MUSTANSIRIYAH UNIVERSITY
COLLEGE OF ENGINEERING
HIGHWAY AND TRANSPORTATION ENGINEERING DEPARTMENT
POSTGRADUATE/MSC
2025– 2026

ADVANCED TRAFFIC ENGINEERING

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LECTURE 2:

HEADWAY MODELS

2.1 RELATION BETWEEN HEADWAY AND CAPACITY

HEADWAY DISTRIBUTION

- **Headway** = the time gap between two consecutive vehicles passing a point.
- Traffic headways can vary, so they are often described by a **probability distribution** $P(h)$.
- The **average headway** is the expected value of this distribution:

$$\langle h \rangle = \int_0^\infty hP(h)\,dh \tag{2.1}$$

This formula calculates the mean headway from the probability distribution.

FLOW AND CAPACITY

- **Flow** (vehicles per unit time) is inversely related to headway:
  - Shorter headways → higher flow.
  - Longer headways → lower flow.
- In a **bottleneck** (like a signal or merge point), the **minimum headway** that drivers can maintain determines the **capacity**.
- Hence, **capacity** C is the reciprocal of the mean headway in the bottleneck:

$$C = \frac{1}{\langle h_{\text{bottleneck}} \rangle} \tag{2.2}$$

The number of vehicles arriving in a certain period could be a useful measure. This holds for instance for traffic lights, where the number of arrivals per red period is relevant. As illustrated in figure 2.1, there could be different lanes for different directions at a traffic light. The idea is that the traffic towards one direction will not block the traffic to other directions, hence, the length should be long enough to allow the number of vehicles in the red period. The average number of vehicles in a red period can be determined from the flow. However, mostly requirements are that in p% of the red times (under a constant demand) the queue should not exceed the dedicated lane. In that case, the headway distribution can form the basis for the calculations.

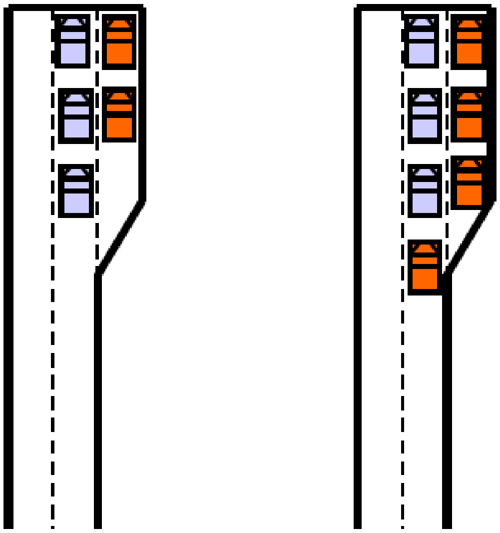


Figure 2.1: A queuing area. The orange coloured vehicles turn right, whereas the blue ones continue straight

PRACTICAL APPLICATION (TRAFFIC LIGHTS EXAMPLE)

- For signalized intersections:
  - The number of vehicles arriving during the **red period** is important.
  - The arrival headway distribution helps predict whether queues will spill over into other lanes.
  - To avoid **queue overflow**, the average headway (and thus the expected arrivals) must be balanced against green time and lane length.

EXAMPLE: CALCULATING CAPACITY FROM AVERAGE HEADWAY

Step 1: Assume an average headway

Suppose the **average headway** at a bottleneck is:

$$\langle h \rangle = 2.5 \text{ seconds/vehicle}$$

Step 2: Use the capacity formula

Capacity is the reciprocal of the mean headway:

$$C = \frac{1}{\langle h_{\text{bottleneck}} \rangle}$$

Step 3: Convert to vehicles per hour

Since headway is in seconds per vehicle, we must convert seconds → hours.

$$C = \frac{3600}{\langle h \rangle}$$

where 3600 = seconds in one hour.

Step 4: Insert the value

$$C = \frac{3600}{2.5} = 1440 \text{ veh/hr}$$

If the average headway is **2.5 seconds/vehicle**, then the maximum capacity is:

$$C = 1440 \text{ vehicles per hour (veh/hr)}$$

For various Ave. headway, the corresponding capacity (veh/hr)

Average Headway (s/veh)	Capacity (veh/hr)
1.5	2400
2	1800
2.5	1440
3	1200
3.5	1028.571

2.2 ARRIVALS PER INTERVAL

OVERVIEW OF ARRIVAL DISTRIBUTIONS

This section describes the number of arrivals per time interval. For different conditions, this distribution is different.

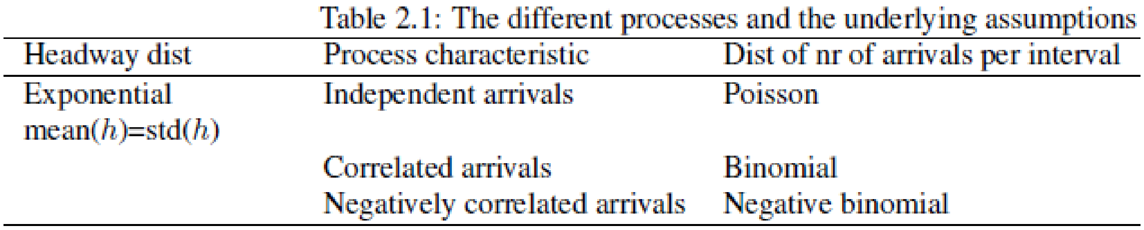


Table 2.1: The different processes and the underlying assumptions

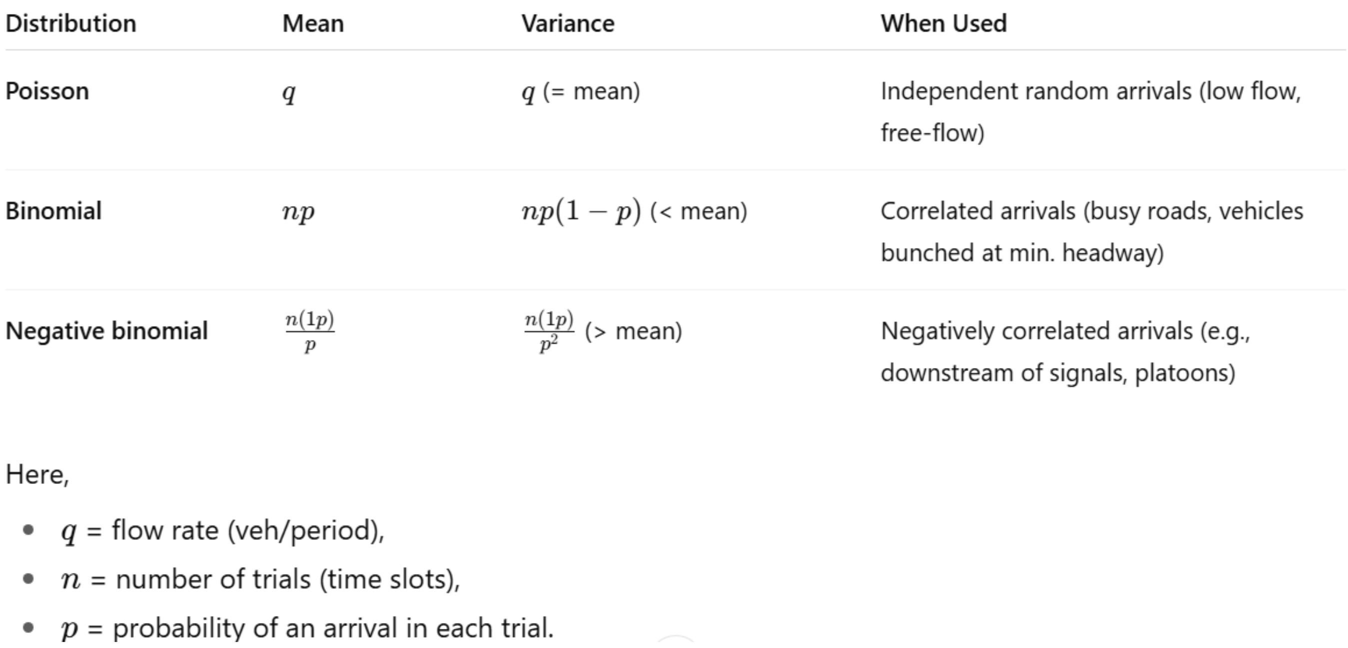
Headway dist	Process characteristic	Dist of nr of arrivals per interval
Exponential mean(h)=std(h)	Independent arrivals	Poisson
	Correlated arrivals	Binomial
	Negatively correlated arrivals	Negative binomial

- **Exponential headway distribution** → independent arrivals → Poisson distribution for number of arrivals.
- **Correlated arrivals** → Binomial distribution.
- **Negatively correlated arrivals** → Negative binomial distribution.

