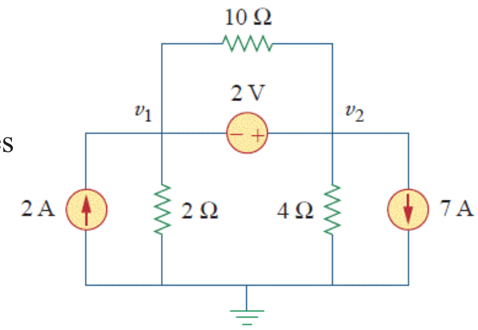
Figure 3.4. Applying KVL to a supernode.

$$\boldsymbol{-v_2 + 5 + v_3 = 0 \quad \Rightarrow \quad v_2 - v_3 = 5} \qquad \text{Eq. (3.12)}$$

Note the following properties of a supernode:

1. The voltage source inside the supernode provides a constraint equation needed to solve the node voltages.
2. A supernode has no voltage of its own.
3. A supernode requires the application of both KCL and KVL.

Example 3.3: For the circuit shown below, find the node voltages.



Solution: The supernode contains the 2 V source,

Nodes 1and 2, and the 10 Ω resistor.

Applying **KCL** to supernode as shown in the **Fig. (a)** gives

$2 = i_1 + i_2 + 7$

$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7 \quad \Rightarrow \quad 8 = 2v_1 + v_2 + 28$

Or $v_2 = -20 - 2v_1$ Eq. (1)

To get the relationship between v_1 and v_1. We apply **KVL** to the circuit in **Fig. (b)**.

$-v_1 - 2 + v_2 = 0 \quad \Rightarrow \quad v_2 = v_1 + 2$ Eq. (2),

From Eq. (1) and Eq. (2), we have

$v_2 = v_2 + 2 = -20 - 2v_1$

Or

$3v_1 = -22 \quad \Rightarrow \quad v_1 = -7.333$ V.

$v_2 = v_1 + 2 = -5.333$ V.

Note that the 10 Ω resistor does not make any difference because it is connected across the supernode

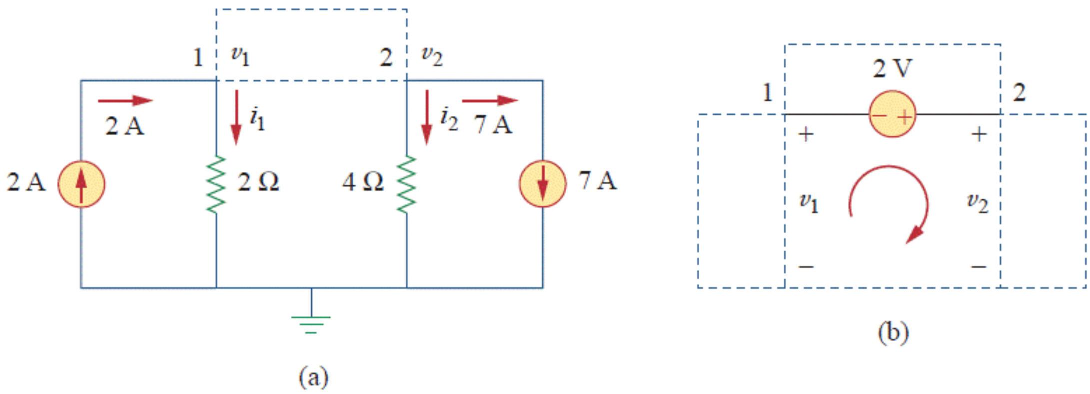
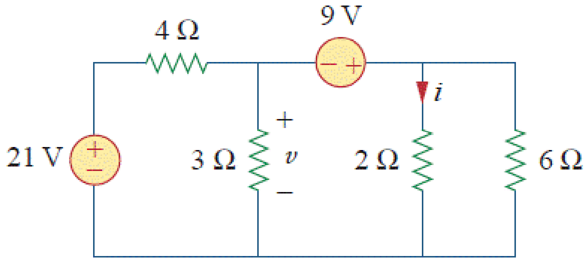


Fig. Applying: (a) KCL to the supernode. (b) KVL to the loop.

Practice Example: Find v and i in the circuit shown below:

Answer: -0.6 V, 4.2 A.



3.3. Mesh Analysis

Mesh analysis provides another general procedure for analyzing circuits, using **mesh currents** as the circuit variables. Using mesh currents instead of element currents as circuit variables is convenient and reduces the number of equations that must be solved simultaneously. Recall that a loop is a closed path with no node passed more than once. A mesh is a loop that does not contain any other loop within it. ***Nodal analysis applies to KCL to find unknown voltages in a given circuit, while mesh analysis applies to KVL to find unknown currents***. Mesh analysis is not quite as general as nodal analysis because it is only applicable to a circuit that is ***planar***. A planar circuit is one that can be drawn in a plane with no branches crossing one another; otherwise, it is ***nonplanar***. Mesh analysis is also known as **loop analysis** or the **mesh-current method**.

To understand mesh analysis, we should first explain more about what we mean by a mesh. A **mesh is a loop which does not contain any other loops within it**.

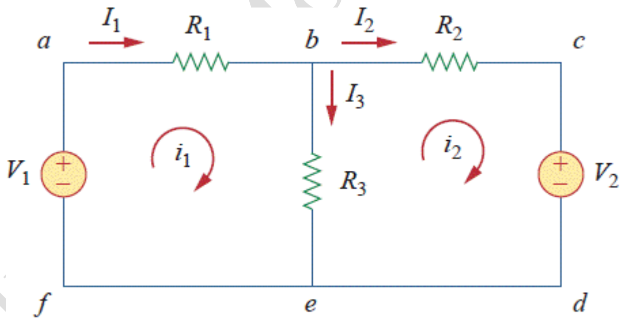


Figure 3.6. A circuit with two meshes.

In **Fig. 3.5**, for example, paths ***abefa*** and ***bcdeb*** are meshes, but path ***abcdefa*** is not a mesh. The current through a mesh is known as ***mesh current***. In mesh analysis, we are interested in applying **KVL** to find the mesh currents in a given circuit.

Steps to Determine Mesh Currents:

1. Assign mesh **currents** $i_1, i_2, \dots, i_n$ to the n meshes.
2. Apply **KVL** to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.
3. Solve the resulting n simultaneous equations to get the mesh currents.

The first step requires that mesh currents i_1 and i_2 are assigned to meshes 1 and 2. Although a mesh current may be assigned to each mesh in an arbitrary direction, it is conventional to assume that each mesh current flows **clockwise**.

As the second step, we apply **KVL** to each mesh. Applying **KVL** to mesh 1, we obtain:

$$-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0 \Rightarrow V_1 - R_1 i_1 - i_1 R_3 + i_2 R_3 = 0$$

$$i_1(R_1 + R_3) - i_2 R_3 = V_1 \qquad \text{Eq. (3.13)}$$

For mech 2, applying **KVL** gives:

$$-i_2 R_2 - R_3(i_2 - i_1) - V_2 = 0 \Rightarrow -i_2 R_2 - i_2 R_3 + i_1 R_3 - V_2 = 0$$

$$i_1 R_3 - i_2(R_2 + R_3) = V_2 \qquad \text{Eq. (3.14)}$$

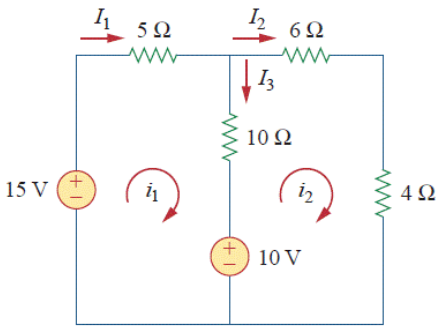
The third step is to solve the mesh currents. Putting Eqs. (3.13) and (3.14) in matrix form yields:

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ R_3 & -(R_2 + R_3) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad \text{Eq. (3.15)}$$

Which can be solved to obtain the mesh currents i_1 and i_2. After finding the mesh current:

$$I_1 = i_1, I_2 = i_2, I_3 = i_1 - i_2 \qquad \text{Eq. (3.16)}$$

Example 3.4: For the circuit in Figure below, find the branch currents I_1, I_2, and I_3 using mechs analysis.



Solution: We first obtain the mesh currents using KVL:

For loop 1:

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

$$5i_1 + 10i_1 - 10i_2 = 5$$

$$3i_1 - 2i_2 = 1 \qquad \text{Eq. (1)}$$

For loop 2:

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0 \quad \Rightarrow \quad 20i_2 - 10i_1 = 10$$

$$-i_1 + 2i_2 = 1 \qquad \text{Eq. (2)}$$

- **Elimination technique**

$2i_1 + 0 = 2 \quad \Rightarrow \quad i_1 = 1$ A

$3 - 2i_2 = 1 \quad \Rightarrow \quad i_2 = 1$ A. Thus,

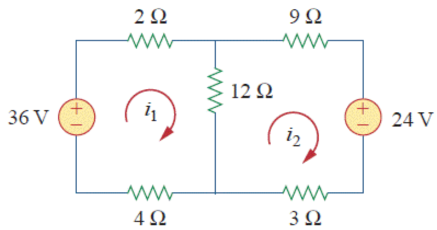
$I_1 = i_1 = 1$ A, $I_2 = i_2 = 1$ A, $I_3 = i_1 - i_2 = 0$

- **Cramer's rule**

$$\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \Delta = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = 4,\ \Delta_1 = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} = 4, \Delta_2 = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = 4,$$

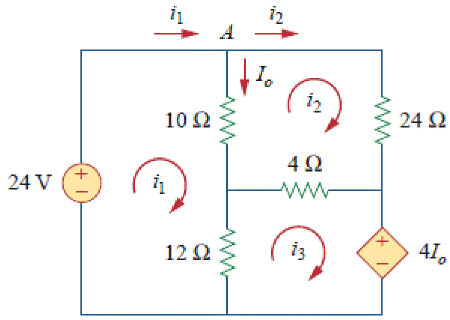
$i_1 = \frac{\Delta_1}{\Delta} = 1$ A, $i_2 = \frac{\Delta_2}{\Delta} = 1$ A.

Practice Example: Calculate the mesh currents i_1 and i_2 of the circuit shown below.



Answer: $i_1 = 2$ A, $i_2 = 0$ A.

Example 3.5: Use mesh analysis to find current I_o in the circuit shown below.



Solution: We apply KVL to the three meshes in turn.

For mesh 1,

$$-24 + 10(i_1 - i_2) + 12(i_1 - i_3) = 0$$

$$22i_1 - 10i_2 - 12i_3 = 24$$

$$11i_1 - 5i_2 - 6i_3 = 12 \qquad \text{Eq. (1)}$$

For mesh 2,

$$10(i_2 - i_1) + 24i_2 + 4(i_2 - i_3) = 0$$

$$-10i_1 + 38i_2 - 4i_3 = 0$$

$$-5i_1 + 19i_2 - 2i_3 = 0 \qquad \text{Eq. (2)}$$

For mesh 3,

$12(i_3 - i_1) + 4(i_3 - i_2) + 4i_o = 0$, **but at node A,** $i_o = i_1 - i_2$

$12(i_3 - i_1) + 4(i_3 - i_2) + 4(i_1 - i_2) = 0$ divide by 4 $\Rightarrow$ $-2i_1 - 2i_2 + 4i_3 = 0$

$$-i_1 - i_2 + 2i_3 = 0 \qquad \text{Eq. (3)}$$

$$\begin{bmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \\ 11 & -5 & -6 \\ -5 & 19 & -2 \end{bmatrix} = 418 - 30 - 10 - 114 - 22 - 50 = 192$$

Similarly we solve Δ_1, Δ_2, and Δ_3

$$\Delta_1 = \begin{bmatrix} 12 & -5 & -6 \\ 0 & 19 & -2 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -5 & -6 \\ 0 & 19 & -2 \\ 0 & -1 & 2 \\ 12 & -5 & -6 \\ 0 & 19 & -2 \end{bmatrix} = 456 - 24 = 432$$

$$\Delta_2 = \begin{vmatrix} 11 & 12 & -6 \\ -5 & 0 & -2 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 11 & 12 & -6 \\ -5 & 0 & -2 \\ -1 & 0 & 2 \\ 11 & 12 & -6 \\ -5 & 12 & -2 \end{vmatrix} = 24 + 120 = 144$$

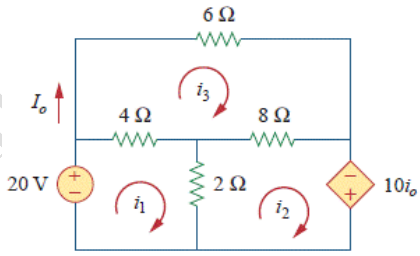
$$\Delta_3 = \begin{vmatrix} 11 & -5 & 12 \\ -5 & 19 & 0 \\ -1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 11 & -5 & 12 \\ -5 & 19 & 0 \\ -1 & -1 & 0 \\ 11 & -5 & 12 \\ -5 & 19 & 0 \end{vmatrix} = 60 + 228 = 288$$

We find $i_1 = \frac{\Delta_1}{\Delta} = \frac{432}{192} = 2.25$ A, $i_2 = \frac{\Delta_2}{\Delta} = \frac{144}{192} = 0.75$ A, $i_3 = \frac{\Delta_3}{\Delta} = \frac{288}{192} = 1.5$ A

$i_o = i_1 - i_2 = 1.5$ A.

Practice Example: Using mesh analysis, find i_o in the circuit shown below.

Answer: -5 A.



3.4. Mesh Analysis with Current Sources

Applying mesh analysis to circuits containing current sources (dependent or independent) may appear complicated. But it is actually much easier than what we encountered in the previous section, because the presence of the current sources reduces the number of equations. Consider the following two possible cases.

Case 1: When a current source exists only in one mesh: Consider the circuit in **Fig. 3.7**, for example. We set $\boldsymbol{i_2 = 5}$ A and write a mesh equation for the other mesh in the usual way; that is,

$$\boldsymbol{-10 + 4i_1 + 6(i_1 - i_2) = 0 \Rightarrow i_1 = -2 \text{ A}} \qquad \text{Eq. (3.17)}$$

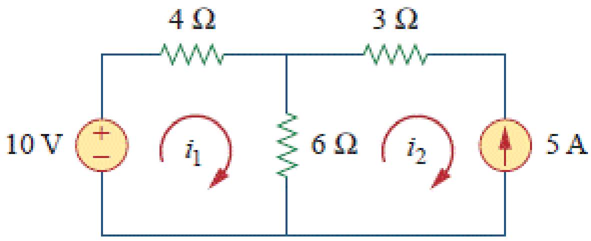


Figure 3.7. A circuit with a current source.

Case 2: When a current source exists between two meshes: Consider the circuit in **Fig. 3.8 (a)**, for example. We create a ***supermesh*** by **excluding the current source and any elements connected in series** with it, as shown in **Fig. 3.8** (b). Thus,

A **supermesh** results when two meshes have a (dependent or independent) current source in common.

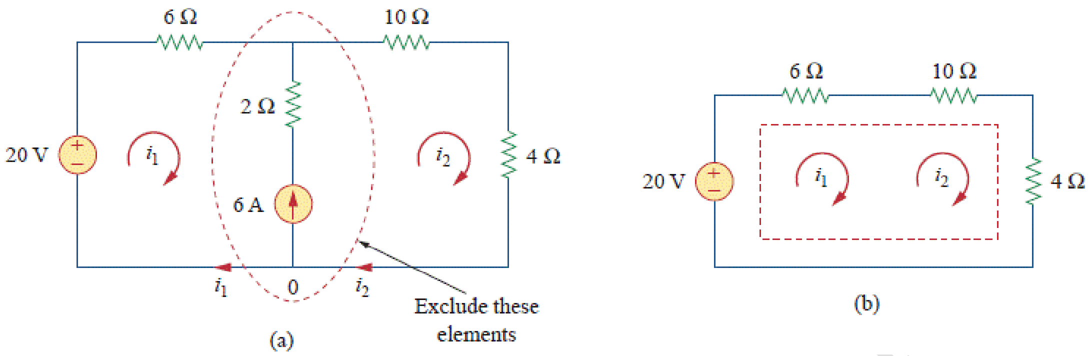


Figure 3.8. A circuit with a current source.

Therefore, applying **KVL** to the supermesh in Fig. (b) gives,

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0 \text{ Or } 6i_1 + 14i_2 = 20 \qquad \text{Eq. (3.18)}$$

We apply **KCL** to a node in the branch where the two meshes intersect. Applying **KCL** to node 0 in **Fig. 3.8 (a)** gives,

$$i_2 = i_1 + 6 \qquad \text{Eq. (3.19)}$$

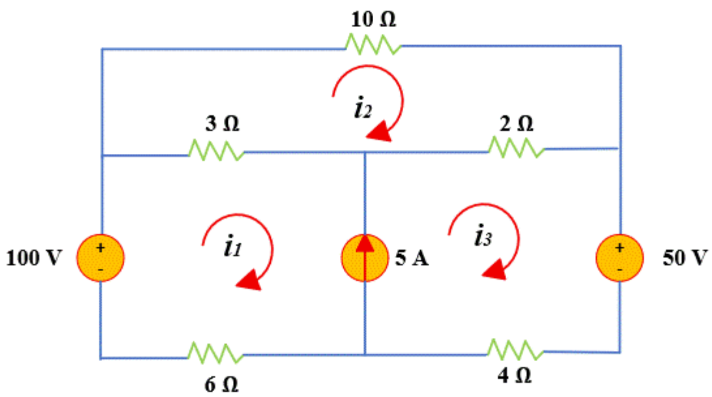
Solving Eqs. (3.18) and (3.19), we get,

$$i_1 = -3.2 \text{ A}, i_2 = 2.8 \text{ A} \qquad \text{Eq. (3.20)}$$

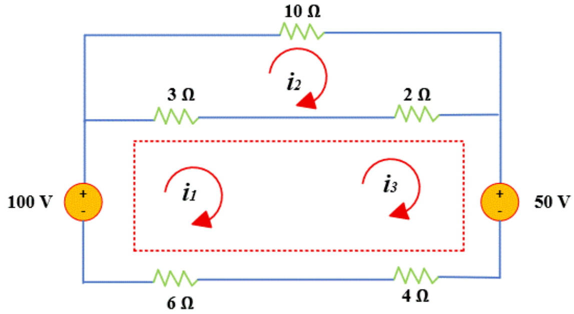
Note the following properties of a supermesh:

1. The **current source** in the **supermesh** provides the **constraint equation** necessary to solve for the mesh currents.
2. A **supermesh** has **no current** of its own.
3. A **supermesh** requires the application of both **KVL** and **KCL**.

<u>**Example 3.6**</u>: For the circuit in shown below, find I_1 to I_3 using mesh analysis.



Solution: We don't know the voltage across the current source. Therefore, we remove the whole branch that includes the current source.



Applying KVL around the supermesh,

$$-100 + 3(i_1 - i_2) + 2(i_3 - i_2) + 50 + 4i_3 + 6i_1$$

$$9i_1 - 5i_2 + 6i_3 = 50 \text{ Eq. (1)}$$

Applying KCL around the supermesh,

$$i_1 = 5 + i_3$$

$$i_1 + 0i_2 - i_3 = 5 \qquad \text{Eq. (2)}$$

Applying KVL around the i_2,

$$10i_2 + 2(i_2 - i_3) + 3(i_2 - i_1) = 0$$

$$-3i_1 + 15i_2 - 2i_3 = 0 \qquad \text{Eq. (3)}$$

$$\begin{bmatrix} 9 & -5 & 6 \\ 1 & 0 & -1 \\ -3 & 15 & -2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 5 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 9 & -5 & 6 \\ 1 & 0 & -1 \\ -3 & 15 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -5 & 6 \\ 1 & 0 & -1 \\ -3 & 15 & -2 \\ 9 & -5 & 6 \\ 1 & 0 & -1 \end{bmatrix} = 0 + 90 - 15 - 0 + 135 - 10 = 200$$

Similarly, we solve Δ_1, Δ_2, and Δ_3

$$\Delta_1 = \begin{bmatrix} 50 & -5 & 6 \\ 5 & 0 & -1 \\ 0 & 15 & -2 \end{bmatrix} = \begin{bmatrix} 50 & -5 & 6 \\ 5 & 0 & -1 \\ 0 & 15 & -2 \\ 50 & -5 & 6 \\ 5 & 0 & -1 \end{bmatrix} = 450 + 750 - 50 = 1150$$

$$\Delta_2 = \begin{bmatrix} 9 & 50 & 6 \\ 1 & 5 & -1 \\ -3 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 50 & 6 \\ 1 & 5 & -1 \\ -3 & 0 & -2 \\ 9 & 50 & 6 \\ 1 & 5 & -1 \end{bmatrix} = -90 + 150 + 90 + 100 = 250$$

$$\Delta_3 = \begin{bmatrix} 9 & -5 & 50 \\ 1 & 0 & 5 \\ -3 & 15 & 0 \end{bmatrix} = \begin{bmatrix} 9 & -5 & 50 \\ 1 & 0 & 5 \\ -3 & 15 & 0 \\ 9 & -5 & 50 \\ 1 & 0 & 5 \end{bmatrix} = 750 + 75 - 675 = 150$$

We find $i_1 = \frac{\Delta_1}{\Delta} = \frac{1150}{200} = 5.75$ A, $i_2 = \frac{\Delta_2}{\Delta} = \frac{250}{200} = 1.25$ A, $v_3 = \frac{\Delta_3}{\Delta} = \frac{150}{200} = 0.75$ A

Chapter Four

Circuit Theorems

4. Introduction

A major advantage of analyzing circuits using Kirchhoff's laws as we did in Chapter 3 is that we can analyze a circuit without tampering with its original configuration. A major disadvantage of this approach is that, for a large, complex circuit, tedious computation is involved. The growth in areas of application of electric circuits has led to an evolution from simple to complex circuits. To handle the complexity, engineers over the years have developed some theorems to simplify circuit analysis. Such theorems include Thevenin's and Norton's theorems. Since these theorems are applicable to ***linear*** circuits, we first discuss the concept of circuit linearity. In addition to circuit theorems, we discuss the concepts of superposition, source transformation, and maximum power transfer in this chapter. The concepts we develop are applied in the last section to source modeling and resistance measurement.

4.1. Superposition

If a circuit has two or more independent sources, one way to determine the value of a specific variable (voltage or current) is to use nodal or mesh analysis as in Chapter 3. Another way is to determine the contribution of each independent source to the variable and then add them up. The latter approach is known as the ***superposition***. The idea of superposition rests on the linearity property.

