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ADVANCED TRAFFIC ENGINEERING

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LECTURE 3:

CUMULATIVE FLOW CURVES

Function $N_x(t)$:

- It represents the number of vehicles that have passed a point x at time t.
- It only used for one traffic direction.
- It is a non-decreasing function (only increases with time since vehicles keep passing).
- Strictly, it is a step function (jumps by 1 with each passing vehicle).
- For practical/analytical purposes, it is often smoothed into a continuous function.

Mathematical Formulation

- The **rate of increase** of Nx(t) equals the **traffic flow**:

$$\frac{dN}{dt} = q \qquad (3.1)$$

Where q is the flow (vehicles per unit time).

- Hence, the **cumulative curve** can be constructed by integrating the flow:

$$N = \int q\,dt \qquad (3.2)$$

This integration gives one **degree of freedom**: the starting value of N.

It can be chosen freely or aligned with other cumulative curves (for different points).

VERTICAL QUEUING MODEL

Concept

- This is a **simplified queuing model** used in traffic flow analysis.
- Assumptions:
  - **Unlimited inflow** of vehicles.
  - **Outflow restricted** by bottleneck capacity.
  - Vehicles unable to pass are stacked "**VERTICALLY**" (do not take physical road space).
- Purpose: To model traffic congestion at bottlenecks in a mathematically tractable way.

Figure 3.1: Illustration of a vertical queue directly visualizes the **vertical queuing model**

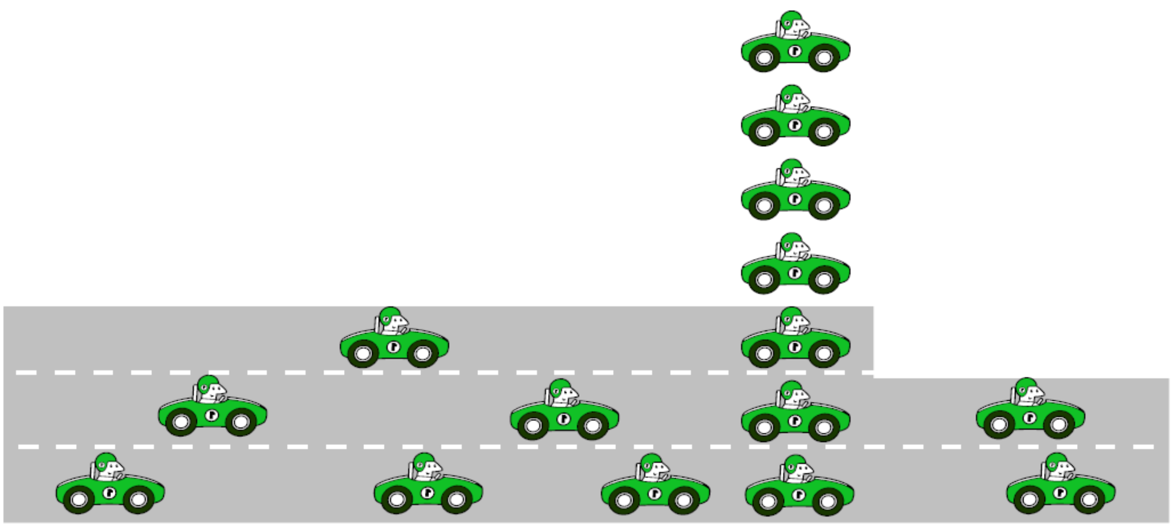


Figure 3.1: Illustration of a vertical queue

In Figure 3.1:

• The cars on the road represent vehicles moving toward a bottleneck.
• The cars stacked upward represent the queue.
• Instead of spreading out horizontally and taking up road space, waiting vehicles are drawn in a vertical column.

Why "Vertical" Queue?

- In reality, vehicles waiting at a bottleneck form **a** horizontal queue (bumper-to-bumper along the road).
- To make the mathematics simpler, we ignore physical space and assume:
  - All waiting vehicles are stored in an abstract "stack."
  - This stack is drawn vertically, as shown.

- Hence, the queue does not affect other vehicles spatially — it only affects delay and flow calculations.

• Unlimited **Inflow:** Vehicles keep arriving at the bottleneck.

• Restricted **Outflow:** Only a limited number can pass per unit time (based on capacity).

• Excess **Vehicles Queue:** Vehicles that cannot pass **stack in the vertical stack**.

• Delay **Representation:** The height of the vertical stack represents the **number of waiting vehicles**, i.e., the **size of the queue**.

DYNAMICS OF THE QUEUE

- **Time is discretized** into steps of size Δt, indexed by t.
- **Demand (inflow)**:
  - Denoted by D.
  - At each time step t, vehicles enter the system at rate D.
- **Stack S:**
  - Represents the number of vehicles waiting (in the "vertical queue").
  - Updated at each half-step (t+1/2) based on inflows and outflows.
- **Outflow:**
  - Limited by bottleneck capacity.
  - Depends on how many vehicles are in the stack and the system's discharge capability.

Initial Condition

- The queue starts empty: $S_0=0$
- **Inflow equation:**

$$q_{\mathrm{in},t} = D \qquad (3.3)$$

This means that at every time step, the inflow equals the external demand.

Intermediate Queue (before outflow):
at time step t+1/2. This intermediate step is the number of vehicles in the queue if there were no outflow, so the inflow is added to the existing stack:

$$S_{t+1/2} = S_t + q_{\mathrm{in}}\,\Delta t \qquad (3.4)$$

• St: vehicles already in the stack at the beginning.

• q_{in}: inflow rate (veh/s or veh/h).

• Δt: time-step length.

• This gives the **intermediate number of vehicles**, assuming no one has left yet.