So, the choice of distribution depends on whether arrivals are independent, positively correlated, or negatively correlated.

SUMMARY OF DISTRIBUTIONS

Table 2.2 gives an overview of the distributions described in this section, and gives some characteristics.



Distribution	Mean	Variance	When Used
Poisson	q	q (= mean)	Independent random arrivals (low flow, free-flow)
Binomial	np	$np(1-p)$ (< mean)	Correlated arrivals (busy roads, vehicles bunched at min. headway)
Negative binomial	$\frac{n(1p)}{p}$	$\frac{n(1p)}{p^2}$ (> mean)	Negatively correlated arrivals (e.g., downstream of signals, platoons)

Here,

- q = flow rate (veh/period),
- n = number of trials (time slots),
- p = probability of an arrival in each trial.

POISSON DISTRIBUTION (INDEPENDENT ARRIVALS)

- Probability of k arrivals in a period:

$$P(X = k) = \frac{q^k}{k!} e^{-q} \tag{2.3}$$

- Works well when flow is **low** and vehicles arrive randomly (no interaction).
- Illustrated in **Figures 2.2 & 2.3**:
  - At low flow (e.g., $q = 280$ veh/h), arrivals fit Poisson well.
  - At higher flow (e.g., $q = 800$ veh/h), independence assumption fails because vehicles follow minimum headways.

Equation (2.3) gives the probability that k vehicles arrive if the average arrival rate per period is q.

P(k) = probability of k arrivals

q= expected number of arrivals (flow × interval length)

for low q(<1)…> P(0) dominates, meaning zero arrivals are most probable

for higher q (>1)…> the peak occur at k=q, meaning the most likely number of arrivals equals the expected flow.

Hence note that one needs to rescale q to units of number of vehicles per aggregation period!

Figure 2.2 shows examples of the Poisson distribution. Note that for low values of the flow (expected value smaller than 1), the probability is decreasing. If the flow is higher, there is a maximum at the number of arrivals which is at a higher value than 1.

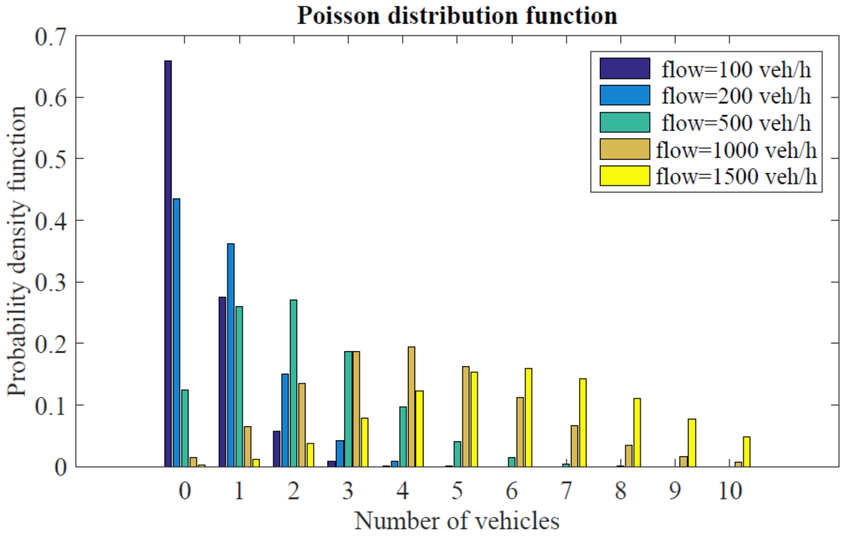
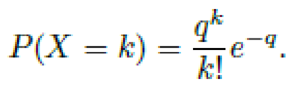


Figure 2.2: Example of the Poisson distribution for different flow values; the flow is indicated in veh/h, and these are the probabilities for arrivals in 15 seconds

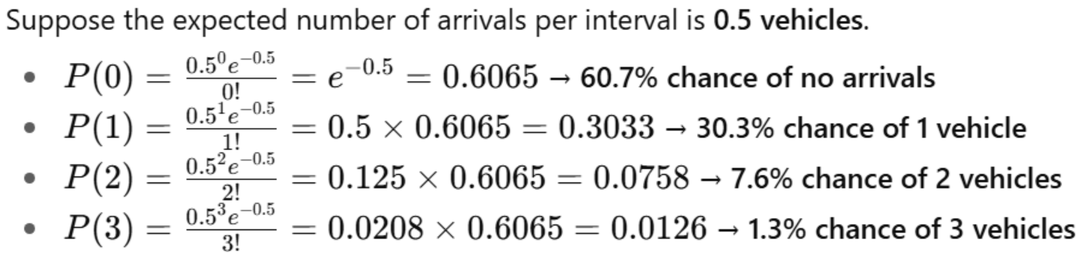
Key Points from the Figure

1. **Low Flow (100–200 veh/h)**
    - The probability is highest at **0 vehicles** (no arrivals).
    - The curve decreases monotonically as the number of vehicles increases.
    - Interpretation: At very low flows, it's most likely that **no car arrives** in a short observation interval.
2. **Moderate Flow (500 veh/h)**
    - The distribution peaks around **1–2 vehicles**.
    - Interpretation: At this flow, it's more common to see **a small number of arrivals**, but the chance of having zero arrivals is still significant.
3. **High Flow (1000–1500 veh/h)**
    - The peak shifts to the right (3–4 or more vehicles).
    - Interpretation: At high flows, it is **unlikely to have zero arrivals**; instead, the most probable outcome is **several vehicles arriving** in the interval.

Example :



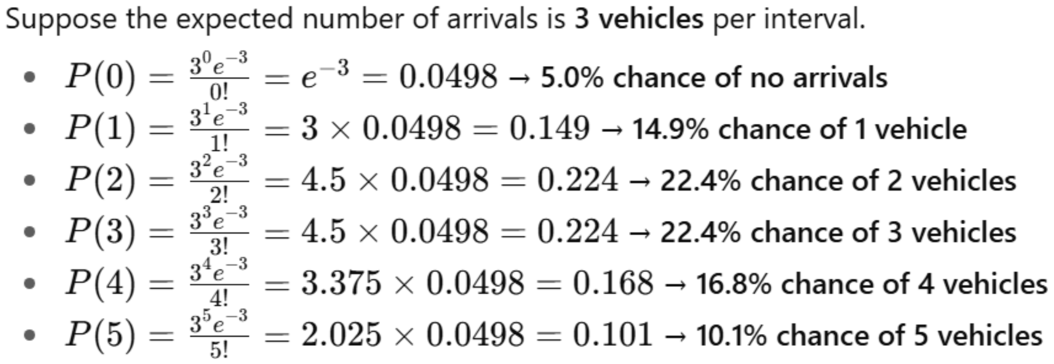
$$P(X = k) = \frac{q^k}{k!}e^{-q}.$$



Suppose the expected number of arrivals per interval is **0.5 vehicles.**

- $P(0) = \frac{0.5^0 e^{-0.5}}{0!} = e^{-0.5} = 0.6065$ → **60.7% chance of no arrivals**
- $P(1) = \frac{0.5^1 e^{-0.5}}{1!} = 0.5 \times 0.6065 = 0.3033$ → **30.3% chance of 1 vehicle**
- $P(2) = \frac{0.5^2 e^{-0.5}}{2!} = 0.125 \times 0.6065 = 0.0758$ → **7.6% chance of 2 vehicles**
- $P(3) = \frac{0.5^3 e^{-0.5}}{3!} = 0.0208 \times 0.6065 = 0.0126$ → **1.3% chance of 3 vehicles**

Here, the **highest probability is at 0 vehicles**. This matches the leftmost blue bars in the figure (flow = 100 veh/h).



Suppose the expected number of arrivals is **3 vehicles** per interval.