The **superposition principle** states that the voltage across (or current through) an element in a linear circuit is the **algebraic sum** of the voltages across (or currents through) that element due to **each independent source acting alone**.

The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately. However, to apply the superposition principle, we must keep two things in mind:

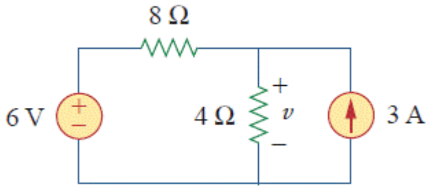
1. We consider one **independent** source at a time while all other independent sources are ***turned off***. This implies that we replace every **voltage source** by **0 V** (or a **short circuit**), and every **current source** by **0 A** (or an **open circuit**). This way we obtain a simpler and more manageable circuit.
2. **Dependent** sources are left intact because they are controlled by circuit variables.

Steps to Apply **Superposition Principle**:

1. Turn off all independent sources except one source. Find the output (voltage or current) due to that active source using the techniques covered in Chapters 2 and 3.
2. Repeat step 1 for each of the other independent sources.
3. Find the total contribution by adding algebraically all the contributions due to the independent sources.

Analyzing a circuit using superposition has **one major disadvantage**: it may very likely involve more work. **However**, superposition does help reduce a complex circuit to simpler circuits through replacement of voltage sources by short circuits and of current sources by open circuits.

Example 4.1: Use the superposition theorem to find v in the circuit shown below.



Solution: Since there are two sources, let

$v = v_1 + v_2$

where v_1 and v_2 are the contributions due to the 6 V voltage source and the 3 A current source, respectively. To obtain v_1, we set the current source to 0. Applying KVL to the loop:

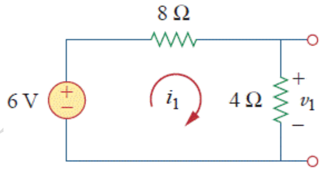
$12i_1 - 6 = 0 \;\Rightarrow\; i_1 = 0.5$ A.

Thus,

$v_1 = 4i_1 = 2$ V

We may also use voltage division to get v_1 by writing:

$v_1 = \frac{4}{4+8}(6) = 2$ V



(a)

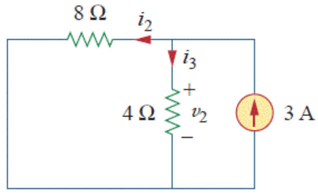
To get v_2, we set the voltage source to 0. Using current division,

$i_3 = \frac{8}{8+4}(3) = 2$ A

Hence,

$v_2 = 4i_3 = 8$ V

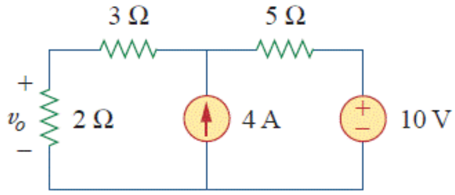
And we find $v = v_1 + v_2 = 2 + 8 = 10$ V.



(b)

Practice Example: Using superposition theorem, find v_o in the circuit shown below.

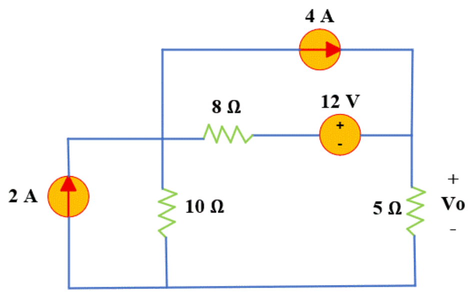
Answer: 6 V.



Example 4.2: Use the superposition theorem to find v_o in the circuit shown below.

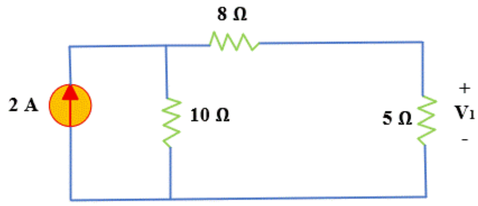
Solution: Let $v_o = v_1 + v_2 + v_3$,

where v_1, v_2, and v_3 are due to the independent sources.



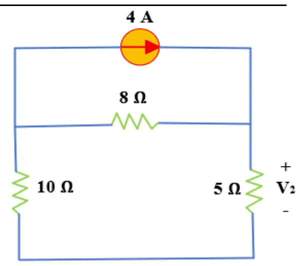
To find v_1, see the figure below,

$v_1 = 5 \times \frac{10}{5+8+10} 2 = 4.3478$ V



To find v_2, see the figure below,

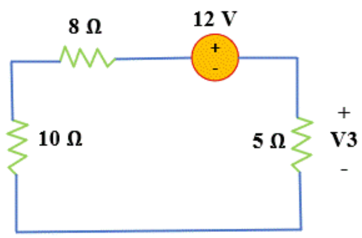
$$v_2 = 5 \times \frac{8}{5+8+10} 4 = 6.956 \text{ V}$$



To find v_3, see the figure below,

$$v_2 = -12\left(\frac{5}{5+8+10}\right) = -2.6087 \text{ V}$$

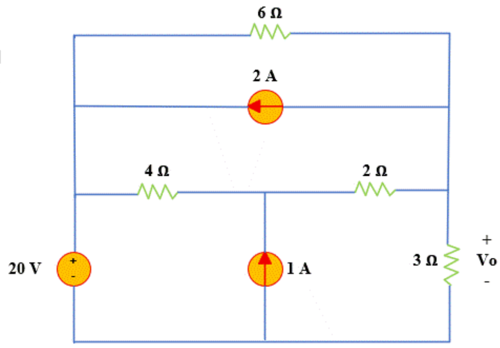
$$v_o = 4.3478 + 6.956 - 2.6087 = 8.6956 \text{ V}$$



Example 4.3: Apply the superposition theorem to find v_o in the circuit shown below.

Solution: Let $v_o = v_{o1} + v_{o2} + v_{o3}$,

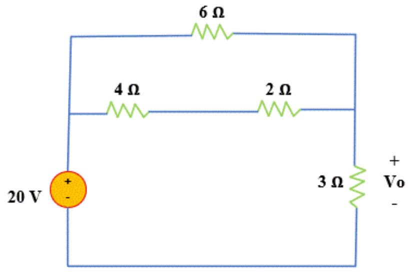
where v_{o1}, v_{o2}, and v_{o3} are due to the independent sources.



To find v_{o1}, see the figure below,

6 || (4+2) = 3 Ω

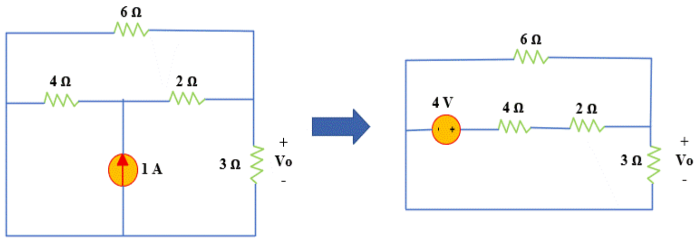
$$v_{o1} = \frac{3}{3+3} 20 = 10 \text{ V}$$



To find v_{o2}, see the figure below,

3 || 6 = 2 Ω

$$v_{o2} = \left[\frac{2}{4+2+2}\right](4) = 1 \text{ V}$$

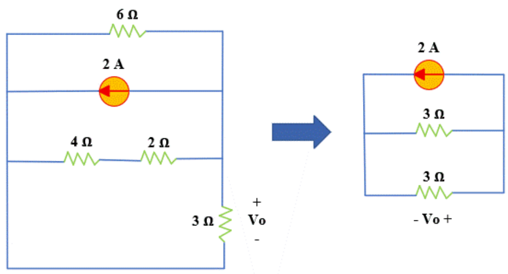


To find v_{o3}, see the figure below,

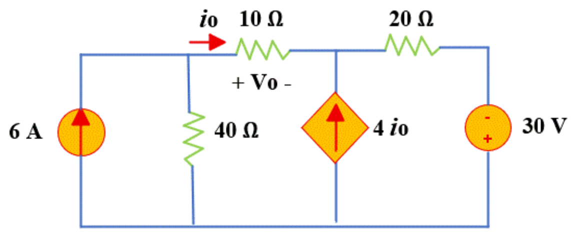
6 || (4+2) = 3 Ω

$$v_{o3} = -2 \times 3 = -6$$

$$v_o = 10 + 1 - 6 = 5 \text{ V}$$

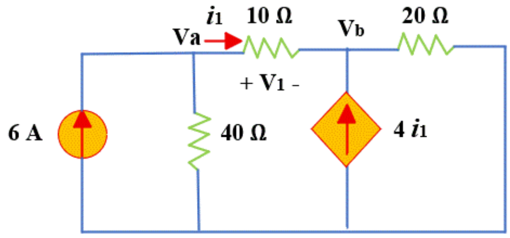


Example 4.4: Use the superposition principle to find v_o in the circuit shown below.



Solution: Let $v_o = v_{o1} + v_{o2}$, where v_{o1} and v_{o2} are due to the 6 A and 30 V sources.

To find v_1, see the figure below,



At node a, Apply KCL

$6 = \frac{v_o}{40} + \frac{v_a - v_b}{10}$

$240 = 5v_a - 4v_b$ Eq. (1)

At node b, Apply KCL

$I_1 + 4I_1 = \frac{(v_b - 0)}{20}$

$\frac{v_a - v_b}{10} + 4\left(\frac{v_a - v_b}{10}\right) = \frac{v_b}{20}$ $\times 20$

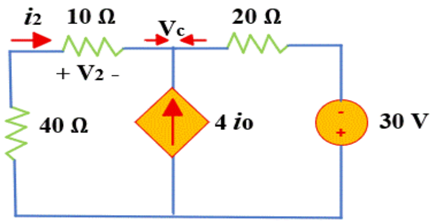
$10v_a - 10v_b = v_b$

$v_a = 1.1v_b$ Eq. (2), substituting (2) into (1)

$240 = 5(1.1v_b) - 4v_b \Rightarrow v_b = 160$ V and so $v_a = 1.1 \times 160 = 176$ V

However, $v_1 = v_a - v_b = 16$ V

To find v_2, consider the figure below,



$\frac{v_c}{50} + 4i_o + \frac{(v_c - 30)}{20} = 0$

But $i_o = \frac{v_c}{50}$

$\frac{v_c}{50} + 4\left(\frac{v_c}{50}\right) + \frac{(v_c - 30)}{20} = 0$

$\frac{5v_c}{50} + \frac{(v_c - 30)}{20} = 0$ $\times 100$

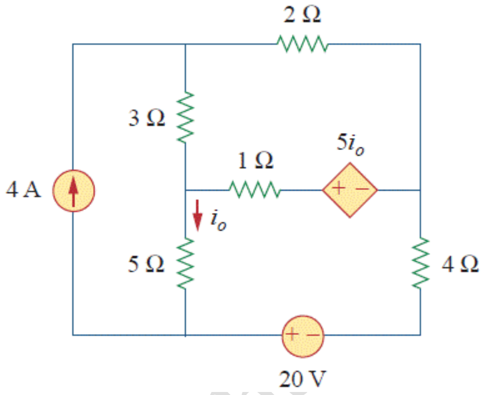
$10v_c + 5v_c - 150 = 0 \Rightarrow v_c = 10$ V

$i_2 = \frac{v_c}{50} = \frac{10}{50} = \frac{1}{5}$

$v_2 = 10i_2 = 2$ V

$v_o = v_1 + v_2 = 16 + 2 = 18$ V and $i_o = \frac{v_o}{10} = 1.8$ A.

Example 4.5: Use the superposition theorem to find i_o in the circuit shown below.



Solution: the circuit in the figure has a dependent source, Which must be left intact. We let,

$\boldsymbol{i_o = i_o' + i_o''}$

where i_o' and i_o'' are due to the 4 A current source and 20 V voltage source. To obtain i_o', we turn off the 20 V. We apply mesh analysis to find i_o', as shown in the below Figure. For loop 1,

$i_1 = 4$ A **Eq. (1)**

But at 0,

$i_1 = i_3 + i_o'$,

$i_o' = i_1 - i_3$ **Eq. (2)**

For loop 2,

$3(i_2 - i_1) + 2i_2 - 5i_o' + 1(i_2 - i_3) = 0$

$-8i_1 + 6i_2 + 4i_3 = 0$, Hence $i_1 = 4$

$6i_2 + 4i_3 = 32$ **Eq. (3)**

For loop 3,

$5(i_3 - i_1) + 1(i_3 - i_2) + 5i_o' + 4i_3 = 0$

$5i_3 - 5i_1 + i_3 - i_2 + 5i_o' + 4i_3 = 0$

$-5i_1 - i_2 + 10i_3 + 5i_1 - 5i_3 = 0$ re-arrange and $i_o' = i_1 - i_3$

$-i_2 + 5\,i_3 = 0$ **Eq. (4)**

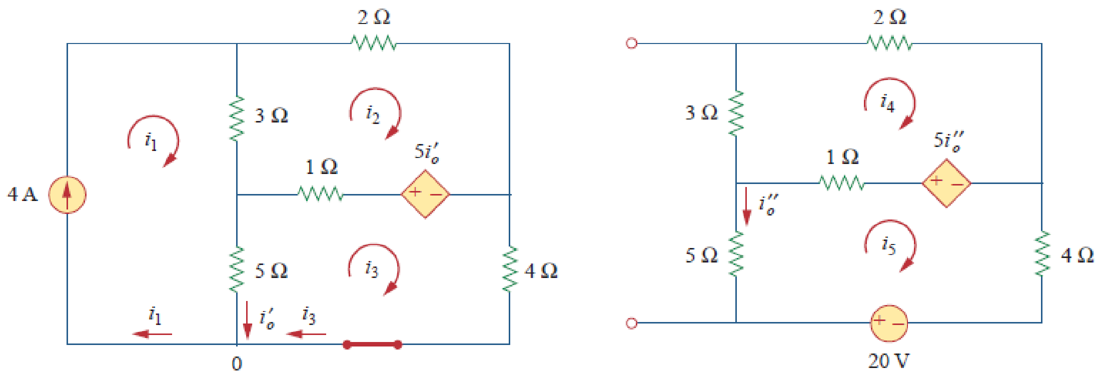
Using Cramer rule, for equations 3 and 4,

$$\Delta = \begin{vmatrix} 6 & 4 \\ -1 & 5 \end{vmatrix} = 30 + 4 = 34,$$

$$\Delta_2 = \begin{vmatrix} 6 & 32 \\ -1 & 0 \end{vmatrix} = 0 + 32 = 32$$

$i_3 = \frac{\Delta_2}{\Delta} = \frac{32}{34} = 0.941$, substitute in equation 2

$i_o' = 4 - 0.941 = 3.058$ A



(a) (b)

To obtain i_o'', we turn of the 4 A current source as shown in figure (b),

For loop 4, applying KVL,

$3i_4 + 2i_4 - 5i_o'' + 1(i_4 - i_5) = 0$

$6i_4 - i_5 - 5i_o'' = 0$ **But** $i_5 = -i_o''$ **Eq. (5)**

$6i_4 + 4i_5 = 0$ **Eq. (6)**

For loop 5, applying KVL,

$5i_5 + 1(i_5 - i_4) + 5i_o'' + 4i_5 - 20 = 0$

$5i_5 + i_5 - i_4 - 5i_5 + 4i_5 = 20$

$-i_4 + 5i_5 = 20$ **Eq. (7)**

Using Cramer rule,

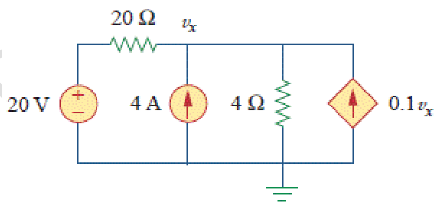
$\Delta = \begin{vmatrix} 6 & 4 \\ -1 & 5 \end{vmatrix} = 30 + 4 = 34, \Delta_2 = \begin{vmatrix} 6 & 0 \\ 1 & -20 \end{vmatrix} = 120 - 0 = 120,$

$i_5 = \frac{120}{34} = 3.529$, **hence** $i_5 = -i_o''$

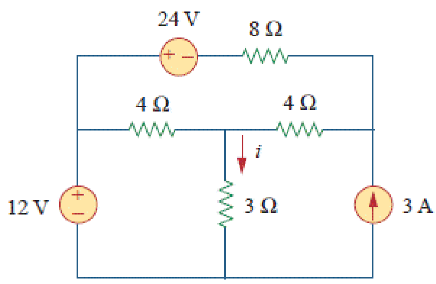
Thus, $i_o = 3.058 + (-3.529) = -0.471$ A

Practice Example: Using superposition theorem, find v_x in the circuit shown below.

Answer: $v_x = 25$ V.



Example 4.6: For the circuit shown below, use the superposition theorem to find i.



Solution: In this case, we have three sources, Let

$i = i_1 + i_2 + i_3$

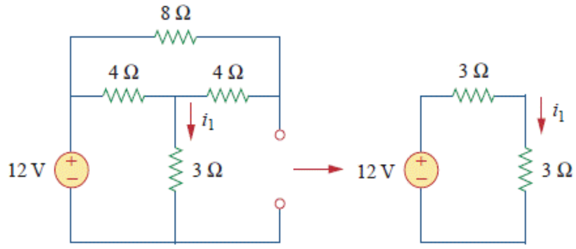
where i_1, i_2, and i_3 are due to the 12 V, 24 V, and 3 A sources.

To get i_1, Consider the circuit given below (a):

4 Ω in series with 8 Ω = 12 Ω.

12 Ω in parallel with 4 Ω = 3 Ω, see Fig. (a).

$i_1 = 12/6 = 2$ A.



(a)

To get i_2, see Fig. (b), Applying mesh analysis,

For loop i_a,

$24 - 8i_a - 4i_a - 4(i_a - i_b) = 0$

$4i_a - i_b = -6$ Eq. (1)

For loop i_b,

$4(i_b - i_a) + 3i_b = 0$

$-4i_a + 7i_b = 0 \Rightarrow i_a = \frac{7}{4} i_b$ Eq. (2)

Eq. (2), substituting in Eq. (1)

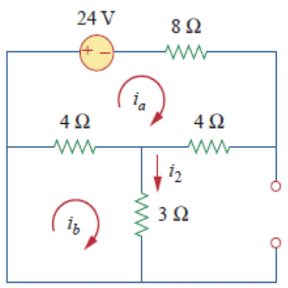
$i_2 = i_b = -1$

To get i_3, see Fig. (c), Applying nodal analysis,

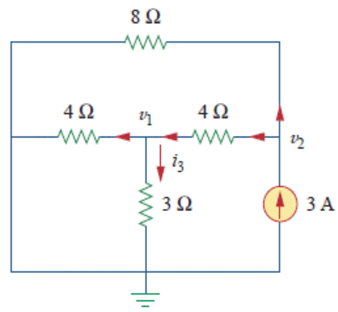
For nodal 1,

$3 = \frac{v_2}{8} + \frac{v_2 - v_1}{4} \Rightarrow 3 = \frac{v_2}{8} + \frac{v_2}{4} - \frac{v_1}{4}$

$24 = 3v_2 - 2v_1$ Eq. (3)



(b)



(c)

For nodal 2,

$\frac{v_2 - v_1}{4} = \frac{v_1}{4} + \frac{v_1}{3} \quad \Rightarrow \quad 3v_2 = 10v_1$

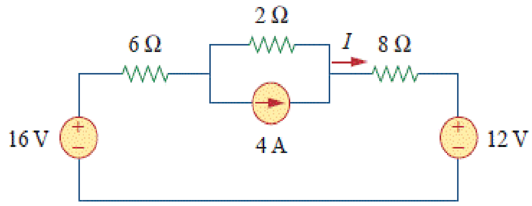
$v_2 = \frac{10v_1}{3}$ Eq. (4), substituting Eq. (4) into Eq. (3)

$v_1 = 3$, hence $i_3 = \frac{v_1}{3} = 1$ A

$i = i_1 + i_2 + i_3 = 2 - 1 + 1 = 2$ A.

Practice Example: Find I in the circuit shown below using the superposition principle.

Answer: $I = 0.75$ A.



4.2. Source Transformation

Source transformation is another tool for simplifying circuits. Basic to these tools is the concept of *equivalence*. We recall that an equivalent circuit is one whose v-i characteristics are identical with the original circuit. It is therefore expedient in circuit analysis to be able to **substitute a voltage source v_s in series with a resistor R for a current source i_s in parallel with a resistor R**, or vice versa, as shown in **Fig. 4.1**. Either substitution is known as a ***source transformation***.

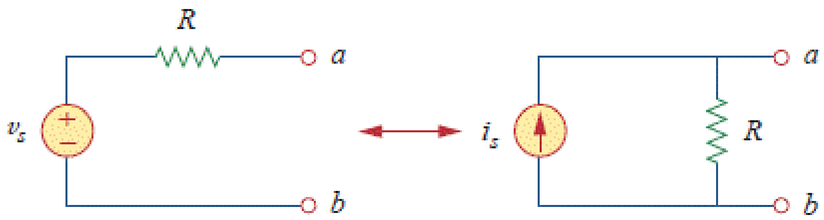


Figure. 4.1. Transformation of independent sources.

If the sources are turned off, the equivalent resistance at terminals a-b in both circuits is R. Also, when terminals a-b are short-circuited, the short-circuit current flowing from a to b is $i_{sc} = v_s/R$ in the circuit on the left-hand side and $i_{sc} = i_s$ for the circuit on the right-hand side. Thus, $\frac{v_s}{R} = i_s$ in order for the two circuits to be equivalent. Hence, source transformation requires that

$$\boldsymbol{v_s = i_s R} \qquad \textbf{or} \qquad \boldsymbol{i_s = \frac{v_s}{R}} \qquad \text{Eq. (4.1)}$$

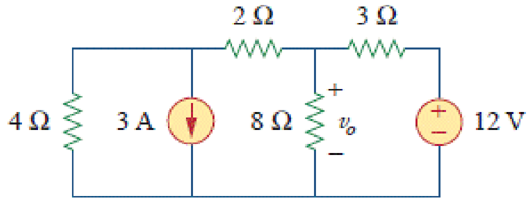
we should keep the following points in mind when dealing with source transformation

1. Note from Fig. 4.1 that the arrow of the current source is directed toward the positive terminal of the voltage source.