Outflow Rule:

The outflow is the smaller of:

- The bottleneck's **capacity**, $C\Delta t$.
- The intermediate stack size $S_{t+1/2}$

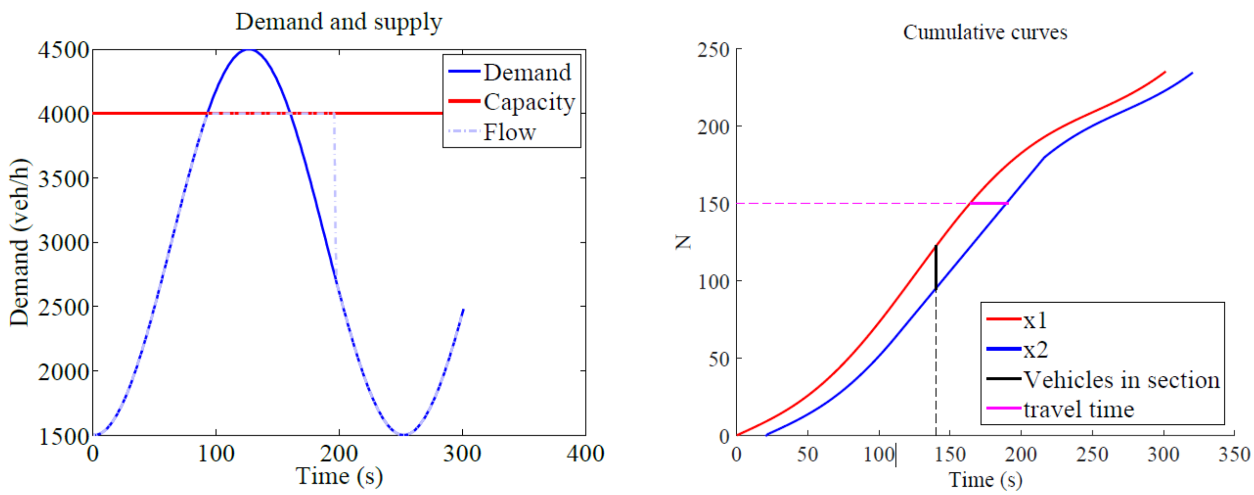
$$q_{\mathrm{out}} = \min\{C\Delta t,\ S_{t+1/2}\} \qquad (3.5)$$

Final Stack (after outflow):

After outflow occurs, the new stack is:

$$Si + 1 = S_{i+1/2} - q_{\text{out}}\Delta t = S_i + (q_{\text{in},i} - q_{\text{out},i})\,\Delta t \qquad (3.6)$$

This recursive relation fully describes how the vertical queue evolves over time.



(a) Demand and supply (b) Cumulative curves

Figure 3.2: Demand and cumulative curves

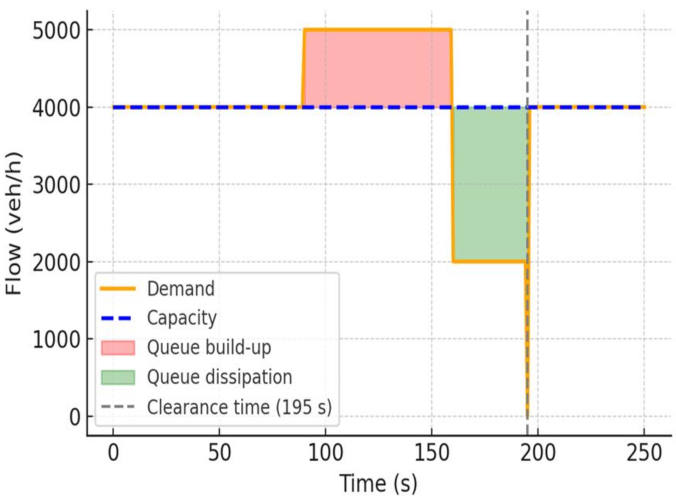
Figure 3.2(a): Demand and Supply

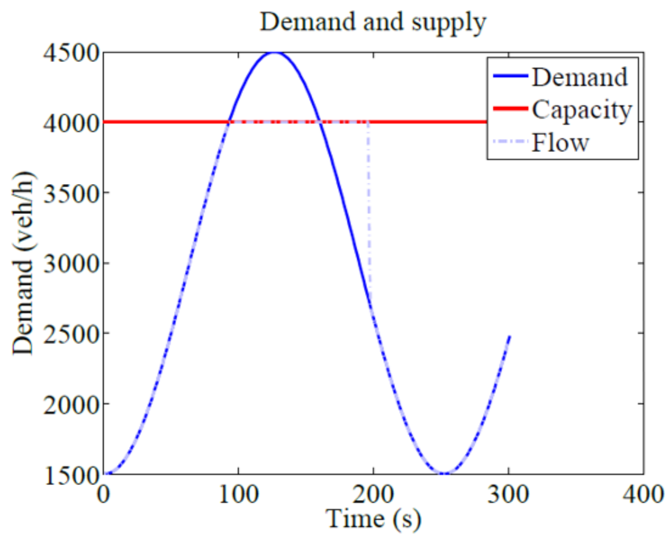
- **Blue curve:** demand profile (vehicles wanting to pass).
- **Red horizontal line:** constant capacity (4000 veh/h).
- **Black curve (Flow):** actual outflow, determined by min (demand, capacity).
- Key: when demand > capacity (90–160 s), a queue forms; when demand < capacity (160–200 s), the queue dissipates
- From 90–160 s, demand > capacity → build-up area = vehicles queued.
- From 160–200 s, capacity > demand → dissipation area = vehicles discharged.
- Flow stays at capacity until the queue is gone, after which outflow matches demand again.

Figure 3.2(b): Cumulative curves

- Plots cumulative vehicles N(t) at two locations, x1 and x2.
- The vertical difference between the red and blue lines → number of vehicles **in the section**.
- The horizontal difference (between x1 and x2) → **travel time** through the section.
- Travel time grows during queue build-up and shrinks back once the queue clears.

- **Demand–supply diagram** shows when queues form and clear.
- **Cumulative curves** translate that into delay and travel time analysis.
- **Vertical stack model** provides the computational framework.