- $P(0) = \frac{3^0 e^{-3}}{0!} = e^{-3} = 0.0498$ → **5.0% chance of no arrivals**
- $P(1) = \frac{3^1 e^{-3}}{1!} = 3 \times 0.0498 = 0.149$ → **14.9% chance of 1 vehicle**
- $P(2) = \frac{3^2 e^{-3}}{2!} = 4.5 \times 0.0498 = 0.224$ → **22.4% chance of 2 vehicles**
- $P(3) = \frac{3^3 e^{-3}}{3!} = 4.5 \times 0.0498 = 0.224$ → **22.4% chance of 3 vehicles**
- $P(4) = \frac{3^4 e^{-3}}{4!} = 3.375 \times 0.0498 = 0.168$ → **16.8% chance of 4 vehicles**
- $P(5) = \frac{3^5 e^{-3}}{5!} = 2.025 \times 0.0498 = 0.101$ → **10.1% chance of 5 vehicles**

Here, the **peak is around 2–3 vehicles**, not at 0. This matches the yellow bars (flow = 1000–1500 veh/h) in your figure.

- At **low q**, the distribution is skewed left (most probable = 0).
- At **high q**, the distribution shifts right (most probable ≈ q).

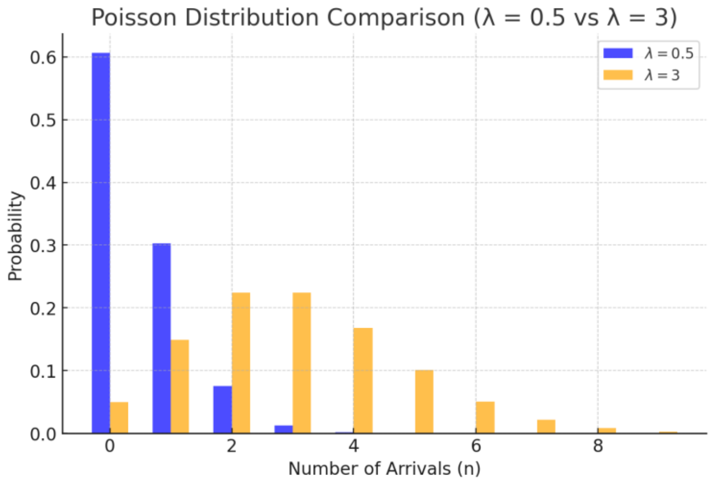


Figure 2.3 shows the best fits of this distribution on real life data. This distribution is accurate if the flow is low, and is not so good if the flow increases. This is because once the flow is low, the assumption of independent arrivals does not hold any more. Once vehicles are bound by the minimum headway, the arrivals are not independent any more. This restriction come more into play once the flow is high.

This figure compares **real-world traffic arrival data (black histograms)** with the **Poisson distribution fit (blue curves)** for different flow levels (q).

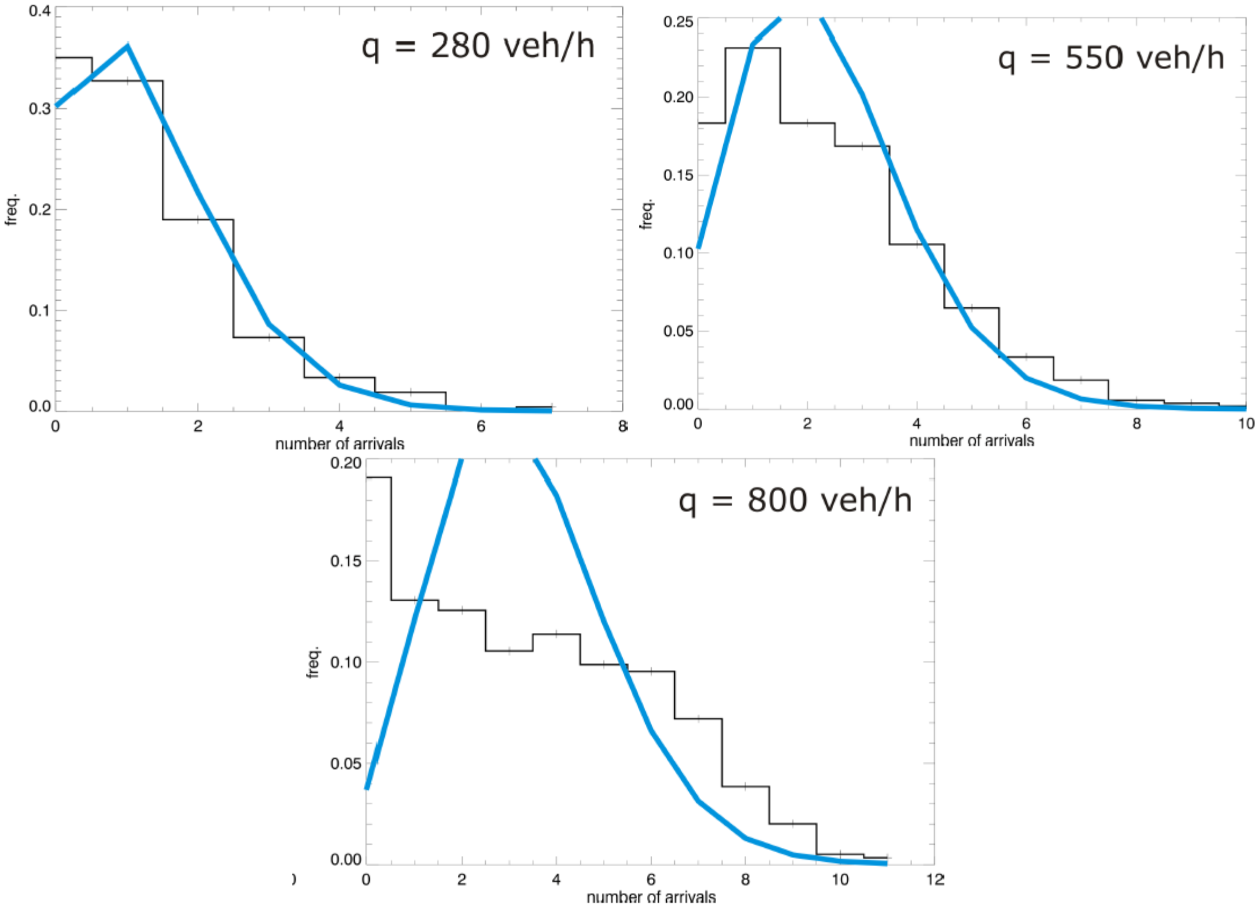


Figure 2.3: Illustration of the number of arrivals from real world data

From figure 2.3, we observe:

• Low flow (q = 280 veh/h)

- The Poisson model fits the data fairly well.
- Most probability mass is around 0–1 arrivals, as expected.
- At low flows, vehicles arrive **independently**, so the Poisson assumption holds.

• Moderate flow (q = 550 veh/h)

- The Poisson fit is still reasonable, but some mismatch starts to appear.
- The histogram shows slightly different behavior compared to the smooth theoretical curve.
- Independence assumption begins to weaken.

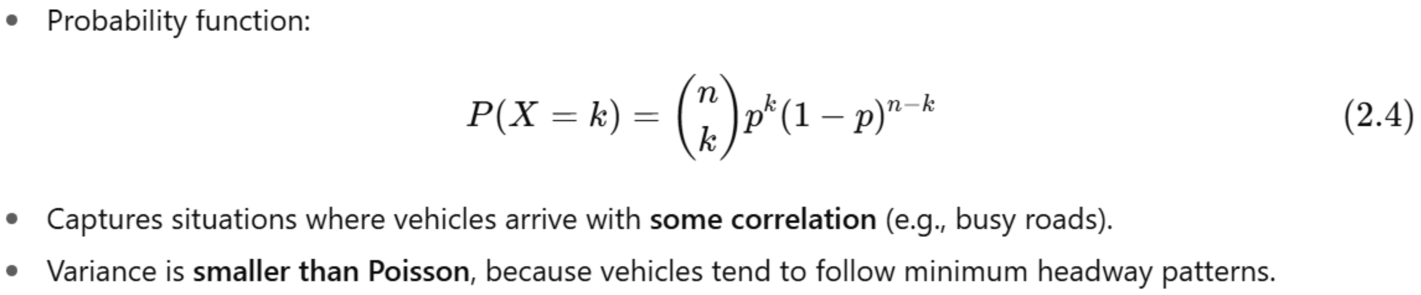
• High flow (q = 800 veh/h)

- The Poisson model is a poor fit.
- Real-world data shows a **flatter, wider distribution** than the Poisson prediction.
- This happens because in real traffic, vehicles cannot arrive completely independently — **minimum headway restrictions** (time gap between cars) force some correlation.

Why Poisson Fails at Higher Flows?

- **Poisson assumption:** Arrivals are **independent** and can occur at any instant.
- **Reality in traffic:**
  - Vehicles must maintain a **minimum headway**
  - As flow increases, cars are **bunched** and influenced by signals, platoons, or congestion.
  - This destroys the independence assumption → the Poisson model underestimates the variance.