2. Note from Eq. (4.1) that source transformation is not possible when $R = 0$, which is the case with an ideal voltage source. However, for a practical, nonideal voltage source,

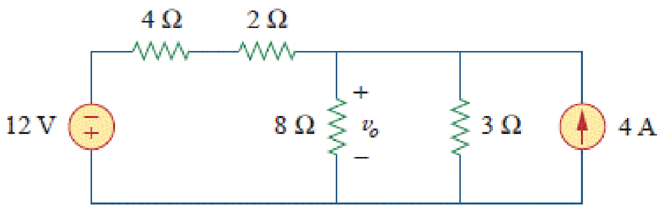
$R \neq 0$. Similarly, an ideal current source with $R = \infty$ cannot be replaced by a finite voltage source. More will be said on ideal and nonideal sources

Example 4.7: Use source transformation to find v_o in the circuit shown below.

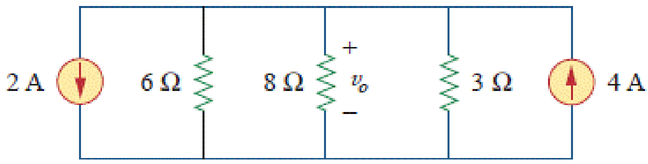


Solution: We first transform the current and voltage sources to obtain the circuit in Fig. (a).

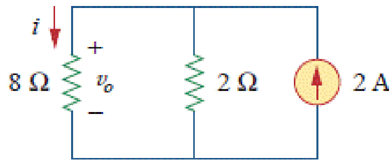
Combining the and resistors in series and transforming the 12-V voltage source gives us Fig. (b). We now combine the 3Ω and 6 Ω resistors in parallel to get 2 Ω in parallel. We also combine the 2-A and 4-A current sources to get a 2-A source. Thus, by repeatedly applying source transformations, we obtain the circuit in Fig. (c).



(a)



(b)



(c)

We use current division in Fig. (c) to get:

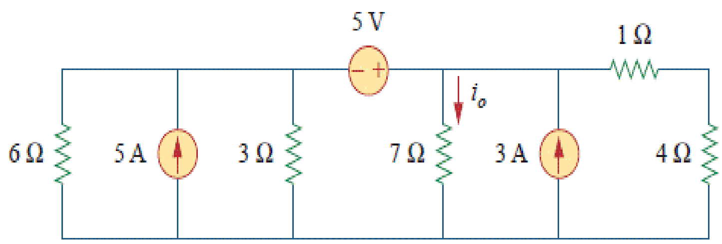
$$i = \frac{2}{2+8}(2) = 0.4 \text{ A, and } v_o = 8i = 3.2 \text{ V}.$$

Alternatively, since the 8 Ω and 2 Ω resistors in Fig. (c) are in parallel, they have the same voltage v_o across them. Hence,

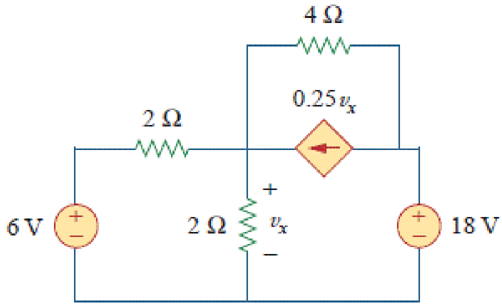
$$v_o = (8 \,||\, 2)(2) = \frac{8\times 2}{10} = 3.2 \text{ V}$$

Practice Example: Find i_o in the circuit shown below using the source transformation.

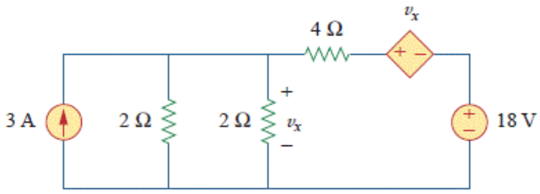
Answer: 1.78 A.



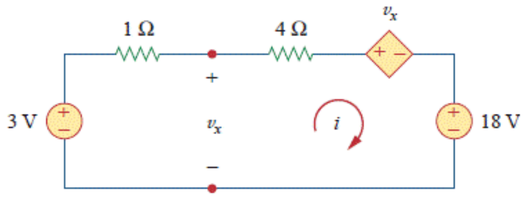
Example 4.8: Find v_x in the figure below using source transformation.



Solution: The circuit in Fig. involves a voltage-controlled dependent current source. We transform this dependent current source as well as the 6-V independent voltage source as shown in Fig. (a). The 18-V voltage source is not transformed because it is not connected in series with any resistor. The two 2 Ω resistors in parallel combine to give 1 Ω a resistor, which is in parallel with the 3-A current source. The current source is transformed into a voltage source as shown in Fig. (b). Notice that the terminals for v_x are intact.



(a)



(b)

Applying KVL around the loop in Fig. (b) gives,

$-3 + 5i + v_x + 18 = 0$ Eq. (1)

Applying KVL to the loop containing only the 3-V voltage source, the 1 Ω resistor, and v_x yields

$-3 + 1i + v_x = 0$

$v_x = 3 - i$ Eq. (2), substituting this equation into (1), we obtain,

$15 + 5i + 3 - i = 0,\ i = -4.5$ A.

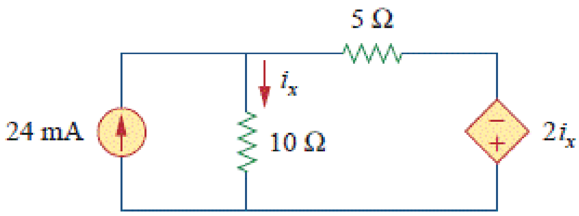
Alternatively, we may apply KVL to the loop containing v_x, the 4 Ω resistor, the voltage-controlled dependent voltage source, and the 18-V voltage source in Fig. (b). We obtain,

$-v_x + 4i + v_x + 18 = 0,\quad i = 4.5$ A.

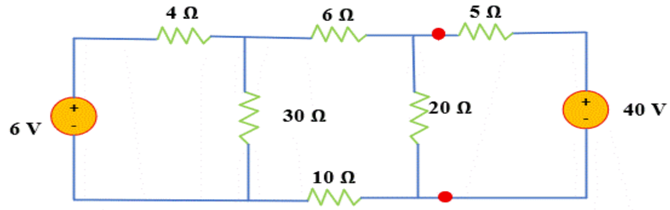
Thus, $v_x = 3 - i = 7.5$ V.

Practice Example: Find i_x in the circuit shown below using the source transformation.

Answer: 7.056 mA.

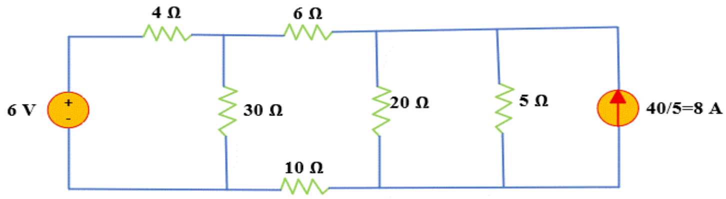


Example 4.9: Using source transformation, find the power associated with the 6 V source.

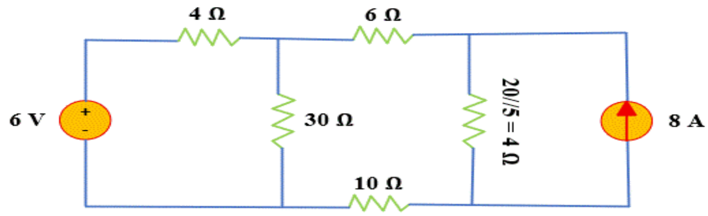


Solution: First, the 40 V in series with 5 Ω

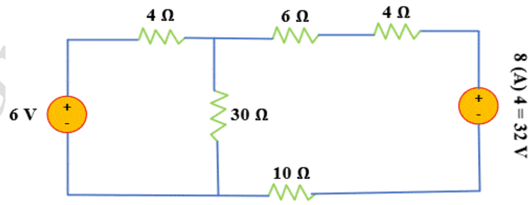
So, the circuit becomes as follows:



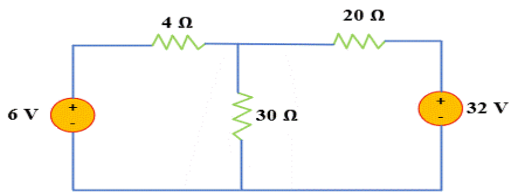
Then, take 5 Ω in parallel with 20 Ω,



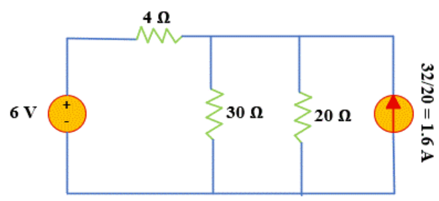
The current 8 A in parallel with 4 Ω



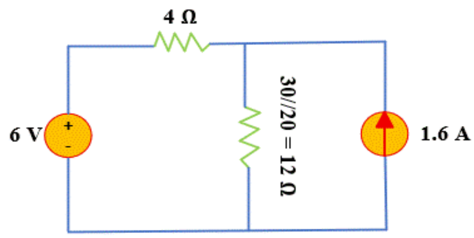
Then, 4 Ω + 6 Ω + 10 Ω in series



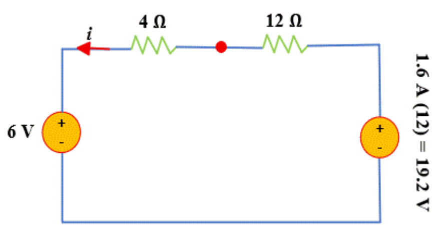
The voltage source 32 V in series with 20 Ω



The 30 Ω in parallel with 20 Ω,



The current source 1.6 A in parallel with 12 Ω



$\text{i} = \frac{19.2-6}{4+12} = 0.825$ A, $\text{V}P_{6V} = vi = 6(0.825) = 4.95$ W

4.3. Thevenin's Theorem

It often occurs in practice that a particular element in a circuit is variable (usually called the *load*) while other elements are fixed. As a typical example, a household outlet terminal may be connected to different appliances constituting a variable load. Each time the variable element is changed, the entire circuit has to be analyzed all over again. To avoid this problem, **Thevenin's theorem** provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit. The ***Thevenin equivalent circuit***; it was developed in 1883 by **M. Leon Thevenin** (1857–1926), a French telegraph engineer.

Thevenin's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{Th} **in series** with a **resistor** R_{Th}, where V_{Th} is the **open-circuit voltage** at the terminals and R_{Th} is the **input** or **equivalent resistance** at the terminals when the **independent** sources are **turned off**.

Steps to Find Thevenin's Equivalent Circuit:

1. Remove the load R_L from the circuit terminals a and b (The load may be a single resistor or another circuit) and redraw the circuit. The two terminals (a and b) have become open-circuited.
2. Calculate R_{Th} by first setting all independent sources to zero (**voltage sources are replaced by short circuits and current sources are replaced by open circuits**) and find the resultant resistance between the network terminals.
3. Calculate E_{Th} by first returning all sources to their original positions and finding the open circuit voltage between the network terminals. If the circuit has more than one source, it may be necessary to use the **superposition theorem**. In that case, it will be necessary to determine the open-circuit voltage due to each source separately and then determine the combined effect.
4. Draw the **Thevenin's equivalent circuit** with R_L from where it was previously removed.
5. Finally, calculate the current flowing through the R_L by the following equation:

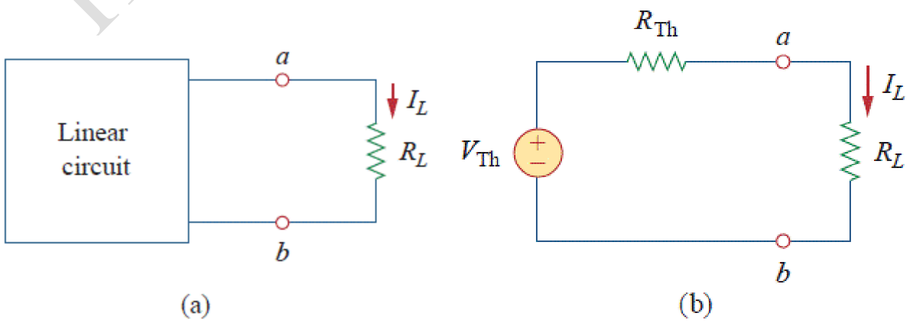
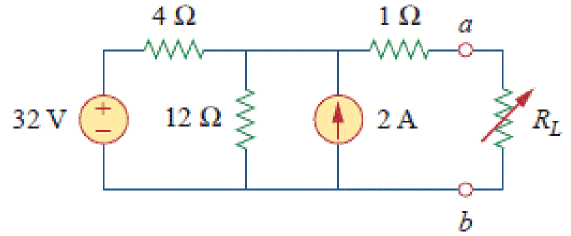


Figure. 4.2. A circuit with a load: (a) original circuit, (b) Thevenin equivalent.

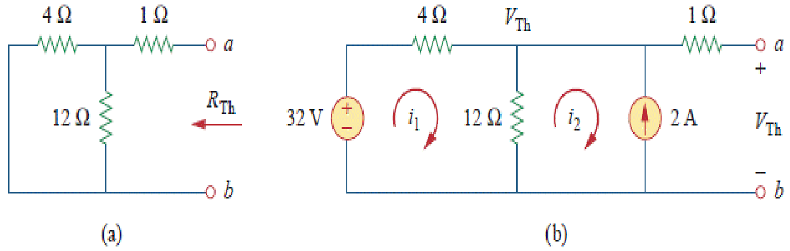
$$I_L = \frac{V_{Th}}{R_L + R_{Th}}, \qquad V_L = I_L R_L = \frac{R_L}{R_L + R_{Th}}(V_{Th}) \qquad \text{Eq. (4.2)}$$

Example 4.10: Find the Thevenin equivalent circuit of the circuit shown in Fig. below, to the left of the terminals $a - b$. Then find the current through $R_L = 6, 16$, and $36\ \Omega$.



Solution:
We find R_{Th} by turning off the 32-V voltage source (replacing it with a short circuit) and the 2-A current source (replacing it with an open circuit). The circuit becomes in below figure.

$R_{Th} = 4 \parallel 12 + 1$
$= \frac{4\times12}{16} + 1 = 4\ \Omega$



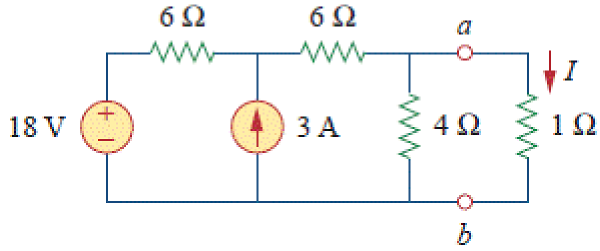
(a) (b)

To find V_{Th}, apply mesh analysis,
$-32 + 4i_1 + 12(i_1 - i_2) = 0$, **but** $i_2 = -2$ A
$i_1 = 0.5$ A.
$V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2) = 30$ V

Or can apply nodal using KCL,
$\frac{32-V_{Th}}{4} + 2 = \frac{V_{Th}}{12} \Rightarrow 96 - 3V_{Th} + 24 = V_{Th} \Rightarrow V_{Th} = 30$ V.

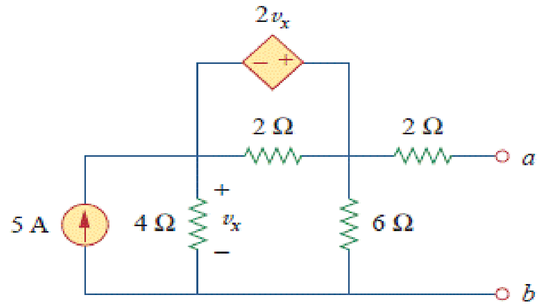
To find I_L through R_L,
$I_L = \frac{V_{Th}}{R_{Th}+R_L} = \frac{30}{4+R_L}$,
when $R_L = 6$ is
$I_L = \frac{30}{10} = 3$ A.
when $R_L = 16$ is
$I_L = \frac{30}{20} = 1.5$ A.
when $R_L = 36$ is
$I_L = \frac{30}{40} = 0.75$ A.

Practice Example: Using Thevenin's theorem, find the equivalent circuit to the left of the terminals in the circuit of figure below. Then find I.

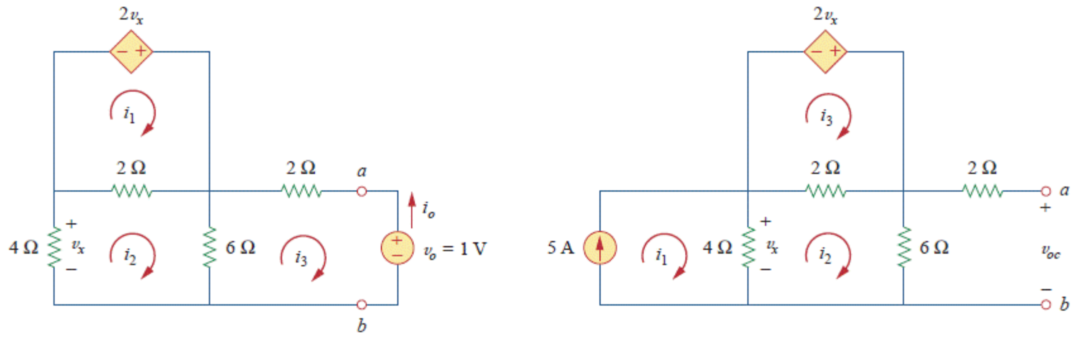


Answer: $V_{Th} = 9$ V, $R_{Th} = 3\ \Omega$, I= 2.25 A.

Example 4.11: Find the Thevenin's equivalent of the circuit shown below at terminals *a*-*b*.



Solution: There is dependent source and we keep it as it.
To find R_{Th}, we may set $v_o = 1$ V connected to the terminal and our goal are to find i_o at terminal. Then, $R_{Th} = 1/i_o$,



(a) (b)

Applying mesh analysis for loop 1, as seen in Fig. (a)

$-2v_x + 2(i_1 - i_2) = 0$

Or $\quad v_x = i_1 - i_2$

But $-4i_2 = v_x = i_1 - i_2$; **thus,** $-4i_2 = i_1 - i_2$

$\boldsymbol{i_1 = -3i_2}$ Eq. (1)

Applying KVL for loop 2

$4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) = 0 \Rightarrow \quad -2i_1 + 12i_2 - 6i_3 = 0$

$-i_1 + 6i_2 - 3i_3 = 0$ Eq. (2)

Applying KVL for loop 3

$6(i_3 - i_2) + 2i_3 + 1 = 0$

$-6i_2 + 8i_3 = -1$ Eq. (3)

Substituting Eq. (1) into Eq. (2),

$3i_2 + 6i_2 - 3i_3 = 0 \quad \Rightarrow \quad \boldsymbol{i_3 = 3i_2}$, substituting into Eq. (3)

$-6i_2 + 24i_2 = -1 \quad \Rightarrow \quad \boldsymbol{i_2 = -\frac{1}{18} = -0.055, i_3 = -0.1666, i_1 = 0.1666}$,

But $i_o = -i_3 = 0.1666$ A

$R_{Th} = \frac{1}{i_o} = 6\ \Omega$

To get V_{Th}, we find v_{oc} in the circuit of Fig. (b). Applying mesh analysis,

For loop 1,

$i_1 = 5$ Eq. (4)

For loop 3,

$-2v_x + 2(i_3 - i_2) = 0 \quad \Rightarrow \quad v_x = i_3 - i_2$ Eq. (5)

For loop 2

$4(i_2 - i_1) + 2(i_2 - i_3) + 6i_2 = 0$

$-4i_1 + 12i_2 - 2i_3 = 0$ Eq. (6)

But $\boldsymbol{4(i_1 - i_2) = v_x}$, solving these equations leads to,

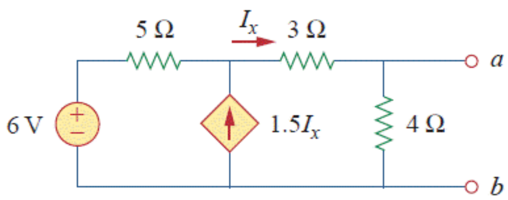
Obtaining $4(i_1 - i_2) = v_x$ in Eq. (5)

$4(i_1 - i_2) = i_3 - i_2$, and $\boldsymbol{i_1 = 5}$, $\Rightarrow$ $i_3 = -3i_2 + 20$, substituting in Eq. (6)

$-20 + 12i_2 + 6i_2 - 40 = 0 \quad \Rightarrow \quad i_2 = 3.33$ A.

$V_{Th} = v_{oc} = 6i_2 = 20$ V.

Practice Example: Using Thevenin's theorem, find the equivalent circuit to the left of the terminals in the circuit of figure below.



Answer: $V_{Th} = 5.33$ V, $R_{Th} = 0.44\ \Omega$

4.4. Norton's Theorem

Norton's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source I_N in parallel with a resistor R_N, where I_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

Thus, the circuit in **Fig. 4.3** (**a**) can be replaced by the one in **Fig. 4.3** (**b**).

$\boldsymbol{R_N = R_{Th}}$

$\boldsymbol{I_N = i_{sc}}$

$\boldsymbol{I_N = \frac{V_{Th}}{R_{Th}}}$

$\boldsymbol{E_{Th} = I_N R_N}$

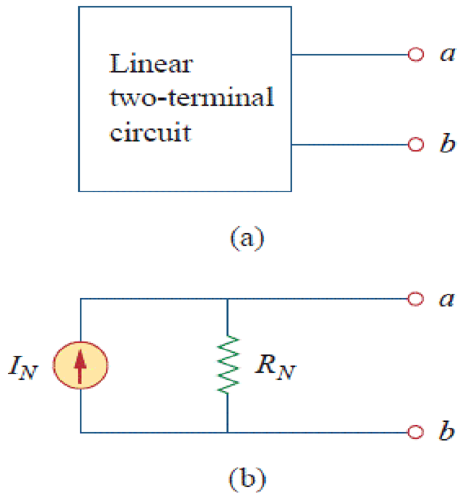
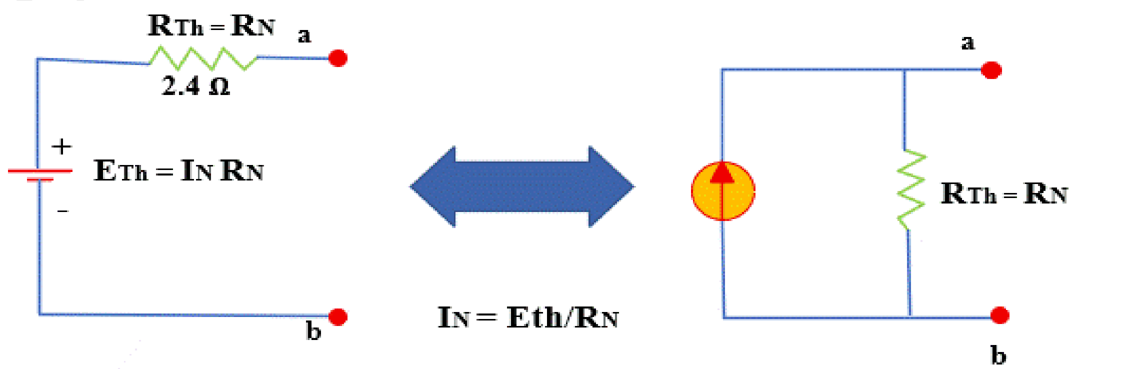


Figure. 4.3. (a) Original circuit, (b) Norton equivalent circuit.