- **Queue build-up area (veh): 19.444**

$$A_{\text{build}} = \frac{(D_{\text{high}} - C)(t_2 - t_1)}{3600} = \frac{(5000 - 4000)(160 - 90)}{3600} = 19.444$$

- **Clearance time:** $t_{\text{clear}} = 195\ \text{s}$

  The queue dissipates under low demand with net rate $C - D_{\text{low}} = 4000 - 2000 = 2000\ \text{veh/h}$:

$$t_{\text{clear}} = t_2 + \frac{A_{\text{build}}}{C - D_{\text{low}}} \times 3600 = 160 + \frac{19.444}{2000} \times 3600 = 195\ \text{s}.$$

- **Dissipation area up to clearance: 19.444 veh** (matches build-up, as required).

Example:

- Time step: Δt=1 minute
- **Capacity (outflow limit):** C=5 veh/min
- **Demand (inflow):**
  - Minutes 1–5: D=8 veh/min (creates a queue)
  - Minute 6 onward: D=3 veh/min (queue dissipates)
- Queue update each minute:

$$q_{\text{in},t} = D_t$$

$$q_{\text{out},t} = \min\left(C,\ S_{t-1} + q_{\text{in},t}\right)$$

$$S_t = S_{t-1} + q_{\text{in},t} - q_{\text{out},t}, \quad S_0 = 0$$

minute t	$q_{in,t}$	$q_{out,t}$	Queue S_t (veh)
1	8	5	3
2	8	5	6
3	8	5	9
4	8	5	12
5	8	5	15
6	3	5	13
7	3	5	11
8	3	5	9
9	3	5	7
10	3	5	5
11	3	5	3
12	3	5	1
13	3	**4** *(limited by what's available)*	0
Total	64	**64**	94

- **Build-up (t = 1–5):** Since $D = 8 > C = 5$, the queue grows by $8 - 5 = 3$ veh/min → $S_5 = 15$ vehicles.
- **Dissipation (t ≥ 6):** With $D = 3 < C = 5$, the queue shrinks by $5 - 3 = 2$ veh/min. Continuous-time clearance time $= \frac{15}{5-3} = 7.5$ min → in discrete minutes it clears by **t = 13**.
- **Growth rate** while congested: $D - C$ veh/min.
- **Clearance time** after demand drops below capacity: $T_{\text{clear}} = \frac{S_{\text{at drop}}}{C - D_{\text{low}}}$.
- **Total delay (veh·min)** over any interval ≈ **area under** S_t (sum of queue sizes each minute).

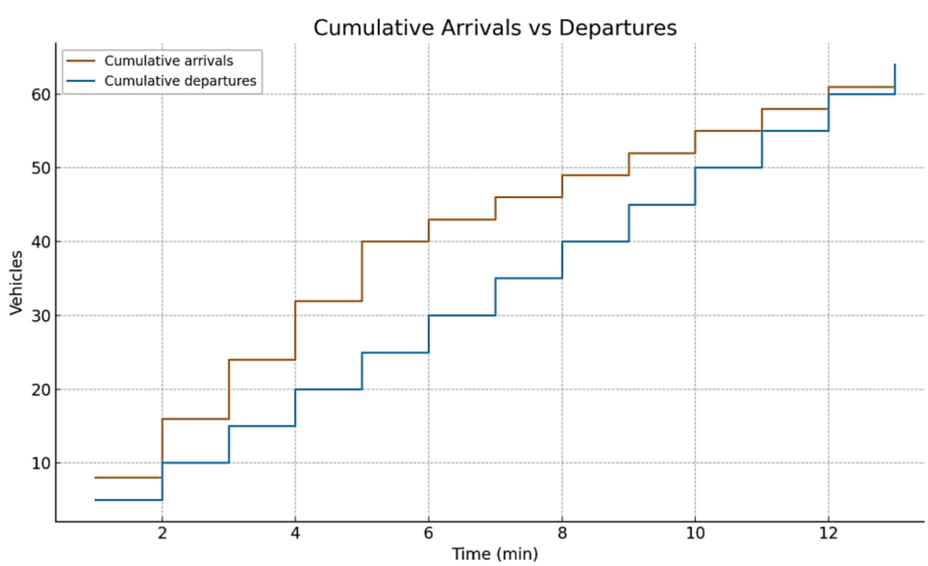
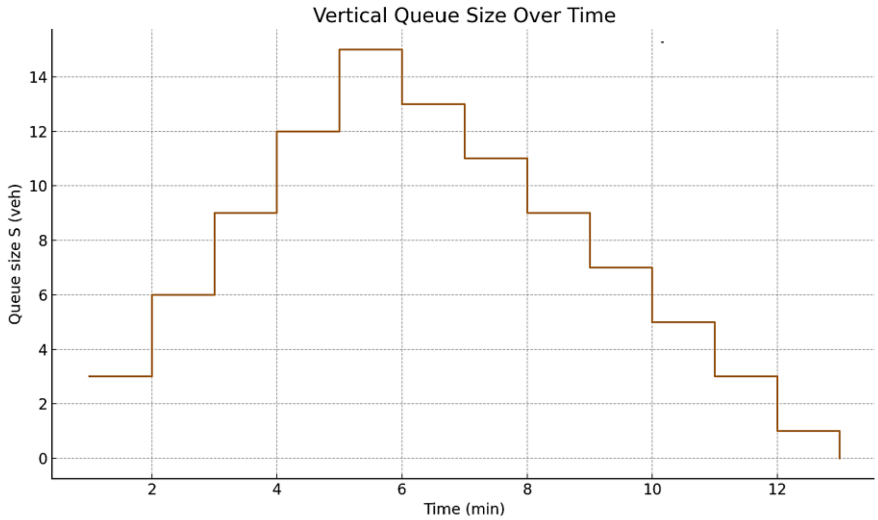


Figure (A)



Figure(B)

• Figure **(A) – Cumulative arrivals vs. departures:** the vertical gap between the two step-curves at any time equals the queue size. You'll see the gap widens during D=8 (mins 1–5), then shrinks once demand drops to 3 and the bottleneck discharges at capacity 5.