BINOMIAL DISTRIBUTION (CORRELATED ARRIVALS)



- Probability function:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad (2.4)$$

- Captures situations where vehicles arrive with **some correlation** (e.g., busy roads).
- Variance is **smaller than Poisson**, because vehicles tend to follow minimum headway patterns.

- The **binomial distribution** models the number of “successes” in n independent trials, each with probability p.

- **Mean:** $\mu = np$
- **Variance:** $\sigma^2 = np(1-p)$
- In the context of arrivals:
    - n = number of "opportunities" (trials) in the interval (e.g., maximum number of vehicles possible).
    - p = probability of one vehicle arriving in each opportunity.
    - k = actual number of arrivals observed.

The idea behind the distribution is that one does n tries, each with an independent success rate of *p*. The number of successes is *k*. The mean of the distribution is *np*, see table 2.2. A certain flow specifies the mean of the arrivals, which hence determines np. This gives a freedom to choose n or p, by which one can match the spread of the function.

The number of observations in the distribution can never exceed n, so an reasonable choice of n would be the interval time divided by the minimum headway. Figure 2.4 shows examples of the binomial distribution function.

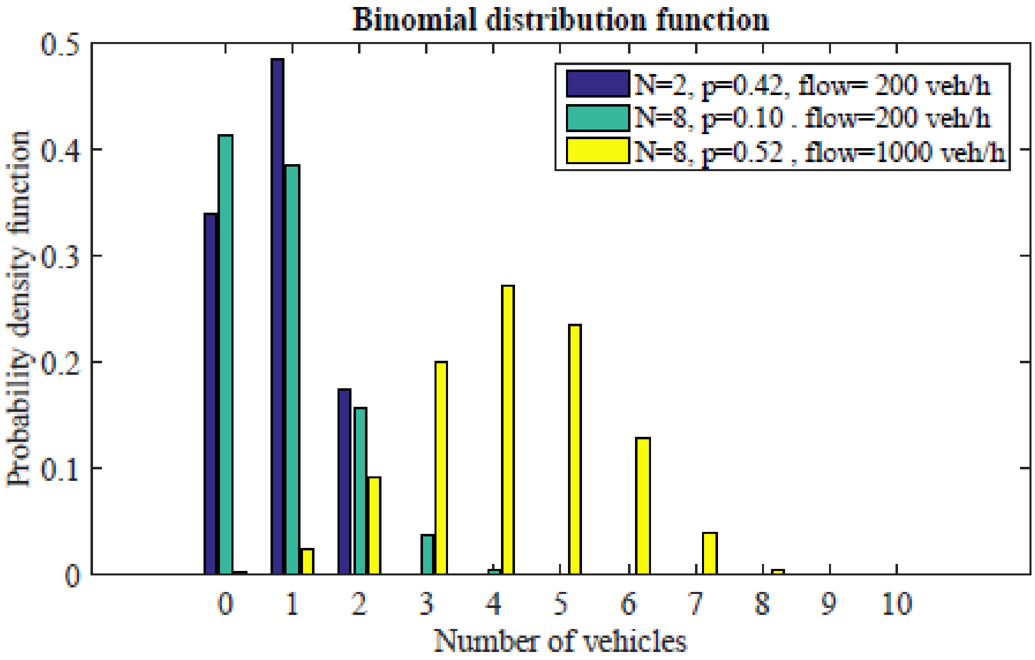
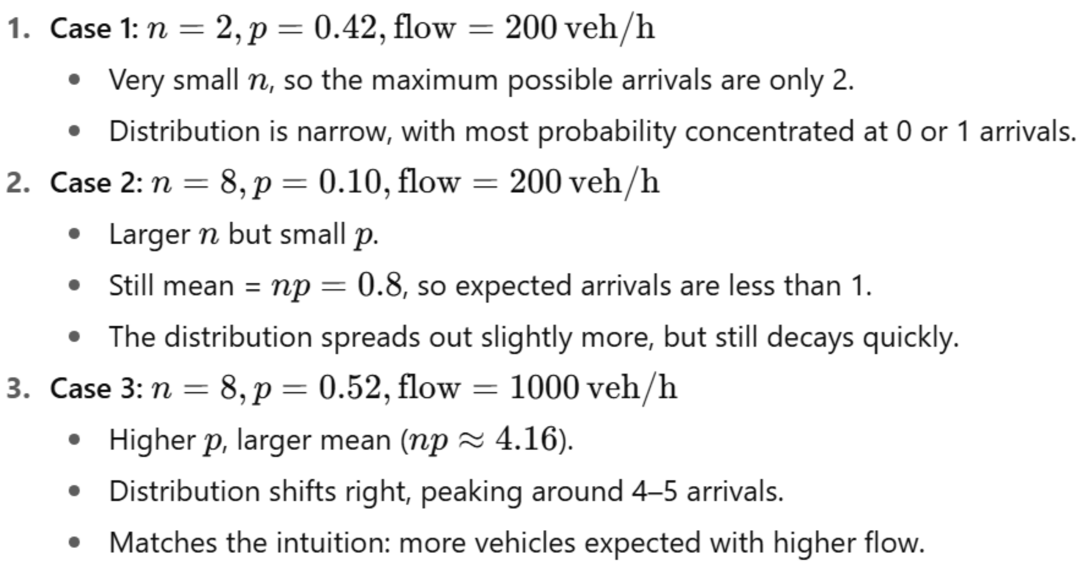


Figure 2.4: Examples of the binomial distribution function for the number of arrivals in an aggregation period

From this figure, we observe:

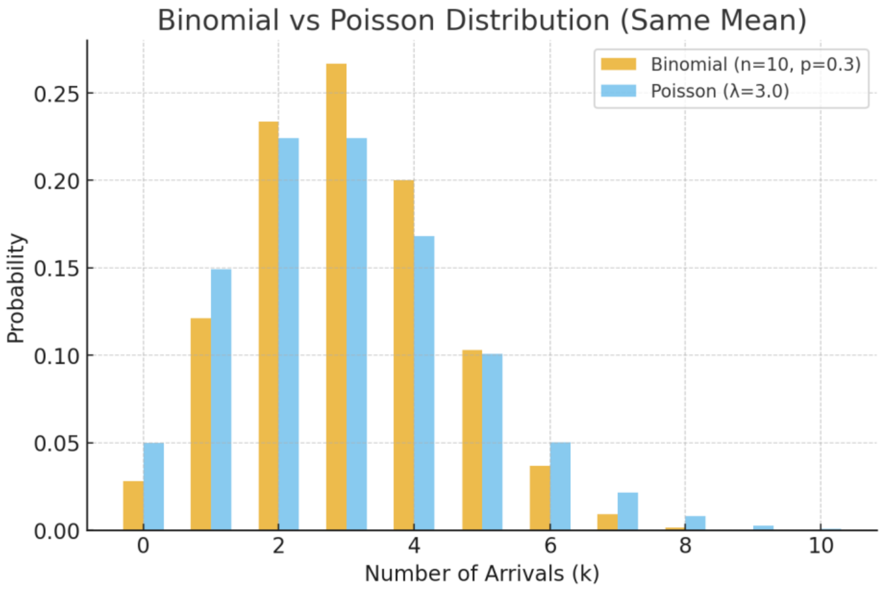


1. Case 1: $n = 2, p = 0.42, \text{flow} = 200\,\text{veh/h}$
   - Very small n, so the maximum possible arrivals are only 2.
   - Distribution is narrow, with most probability concentrated at 0 or 1 arrivals.
2. Case 2: $n = 8, p = 0.10, \text{flow} = 200\,\text{veh/h}$
   - Larger n but small p.
   - Still mean = $np = 0.8$, so expected arrivals are less than 1.
   - The distribution spreads out slightly more, but still decays quickly.
3. Case 3: $n = 8, p = 0.52, \text{flow} = 1000\,\text{veh/h}$
   - Higher p, larger mean ($np \approx 4.16$).
   - Distribution shifts right, peaking around 4–5 arrivals.
   - Matches the intuition: more vehicles expected with higher flow.

<u>Why Binomial is Useful in Traffic</u>

- Unlike Poisson (which allows an infinite number of arrivals), **Binomial has an upper bound of n**.
- A **reasonable choice of n** = (interval time ÷ minimum headway).
  - Example: if interval = 60 seconds, minimum headway = 6 s → max 10 vehicles → n=10.
- This makes the Binomial **more realistic**, since in real traffic, the number of vehicles is limited by headway constraints.