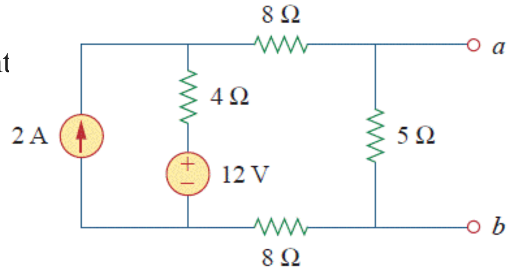
Thevenin equivalent circuit

Norton equivalent circuit

The following steps provide a technique which allows the conversion of any circuit into its Norton equivalent:

1. Remove the **load** from the circuit.
2. Label the resulting two terminals. We will label them as ***a* and *b***, although any notation may be used.
3. Set all sources to **zero**. As before, voltage sources are set to zero by replacing them with short circuits and current sources are set to zero by replacing them with open circuits.
4. Determine the **Norton equivalent resistance**, $\boldsymbol{R_N}$, by calculating the resistance seen between terminals *a* and *b*. It may be necessary to redraw the circuit to simplify this step.
5. Replace the sources removed in **Step 3**, and determine the current which would occur in a short if the short were connected between terminals *a* and *b*. If the original circuit has more than one source, it may be necessary to use the superposition theorem. In this case, it will be necessary to determine the short-circuit current due to each source separately and then determine the combined effect. The resulting short-circuit current will be the value of the Norton current $\boldsymbol{I_N}$.
6. Sketch the Norton equivalent circuit using the **resistance** determined in **Step 4** and the **current** calculated in **Step 5**. As part of the resulting circuit, include that portion of the network removed in **Step 1**. The Norton equivalent circuit may also be determined directly from the **Thevenin equivalent circuit by using source conversion technique**.

Example 4.12: Find the Norton equivalent circuit of the circuit shown below at terminals *a-b*.



Solution: we find R_N similar to R_{Th}, set independent Sources to zero, see Fig. (a) below.

$R_N = 5 \parallel (8+4+8) = 5 \parallel 20 = \frac{20\times5}{25} = 4\ \Omega$

To find I_N, we short-circuit terminals a and b.

We ignore 5 Ω as it has been short-circuited.

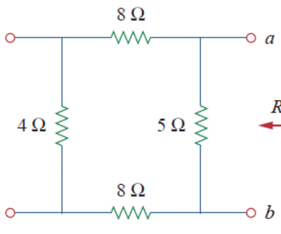
Applying mesh analysis,

$i_1 = 2$ A Eq. (1)

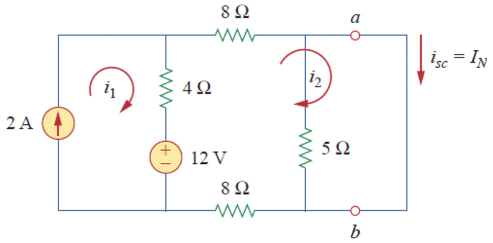
$4(i_2 - i_1) + 16i_2 - 12 = 0$

$-4i_1 + 20i_2 - 12 = 0$ Eq. (2)

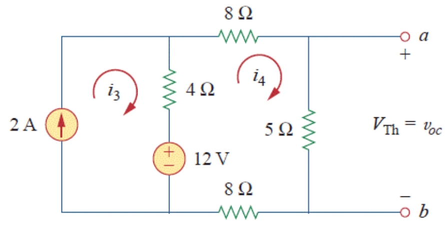
So, $i_2 = 1$ A $= i_{sc} = I_N$



(a)



(b)



(c)

Alternatively, we can find

$I_N = \frac{V_{Th}}{R_{Th}}$ apply mesh analysis

$i_3 = 2$ A Eq. (3)

$25i_4 - 4i_3 - 12 = 0$ Eq. (4)

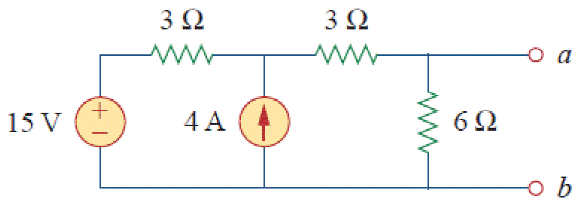
$i_4 = 0.8$ A

$v_{oc} = V_{Th} = 5i_4 = 4$ V

$I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1$ A

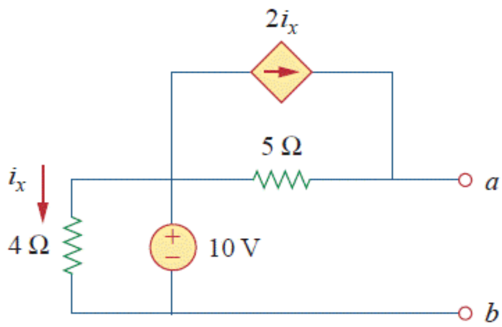
From Fig. (c), we can find $R_{Th} = \frac{v_{oc}}{i_{sc}} = \frac{4}{1} = 4\ \Omega$

Practice Example: Find the Norton equivalent circuit for the circuit shown below, at terminals *a*-*b*.



Answer: $R_N = 3\ \Omega$, $I_N = 4.5$ A.

Example 4.13: Using Norton's theorem, find R_N and I_N of the circuit shown below at terminals *a*-*b*.



Solution: To find R_N, we set independent voltage source to zero and connect a voltage source of $v_o = 1$ V. we ignore 4 Ω as it is short-circuited. Hence $i_x = 0$.

At node *a*, $i_o = \frac{1v}{5\Omega} = 0.2$ A.

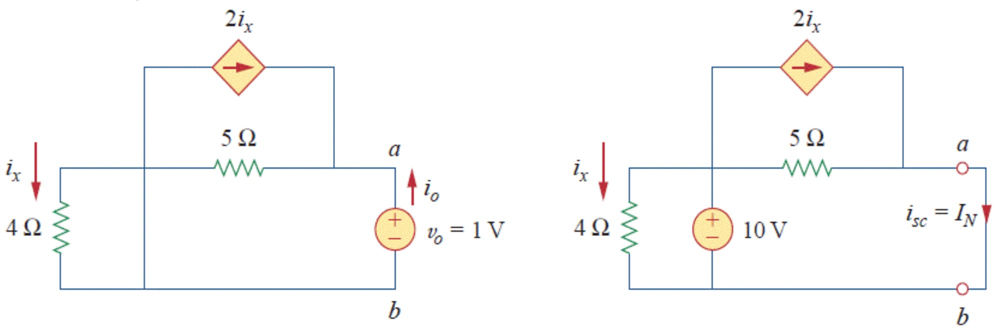
$R_N = \frac{v_o}{i_o} = \frac{1}{0.5} = 5\ \Omega$

To find I_N, we short-circuit terminals *a* and *b* and find current i_{sc},

$i_x = \frac{10}{4} = 2.5$ A.

At node *a*, KCL gives,

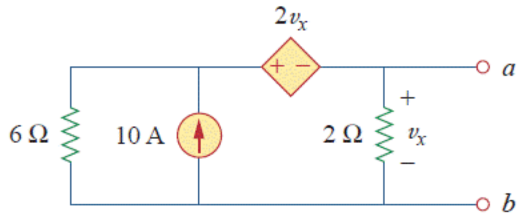
$i_{sc} = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7\text{ A} = I_N$.



(a) (b)

Practice Example: Find the Norton equivalent circuit for the circuit shown below, at terminals *a*-*b*.

Answer: $R_N = 1\ \Omega$, $I_N = 10$ A.



4.5. Maximum Power Transfer

The Thevenin equivalent is useful in finding the maximum power a linear circuit can deliver to a load. We assume that we can adjust the load resistance R_L. If the entire circuit is replaced by its Thevenin equivalent except for the load, as shown in **Fig. 4.3**, the power delivered to the load is

$$\boldsymbol{p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th}+R_L}\right)^2 R_L} \qquad \text{Eq. (4.3)}$$

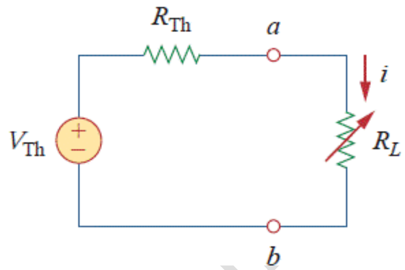


Figure. 4.3. The circuit is used for maximum power transfer.

For a given circuit, V_{Th} and R_{Th} are fixed. By varying the load resistance R_L the power delivered to the load varies as sketched in Figure. We notice from **Fig. 4.4** that the power is small for small or large values of R_L but maximum for some value of R_L between 0 and ∞. We now want to show that this maximum power occurs when R_L is equal to R_{Th}. This is known as the ***maximum power theorem***.

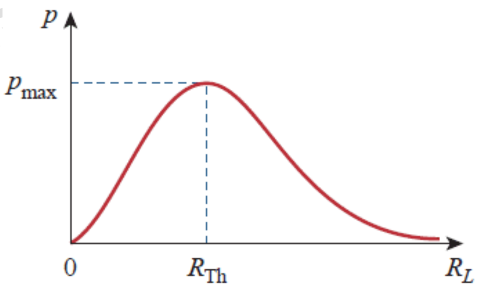


Figure. 4.4. Power delivered to the load as a function of R_L.

Maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$).

To prove the maximum power transfer theorem, we differentiate p in **Eq. (4.3)** with respect R_L to and set the result equal to zero. We obtain,

$$\frac{dp}{dR_L} = V_{Th}^2 \left[\frac{(R_{Th}+R_L)^2 - 2R_L(R_{Th}+R_L)}{(R_{Th}+R_L)^2}\right]$$

$$= V_{Th}^2 \left[\frac{R_{Th}+R_L-2R_L}{(R_{Th}+R_L)^3}\right]$$

$$0 = (R_{Th} + R_L - 2R_L) = (R_{Th} - R_L)$$

$$\text{So, } \boldsymbol{R_L = R_{Th}} \qquad \text{Eq. (4.4)}$$

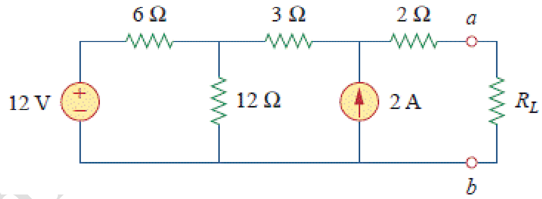
showing that the maximum power transfer takes place when the load resistance R_L equals the Thevenin resistance R_{Th}. We can readily confirm that Eq. (4.4) gives the maximum power by showing that $\frac{d^2p}{dR_L{}^2} < 0$.

The maximum power transferred is obtained by substituting **Eq. (4.4)** into **Eq. (4.3)**, for

$$\boldsymbol{p_{max} = \frac{V_{Th}{}^2}{4R_{Th}}} \qquad \text{Eq. (4.5)}$$

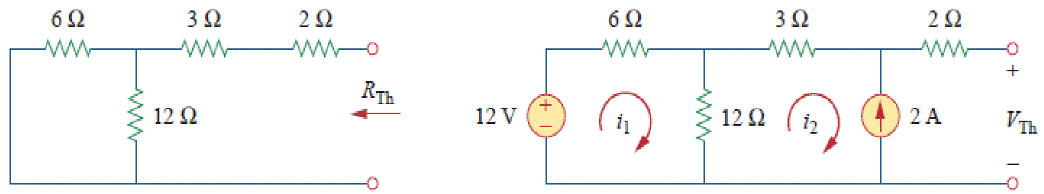
Equation (4.5) applies only when $R_L = R_{Th}$. When $R_L \neq R_{Th}$, we compute the power delivered to the load using **Eq. (4.3)**.

<u>Example 4.14</u>: Find the value of R_L for maximum power transfer in the circuit of fig. below. Find the maximum power.



<u>Solution</u>: We find R_{Th} and V_{Th},

$R_{Th} = 2 + 3 + 6 \parallel 12 = 5 + \frac{6\times12}{18} = 9\ \Omega.$



(a) (b)

To find V_{Th}, applying mesh analysis,

$-12 + 18i_i - 12i_2 = 0,$

$i_2 = -2$ A

Therefore, $i_1 = -2/3$. Applying KVL around the outer loop to get V_{Th} across terminals *a-b*.

$-12 + 6i_1 + 3i_2 + 2(0) + V_{Th} = 0$

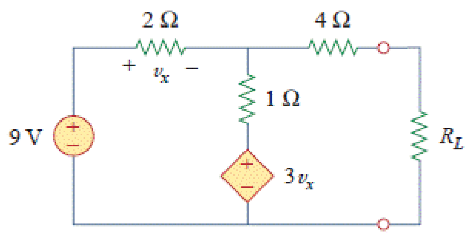
$V_{Th} = 22$ V.

Maximum power transfer is $R_L = R_{Th} = 9\ \Omega$. So,

$p_{max} = \frac{V_{Th}{}^2}{4R_{Th}} = \frac{22^2}{4\times9} = 13.44$ W.

<u>Practice Example</u>: Determine the value of R_L that will draw the maximum power from the rest of the circuit in Fig. below. Calculate the maximum power.

<u>Answer</u>: $R_N = 4.22\ \Omega$, $p_{max} = 2.901$ W.



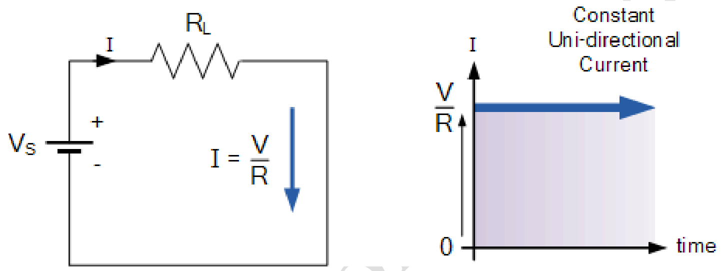
Chapter Five

Single-phase A.C. Circuits

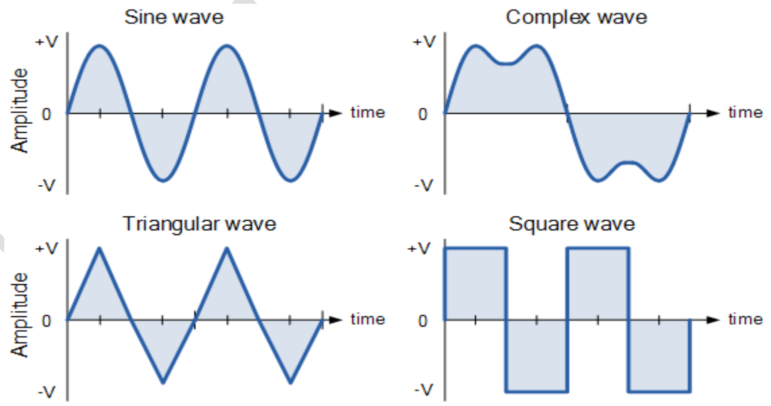
5. A.C. Circuits

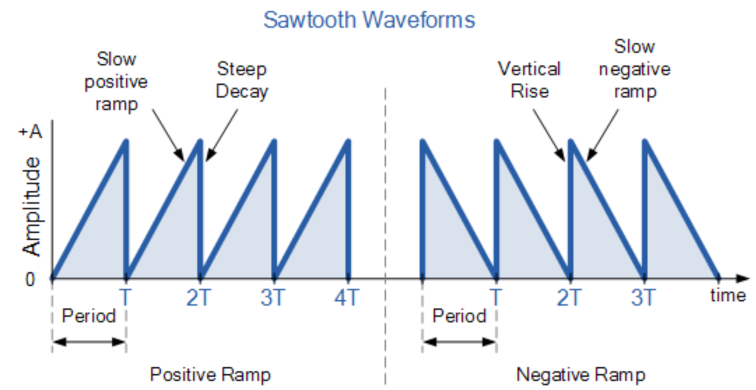
The flow of electricity can be done in two ways like **AC** (**alternating current**) and **DC** (**direct current**). Electricity can be defined as the flow of electrons throughout a conductor such as a wire. The main disparity among AC & DC mainly lies within the direction where the electrons supply. In direct current, the flow of electrons will be in a single direction & in the alternating current; the flow of electrons will change directions like going forward & then going backward.

A DC voltage or current has a **fixed magnitude** (**amplitude**) and a definite direction associated with it. And do not change their values with regards to time, they are constant values flowing in a continuous steady state direction.



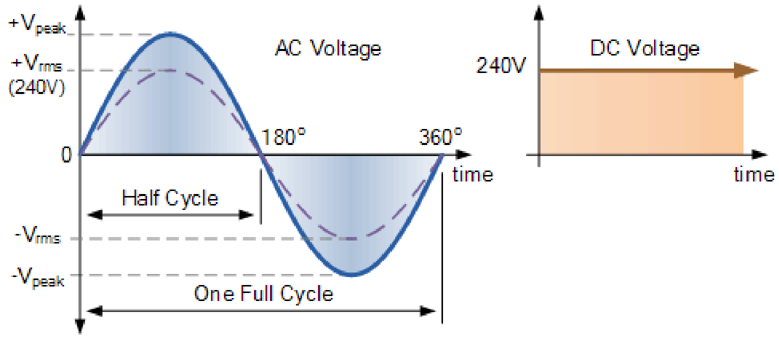
The term ***alternating*** indicates only that the waveform alternates between two prescribed levels in a set time sequence.





5.1. The Sinusoidal Source

Circuits with alternating current (AC) are functions whose values vary in both magnitude and direction.



Definitions: A few basic terms will be defined in this section which can be applied to any waveform.

- A **path** plotted as a function of some variable such as time, position, degree, radian, temperature, and so on.
- ***Periodic waveform***: a waveform that continually repeats itself after the same time interval.
- ***Period*** **(*T*):** the time interval between successive repetitions of a periodic waveform or the time of one cycle.
- ***Cycle:*** the portion of a waveform contained in one period at a time.
- ***Frequency*** **(*f*):** the number of cycles that occur in 1 second, the unit used for measuring frequency is cycle per second or hertz (Hz).

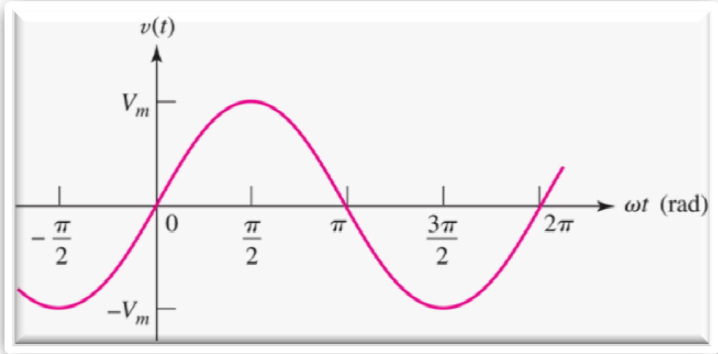
$$f = \frac{1}{T}, \qquad f = Hz, \qquad T = second\ (s)$$

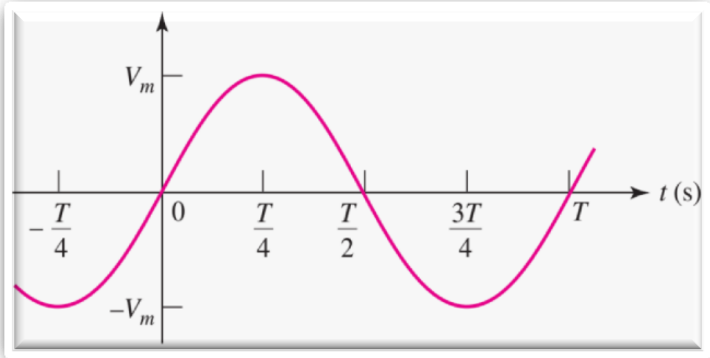
- ***Instantaneous value***: the magnitude of the waveform at any instant of time.
- ***Amplitude or Peak value***: the maximum value of a waveform.
- ***Angular velocity (w)***: the velocity with which the radius vector rotates about the center.

A sinusoidal current source (independent or dependent) produces a current that varies sinusoidally with time.

$$v(t) = V_m \sin \omega t$$

Where;
V_m is the amplitude of the sinusoid
ω is the angular frequency in radians/s
ωt is the argument of the sinusoid





The function repeats itself every 2π radians, and its period is therefore 2π radians. A sine wave having a period T must execute 1/T period each second; its frequency f is 1/T hertz, abbreviated Hz. Thus,

$$f = \frac{1}{T}, \qquad \omega T = 2\pi, \qquad \omega = 2\pi f$$

1 hertz (Hz) = 1 cycle per second (c/s)

$$\text{Radians} = \left(\frac{\pi}{180^\circ}\right) \times (\text{degrees})$$

$$\text{Degrees} = \left(\frac{180^\circ}{\pi}\right) \times (\text{radians})$$

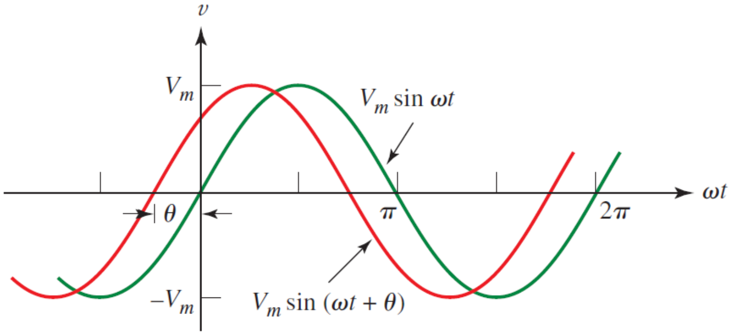
A more general form of the sinusoid,

$$v(t) = V_m \sin(\omega t + \theta)$$

Includes a phase angle θ in its argument. So, we can say that:

- $V_m \sin(\omega t + \theta)$ ***leads*** $V_m \sin(\omega t)$ by θ rad.
- $V_m \sin(\omega t)$ ***lags*** $V_m \sin(\omega t + \theta)$ by θ rad.