• Figure **(B) – Queue size St:** grows linearly to 15 veh by min 5, then dissipates linearly and clears by min 13.

(a) Total delay (veh·min)

Using the previous run (Δt=1min), the total delay is the area under the queue:

$$D_{\text{tot}} = \sum_t S_t \, \Delta t$$

D_{tot}=94 veh·min.

(b) Average delay per vehicle

By min 13, **64** vehicles have departed, so

$$\bar{d} = \frac{94}{64} \approx \mathbf{1.47} \text{ min/veh.}$$

Quick cross-check: after the demand drop (from 8→3 at t=6), clearance rate is C−D=5−3=2 veh/min.

With a peak queue of 15 veh, continuous-time clearance ≈ 15/2=7.5 min → matches the discrete clearing by minute 13.

(C) Incident Scenario (Temporary Capacity Drop)

Let capacity fall to **2 veh/min** during minutes **4–6**, then return to 5 veh/min (demand stays 8 for mins 1–5, then 3).

Key outcomes:

- Peak queue: **22 veh** (at minute 6).
- Queue clears by **minute 17** (vs. 13 without incident).

- Total delay: **186 veh·min** (≈ **2×** larger than the base case).
- Average delay: $\bar{d} = \frac{186}{76} \approx \mathbf{2.45}\ \mathrm{min/veh}$

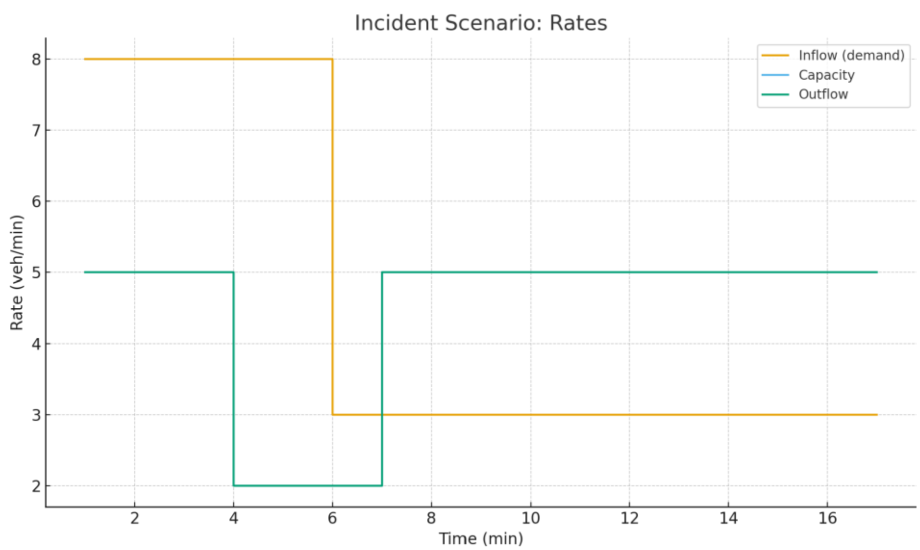
(76 vehicles served by clearance time).

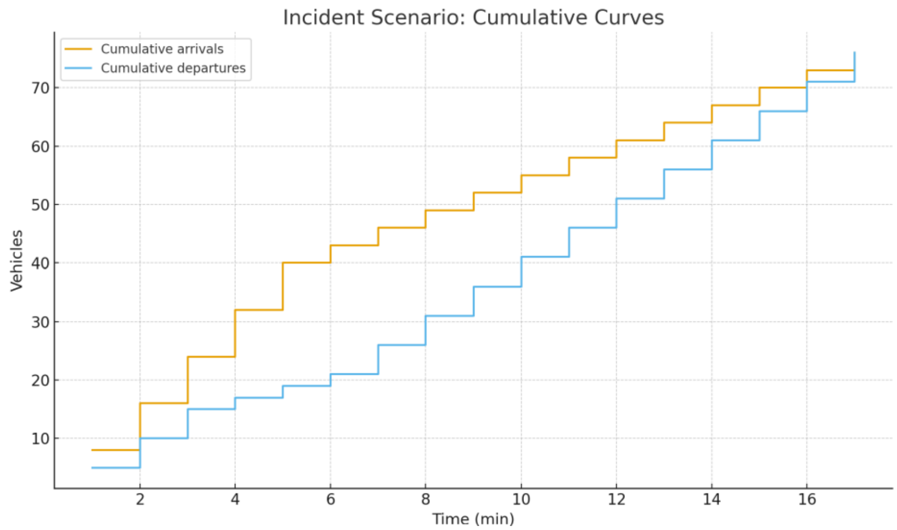
• Total **delay** scales with both the **height** (queue size) and **length** (duration) of the queue.

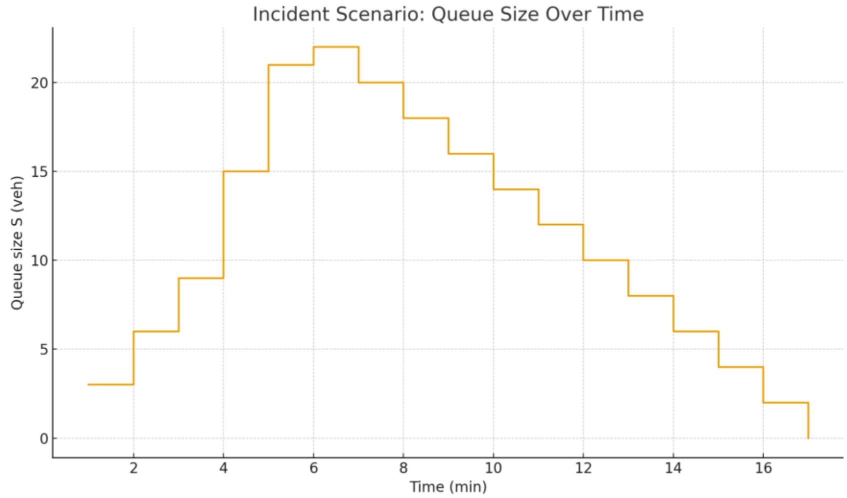
• A short, sharp **capacity dip** can nearly **double** delay and add several minutes to **clearance time**, even after capacity recovers.