- **Poisson** is a good model when flows are low and arrivals are independent.
- **Binomial** provides a **bounded and more flexible model**, especially when minimum headway effects need to be considered.

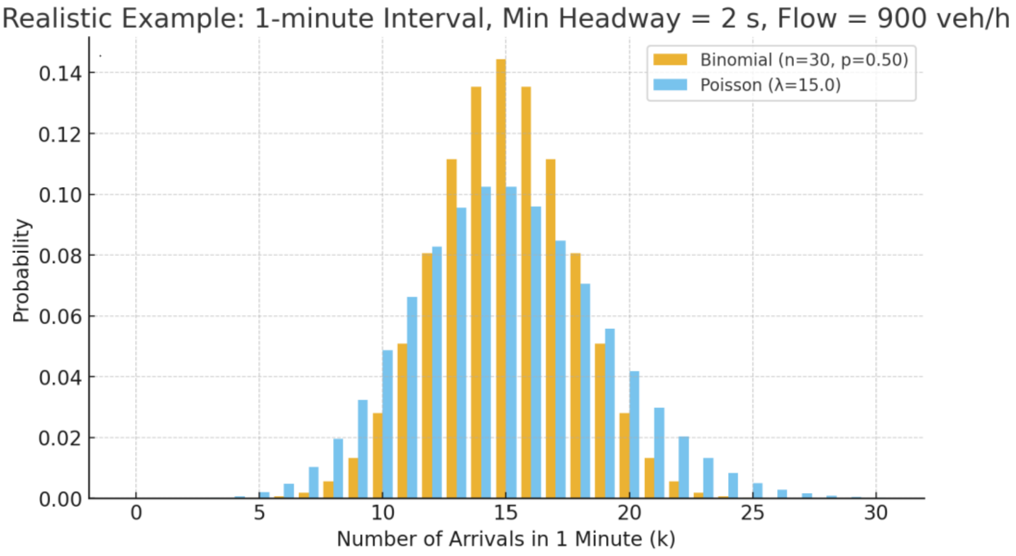


$\lambda = q$

In this figure, the **Binomial is compared with Poisson comparison** (both with the same mean, q = 3):

- **Binomial (yellow, n=10, p=0.3):**
    - Limited to a maximum of **10 arrivals** (bounded).
    - Probabilities are slightly more "concentrated" around the mean.
- **Poisson (blue, q=3):**
    - Theoretically unbounded (can exceed 10 arrivals, though with small probability).
    - Has a slightly longer "tail" to the right.

- **When n is large and p is small**, the Binomial approaches the Poisson distribution.
- But for traffic modeling, **Binomial is more realistic** since the maximum number of vehicles in an interval is capped by the **minimum headway**.
- Poisson may **overestimate rare high-arrival cases**, while Binomial respects physical limits.



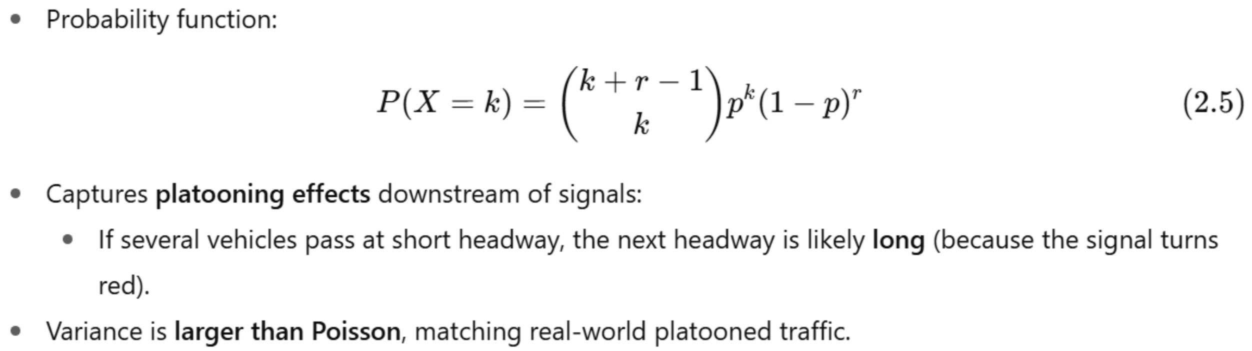
Here's a realistic comparison using a **1-minute interval** and **minimum headway = 2 s**:

- Physical cap ⇒ $n = 60/2 = 30$ vehicles per minute.
- Choose a flow $q = 900$ veh/h ⇒ mean per minute $= 15$.
- Binomial: $n = 30$, $p = 15/30 = 0.5$ (so mean $np = 15$).
- Poisson: $q = 15$.

The chart shows:

- **Binomial** (bounded at 30) is slightly tighter around the mean.
- **Poisson** has a longer right tail and assigns small probability beyond the physical cap.

NEGATIVE BINOMIAL DISTRIBUTION (NEGATIVELY CORRELATED ARRIVALS)

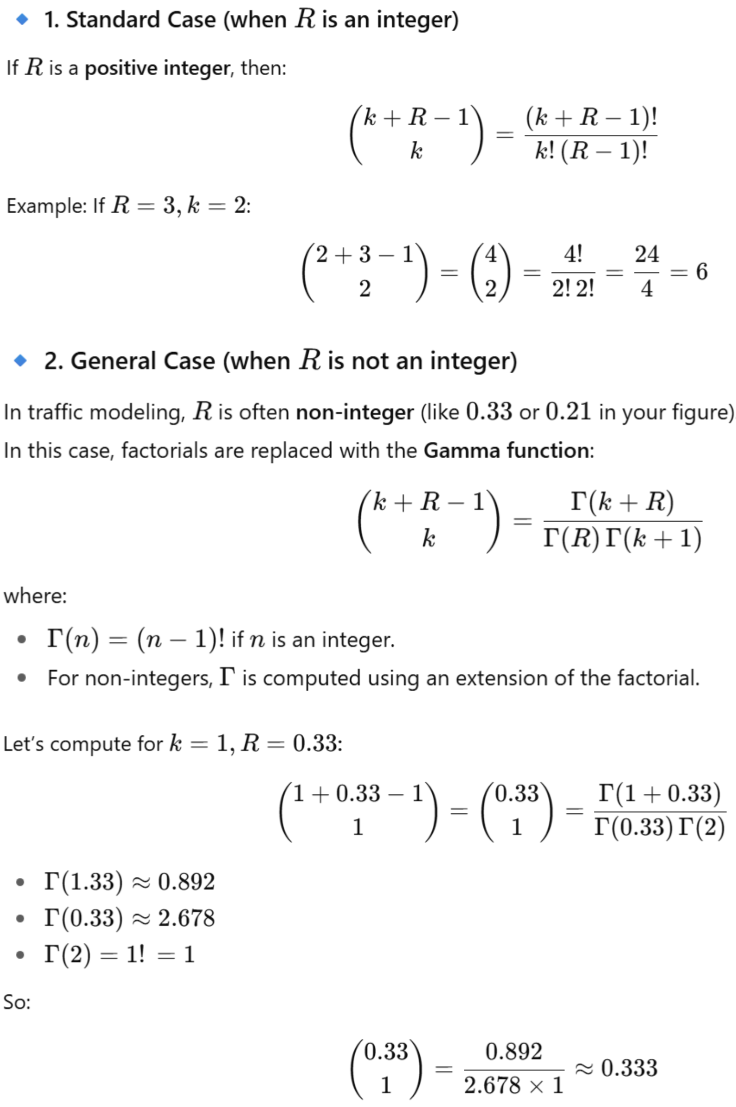


- Probability function:

$$P(X = k) = \binom{k + r - 1}{k} p^k (1 - p)^r \tag{2.5}$$

- Captures **platooning effects** downstream of signals:
  - If several vehicles pass at short headway, the next headway is likely **long** (because the signal turns red).
- Variance is **larger than Poisson**, matching real-world platooned traffic.

$\binom{k + r - 1}{k}$ is a **generalized binomial coefficient** that appears in the formula of the negative binomial distribution. Let's break it down:



◆ **1. Standard Case (when R is an integer)**

If R is a **positive integer**, then:

$$\binom{k + R - 1}{k} = \frac{(k + R - 1)!}{k!\,(R - 1)!}$$

Example: If $R = 3, k = 2$:

$$\binom{2 + 3 - 1}{2} = \binom{4}{2} = \frac{4!}{2!\,2!} = \frac{24}{4} = 6$$

◆ **2. General Case (when R is not an integer)**

In traffic modeling, R is often **non-integer** (like 0.33 or 0.21 in your figure).
In this case, factorials are replaced with the **Gamma function**:

$$\binom{k + R - 1}{k} = \frac{\Gamma(k + R)}{\Gamma(R)\,\Gamma(k + 1)}$$

where:

- $\Gamma(n) = (n - 1)!$ if n is an integer.
- For non-integers, Γ is computed using an extension of the factorial.