Thus, ***leading or lagging***, we say that the sinusoids are ***out of phase***. If the phase angles are ***equal***, the sinusoids are said to be ***in phase***.



Some geometric relations:

1. $\sin(-\omega t) = -\sin(\omega t)$
2. $\cos(-\omega t) = \cos(\omega t)$
3. $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
4. $\cos(a \pm b) = \cos a \cos b \pm \sin a \sin b$
5. $\sin\left(\omega t \pm \frac{\pi}{2}\right) = \pm\cos(\omega t)$
6. $\cos\left(\omega t \pm \frac{\pi}{2}\right) = \mp\sin(\omega t)$
7. $\sin(\omega t \pm \pi) = -\sin(\omega t)$
8. $\cos(\omega t \pm \pi) = -\cos(\omega t)$
9. $\sin^2(\omega t) = \frac{1}{2}(1 - \cos(2\omega t))$
10. $\cos^2(\omega t) = \frac{1}{2}(1 + \cos(2\omega t))$

Example 5.1: What is the period T and the frequency *f* of a period current that has 42 cycles in 50 ms.

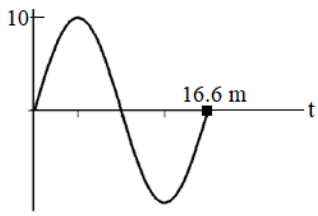
Solution: period T is the time of one cycle, $T = \frac{time\ t}{cycle} = \frac{50\ m}{42} = 1.19$ m

$f = \frac{1}{T} = \frac{1}{1.19\ m} = 840$ Hz.

Example 5.2: Sketch $V = \sin(377t)$ against time in seconds.

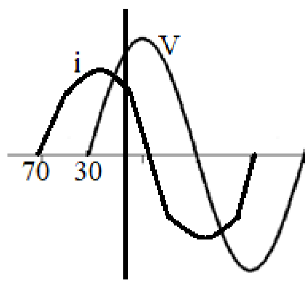
Solution: $\omega = 377 = 2\pi f \quad \Rightarrow f = \frac{377}{2\times\pi} = \frac{377}{2\times 3.14} = 60$ Hz

$T = \frac{1}{f} = \frac{1}{60} = 0.0166 = 16.6$ ms.



Example 5.3: Find the phase relations for the following waveform.

a. $V = 10\sin(\omega t + 30^{\circ})$
   $i = 5\sin(\omega t + 70^{\circ})$
b. $V = -2\cos(\omega t - 60^{\circ})$
   $i = 3\sin(\omega t - 150^{\circ})$
c. $V = 5\cos(300t - 10^{\circ})$
   $i = 10\sin(600t + 30^{\circ})$



Solution:

a. i leads V by 40°
   Or V lags by i 40°
b. $V = 2\sin\left(\omega t - \frac{\pi}{2} - 60^{\circ}\right) = 2\sin(\omega t - 150^{\circ})$
   So, V and i are in phase (phase shift = 0)
c. No phase relation between V and i.

Example 5.4: An alternative voltage of frequency 60 Hz has a maximum value of 120 V. Write down the equation for its instantaneous value. Taking time from instant the current is zero and is becoming positive, find (a) the instantaneous value after 2.8 ms, and (b) the time taken to reach 96 A for the first time.

Solution:

$V = 120\sin(2\pi f t) = 120\sin(377t)$

a. $V = 120\sin(377 \times 2.8 \times 10^{-3}) = 104.4$ V.

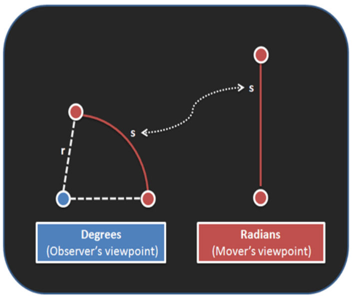
b. $96 = 120\sin(377 \times t) \quad \Rightarrow \quad \sin(377 \times t) = 96/120$

$\sin(377 \times t) = 0.8 \quad \Rightarrow \quad (377 \times t) = sin^{-1}0.8$

$(377 \times t) = 53.130 \times \left(\frac{\pi}{180^\circ}\right) \Rightarrow \quad (377 \times t) = 0.927$

$t = 0.00246$ second.

Degrees vs. Radians

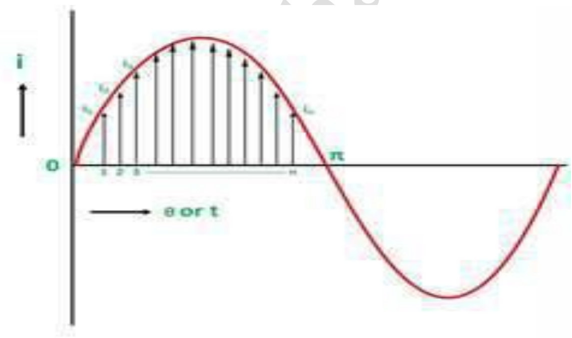


5.2. Average and rms values of waveforms

5.2.1. Average value

The ***average value*** of a periodic waveform whether it is a sine wave, square wave or triangular waveform is defined as: "**Average value is defined as the area under the curve divided by the baseline of the curve**". In other words, the average of all the instantaneous values along time axis with time being one full period, (T).

For symmetrical waves like sinusoidal current or voltage waveform, the positive half cycle will be exactly equal to negative half cycle. Therefore, the average value over a complete cycle will be zero. The work is done by both positive and negative cycle and hence the average value is determined without considering the signs. So, the only positive half cycle is considered to determine the average value of alternating quantities of sinusoidal waves.



$$\textbf{Average} = \frac{\textbf{area under curve}}{\textbf{lenght of base}}$$

$$= \frac{1}{T}\int_0^T f(t).dt \qquad \textbf{Or} \qquad \frac{\int_0^T f(t).dt}{T}$$

Example 5.5: Find the average value of the signal $i = I_m \sin(\omega t)$.

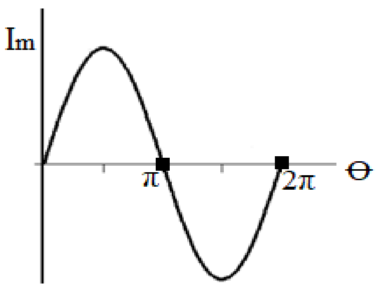
Solution: one period time $\omega t = \pi$

$T = 2\pi$ and $\frac{T}{2} = \pi$

For complete cycle, average = 0.

For half cycle, average $= \frac{\int_0^{\frac{T}{2}} f(t).dt}{\frac{T}{2}} = \frac{\int_0^{\pi} I_m \sin(\omega t)\,.\,dt}{\pi}$

$\frac{I_m[-\cos(\omega t)]_0^{\pi}}{\pi} = \frac{I_m[-(-\cos(\pi) - \cos(0))]}{\pi} = \frac{I_m[-(-1-1)]}{\pi} = \frac{2I_m}{\pi} = 0.637$



5.2.2. Root Mean Square (RMS)

"**RMS**" stands for "**Root-Mean-Squared**", also called the **effective or heating** value of alternating current, which would provide the same amount of heat generation in a resistor as the AC voltage would if applied to that same resistor.

Suppose I to be the value of **D.C. current** passing through the resistance R and producing a heating effect equal to the average heating effect of the **A.C. current**.

$$P_{D.C.} = P_{A.C.}$$

$$I_{DC}^2 R = I_{rms}^2 R = \frac{1}{T}\int_0^T Ri(t)^2 \, .dt$$

$$I_{eff} = I_{rms} = \sqrt{\frac{1}{T}\int_0^T Ri(t)^2 \, .dt}$$

$$RMS = \sqrt{\frac{\int_0^{2\pi} f(t)^2 \, .dt}{2\pi}}$$

- For any periodic function $X(t)$, the rms value is given by,

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2 dt}$$

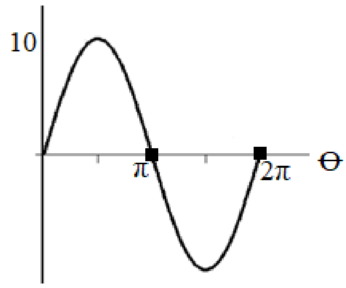
- For the sinusoid $i(t) = I_m \cos(\omega t + \theta)$, $T = \frac{2\pi}{\omega}$, the rms value of $i(t)$ is

$$I_{rms} = \sqrt{\frac{1}{T}\int_{t_o}^{t_o+T} {I_m}^2 \cos^2(\omega t + \theta). dt}$$

$$I_{rms} = I_m \sqrt{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \left[\frac{1}{2} + \frac{1}{2}\cos(2\omega t + \theta)\, dt\right]} = I_m \sqrt{\frac{\omega}{4\pi} \times \frac{2\pi}{\omega}}$$

- **For sinusoidal current or voltage**

$$I_{rms} = \frac{I_m}{\sqrt{2}}, \qquad V_{rms} = \frac{V_m}{\sqrt{2}}$$



Example 5.6: Find the average value of the signal $I_m \sin(\theta)$

Solution:

$$I_{rms} = \sqrt{\frac{\int_0^{\pi}[I_m \sin(\theta)]^2 .d\theta}{\pi}} = \sqrt{\frac{100\int_0^{\pi}[I_m \sin(\theta)]^2 .d\theta}{\pi}}$$

$$= \sqrt{\frac{100\int_0^{\pi}\frac{1}{2}(1-\cos(2\theta)).d\theta}{\pi}} = \sqrt{\frac{100\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}}{2\pi}}$$

$$= \sqrt{\frac{100\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]}{2\pi}} = \sqrt{\frac{100\left[\left(\pi - \frac{1}{2}(0)\right) - \left(0 - \frac{1}{2}(0)\right)\right]}{2\pi}}$$

$$\sqrt{\frac{100[(\pi - 0) - (0 - 0)]}{2\pi}} = \sqrt{\frac{100[\pi]}{2\pi}} = \sqrt{\frac{100}{2}} = 7.07$$

5.3. Series A.C. circuits

5.3.1. AC Sources

An AC circuit consists of circuit elements and a power source that provides an alternating voltage Δv. This time-varying voltage is described as:

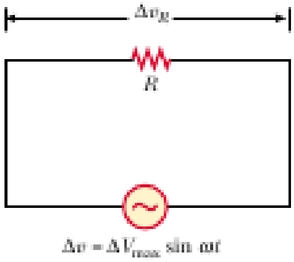
$$\Delta v = \Delta V_{max} \sin \omega t$$

where ΔV_{max} is the maximum output voltage of the AC source, or the voltage amplitude. There are various possibilities for AC sources, including generators and electrical oscillators. In a home, each electrical outlet serves as an AC source.

The units of AC are cycles per second $f = \frac{1}{T} = \frac{\omega}{2\pi}$, **Or** $\omega = 2\pi f$.

5.3.2. Resisters in an AC Circuit

Consider a simple AC circuit consisting of a resistor and an AC source, as shown in Figure below. At any instant, the algebraic sum of the voltages around a closed loop in a circuit must be zero (Kirchhoff's loop rule). Therefore,



$$\Delta v + \Delta v_R = 0$$

So that the magnitude of the source voltage equals the magnitude of the voltage across the resistor:

$$\Delta v = \Delta v_R = \Delta V_{max} \sin \omega t \qquad \text{Eq. (1)}$$

where Δv_R is the instantaneous voltage across the resistor, and $R = \frac{\Delta v}{I}$. The instantaneous current in the resistor is:

$$i_R = \frac{\Delta v_R}{R} = \frac{\Delta V_{max} \sin \omega t}{R} = I_{max} \sin \omega t$$

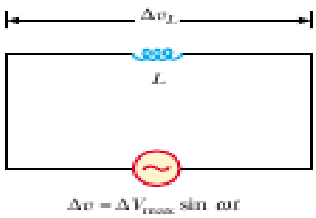
where I_{max} is the maximum current:

$I_{max} = \frac{\Delta V_{max} \sin \omega t}{R}$, substituted in Eq. (1) we get:

$$\Delta v_R = I_{max} R \sin \omega t$$

5.3.3. Inductors in an AC Circuit

Consider an AC circuit consisting only of an inductor connected to the terminals of an AC source, as shown in Figure below. If $\Delta v_L = \varepsilon_L = -L\frac{dI}{dt}$ is the self-induced instantaneous voltage across the inductor. Then Kirchhoff's loop rule applied to this circuit gives,



$\Delta v + \Delta v_L = 0,$ Or $\Delta v - L\frac{dI}{dt} = 0$

When we substitute $\Delta v = \Delta V_{max} \sin \omega t$, we obtain:

$\Delta V_{max} \sin \omega t - L\frac{dI}{dt} = 0 \quad \Rightarrow \quad \Delta V_{max} \sin \omega t = L\frac{dI}{dt}$

Solving this equation for dI, we find that:

$dI = \frac{\Delta V_{max} \sin \omega t}{L}$

Integrating this expression gives the instantaneous current I_L in the inductor as a function of time:

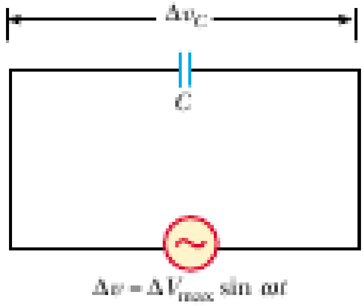
$I_L = \frac{\Delta V_{max}}{L}\int \sin \omega t\, dt = -\frac{\Delta V_{max}}{L}\cos \omega t$

We give the inductor reactance $X_L = \omega C$, because this function varies with frequency. We can write Eq. as:

$I_{max} = X_L \Delta V_{max}$

5.3.4. Capacitors in an AC Circuit

Figure below shows an AC circuit consisting of a capacitor connected across the terminals of an AC source. Kirchhoff's loop rule applied to this circuit gives,



$\Delta v + \Delta v_C = 0$

$\Delta v = \Delta v_C = \Delta V_{max} \sin \omega t$ Eq. (1)

where Δv_C is the instantaneous voltage across the capacitor.

We know from the definition of the capacitance that $q/\Delta v_C$

$q = C\Delta V_{max} \sin \omega t$ Eq. (2)

where q is the instantaneous charge on the capacitor.

Because $I_C = dq/dt$ differentiating Eq. (2) with Respect for time gives the instantaneous current in the circuit:

$I_C = \frac{dq}{dt} = \omega C \Delta V_{max} \sin \omega t$

Using trigonometric identity

$\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$

$I_C = \frac{dq}{dt} = \omega C \Delta V_{max} \sin\left(\omega t + \frac{\pi}{2}\right)$ Eq. (3)

For a sinusoidally applied voltage, the current always leads the voltage across a capacitor by 90°.

From Eq. (3), we see that the current in the circuit reaches its maximum value when $\cos \omega t = 1$:

$I_{max} = \omega L \Delta V_{max} = \frac{\Delta V_{max}}{(\frac{1}{\omega C})}$ Eq. (4)

We give the capacitive reactance $X_C = \frac{1}{\omega C}$, because this function varies with frequency. We can write Eq. (4) as:

$I_{max} = \frac{\Delta V_{max}}{X_C}$ Eq. (5)

Combining Eq. (5) and Eq. (1), we can express the instantaneous voltage across the capacitor as:

$\Delta v = \Delta v_C = \Delta V_{max} \sin \omega t = I_{max} X_C \sin \omega t$

Example 5.7: An **8 μF capacitor is connected to the terminal of a 60 Hz AC source whose rms voltage is 150 V. Find the capacitive reactance and the rms current in the circuit.**

Solution:

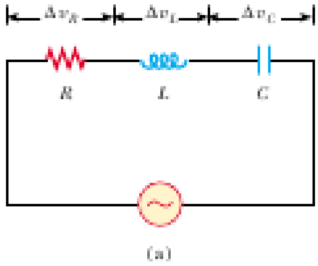
$$\omega = 2\pi f = 2\pi \times 60Hz = 377\ sec^{-1}$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(377\ sec^{-1})(8\times10^{-6}F)} = 332\ \Omega$$

$$I_{rms} = \frac{\Delta V_{rms}}{X_C} = \frac{150V}{332\Omega} = 0.452\ \text{A}$$

5.3.5. The RLC Series Circuit and Resonance

Figure below shows a circuit that contains a resistor, an inductor, and a capacitor connected in series across an alternating voltage source. As before, we assume that the applied voltage varies sinusoidally with time. It is convenient to assume that the instantaneous applied voltage is given by:



$$\Delta v = \Delta V_{max} \sin \omega t$$

While the current varies as:

$$i = \Delta I_{max} \sin(\omega t - \theta)$$

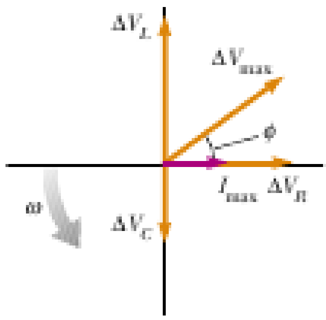
where θ is some phase angle between the current and the applied voltage. The current at all points in a series AC circuits have the same amplitude and phase. We can express the instantaneous voltages across the three circuit elements as:

$$\Delta_{VR} = I_{max} R \sin \omega t = \Delta V_R \sin \omega t$$

$$\Delta_{VL} = I_{max} X_L \sin\left(\omega t + \frac{\pi}{2}\right) = \Delta V_L \cos \omega t$$

$$\Delta_{VC} = I_{max} X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -\Delta V_R \cos \omega t$$

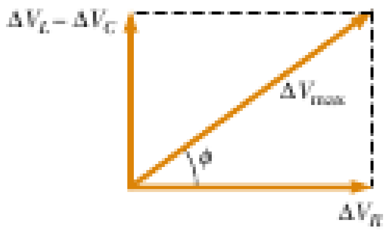
where Δ_{VR}, Δ_{VL}, and Δ_{VC} are the maximum voltage values across the elements: $\Delta_{VR} = I_{max}R$, $\Delta_{VL} = I_{max}X_L$, and $\Delta_{VC} = I_{max}X_C$.



At this point, we could proceed by noting that the instantaneous voltage Δv across the three elements equals the sum:

$$\Delta v = \Delta_{VR} + \Delta_{VL} + \Delta_{VC}$$

In the right figure, we see that the vector sum of voltage amplitudes Δ_{VR}, Δ_{VL}, and Δ_{VC} equals a phasor whose length is maximum applied voltage Δv_{max} and which makes an angle θ with the current phasor I_{max}.



The voltage phasors Δ_{VL} and Δ_{VC} are in opposite directions Along the same line, we can construct the different phasors: $\Delta_{VL} - \Delta_{VC}$, which is perpendicular to the phasor Δ_{VR}.

From either one of the right triangles:

$$\Delta V_{max} = \sqrt{{\Delta_{VR}}^2 + (\Delta_{VL} - \Delta_{VC})^2} = \sqrt{(I_{max}R)^2 + (I_{max}X_L - I_{max}X_C)^2}$$

$$\Delta V_{max} = I_{max}\sqrt{(R)^2 + (X_L - X_C)^2}$$

Therefore, we can express the maximum current as:

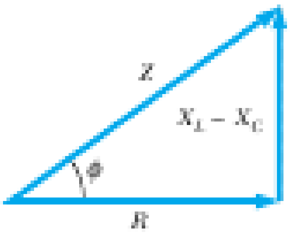
$$\boldsymbol{I_{max}} = \frac{\Delta V_{max}}{\sqrt{(R)^2+(X_L-X_C)^2}}$$

where $\boldsymbol{Z} = \sqrt{(\boldsymbol{R})^2 + (\boldsymbol{X_L} - \boldsymbol{X_C})^2}$ is called the impedance.

$$\Delta \boldsymbol{V_{max}} = \boldsymbol{I_{max} Z}$$

Find that the phase angle θ between the current and voltage is:

$$\tan^{-1}\boldsymbol{\theta} = \left(\frac{X_L-X_C}{R}\right)$$



The **resonance** condition for the **Series RLC** circuit is given by, $\boldsymbol{\theta} = \boldsymbol{0}$, which implies: $\boldsymbol{X_L} = \boldsymbol{X_C}$ from which we obtain

$$\boldsymbol{\omega_o L} = \boldsymbol{1/\omega_o C}$$

The **resonant frequency is** $\boldsymbol{\omega_o} = \boldsymbol{1/\sqrt{LC}}$

Example 5.7: A series **RLC AC** circuit has $R = 425\ \Omega$, $L = 1.25$ H, $C = 3.5\ \mu F$, $\omega = 377\ sec^{-1}$, and $V_{max} = 150$ V. (**A**) Determine the inductive reactance, the capacitive reactance, and the impedance of the circuit, (**B**) Find the maximum current in the circuit. (**C**) Find the phase angle between the current and voltage. (**D**) Find both the maximum voltage and the instantaneous voltage across each element.

Solution: The reactance is $X_L = \omega L = 471\ \Omega$, $X_C = \frac{1}{\omega C} = 758\ \Omega$.

The impedance is:

$$Z = \sqrt{(R)^2 + (X_L - X_C)^2} = Z = \sqrt{(425)^2 + (471 - 758)^2} = 513\ \Omega$$

$$\theta = \tan^{-1}\left(\frac{X_L-X_C}{R}\right) = \tan^{-1}\left(\frac{471-758}{425}\right) = -34^\circ$$

$$\Delta V_R = I_{max} R = (0.292\ A)\ (425\ \Omega) = 124\ \text{V}$$

$$\Delta V_L = I_{max} X_L = (0.292\ A)\ (471\ \Omega) = 138\ \text{V}$$

$$\Delta V_C = I_{max} X_C = (0.292\ A)\ (758\ \Omega) = 221\ \text{V}$$

$$\Delta_{VR} = I_{max} R \sin\omega t = \Delta V_R \sin\omega t = (124\ V)\ sin377t$$

$$\Delta_{VL} = I_{max} X_L \sin\left(\omega t + \frac{\pi}{2}\right) = \Delta V_L \cos\omega t = (138\ V)\ cos377t$$

$$\Delta_{VC} = I_{max} X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -\Delta V_R \cos\omega t = (-221\ V)\ cos377t$$

Example 5.8: Consider a **series RLC** circuit for which $\mathbf{R} = \mathbf{150\ \Omega}$, $\boldsymbol{L} = \mathbf{20\ mH}$, $\Delta \boldsymbol{V_{rms}} = \mathbf{20}$ V, and $\boldsymbol{\omega} = \mathbf{5000}\ \boldsymbol{sec^{-1}}$. Determine the value of the capacitance for which the current is a maximum.