• For planning: reducing the duration of the capacity loss (e.g., quicker incident response) is often as valuable as increasing nominal capacity.

Min.	Demand q_{in}_veh per min	Capacity veh per min	Outflow q_{out} veh per min	Queue S veh	Cum arrivals	Cum departures
1	8	5	5	3	8	5
2	8	5	5	6	16	10
3	8	5	5	9	24	15
4	8	2	2	15	32	17
5	8	2	2	21	40	19
6	3	2	2	22	43	21
7	3	5	5	20	46	26
8	3	5	5	18	49	31
9	3	5	5	16	52	36
10	3	5	5	14	55	41
11	3	5	5	12	58	46
12	3	5	5	10	61	51
13	3	5	5	8	64	56
14	3	5	5	6	67	61
15	3	5	5	4	70	66
16	3	5	5	2	73	71
17	3	5	5	0	76	76







3.3 TRAVEL TIMES, DENSITIES, AND DELAYS

Purpose

- Shows how to compute **travel time** and **delay** from **cumulative curves** (like Fig. 3.2(b)).
- Assumption: **no spillback** (so vertical queue assumption still valid).
- If spillback occurs → must use **shockwave theory** (Chapter 5).

CONSTRUCTION OF CUMULATIVE CURVES

- Use equation (3.2):

$$N = \int q \, dt$$

- Inflow curve: cumulative number of vehicles entering at x1. Use Eq 3.3

• Outflow curve: cumulative number of vehicles exiting at x2. Use Eq 3.3

• These are the two curves shown in Fig. 3.2(b).

• The **vertical distance** between curves = vehicles in section.

• The **horizontal distance** = travel time of specific vehicles.

TRAVEL TIMES, NUMBER OF VEHICLES IN THE SECTION

1. **Vertical interpretation (densities):**

- At time t, the vertical gap ΔN between inflow and outflow = number of vehicles between x1 and x2.
- Multiplying ΔN by dt and integrating over time gives **total travel time (veh·time):**

$$tt = \int \Delta N \, dt \qquad (3.7)$$

2. **Horizontal interpretation (travel time per vehicle):**

- Consider the 150th vehicle.
- Its entry time = when inflow curve reaches N=150.
- Its exit time = when outflow curve reaches N=150.
- The horizontal gap between the two = travel time of that vehicle.
- Adding up over all vehicles:

$$tt = \sum_i t_i \qquad (3.8)$$

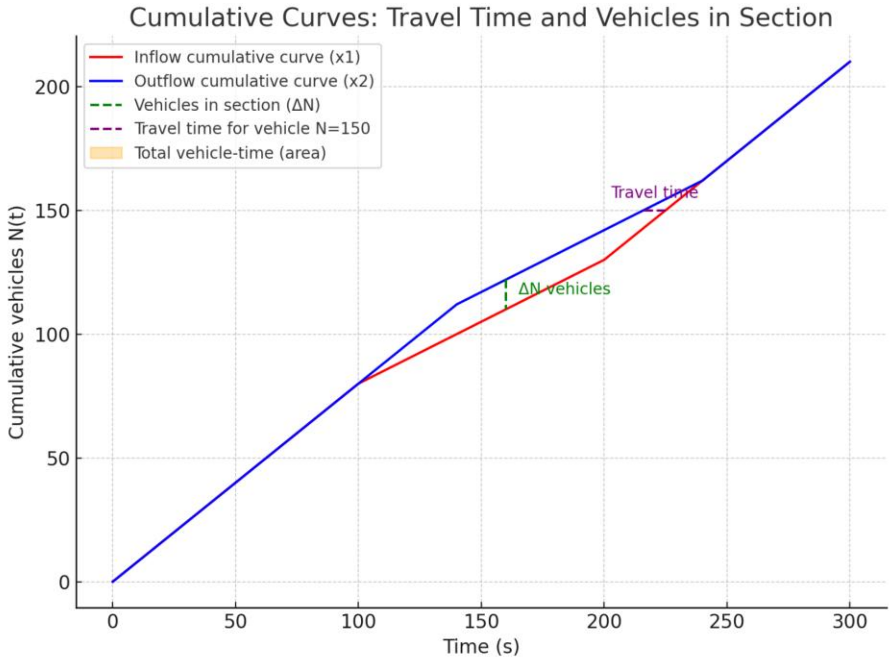
- In continuous form:

$$tt = \int t \, dt_i \qquad (3.9)$$

- **Total time spent by all vehicles = area between inflow and outflow cumulative curves.**

In summary:

- *Vertical difference* between cumulative curves → number of vehicles in section at time ttt.
- *Horizontal difference* → travel time of individual vehicles.
- *Area between curves* → total vehicle-time (or delay).



This figure shows the schematic of **cumulative curves** with key interpretations:

- **Red curve:** inflow cumulative vehicles at upstream point x1.
- **Blue curve:** outflow cumulative vehicles at downstream point x2.
- **Green vertical line (ΔN)**: number of vehicles in the section at time t.
- **Purple horizontal line:** travel time of the 150th vehicle.
- **Orange shaded area:** total vehicle-time (or delay), i.e. the area between inflow and outflow curves.

This figure makes it clear that:

- Vertical gap = vehicles present.
- Horizontal gap = individual travel time.
- Area = total delay.