Let's compute for $k = 1, R = 0.33$:

$$\binom{1 + 0.33 - 1}{1} = \binom{0.33}{1} = \frac{\Gamma(1 + 0.33)}{\Gamma(0.33)\,\Gamma(2)}$$

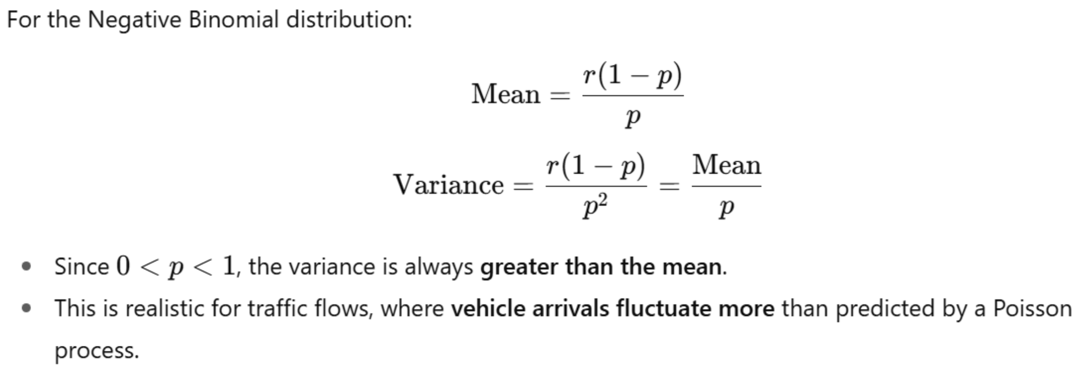
- $\Gamma(1.33) \approx 0.892$
- $\Gamma(0.33) \approx 2.678$
- $\Gamma(2) = 1! = 1$

So:

$$\binom{0.33}{1} = \frac{0.892}{2.678 \times 1} \approx 0.333$$

What the Negative Binomial Distribution Represents

- It models the number of **successes (vehicles observed)** before a fixed number of **failures (gaps or missed arrivals)** occurs.
- Parameters:
  - **r (R in figure)** = number of failures before stopping.
  - **p** = probability of success in each trial.
  - **X** = random variable indicating the number of successes (vehicles).
- This makes it flexible compared to the Poisson distribution since the **variance can be larger than the mean**, which fits traffic flow better (especially near signals).



For the Negative Binomial distribution:

$$\text{Mean} = \frac{r(1-p)}{p}$$

$$\text{Variance} = \frac{r(1-p)}{p^2} = \frac{\text{Mean}}{p}$$

- Since $0 < p < 1$, the variance is always **greater than the mean**.
- This is realistic for traffic flows, where **vehicle arrivals fluctuate more** than predicted by a Poisson process.

This distribution describes when one observes individual and independent process with a success rate of p. One observes so until r failures are observed. X is the stochastic indicating how many successes are observed.

Figure 2.5 shows the value of this function for different parameter sets. Note that the variance (and hence standard deviation) be set independently from the mean, like in the binomial distribution function. For this distribution (see table 2.2), the mean is given by n(1p)/p, and the variance is given by $n(1p)/p^2$, which is the mean divided by p.

Since p is a probability and has a value between 0 and 1, we can derive that the variance is larger than the mean. A larger variance is what one would intuitively expect downstream of a signalised intersection. This characteristic can be used to have an idea of the distribution to use.

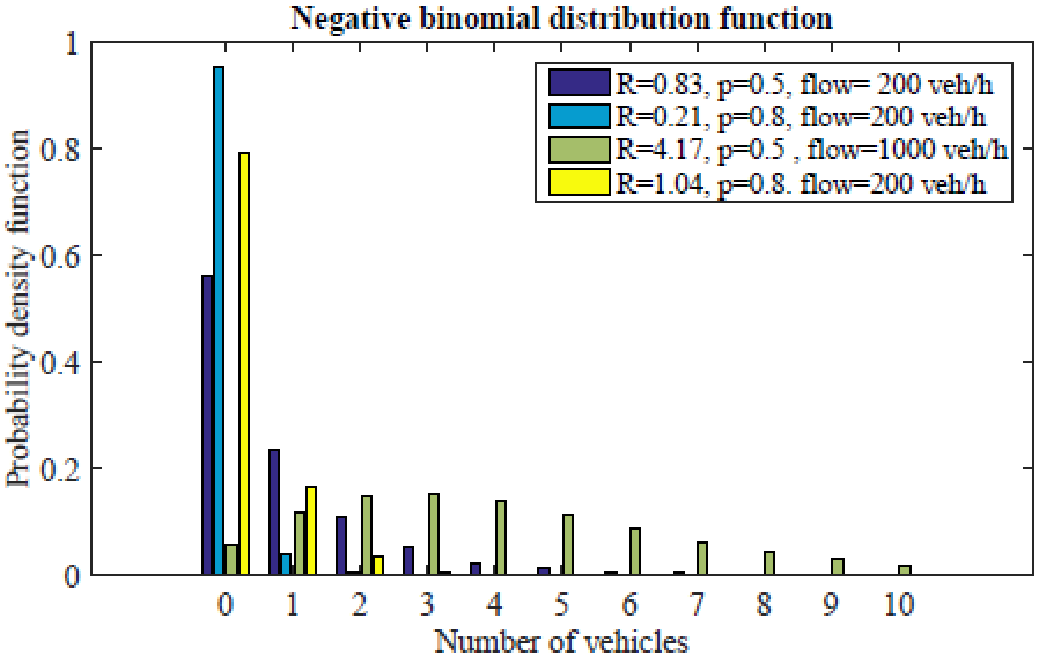


Figure 2.5: Example of the negative binomial function for the number of arrivals in an aggregation period

In figure 2.5, we observe

- **X-axis (Number of vehicles):** Number of observed arrivals in a short interval.
- **Y-axis (Probability density function):** Likelihood of that number occurring.
- Different bars correspond to different parameter settings (R,p,flow)

Examples from legend:

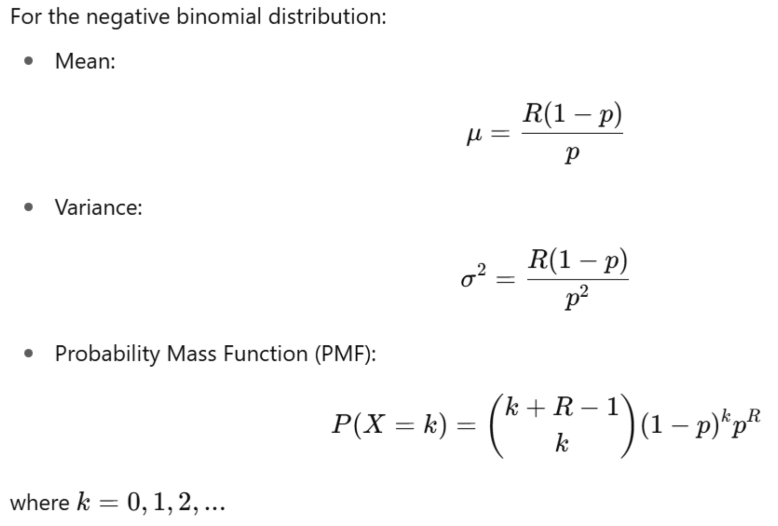
1. **R = 0.33, p = 0.5, flow = 200 veh/h (dark blue):**
   High probability for **0 vehicles**, quickly declines.
2. **R = 0.21, p = 0.8, flow = 200 veh/h (teal):**
   Even higher probability mass at **0 vehicles**, meaning more "gaps."
3. **R = 4.17, p = 0.5, flow = 1000 veh/h (green):**
   Much flatter distribution, indicating **more spread** (higher mean and variance).
   This matches **heavier traffic flows** with many vehicles observed.
4. **R = 1.04, p = 0.8, flow = 200 veh/h (yellow):**
   Still concentrated near 0, but with slightly higher spread than teal.