Solution:

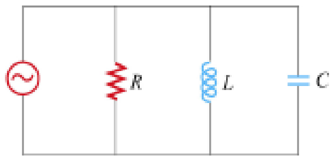
$$\boldsymbol{\omega_o} = \frac{\mathbf{1}}{\sqrt{LC}} = \frac{\mathbf{1}}{\omega_o{}^2 L} = \frac{1}{(5000\ sec^{-1})^2(20\times10^{-3})} = 2\mu F$$

5.3.6. Parallel RLC Circuit and Resonance

Consider the parallel RLC circuit shown in Figure below. The AC voltage source is:

$\Delta v = \Delta V_{max} \sin \omega t$

Unlike the series RLC circuit, the instantaneous voltages across all three circuit elements R, L and C are the same, and each voltage is in phase with the current through the resistor. However, the currents through each element will be different. The current in the resistor is:



$$I_R = \frac{\Delta V_R}{R} = \frac{\Delta V_{max} \sin \omega t}{R} = I_{max} \sin \omega t$$

The voltage across the inductor is:

$$\Delta V_L = \Delta V_{max} \sin \omega t = L\frac{dI}{dt}$$

$$\int_0^{I_L} dI = \int_0^{t} \frac{V_{max}}{L} \sin \omega t \, dt = -\frac{V_{max}}{\omega L} \cos\omega t$$

$$I_L = \frac{V_{max}}{X_L} \sin\left(\omega t - \frac{\pi}{2}\right) = I_{max} \sin\left(\omega t - \frac{\pi}{2}\right)$$

The voltage across the capacitor is:

$\Delta V_C = \Delta V_{max} \sin \omega t$, which implies:

$$I_C = \frac{dq}{dt} = \omega C \Delta V_{max} \sin \omega t = \frac{\Delta V_{max}}{X_C} \sin\left(\omega t + \frac{\pi}{2}\right)$$

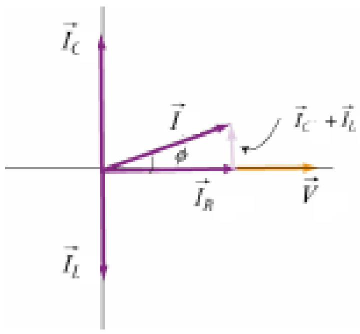
$$I_C = I_{max} \sin\left(\omega t + \frac{\pi}{2}\right)$$

Using **Kirchhoff's rule**, the total current in the circuit is simply the sum of all three currents:

$$I = I_R + I_L + I_C$$

$$I = I_{max} \sin\omega t + I_{max} \sin\left(\omega t - \frac{\pi}{2}\right) + I_{max} \sin\left(\omega t + \frac{\pi}{2}\right)$$

From the phasor diagram the maximum amplitude of the total current can be obtained as:



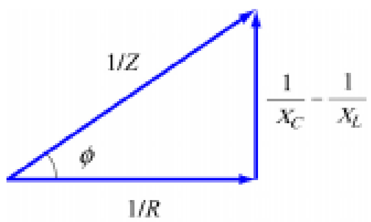
$$I = \sqrt{{I_R}^2 + (I_L - I_C)^2}$$

$$I = V_{max}\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

$$I = V_{max}\sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2}$$

$$I = \frac{V_{max}}{Z}$$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2}$$



$$\tan^{-1}\theta = \frac{(I_C - I_L)}{I_R} = \frac{\left(\frac{V_{max}}{X_C} - \frac{V_{max}}{X_L}\right)}{\frac{V_{max}}{R}} = R\left(\frac{1}{X_C} - \frac{1}{X_L}\right)$$

The resonance condition for the **parallel RLC** circuit is given by $\boldsymbol{\theta = 0}$, which implies:

$$\frac{1}{X_C} = \frac{1}{X_L}$$

The **resonant frequency is:** $\boldsymbol{\omega_o = \frac{1}{\sqrt{LC}}}$

$$\boldsymbol{\omega_o = \frac{1}{\sqrt{LC}}}$$

5.3.7. Power in an AC Circuit

For the **RLC**, we can express the instantaneous power $\boldsymbol{P}$ as:

$$\boldsymbol{P = i\Delta v = I_{max}\, sin(\omega t - \theta)\Delta V_{max} sin\omega t}$$

$$\boldsymbol{P = I_{max}\Delta V_{max}\, sin\omega t\, sin(\omega t - \theta)}$$

$$\boldsymbol{sin(\omega t - \theta) = sin\omega t\, cos\theta - cos\omega t\, sin\theta}$$

$$\boldsymbol{P = I_{max}\, \Delta V_{max}\, sin\omega t\, (sin\omega t\, cos\theta - cos\omega t\, sin\theta)}$$

$$\boldsymbol{P = I_{max}\, \Delta V_{max}\, sin^2\omega t\, cos\theta - I_{max}\, \Delta V_{max}\, sin\omega t\, cos\omega t\, sin\theta}$$

The time average of the second term on the right is identically zero because $\boldsymbol{sin\omega t\, cos\omega t = \frac{1}{2} sin2\omega t}$ and the average value of $\boldsymbol{\frac{1}{2} sin2\omega t}$ is zero and the average value of $\boldsymbol{sin^2\omega t = \frac{1}{2}}$. Therefore, we can express the average power $\boldsymbol{P_{av}}$ as:

$$\boldsymbol{P_{av} = \frac{1}{2} I_{max}\, \Delta V_{max}\, cos\theta}$$

It is convenient to express the average power in terms of rms current and rms voltage defined by $\boldsymbol{I_{rms} = \frac{I_{max}}{\sqrt{2}}}$ and $\boldsymbol{\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}}}$

$$\boldsymbol{P_{av} = I_{rms}\, \Delta V_{rms}\, cos\theta} \qquad \text{Eq. (1)}$$

where the quantity $\boldsymbol{cos\, \theta}$ called the power factor and the maximum voltage across the resistor is given by $\boldsymbol{\Delta V_R = \Delta V_{max}\, cos\theta = I_{max}\, R}$ and $\boldsymbol{cos\theta = \frac{I_{max}\, R}{\Delta V_{max}}}$, substituted in **Eq. (1)**, we get:

$$\boldsymbol{P_{av} = I_{rms}\, \Delta V_{rms}\, cos\theta = I_{rms}\left(\frac{\Delta V_{max}}{\sqrt{2}}\right)\frac{I_{max}\, R}{\Delta V_{max}} = I_{rms}\frac{I_{max}\, R}{\sqrt{2}}}$$

$$\boldsymbol{I_{max} = \sqrt{2}\, I_{rms}}$$

$$\boldsymbol{P_{av} = I^2{}_{rms}\, R}$$

<u>**Example 5.9**</u>: A series **RLC AC** circuit has $\boldsymbol{R = 425\Omega, L = 125\, H. C = 3.5\mu F, \omega = 377\, sec^{-1}}$, and $\boldsymbol{V_{max} = 150}$ **V.** Calculate the average power delivered to the series RLC circuit.

<u>**Solution**</u>:

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = \frac{150V}{\sqrt{2}} = 106 \text{ V}$$

$$\Delta I_{rms} = \frac{\Delta I_{max}}{\sqrt{2}} = \frac{0.292\, A}{\sqrt{2}} = 0.206 \text{ A}$$

$$P_{av} = I_{rms}\, \Delta V_{rms}\, cos\theta$$

$$\theta = -34^{\circ} \quad \Rightarrow \quad cos - 34^{\circ} = 0.829$$

$$P_{av} = (0.206\, A)(106\, V)(0.829) = 18.1 \text{ W.}$$

Chapter Six

Phasors in AC Circuits

6. Phasors

6.1. Introduction

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than **sine** and **cosine** functions.

A **phasor** is a complex number that represents the amplitude and phase of a sinusoid.

Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources; solutions of such circuits would be intractable otherwise. The notion of solving ac circuits using phasors was first introduced by Charles Steinmetz in 1893.

A complex number z can be written in rectangular form as:

$$z = x + jy$$

where $j = \sqrt{-1}$, x is real part of z; y is the imaginary part of z.

$j^2 = -1$

The complex number z can be written in **polar** or **exponential** form us:

$z = r\angle\phi = re^{j\phi}$

where r is the magnitude of z, and ϕ is the phase of z. The z can be represented in three ways:

$z = x + jy$ **Rectangular form**

$z = r\angle\phi$ **Polar form**

$z = re^{j\phi}$ **Exponential form**

The relationship between the rectangular form and the polar form is shown in Figure below, where x axis represents the real part and the y axis represents the imaginary part of a complex number. Given x and y, we can get r and ϕ.

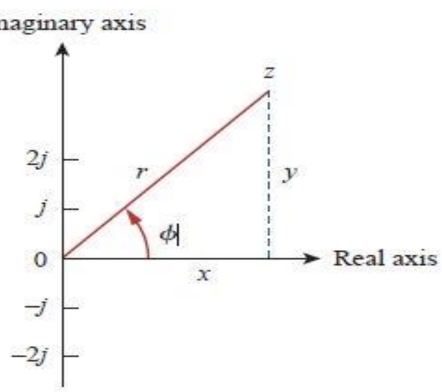
$r = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}\frac{y}{x}$

if we know r and ϕ, we can obtain x and y, as:

$x = r\ \cos\phi$, $y = r\ \sin\phi$

Thus, z may be written as:

$z = x + jy = r\angle\phi = r(\cos\phi + j\ \sin\phi)$



6.2. Phasors Operations

The addition and subtraction of complex numbers are better performed in rectangular form: multiplication and division are better done in polar form. Given the complex numbers:

$z = x + jy = r\angle\phi$, $z_1 = x_1 + jy_1 = r_1\angle\phi_1$, $z_2 = x_2 + jy_2 = r_2\angle\phi_2$

The following operations are important:

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication:

$$z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_1$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \phi_1 - \phi_1$$

Reciprocal:

$$\frac{1}{z} = \frac{1}{r} \angle -\phi$$

Square Root:

$$\sqrt{z} = \sqrt{r} \angle \phi/2$$

Complex Conjugate:

$$z^* = x - jy = r \angle -\phi = re^{-j\phi}$$

From **Reciprocal** Equation, we get:

$$\frac{1}{j} = -j$$

Note: $j = \angle 90^{\circ}$

The idea of phasor representation is based on Euler's identity. In general:

$$re^{\pm j\phi} = \cos\phi + j\sin\phi$$

which shows that we may regard $\cos\phi$ and $\sin\phi$ as the **real** and **imaginary** parts of $e^{j\phi}$; we may write:

$$\cos\phi = Re(e^{j\phi})$$

$$\sin\phi = Im(e^{j\phi})$$

For given sinusoid

$$v(t) = V_m \cos(\omega t + \phi) = Re\left(V_m e^{j(\omega t + \phi)}\right) \qquad \text{Or}$$

$$v(t) = Re\left(V_m e^{j\phi} e^{j\omega t}\right)$$

Thus,

$$v(t) = Re(V e^{j\omega t})$$

where

$$V = V_m e^{j\phi} = V_m \angle \phi$$

V is thus the ***phasor representation*** of sinusoid $v(t)$.

For example, phasors $V = V_m \angle \phi$ and $I = I_m \angle -\phi$ are graphically represented in Fig. below. Such graphical representation of phasors is known as phasor diagram.

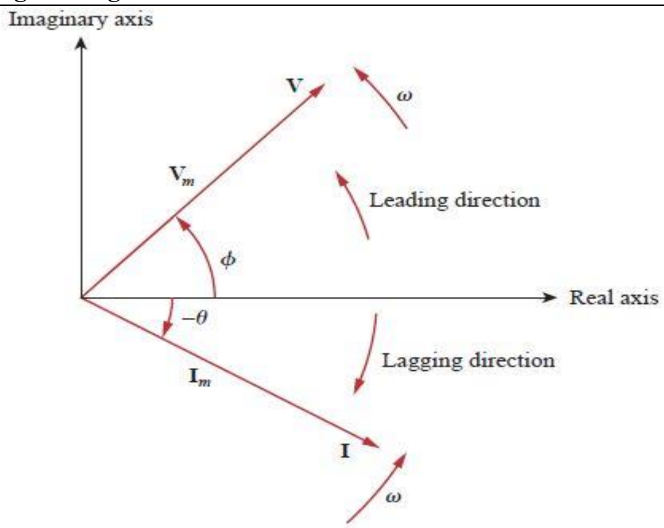


Fig. A phasor diagram showing $V = V_m\angle\phi$ and $I = I_m\angle-\phi$.

The transform of sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$\boldsymbol{v(t) = V_m\cos(\omega t+\phi)} \quad \Leftrightarrow \quad \boldsymbol{V = V_m\angle\phi}$$

Table. Sinusoid-phasor transformation.

Time domain representation	Phasor domain representation
$V_m\cos(\omega t+\phi)$	$V_m\angle\phi$
$V_m\sin(\omega t+\phi)$	$V_m\angle\phi-90^\circ$
$I_m\cos(\omega t+\phi)$	$I_m\angle\phi$
$I_m\sin(\omega t+\phi)$	$I_m\angle\phi-90^\circ$

The frequency is not explicitly (or time) factor $e^{j\phi}$ is supposed to and frequently is not shown in the phasor domain representation because ω is constant. However, the response depends on ω. For this reason, the phasor domain is also known as the ***frequency domain***. Since $v(t) = Re(Ve^{j\omega t}) = V_m\cos(\omega t+\phi)$, so that

$$\frac{dv}{dt} = -\omega V_m\sin(\omega t+\phi) = \omega V_m\cos(\omega t+\phi+90^\circ)$$

$$Re\left(V_m e^{j\omega t}e^{j\phi}e^{j90^\circ}\right) = Re(j\omega V e^{j\omega t})$$

This shows that the derivative $v(t)$ is transformed into the phasor domain as $j\omega V$,

$$\frac{dv}{dt} \quad \Leftrightarrow \quad j\omega V$$

(Time domain) **(Phasor domain)**

Similarly, the integral of $v(t)$ is transformed to the phasor domain as $V/j\omega$,

$$\int v\,dt \quad \Leftrightarrow \quad \frac{V}{j\omega}$$

(Time domain) **(Phasor domain)**

The previous equations of derivative and integral are useful in finding the steady-state solution, which does not require knowing the initial values of the variable involved. This is one of the important applications of phasors.

The difference between $\boldsymbol{v}(\boldsymbol{t})$ and $\boldsymbol{V}$ should be emphasized:

1. $\boldsymbol{v}(\boldsymbol{t})$ is the **instantaneous or time domain** representation, while $\boldsymbol{V}$ is the **frequency or phasor domain** representation.
2. $\boldsymbol{v}(\boldsymbol{t})$ is time dependent, while V is not.
3. $\boldsymbol{v}(\boldsymbol{t})$ is always real with no complex terms, while $\boldsymbol{V}$ is generally complex.

Example: Evaluate these complex numbers:

(a) $(40\angle 50^{\circ} + 20\angle -30^{\circ})^{1/2}$

(b) $\frac{10\angle -30^{\circ} + (3-j4)}{(2+j4)(3-j5)^{*}}$

Solution:

(a) Using polar to rectangular transformation,

$40\angle 50^{\circ} = 40(\cos 50^{\circ} + j\sin 50^{\circ}) = 25.71 + j30.64$

$20\angle -30^{\circ} = 20[\cos(-30^{\circ}) + j\sin(-30^{\circ})] = 17.32 - j10$

Adding them up gives

$40\angle 50^{\circ} + 20\angle -30^{\circ} = 43.03 + j20.64 = 47.72\angle 25.63^{\circ}$

$(40\angle 50^{\circ} + 20\angle -30^{\circ})^{1/2} = 6.91\angle 12.81^{\circ}$

(b) Using polar to rectangular transformation, addition, multiplication, and division,

$\frac{10\angle -30^{\circ} + (3-j4)}{(2+j4)(3-j5)^{*}} = \frac{8.66-j5+(3-j4)}{(2+j4)(3+j5)} = \frac{11.66-j9}{-14+j22} = \frac{14.73\angle -37.66^{\circ}}{26.08\angle 122.47^{\circ}}$

$= 0.565\angle -160.13^{\circ}$

Practice Example: Evaluate the following complex numbers:

(a) $[(5+j2)(-1+j4) - 5\angle 60^{\circ}]^{*}$

(b) $\frac{10+j5+3\angle 40^{\circ}}{-3+j4} + 10\angle 30^{\circ} + j5$

Answer: (a) -15.5- j 13.67, (b) 8.293+ j7.2

Example: Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40^{\circ})$ A

(b) $v = -4\sin(30t + 50^{\circ})$ V

Solution:

(a) $i = 6\cos(50t - 40^{\circ})$ has the phasor, so $I = 6\angle -40^{\circ}$ A

(b) Since $-\sin A = \cos(A + 90^{\circ})$,

$v = -4\sin(30t + 50^{\circ}) = 4\cos(30t + 50^{\circ} + 90^{\circ}) = 4\cos(30t + 140^{\circ})$ V

The phasor $V = 4\angle 140^{\circ}$ V

Practice Example: Transform these sinusoids to phasors:

(a) $v = 7\cos(2t + 40^{\circ})$ V

(b) $i = -4\sin(10t - 10^{\circ})$ A

Answer: (a) $= 7\angle 40^{\circ}$ V, (b) $I = 4\angle 100^{\circ}$ A

Example: find the sinusoids represented by these phasors:

(a) $I = -3 + j4$ A

(b) $V = j8e^{-j20^{\circ}}$ V

Solution:

(a) $I = -3 + j4 = \sqrt{(-3)^2 + (4)^2} = 5$, $\theta = \tan^{-1}(\frac{4}{-3}) = -53.123^{\circ}$,

$\theta = 180^{\circ} - 53.123^{\circ} = 126.87^{\circ}$,

$5\angle 126.87^{\circ}$ transforming this into the time domain gives,

$i(t) = 5\cos(\omega t + 126.87^{\circ})$ A

(b) $j = 1\angle 90^{\circ}$

$V = j8\angle -20^{\circ} = (1\angle 90^{\circ})(8\angle -20^{\circ}) = 8\angle 70^{\circ}$ V, converting to time domain

$v(t) = 8\cos(\omega t + 70^{\circ})$ V

Practice Example: Find the sinusoids corresponding to these phasors:

(a) $V = -10\angle 30^{\circ}$ V

(b) $I = j(5 - j12)$ A

Answer: (a) $v(t) = 10\cos(\omega t + 210^{\circ})$ V or $v(t) = 10\cos(\omega t - 150^{\circ})$ V.

(b) $i(t) = 13\cos(\omega t + 22.62^{\circ})$ A.

Example: Given $i_1(t) = 4\cos(\omega t + 30^{\circ})$ A and $i_2(t) = 5\sin(\omega t - 20^{\circ})$ A find their sum.

Solution: Using phasors for summing sinusoids of the same frequency.

$I_1 = 4\angle 30^{\circ}$.

The $i_2(t)$ in cosine form, the rule for converting sine to cosine is to subtract 90°, so

$i_2(t) = 5\cos(\omega t - 20^{\circ} - 90^{\circ}) = 5\cos(\omega t - 110^{\circ}) = 5\angle -110^{\circ}$.

$I = I_1 + I_2 = 4\angle 30^{\circ} + 5\angle -110^{\circ}$

$3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698$

$= 3.218\angle -56.97^{\circ}$ A.

Transforming this to time domain,

$i(t) = 3.218\cos(\omega t - 56.97^{\circ})$ A

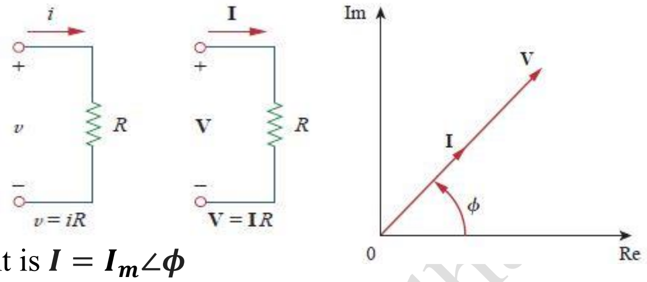
Practice Example: If $v_1 = -10\sin(\omega t - 30^{\circ})$ V and $v_2 = 20\sin(\omega t - 45^{\circ})$ V, find $v = v_1 + v_2$

Answer: $v(t) = 12.158\cos(\omega t + 55.95^{\circ})$ V.

6.3. Phasor Relationships for Circuit Elements

One may legitimately ask how we apply this to circuits involving the passive elements *R, L*, and *C*. What we need to do is transform the voltage-current relationship from the time domain to the frequency domain for each element.

If the current through a resistor R is $\boldsymbol{i = I_m \cos(\omega t + \phi)}$, the voltage across it is given by Ohm's law as:



$$\boldsymbol{v = iR = RI_m \cos(\omega t + \phi)}$$

The phasor form of this voltage is

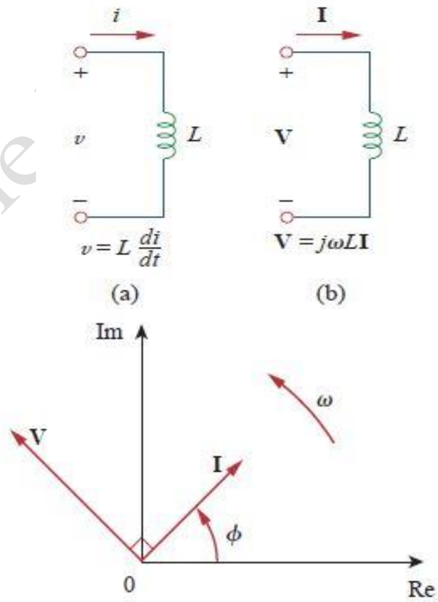
$$\boldsymbol{V = RI_m \angle \phi}$$

But the phasor representation of the current is $\boldsymbol{I = I_m \angle \phi}$

$$\boldsymbol{V = RI}$$

For the indicator L, assume the current through it is

$\boldsymbol{i = I_m \cos(\omega t + \phi)}$. The voltage across the indicator is



$$\boldsymbol{v = L\frac{di}{dt} = -\omega L I_m \cos(\omega t + \phi)},$$

Since $-\sin = A = \cos(A + 90^{\circ})$, the voltage is

$$\boldsymbol{v = \omega L I_m \sin(\omega t + \phi + 90^{\circ})}$$

Which transforms to the phasor

$$\boldsymbol{V = \omega L I_m e^{j(\phi + 90^{\circ})} = \omega L I_m e^{j\phi} e^{j90^{\circ}} = \omega L I_m \angle \phi + 90^{\circ}}$$

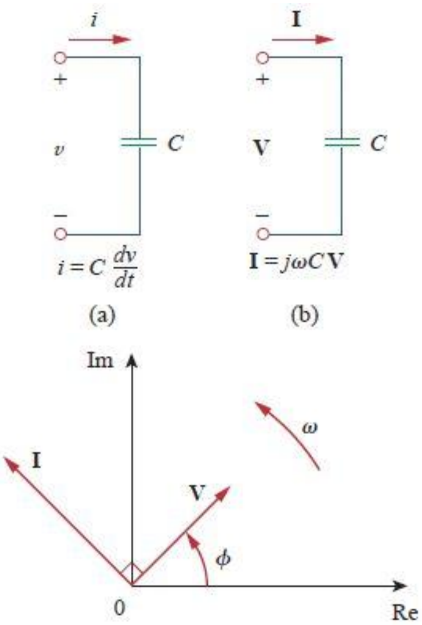
But $\boldsymbol{I_m \angle \phi} = I$, and $\boldsymbol{e^{j90^{\circ}}} = \boldsymbol{j}$. Thus,

$$\boldsymbol{V = j\omega L I}$$

For capacitor C, the voltage across it is $\boldsymbol{v = V_m \cos(\omega t + \phi)}$. Current through C is

$\boldsymbol{i = C\frac{dv}{dt}}$ **Thus**, the same previous steps, we can obtain,

$$\boldsymbol{I = j\omega C V} \quad \Longrightarrow \quad \boldsymbol{V = \frac{I}{j\omega C}}$$



Summary of voltage-current relationships

Element	Time domain	Frequency domain
$\boldsymbol{R}$	$\boldsymbol{v = Ri}$	$\boldsymbol{V = RI}$
$\boldsymbol{L}$	$\boldsymbol{v = L\frac{di}{dt}}$	$\boldsymbol{V = j\omega L I}$
$\boldsymbol{C}$	$\boldsymbol{i = C\frac{dv}{dt}}$	$\boldsymbol{V = \frac{I}{j\omega C}}$

Example: the voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H indicator. Find the steady-state current through the inductor.