DELAYS

Concept

- **Travel time** = actual time a vehicle spends in the section.
- **Free-flow travel time** = time without congestion.
- **Delay** = extra time:

$$\text{Delay} = t_i - t_{\text{free flow}}$$

So, each vehicle's delay = difference between actual outflow curve and the free-flow shifted curve.

Graphical interpretation

- **Step 1:** Take the **outflow cumulative curve** and shift it **left** by the free-flow travel time.
- **Step 2:** Now, compare inflow curve vs. shifted outflow curve:
  - If curves overlap → no delay (vehicles pass unhindered).
  - Horizontal gap → **individual vehicle delay**.
  - Vertical gap → **queue size** (number of vehicles waiting).
  - Area between curves → **total delay**.

Equations

- **Total delay as sum of extra time across all vehicles:**

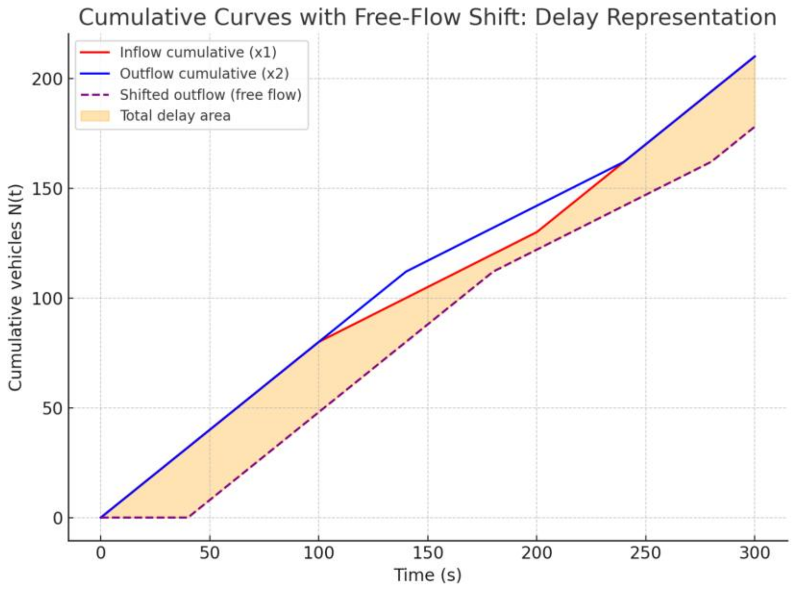
$$D = \int \left(t_i - t_{\text{free flow}}\right) di \tag{3.10}$$

- **Equivalent formulation (using number in queue $N_{\text{queue}}(t)$):**

$$D = \int N_{\text{queue}}(t)\, dt \tag{3.11}$$

So total delay = **area under the queue size curve over time**.

Travel times = area between inflow & outflow cumulative curves.
Delays = area between inflow & free-flow-shifted outflow cumulative curve



This figure shows the schematic for **delay calculation using cumulative curves**:

- **Red curve:** inflow cumulative (vehicles entering at x1).
- **Blue curve:** outflow cumulative (vehicles leaving at x2).
- **Purple dashed curve:** outflow shifted left by free-flow travel time.
- **Orange shaded area: total delay**, i.e. the area between inflow and shifted outflow.

This matches equations (3.10)–(3.11):

- Delay = sum of extra times = area between inflow and shifted outflow.
- Equivalently, delay = integral of queue length $N_{\text{queue}}(t)$ over time.

SLANTED CUMULATIVE CURVES

Slanted cumulative curves are derived by subtracting a **reference flow q_0** (often chosen as the **capacity**) from the inflow/outflow cumulative functions:

$$\tilde{N} = \int (q - q_0)\, dt = \int q dt - \int q_0 dt$$

So instead of raw cumulative counts, we track the **excess (or deficit) relative to capacity flow**.

- **Reference flow q_0**: best chosen as **capacity flow**.
- This transformation **amplifies differences** from capacity, making queues and delays easier to see.
- If demand < capacity → $\tilde{N}$ is negative (system underloaded).
- If demand > capacity → $\tilde{N}$ increases (queue grows).
- When demand returns below capacity → $\tilde{N}$ declines back until queues are dissolved.

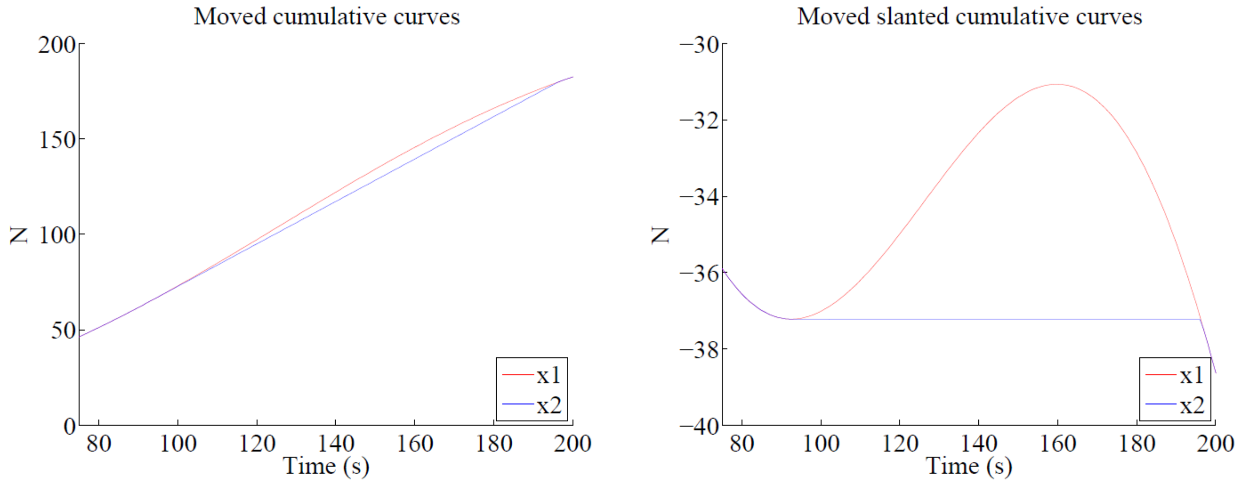
Application to Queues and Delays

- **Figure 3.3(b)** shows the **slanted cumulative curves** for the same case as in 3.3(a).
- The **vertical distance** between the slanted inflow and outflow curves = queue size N_{queue}.
- The **area between curves** = total delay D.
- Equations (3.10) and (3.11) still apply in this transformed view.