Traffic Engineering Meaning

- **Downstream of a signalized intersection**, vehicle arrivals are **not purely random (Poisson)** but more variable.
- Negative binomial allows variance > mean, capturing:
  - **Platooning of vehicles** (several arriving at once after a green phase).
  - **Long gaps** when the signal is red.
- Therefore, this distribution is useful to describe **realistic headways and arrival patterns**.

Example:

Case 1: R=0.33, p=0.5



For the negative binomial distribution:

- Mean:

$$\mu = \frac{R(1-p)}{p}$$

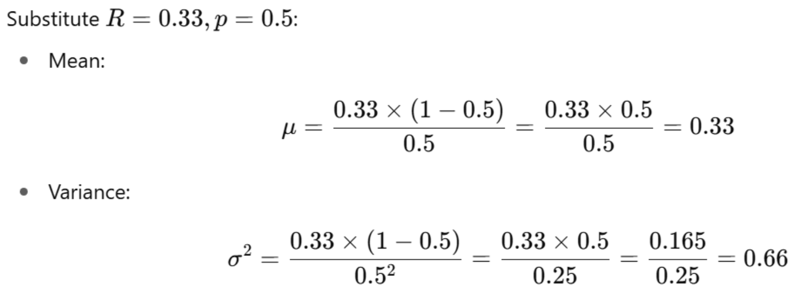
- Variance:

$$\sigma^2 = \frac{R(1-p)}{p^2}$$

- Probability Mass Function (PMF):

$$P(X=k) = \binom{k+R-1}{k}(1-p)^k p^R$$

where $k = 0, 1, 2, \ldots$



Substitute $R = 0.33, p = 0.5$:

- Mean:

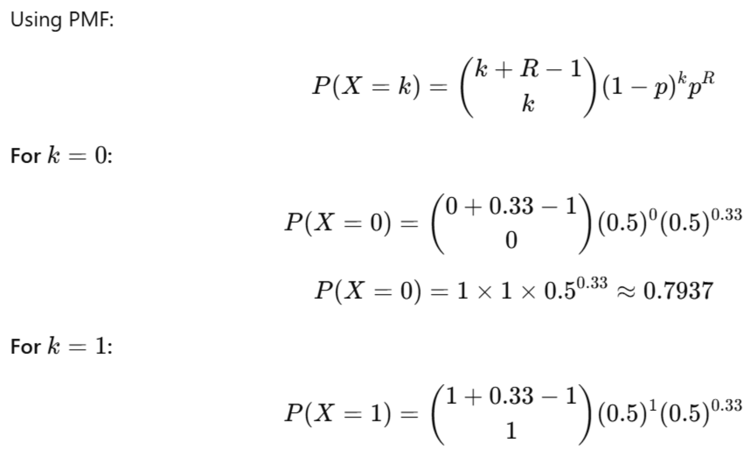
$$\mu = \frac{0.33 \times (1-0.5)}{0.5} = \frac{0.33 \times 0.5}{0.5} = 0.33$$

- Variance:

$$\sigma^2 = \frac{0.33 \times (1-0.5)}{0.5^2} = \frac{0.33 \times 0.5}{0.25} = \frac{0.165}{0.25} = 0.66$$

So mean = **0.33 vehicles**, variance = **0.66 vehicles²**.

Variance > mean → more spread (as expected).



Using PMF:

$$P(X=k) = \binom{k+R-1}{k}(1-p)^k p^R$$

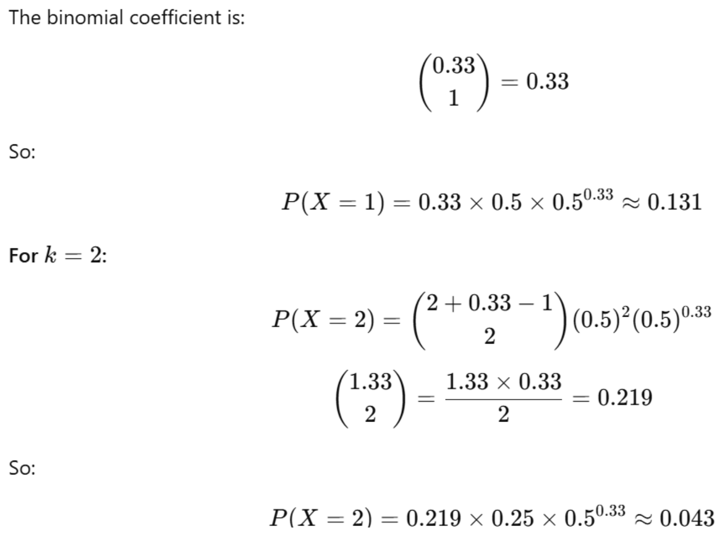
For $k = 0$:

$$P(X=0) = \binom{0+0.33-1}{0}(0.5)^0(0.5)^{0.33}$$

$$P(X=0) = 1 \times 1 \times 0.5^{0.33} \approx 0.7937$$

For $k = 1$:

$$P(X=1) = \binom{1+0.33-1}{1}(0.5)^1(0.5)^{0.33}$$



The binomial coefficient is:

$$\binom{0.33}{1} = 0.33$$

So:

$$P(X = 1) = 0.33 \times 0.5 \times 0.5^{0.33} \approx 0.131$$

For $k = 2$:

$$P(X = 2) = \binom{2 + 0.33 - 1}{2} (0.5)^2 (0.5)^{0.33}$$

$$\binom{1.33}{2} = \frac{1.33 \times 0.33}{2} = 0.219$$

So:

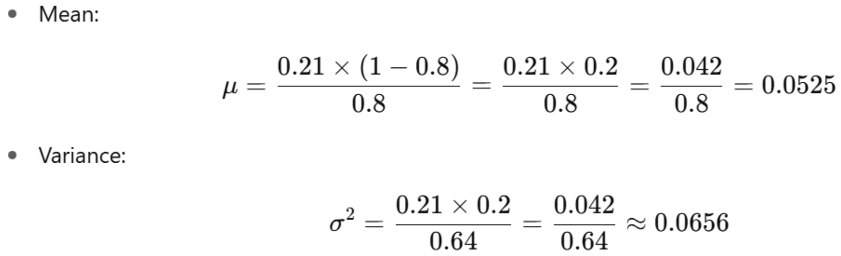
$$P(X = 2) = 0.219 \times 0.25 \times 0.5^{0.33} \approx 0.043$$

Interpretation

- P(X=0)≈0.79→ about 79% chance of **no vehicles**.
- P(X=1)≈0.13→ about 13% chance of **1 vehicle**.
- P(X=2)≈0.04→ about 4% chance of **2 vehicles**.

This matches the figure: **most of the mass is at 0 vehicles** (dark blue bar is tallest at 0).

Case 2 (R=0.21,p=0.8)



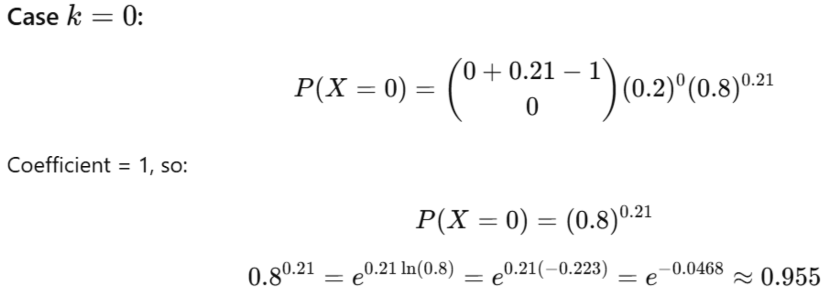
- Mean:

$$\mu = \frac{0.21 \times (1 - 0.8)}{0.8} = \frac{0.21 \times 0.2}{0.8} = \frac{0.042}{0.8} = 0.0525$$

- Variance:

$$\sigma^2 = \frac{0.21 \times 0.2}{0.64} = \frac{0.042}{0.64} \approx 0.0656$$

Mean ≈ **0.053**, Variance ≈ **0.066** → both very small, meaning almost always **0 vehicles**.