Solution: for the inductor, $V = j\omega LI$, where ω=60 rad/s and V=12∠45° V.

$$I = \frac{V}{j\omega L} = \frac{12\angle 45°}{j60\times 0.1} = \frac{12\angle 45°}{6\angle 90°} = 2\angle -45°\text{A}$$

Convert this to the time domain,

$i(t) = 2\cos(60t - 45°)$ A.

Example: the voltage $v = 10\cos(100t + 30°)$ is applied to a 50 μF capacitor. Find the current through the capacitor.

Answer: $50\cos(100t - 120°)$ mA.

6.4. Impedance and Admittance

The voltage-current relations for the three passive elements as:

$$\boldsymbol{V = RI}, \qquad \boldsymbol{V = j\omega LI}, \qquad \boldsymbol{V = \frac{I}{J\omega C}}$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as:

$$\boldsymbol{Z = \frac{V}{I}} \qquad \text{Or} \qquad \boldsymbol{V = ZI}$$

where Z is a frequency-dependent quantity known as ***impedance***, measured in ohms.

The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms (Ω).

Table. Impedances and admittances of passive elements

Element	Impedance	Admittance
R	$Z = R$	$Y = \frac{1}{R}$
L	$Z = j\omega L$	$Y = 1/j\omega L$
C	$Z = 1/j\omega C$	$Y = j\omega C$

As a complex quantity, the impedance may be expressed in rectangular form as:

$Z = R + jX$

where $R = Re\, Z$ is the resistance and $X = I_m\, Z$ is the reactance. The reactance X may be positive or negative. We say that the impedance is inductive when X is positive or capacitive when X is negative. The impedance $Z = R + jX$ is said to be ***inductive*** or ***lagging*** since current lags voltage, while impedance $Z = R - jX$ is ***capacitive*** or

leading because current leads voltage. The impedance, resistance, and reactance are all measured in ohms. The impedance may also be expressed in **polar** form as:

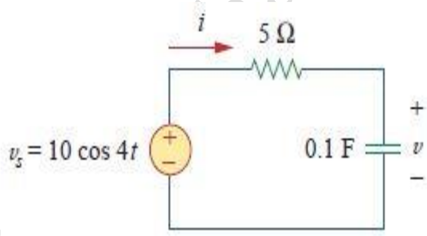
$$\boldsymbol{Z = |Z|\angle\phi}$$

We infer that $\boldsymbol{Z = RjX = |Z|\angle\phi}$

where $\boldsymbol{|Z| = \sqrt{R^2 + X^2}}$, $\quad \boldsymbol{\phi = \tan^{-1}\frac{X}{R}}$

and $\boldsymbol{R = |Z|\cos\phi}$, $\quad \boldsymbol{X = |Z|\sin\phi}$

<u>Example:</u> Find $v(t)$ and $i(t)$ in the circuit shown below:



<u>Solution:</u> from the voltage source $V_s = 10\cos 4t$, $\omega = 4$,

$V_s = 10\angle 0^\circ$ V

The impedance is $Z = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4\times 0.1} = 5 - j2.5\ \Omega$

Hence the current $I = \frac{V_s}{Z} = \frac{10\angle 0^\circ}{5-j2.5} = \frac{10(5+j2.5)}{5^2+2.5^2} = \frac{50}{31.25} + j\frac{25}{31.25}$

Magnitude: $|I| = \sqrt{1.6^2 + 0.8^2} = 1.789$, Angle $\phi = \tan^{-1}\frac{0.8}{1.6} = 26.57^\circ$

$= 1.6 + j0.8 = 1.789 + \angle 26.57^\circ$ A

The voltage across the capacitor is

$j = \angle 90^\circ$

$V = IZ_C = \frac{I}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4\times 0.1} = \frac{1.789\angle 26.57^\circ}{0.4\times\angle 90^\circ} = \frac{1.789}{0.4}\angle 26.57^\circ - \angle 90^\circ =$
$4.47\angle{-63.43^\circ}$V

Converting I and V to the time domain, we get:

$i(t) = 1.789\cos(4t + 26.57^\circ)$ A

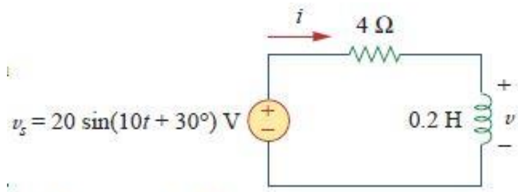
$v(t) = 4.47\cos(4t - 26.43^\circ)$ V

Notice that $i(t)$ leads $v(t)$ by 90° as expected.

<u>Practice Example:</u> Find $v(t)$ and $i(t)$ in the circuit shown below:

<u>Answer:</u> $8.944\sin(10t + 93.43^\circ)$ V

$4.472\sin(10t + 3.43^\circ)$ A.



6.5. Kirchhoff's Laws in the Frequency Domain

For KVL, let $\boldsymbol{v_1, v_2, \ldots, v_n = 0}$ be the voltage around a closed loop.

In the sinusoidal steady state, each voltage may be written in cosine form, as:

$$\boldsymbol{V_{m1}\cos(\omega t + \phi_1) + V_{m2}\cos(\omega t + \phi_2) + \cdots + V_{mn}\cos(\omega t + \phi_n)} = 0$$

This can be written as:

$$\boldsymbol{Re(V_{m1}e^{j\theta_1}e^{j\omega t}) + Re(V_{m2}e^{j\theta_2}e^{j\omega t}) + \cdots + Re(V_{mn}e^{j\theta_n}e^{j\omega t})} = 0$$

Or

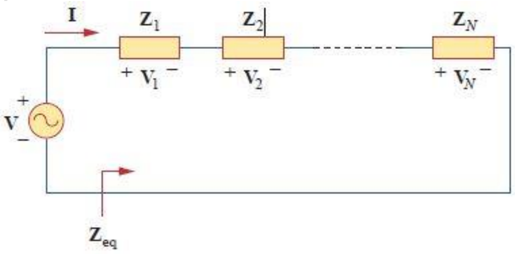
$$\boldsymbol{Re[(V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2} + \cdots + V_{mn}e^{j\theta_n})e^{j\omega t}] = 0}$$

Similar to the **current** use, **KCL** is held in the frequency domain.

6.6. Impedance Combinations

Consider the N series-connected impedances shown in figure below. The same current flows through the impedance. Applying KVL around the loop gives,

$$\boldsymbol{V = V_1, V_2, \ldots, V_N = I(Z_1, Z_2, \ldots, Z_N)}$$

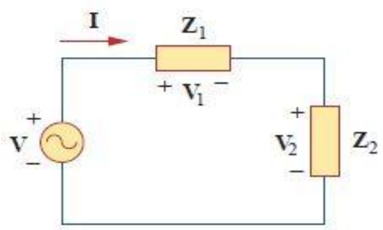


The equivalent impedance at the input terminals is

$$\boldsymbol{Z_{eq} = \frac{V}{I} = Z_1, Z_2, \ldots, Z_N}$$

If $N = 2$, as shown in the figure below, the current through the impedances is,

$\boldsymbol{I = \frac{V}{Z_1 + Z_2}}$, since $\boldsymbol{V_1 = Z_1 I}$ and $\boldsymbol{V_2 = Z_2 I}$**,** then

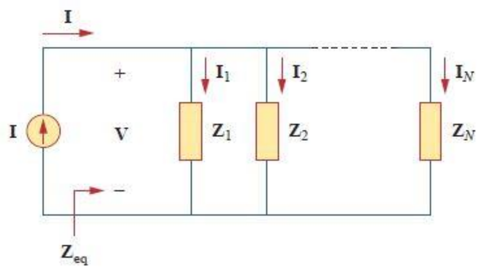


$$\boldsymbol{V_1} = \frac{\boldsymbol{Z_1}}{\boldsymbol{Z_1 + Z_2}}V, \qquad \boldsymbol{V_2} = \frac{\boldsymbol{Z_2}}{\boldsymbol{Z_1 + Z_2}}V$$

Which is the ***voltage-division*** relationship

Similar for $\boldsymbol{N}$ parallel-connected impedances shown in figure below. Applying KCL at the top node,

$$\boldsymbol{I = I_1 + I_2 + \cdots + I_N = I(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N})}$$

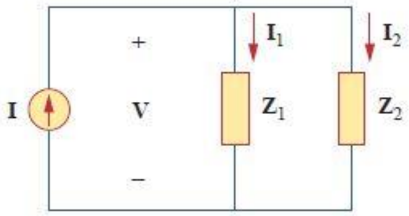


The equivalent impedance is

$$\boldsymbol{\frac{1}{Z_{eq}} = \frac{1}{V} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}}$$

When $N = 2$, as shown in figure below, the equivalent impedance becomes,

$$\boldsymbol{Z_{eq}} = \frac{1}{Y_{eq}} = \frac{1}{Y_1+Y_2} = \frac{1}{\frac{1}{Z_1}+\frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1+Z_2}$$



Also, since

$$\boldsymbol{V = IZ_{eq} = I_1 Z_1 = I_2 Z_2}$$

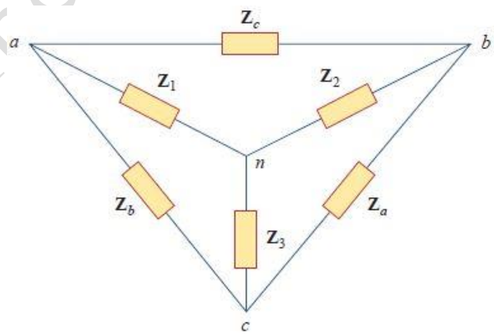
The current in the impedances is:

$$\boldsymbol{I_1} = \frac{Z_2}{Z_1+Z_2}\boldsymbol{I}, \qquad \boldsymbol{I_2} = \frac{Z_1}{Z_1+Z_2}$$

Which is the ***current-division*** principle.

The delta-to-wye and wye-to-delta transformations that we applied to resistance circuits are also valid for impedances. With reference to figure below, the conversion formulas are as follows:

***Y-Δ conversion*:**



$$\boldsymbol{Z_a} = \frac{Z_1 Z_2+Z_2 Z_3+Z_3 Z_1}{Z_1}$$

$$\boldsymbol{Z_b} = \frac{Z_1 Z_2+Z_2 Z_3+Z_3 Z_1}{Z_2}$$

$$\boldsymbol{Z_c} = \frac{Z_1 Z_2+Z_2 Z_3+Z_3 Z_1}{Z_3}$$

***Δ- Y conversion*:**

$$\boldsymbol{Z_1} = \frac{Z_b Z_c}{Z_a+Z_b+Z_c}$$

$$\boldsymbol{Z_2} = \frac{Z_c Z_c}{Z_a+Z_b+Z_c}$$

$$\boldsymbol{Z_3} = \frac{Z_a Z_b}{Z_a+Z_b+Z_c}$$

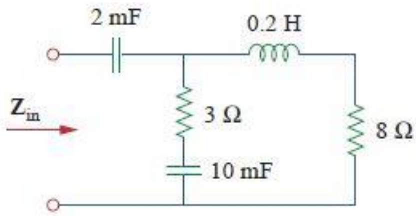
A ***delta*** or ***wye*** circuit is said to be ***balanced*** if it has equal impedances in all three branches.

When a ***Δ- Y*** is balanced, the previous equations become:

$$\boldsymbol{Z_\Delta = 3Z_Y} \qquad \textbf{Or } \boldsymbol{Z_Y = \frac{1}{3}Z_\Delta}$$

where $\boldsymbol{Z_Y = Z_1 = Z_2 = Z_3}$ and $\boldsymbol{Z_\Delta = Z_a = Z_b = Z_c}$.

Example: Find the input impedance of the circuit in figure below at $\omega = 50$ rad/s.



Solution: Let,

Z_1= Impedance of 2-mF capacitors

Z_2= Impedance of the 3 Ω resistor in series 10 mF

Z_3= Impedance of the 0.2 H inductor in series with 8 Ω resistor.

$$\boldsymbol{Z_1} = \frac{1}{j\omega C} = \frac{1}{j50\times2\times10^{-3}} = -j10\ \Omega$$

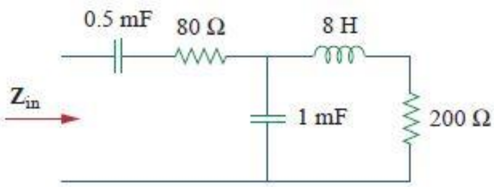
$$\boldsymbol{Z_2} = \boldsymbol{3} + \frac{1}{j\omega C} = 3 + \frac{1}{j50\times10\times10^{-3}} = (3 - j2)\ \Omega$$

$$\boldsymbol{Z_3} = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

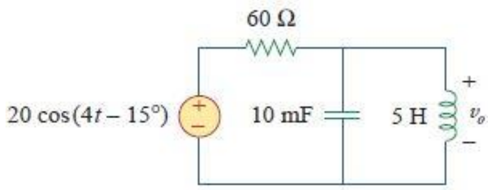
Input impedance is $\boldsymbol{Z_{in}} = Z_1 + Z_2||Z_3 = -j10 + \frac{(3-j2)(8+j10)}{11+j8} = 3.22 - j11.07\ \Omega$

Example: Find the input impedance of the circuit in figure below at $\omega = 50$ rad/s.

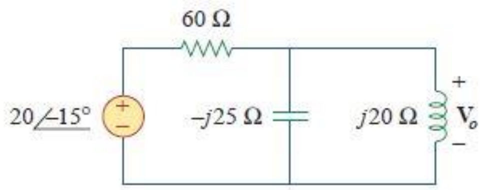
Answer: $129.52 - j295$



Example: Determine the $v_o(t)$ in the circuit of figure below.



Solution: First, transform the time domain to frequency domain, as shown in figure below.



$v_s = 20\cos(4t - 15^\circ) \implies v_s = 20\angle{-15^\circ}$ V, $\omega = 4$

$$10\text{ mF} \implies \frac{1}{j\omega C} = \frac{1}{j4\times10\times\times10^{-3}} = -j25\ \Omega$$

$$5\text{ H} \implies j\omega C = j4 \times 5 = j20\ \Omega$$

Z_1= Impedance of the 60 Ω resistor

Z_2= Impedance of the parallel combination of the 10-mF capacitor and the 5 H inductor.

Z_1= 60 Ω

$$Z_2 = -j25\ ||\ j20 = \frac{-j25\times j20}{-j25+j20} = j100\ \Omega$$

Using voltage divider principle,

$$V_o = \frac{Z_2}{Z_1+Z_2}V_s = \frac{j100}{60+j100}(20\angle{-15^\circ}) = (0.8575\angle 30.96^\circ)(20\angle{-15^\circ}) = 17.1\angle 15.96^\circ$$

We convert this to the time domain and obtain

$v_o(t) = 17.15\cos(4t - 15.96^\circ)$ V.

Explain transformation

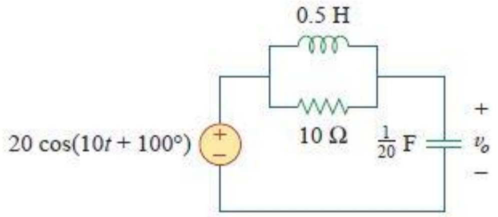
$60 + j100 = \sqrt{60^2 + 100^2} = \sqrt{13600} = 116.62$ Magnitude

$\tan^{-1}\frac{100}{60} = 59.04^{\circ}$ Phase

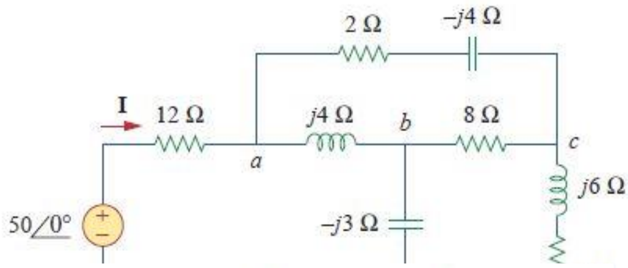
So, $\frac{j100}{60+j100} = \frac{100\angle 90^{\circ}}{116.62\angle 59.04^{\circ}} = 0.8575\angle 30.96^{\circ}$

Practice Example: Determine the $v_o(t)$ in the circuit of figure below.

Answer: $v_o(t) = 14.142\cos(10t - 35^{\circ})$.



Example: Find current I in the circuit of figure below.



Solution:

The delta network connected to nodes:

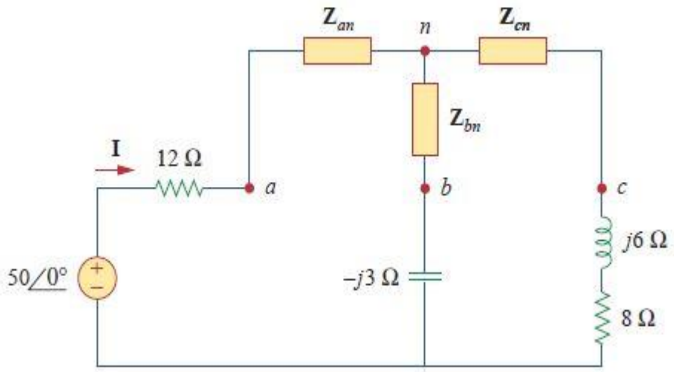
a, b, and c can be converted to the Y network as shown in the figure below. Computes Y as

Δ-Y conversion:

$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$

$Z_2 = \frac{Z_c Z_c}{Z_a + Z_b + Z_c}$

$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$



$Z_{an} = \frac{j4(2-j4)}{j4+2-j4+8} = \frac{4(4+2j)}{10} = 1.6 + j0.8\ \Omega$

$Z_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega$, $Z_{cn} = \frac{8(2-j4)}{10} = (1.6 - j3.2)\ \Omega$

The total imepdance at the source terminal is

$Z = 12 + Z_{an} + (Z_{bn} - j3)||(Z_{cn} + j6 + 8)$

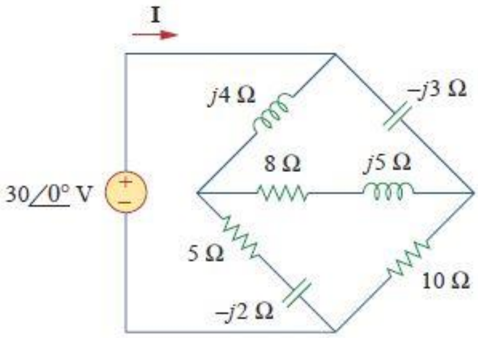
$= 12 + 1.6 + j0.8 + (j0.2)||(9.6 + j2.8) \implies 13.6 + j0.8 + \frac{j0.2(9.6+j2.8)}{9.6+j3}$

$= 13.6 + j1 = 13.64\angle 4.204^{\circ}$

The desired current is

$I = \frac{V}{Z} = \frac{50\angle 0^{\circ}}{13.64\angle 4.204^{\circ}} = 3.666\angle -4.204^{\circ}$.

Practice Example: Determine the I in the circuit of figure below.



Answer: $6.364\angle 3.8^{\circ}$

Chapter Seven

Network Theorem in AC

7. Sinusoidal Steady State Analysis

7.1. Introduction

We want to see how the mesh analysis, nodal analysis, Thevenin's theorem, Norton's theorem, and source transformations are applied in analyzing AC circuit.

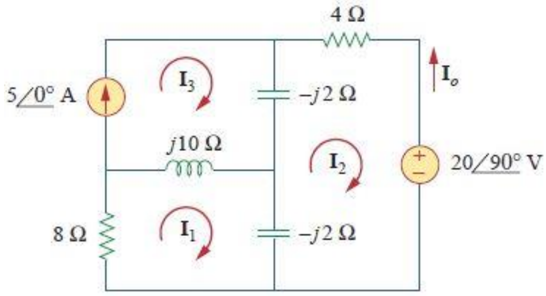
Analyzing AC circuits usually requires three steps:

1. Transform the circuit to phasor or frequency domain.
2. Solving problems using techniques (mesh analysis, nodal analysis, superposition, etc.)
3. Transform the resulting phasor to the time domain.

7.2. Mesh Analysis

Kirchhoff's voltage law (KVL) forms the basis of mesh analysis.

Example: Determine current I_o in the circuit of Fig. below using mesh analysis.