Why Use Slanted Cumulative Curves?

- Easier to **detect capacity changes**:
  - If slope of the slanted outflow curve changes → capacity drop or recovery detected.
- More robust for **longer observation windows** because baseline trends are canceled out.
- Useful in **empirical data analysis** (loop detectors, field counts) where absolute counts are large but differences matter.

Slanted cumulative curves are just **regular cumulative curves with a reference slope (capacity) removed**.
They simplify the visualization of queues and delays, and highlight capacity changes directly.



(a) Moved cumulative curves (b) Slanted cumulative curves

Figure 3.3: Determining the delay and the flows from cumulative curves

Figure 3.3(a) – Moved cumulative curves

- Shows **inflow (x1)** and **outflow (x2)** after the outflow curve has been shifted left by the free-flow travel time.
- If demand = free-flow, the two curves overlap → no delay.
- The horizontal distance between these shifted curves gives **delay per vehicle**.
- The vertical distance represents the **number of queued vehicles**.
- The area between the two curves gives **total delay** (as per equations 3.10–3.11).

Figure 3.3(b) – Moved slanted cumulative curves

- Same concept, but using **slanted cumulative curves** instead of vertical ones.
- Advantage: makes it easier to read **flows** directly.
- The difference between inflow and shifted outflow gives delay, similar to part (a).
- Vertical cumulative curves (3.3a).
- Slanted cumulative curves (3.3b).

Both lead to the same interpretation:

- **Delay = area between inflow curve and free-flow-shifted outflow curve**.

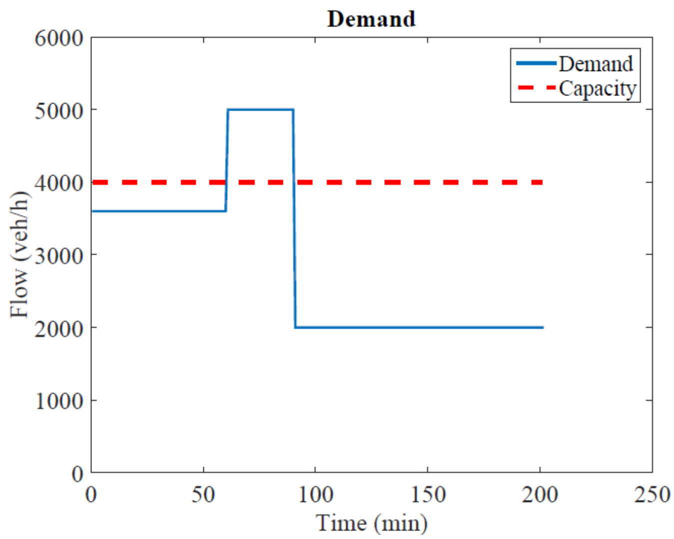


Figure 3.4: Demand and capacity

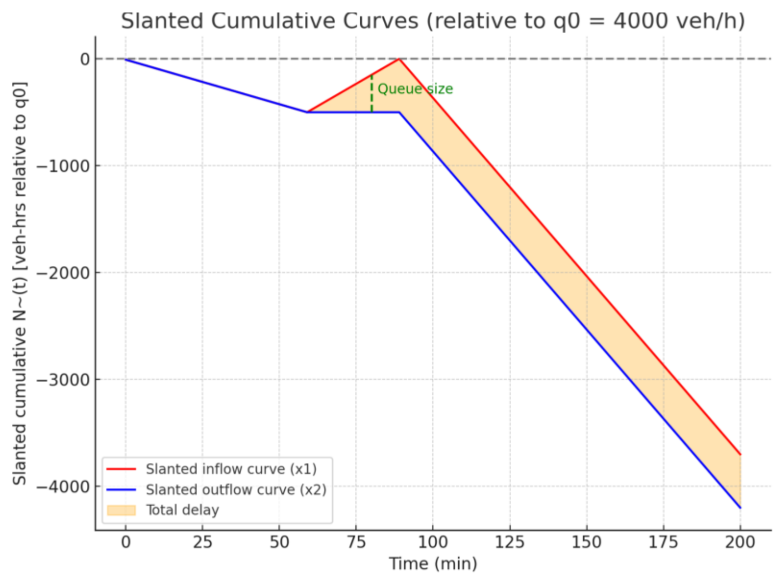
Figure 3.4: Demand and Capacity

- **Blue line:** time-varying demand.
- **Red dashed line:** constant capacity (4000 veh/h).
- 0–60 min: Demand below capacity (≈ 3500 veh/h) → no queue forms, system underloaded.
- 60–90 min: Demand rises to 5000 veh/h (> capacity) → queue builds.
- After 90 min: Demand drops sharply to 2000 veh/h (< capacity) → queue dissipates as capacity exceeds demand.

How it links to slanted cumulative curves

- Using this demand profile, slanted cumulative curves subtract the capacity reference slope (q_0=4000 veh/h).
- When demand < capacity → slanted cumulative curve decreases (negative slope).
- When demand > capacity → slanted cumulative curve increases (positive slope).
- This makes it visually clear when the system is **overloaded or underloaded**, and the vertical gaps directly give queue length.

With **slanted cumulative curves**, the queue build-up (60–90 min) and dissipation (90–200 min) can be analyzed using vertical distances (queue size) and shaded areas (total delay).