Solution: Applying KVL to mesh 1, we obtain,

$(8+j10-j2)I_1-(-j2)I_2-j10I_3=0$ Eq. (1)

For mesh 2,

$(4-j2-j2)I_2-(-j2)I_1-(-j2)I_3+20\angle 90^{\circ}=0$ **Eq. (2)**

Note: $20\angle 90^{\circ}=j20$

For mesh 3, $I_3=5$. Substituting this in previous equations, as follows:

$(8+j8)I_1+j2I_2=j50$ **Eq. (3)**

$j2I_1+(4-j4)I_2=-j20-j10$ **Eq. (4)**

From Eqs. (3) and (4) can be put in matrix form as:

$$\begin{bmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}=\begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

$$\Delta=\begin{bmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{bmatrix}=34(1+j)(1-j)+4=68$$

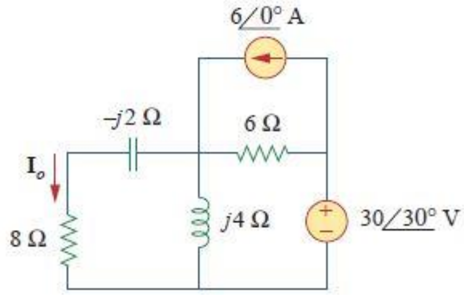
$$\Delta_2=\begin{bmatrix} 8+j8 & j50 \\ j2 & -j30 \end{bmatrix}=340-j240=416.17\angle -35.22^{\circ}\text{ A}$$

$$\mathrm{I}_2=\frac{\Delta_2}{\Delta}=\frac{416.17\angle -35.22^{\circ}}{68}=6.12\angle -35.22^{\circ}\text{ A}$$

$$\mathrm{I}_0=-\mathrm{I}_2=416.17\angle 144.78^{\circ}\text{ A}$$

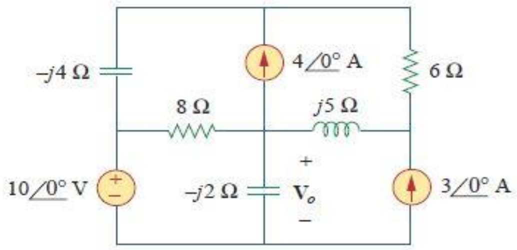
Because multiplying by −1 in phasor form adds 180° to the angle.

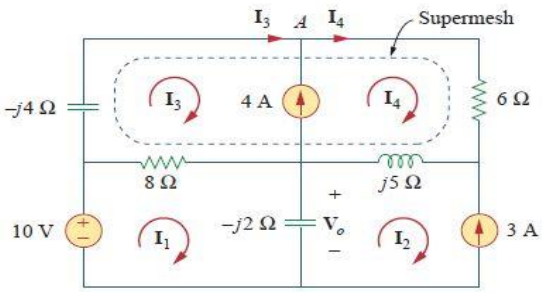
Practice Example: Find I_o using mesh analysis.



Answer: $3.582\angle 65.45^{\circ}$

Example: Solve for V_o in the circuit of Fig. below using mesh analysis.





Solution: From figure below, a supermesh between Meshes 3 and 4 due to the current source. KVL gives,

For mesh 1,

$$-10 + (8 - j2)I_1 - (-j2)I_2 - 8I_3 = 0$$

Simplified as:

$$(8 - j2)I_1 + j2I_2 - 8I_3 = 10 \qquad \textbf{E1. (1)}$$

For mesh 2,

$$I_2 = -3 \qquad \textbf{Eq. (2)}$$

For supermesh,

$$(8 - j2)I_3 - 8I_1 + (6 + j5)I_4 - j5I_2 = 0 \qquad \textbf{Eq. (3)}$$

In the supermesh 3 and 4:

$$I_4 = I_3 + 4 \qquad \textbf{Eq. (4)}$$

Combining Eq. (1) and (2),

$$(8 - j2)I_1 - 8I_3 = 10 + j6 \qquad \textbf{Eq. (5)}$$

Combining Eqs (2), (3), and (4):

$$-8I_1 + (8 - j2)I_3 + 24 + j20 + 6I_3 + (j5)I_3 + j15 = 0$$

$$-8I_1 + (14 + j3)I_3 = -24 - j35 \qquad \textbf{Eq. (6)}$$

From Eqs. (5) and (6) can be put in matrix form as:

$$\begin{bmatrix} 8 - j2 & -8 \\ -8 & 14 + j3 \end{bmatrix}\begin{bmatrix} I_1 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 + j6 \\ -24 - j35 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 8 - j2 & -8 \\ -8 & 14 + j3 \end{bmatrix} = 112 + j24 - j28 + 6 - 64 = 54 - j4 = 2932\angle 4.32^{\circ}$$

$$\Delta_1 = \begin{bmatrix} 10 + j6 & -8 \\ -24 - j35 & 14 + j3 \end{bmatrix} = 140 + j30 + j84 - 18 - 192 - j280$$

$$= -70 - j166 = 32456\angle 67.14^{\circ}$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{32456\angle 67.14^{\circ}}{2932\angle 4.32^{\circ}} = 11.08\angle 62.82^{\circ}$$

$V_0 = -j2(I_1 - I_2) = -j2(11.08\angle 62.82^{\circ} + 3)$, **Convert** $11.08\angle 62.82^{\circ}$ in rectangular form $11.08(\cos 62.82^{\circ} + j \sin 62.82^{\circ}) = 11.08(0.4536 + j0.8915)$

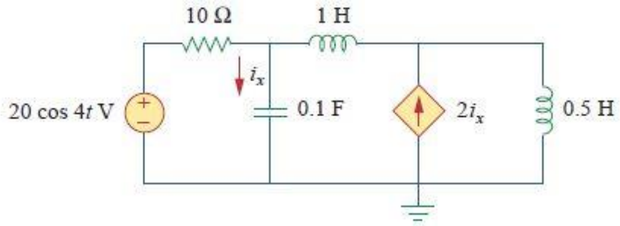
$$(5.02 + j9.89) + 3 = 8.02 + j9.89$$

$$-j2(8.02 + j9.89) = -j16.4 + 19.78 = 25.47\angle -39.2^{\circ} \text{ V}$$

7.3. Nodal Analysis

The basis of nodal analysis is Kirchhoff's current law. We can analyze AC circuits by nodal analysis. The following example illustrates this.

Example: Find i_x in the circuit of Fig. below using nodal analysis.



Solution: We first convert the circuit to the Frequency domain:

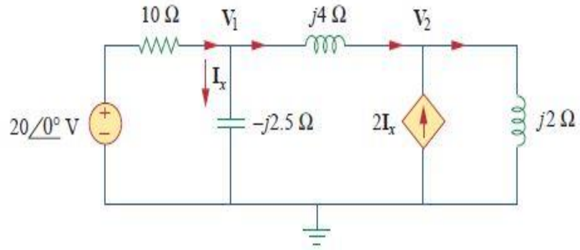
$20\cos 4t \;\Longrightarrow\; 20\angle 0^\circ,\;\omega = 4$ rad/s

$1\text{ H} \;\Longrightarrow\; j\omega L = j4$

$0.5\text{ H} \;\Longrightarrow\; j\omega L = j2$

$0.1\text{ F} \;\Longrightarrow\; \frac{1}{j\omega C} = -\frac{j}{wC} = -j2.5,\ \frac{\mathbf{1}}{\boldsymbol{j}} = -\boldsymbol{j} \;\Longrightarrow\; \boldsymbol{j(-j) = -j^2 = -(-1) = 1}$

Thus, the frequency of domain equivalent circuits is as shown below.



Applying KCL at node 1,

$\frac{20-V_1}{10} = \frac{V_1}{-j2.5} + \frac{V_1-V_2}{j4}$

$2 - 0.1V_1 = j0.4V_1 - 0.25jV_1 + 0.25jV_2$

$(0.1 + j0.15j)V_1 + j0.25V_2 = 2$ simplified $\times 10$

$(1 + j1.5)V_1 + j2.5V_2 = 20$ Eq. (1)

At node 2,

$2I_x + \frac{V_1-V_2}{j4} = \frac{V_2}{j2}$, **But** $I_x = \frac{V_1}{-j2.5}$ substituting this gives,

$\frac{2V_1}{-j2.5} + \frac{V_1-V_2}{j4} = \frac{V_2}{j2} \;\Longrightarrow\; -0.8jV_1 + 0.25j(V_1 - V_2) = 0.5jV_2 \;\Longrightarrow -0.55V_1 = 0.75V_2$

$11V_1 + 15V_2 = 0$ Eq. (2) this is $\times\, 100$ then $\div\, 5$

From Eqs. (5) and (6) can be put in matrix form as:

$$\begin{bmatrix} 1+j1.5 & j2.5 \\ 11 & 15 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 1+j1.5 & j2.5 \\ 11 & 15 \end{bmatrix} = 15 - j5$$

$$\Delta_1 = \begin{bmatrix} 20 & j2.5 \\ 0 & 15 \end{bmatrix} = 300\ ,\ \Delta_2 = \begin{bmatrix} 1+j1.5 & 20 \\ 11 & 0 \end{bmatrix} = -220$$

$$V_1 = \frac{\Delta_1}{\Delta} = \frac{300}{15-j5} = 18.97\angle 18.43^\circ \text{ V}$$

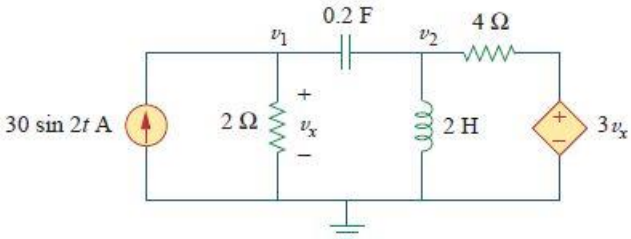
$$V_2 = \frac{\Delta_2}{\Delta} = \frac{-220}{15-j5} = 13.91\angle 198.3^\circ \text{ V}$$

The current I_x is given by $I_x = \frac{V_1}{-j2.5} = \frac{18.97\angle 18.43^\circ}{2.5\angle -90^\circ} = 7.59\angle 108.4^\circ$

Convert to time domain

$I_x = 7.59\cos(4t + 108.4^\circ)$ A

Example: Using nodal analysis, find v_1 and v_2 in the circuit of Fig. below.



Answer:

$v_1(t) = 33.96\sin(2t + 60.01^\circ)$ V

$v_2(t) = 99.06\sin(2t + 57.12^\circ)$ V

7.4. Thevenin Analysis

Thevenin's and Norton's theorems are applied to ac circuits in the same way as they are to dc circuits. The only additional effort arises from the need to manipulate complex numbers. The frequency domain version of a **Thevenin** equivalent circuit is depicted in Fig. below, where a linear circuit is replaced by a voltage source in series with an impedance. The **Norton** equivalent circuit is illustrated in Fig. below, where a linear circuit is replaced by a current source in parallel with an impedance. Keep in mind that the two equivalent circuits are related just as in source transformation. $V_{Th} = Z_N I_N$ is the open-circuit voltage while I_N is the short-circuit current.

$\mathbf{V_{Th} = Z_N I_N}$, $\mathbf{Z_{Th} = Z_N}$

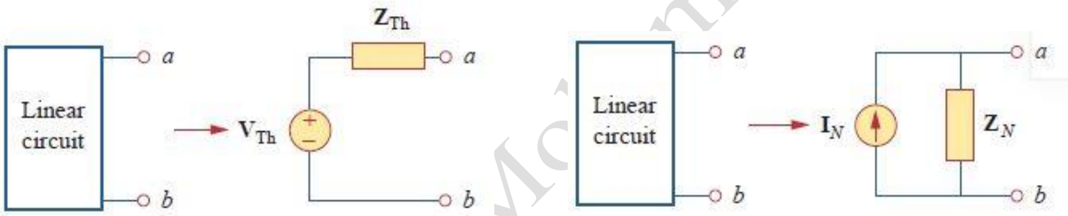
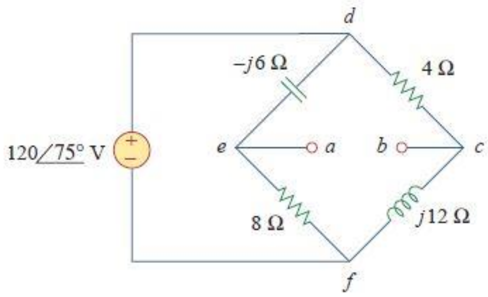


Figure. (a) Thevenin equivalent (b) Norton equivalent

If the circuit has sources operating at different frequencies, the Thevenin or Norton equivalent circuit must be determined at each frequency. This leads to entirely different equivalent circuits, one for each frequency, not one equivalent circuit with equivalent sources and equivalent impedances.

Example: Obtain the Thevenin equivalent at terminals *a-b* of the circuit below.



Solution: Find Z_{Th} by setting the voltage source to Zero, as seen in Fig. below.

$Z_1 = -j6||8 = \frac{-6j\times 8}{8-j6}$, multiply by $8 + j6$

$= \frac{-j48(8+j6)}{8^2+6^2} = \frac{-j384-288}{100}$

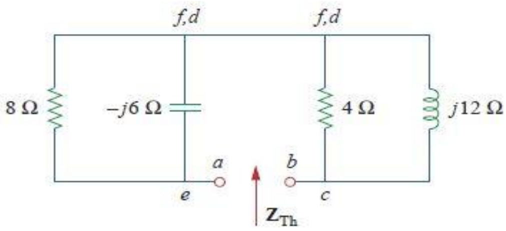
$= 2.88 - j3.84\ \Omega$

$Z_2 = 4||j12 = \frac{j12\times 4}{4+j12}$, multiply by $4 - j12$

$\frac{48j(4-j12)}{16+144} = \frac{576+j192}{160}$

$= 3.6 + j1.2\ \Omega$

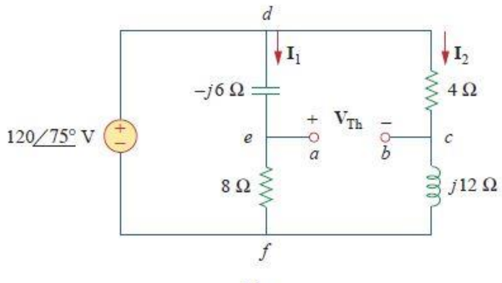
$Z_{Th} = Z_1 + Z_2 = 2.88 - j3.84 + 3.6 + j1.2$



(a)

$Z_{Th} = 6.48 - j2.64\ \Omega$

To find V_{Th}, consider the circuit below. The currents I_1 an



(b)

$I_1 = \frac{120\angle 75^\circ}{8-j6}$ A, $I_1 = \frac{120\angle 75^\circ}{4+j12}$ A

Applying KVL around loop *bcdeab* gives:

$V_{Th} - 4I_2 + (-j6)I_1 = 0$

$V_{Th} = \mathbf{4}I_2 + j6I_1$

$= \frac{480\angle 75^\circ}{4+j12} + \frac{720\angle 75^\circ + 90^\circ}{8-j6}$

$\frac{480\angle 75^\circ}{4+j12} = \frac{480\angle 75^\circ}{12.649\angle 71.565^\circ} = 37.95\angle 3.43^\circ$

$\frac{720\angle 165^\circ}{8-j6} = \frac{720\angle 165^\circ}{10\angle -36.87^\circ} = 72\angle 201.87^\circ$

$= 37.95\angle 3.43^\circ + 72\angle 201.87^\circ$

$= 37.95(\cos 3.43^\circ + j\sin 3.43^\circ) + 37.95(\cos 201.87^\circ + j\sin 201.87^\circ)$

$= 37.88 + j2.27 + (-67.02) - j26.32 = -29.14 - j24.04$, convert to polar

$r = \sqrt{(-29.14^2) + (-24.05^2)} = \sqrt{1427.5} = 37.78$

$\theta = \tan^{-1}\left(\frac{-24.05}{-29.14}\right) = 39.2^\circ$

Both parts are negative = 3rd quadrant, thus add 180°

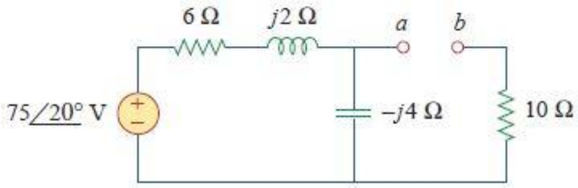
$V_{Th} = 37.78\angle 219.2^\circ$

<u>Practice Example</u>: Find the Thevenin equivalent at terminals *a*-*b* of the circuit below.

<u>Answer</u>:

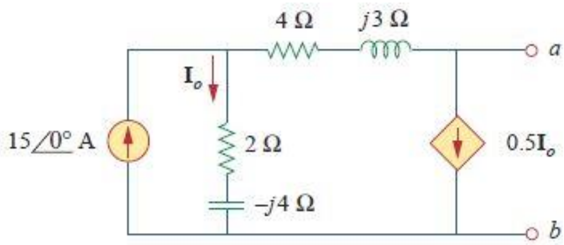
$Z_{Th} = 12.4 - j3.2\ \Omega$

$V_{Th} = 47.42\angle -51.57^\circ$



<u>Example</u>: Obtain the Thevenin equivalent at terminals *a*-*b* of the circuit below.

<u>Solution</u>:



To find V_{Th}, we apply KCL at node in Fig. below,

$15 = I_o + 0.5I_o$

$I_o = 10$ A

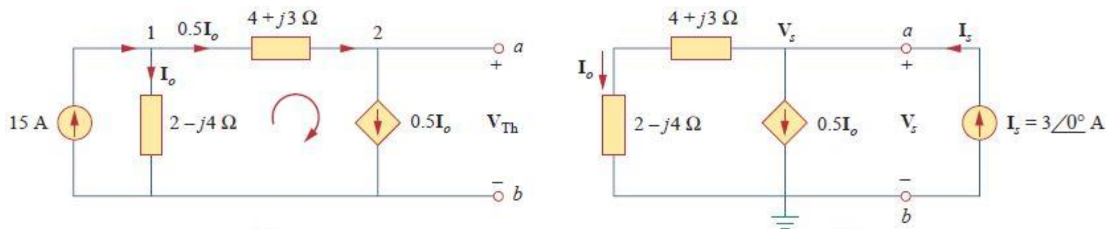
Applying KVL to the loop on the right-hand side in Fig. b, we obtain:

$-I_o = (2 - j4) + 0.5I_o(4 + 3) + V_{Th} = 0$

$V_{Th} = 10(2 - 4j) - 5(4 + 3j) = -j55$

Thus, the Thevenin voltage is

$V_{Th} = 55\angle -90^\circ$ V



(a) (b)

To obtain Z_{Th}, we remove the independent source. Due to the presence of the dependent current source. We connect a 3-A current source (3 is an arbitrary value chosen for convenience here, a number divisible by the sum of current leaving the node) to terminals *a-b* as shown in Fig. b. At node, KCL gives,

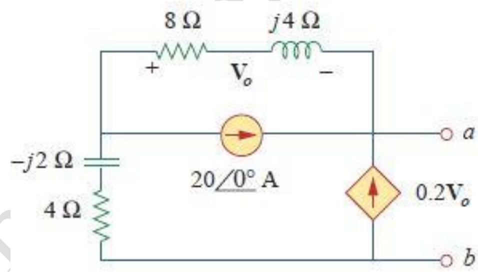
$3 = I_o + 0.5I_o$

$I_o = 2$ A

$V_s = I_o(4 + j3 + 2 - j4) = 2(6 - j)$

The Thevenin impedance is $Z_{Th} = \frac{V_s}{I_s} = \frac{2(6-j)}{3} = 4 - j0.6667\ \Omega$

Practice Example: Determine the Thevenin equivalent of the circuit in Fig. below as seen from the terminals *a-b*.



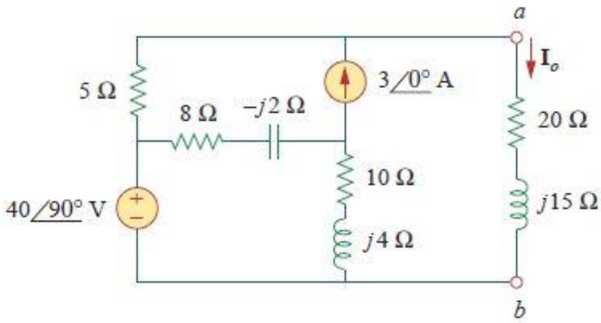
Answer:

$Z_{Th} = 4.473\angle{-7.64^\circ}\ \Omega$

$V_{Th} = 29.4\angle 72.9^\circ$ V.

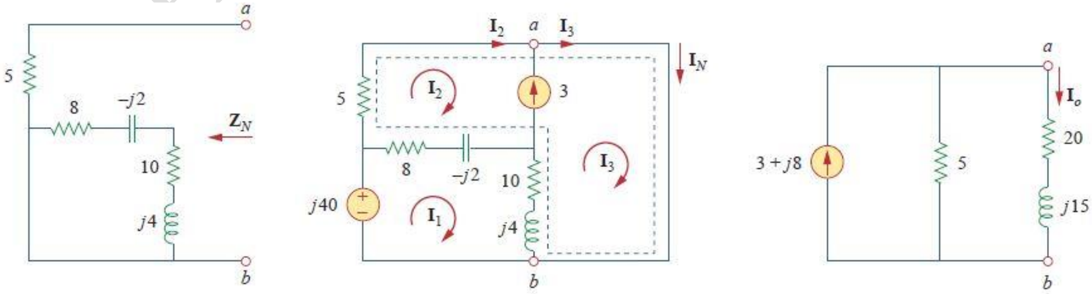
7.5. Norton Analysis

Example: Obtain current I_o in Fig. below using Norton's theorem.



Solution: Our first objective is to find the Norton equivalent at terminals a-b. Z_N is found in the same way as Z_{Th}. We set the sources to zero as shown in Fig. a. as evident from figure, the $(8 - j2)$ and $(10 + j4)$ impendences are short circuited, so that

$Z_N = 5\ \Omega$



(a) (b) (c)

To get I_N, we short-circuit to find the Norton equivalent at terminal *a-b* in the Fig. b and apply mesh analysis. Notice that meshes 2 and 3 form a supermesh because of the current source linking them. For mesh 1,

$-j40 + (18 + j2)I_1 - (8 - 2j)I_2 - (10 + j4)I_3 = 0$ Eq. 1

For supermesh,

$(13 - j2)I_2 + (10 + 4j)I_3 - (18 + j2)I_1 = 0$ Eq. 2

At node a, due to the current source between meshes 2 and 3,

$I_3 = I_2 + 3$ Eq. 3

Adding Eqs. 1 and 2 give

$-40j + 5I_2 = 0$

$I_2 = j8$

From Eq. 3,

$I_3 = I_2 + 3 = 3 + j8$

The Norton current is

$I_N = I_3 = 3 + j8$ A

Fig. c shows the Norton equivalent circuit along with the impedance at terminals *a-b*.

by current division,

$I_o = \frac{5}{5+20+j15} I_N = \frac{3+j8}{5+j3} = 1.465\angle 38.48^\circ$ A

Practice Example: Determine the Norton equivalent of the circuit in Fig. below as seen from the terminals *a-b*. Use the equivalent to find I_o.

Answer:

$Z_N = 3.176 + j0.706\ \Omega$

$I_N = 4.198\angle -32.68^\circ$ A.

$I_o = 985.5\angle -2.101^\circ$ mA.