Here's the **slanted cumulative curves schematic** corresponding to the demand profile from Figure 3.4:

- **Red line:** slanted inflow curve (relative to capacity q0=4000 veh/h).
- **Blue line:** slanted outflow curve.
- **Green dashed line:** vertical gap = **queue size**.
- **Orange shaded area: total delay** (area between inflow and outflow).
- The downward slope before 60 min shows demand < capacity; upward slope during 60–90 min shows demand > capacity (queue builds); after 90 min the slope reverses as demand drops below capacity (queue dissipates).

  This is exactly how slanted cumulative curves make queues and delays visually clear.

EXAMPLE:

$$q_{\text{in}}(t) = \begin{cases} 3600 \text{ veh/h} & t < 1 \text{ h} \\ 5000 \text{ veh/h} & 1 \leq t < 1.5 \text{ h} \\ 2000 \text{ veh/h} & t \geq 1.5 \text{ h} \end{cases} \quad \text{with capacity } C = 4000 \text{ veh/h}.$$

1) Construct the (moved/translated) cumulative curves

Answer:

- **Inflow curve**: slope = demand.
- **Outflow curve**: slope = min(demand, capacity).
  t<1 hr: demand < capacity ⇒ outflow slope = demand.
  1–1.5 : demand > capacity ⇒ outflow slope = capacity.
  After 1.5 h: demand < capacity and queue exists ⇒ outflow stays at capacity until curves meet; afterward it follows inflow.

2) First vehicle that encounters delay (vehicle index N)

Delay starts when demand first exceeds capacity ⇒ at **1 h**.

$$N = \int_0^{1\text{ h}} 3600\, dt = \boxed{3600 \text{ vehicles}}.$$

3) Time at which delay is largest

Queue builds while demand > capacity ⇒ until **1.5 h**.

$\boxed{t = 1.5 \text{ h}}$.

4) Maximum number of vehicles in the queue

Build period length = 0.5 h with net rate $5000 - 4000 = 1000$ veh/h:

$$\boxed{N_{\text{queue,max}} = 1000 \times 0.5 = 500 \text{ veh}}.$$

5) Vehicle number with the largest delay

It is the vehicle entering exactly at the end of the build period:

$$N = 3600 + 0.5 \times 5000 = \boxed{6100}.$$

6) That vehicle's delay

To serve 2500 vehicles that arrived during the build (from 1.0–1.5 h) at capacity 4000:

$$\text{service time} = \frac{2500}{4000} \text{ h} = 0.625 \text{ h} = 37.5 \text{ min.}$$

That vehicle entered 0.5 h (=30 min) after 1 h, so delay:

$$\boxed{37.5 - 30 = 7.5 \text{ min}}.$$

7) Time the queue is fully cleared

Dissipation rate after 1.5 h: $4000 - 2000 = 2000$ veh/h.

Clearance time for 500 vehicles: $500/2000 = 0.25$ h = 15 min.

Queue clears at **1.5 h + 0.25 h = 1.75 h = 1 h 45 min**.

(*Sometimes written "1:45 h"; not 1.45 as a decimal.)

8) Last vehicle that encounters delay (vehicle index)

Vehicles delayed are those between first delayed (3600) and last at clearance.

Vehicles that exit during the 15-min discharge window = $2000 \times 0.25 = 500$.

Add to the index at end of build:

$$\boxed{N = 6100 + 500 = 6600}.$$

9) Total delay (veh·h)

Triangle area between cumulative inflow and outflow:

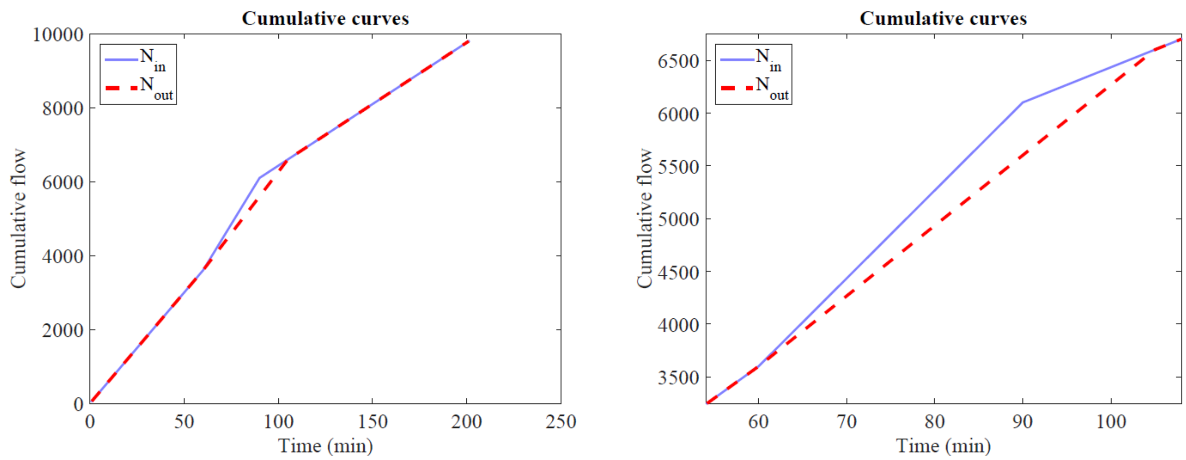
- Height = max queue = 500 veh
- Base = build time + clearance time = $30 + 15 = 45$ min $= 0.75$ h

$D = \frac{1}{2} \times 500 \times 0.75 = \boxed{187.5 \text{ veh/h}}$

10) Average delay of delayed vehicles

Delayed vehicles = $6600 - 3600 = 3000$.

$$\bar{d} = \frac{187.5}{3000} = 0.0625 \text{ h} = \boxed{3.75 \text{ min/veh}}.$$



(a) Overview of the cumulative curves (b) More detailed view of the cumulative curves

Figure 3.5: Cumulative curves for the example

<u>References</u>

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