Case $k = 0$:

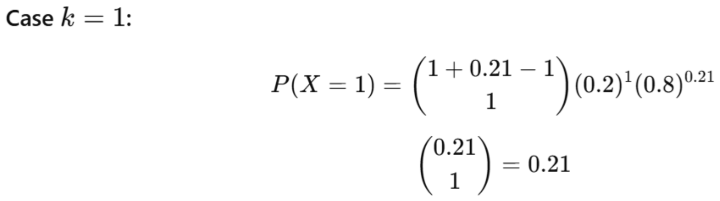
$$P(X = 0) = \binom{0 + 0.21 - 1}{0} (0.2)^0 (0.8)^{0.21}$$

Coefficient = 1, so:

$$P(X = 0) = (0.8)^{0.21}$$

$$0.8^{0.21} = e^{0.21 \ln(0.8)} = e^{0.21(-0.223)} = e^{-0.0468} \approx 0.955$$

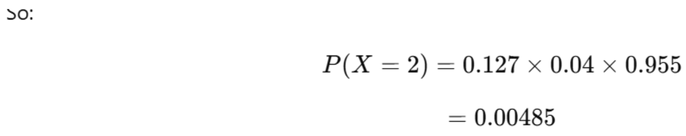
P(X=0)≈0.955.



Case $k = 1$:

$$P(X = 1) = \binom{1 + 0.21 - 1}{1}(0.2)^1(0.8)^{0.21}$$

$$\binom{0.21}{1} = 0.21$$

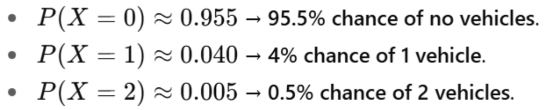


So:

$$P(X = 2) = 0.127 \times 0.04 \times 0.955$$

$$= 0.00485$$

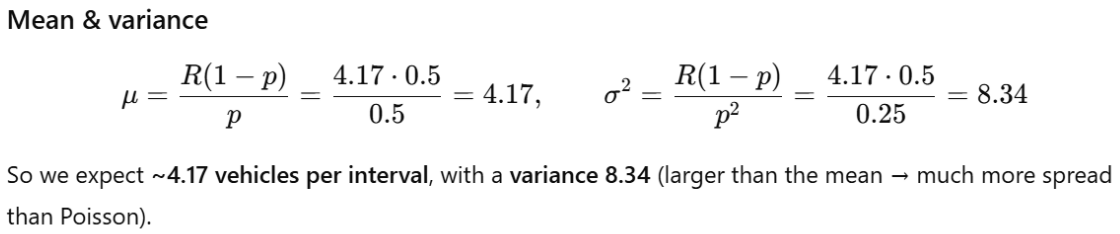
P(X=2)≈0.005.



- $P(X = 0) \approx 0.955$ → **95.5% chance of no vehicles.**
- $P(X = 1) \approx 0.040$ → **4% chance of 1 vehicle.**
- $P(X = 2) \approx 0.005$ → **0.5% chance of 2 vehicles.**

This matches the teal bar in the figure — **almost all probability at 0 vehicles** when p=0.8

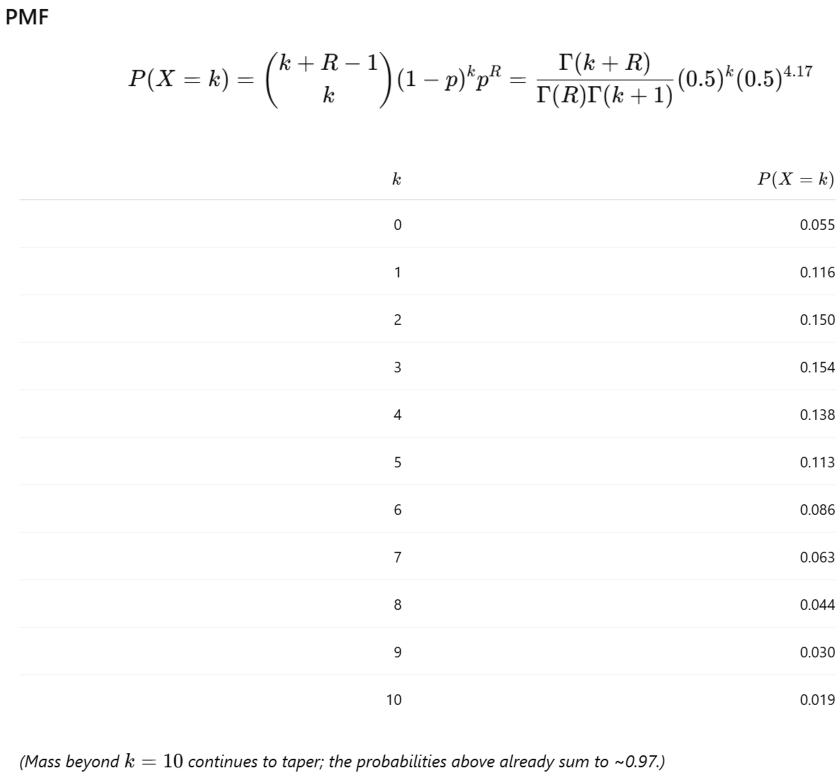
Case 3: R=4.17, p=0.5 (the green series).



Mean & variance

$$\mu = \frac{R(1-p)}{p} = \frac{4.17 \cdot 0.5}{0.5} = 4.17, \qquad \sigma^2 = \frac{R(1-p)}{p^2} = \frac{4.17 \cdot 0.5}{0.25} = 8.34$$

So we expect **~4.17 vehicles per interval**, with a **variance 8.34** (larger than the mean → much more spread than Poisson).



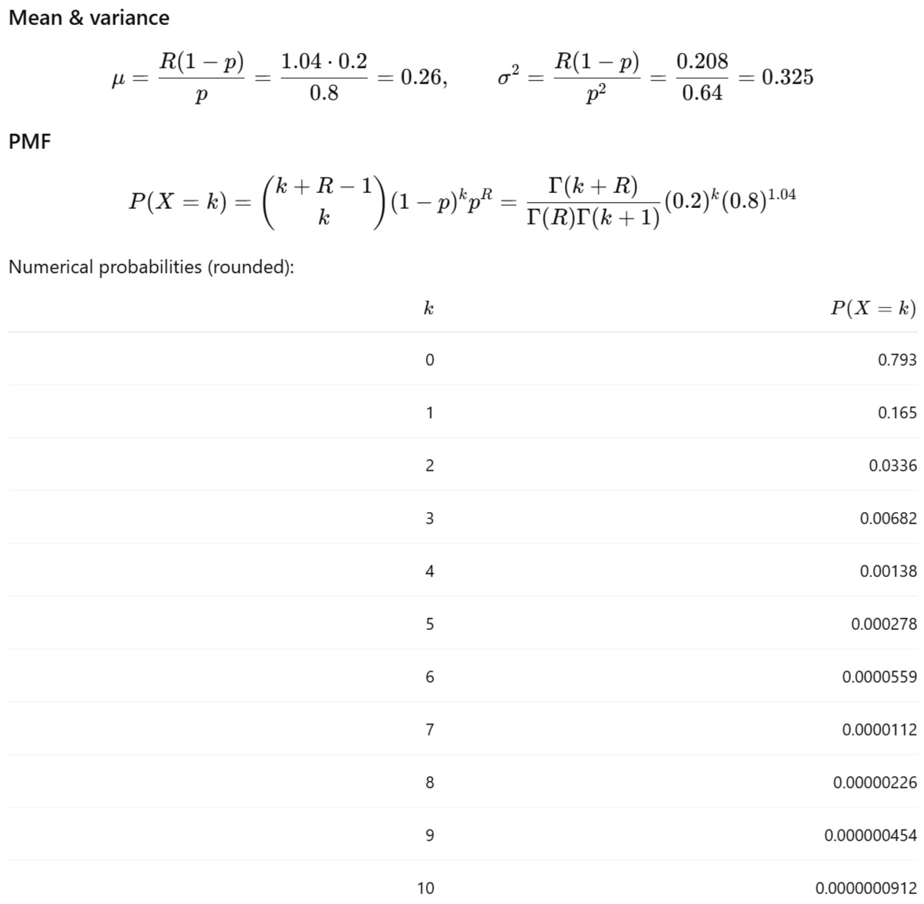
PMF

$$P(X = k) = \binom{k + R - 1}{k}(1-p)^k p^R = \frac{\Gamma(k + R)}{\Gamma(R)\Gamma(k+1)}(0.5)^k(0.5)^{4.17}$$

k	$P(X = k)$
0	0.055
1	0.116
2	0.150
3	0.154
4	0.138
5	0.113
6	0.086
7	0.063
8	0.044
9	0.030
10	0.019

(Mass beyond $k = 10$ continues to taper; the probabilities above already sum to ~0.97.)

Case 4: R=1.04, p=0.8 (yellow series).



Mean & variance

$$\mu = \frac{R(1-p)}{p} = \frac{1.04 \cdot 0.2}{0.8} = 0.26, \qquad \sigma^2 = \frac{R(1-p)}{p^2} = \frac{0.208}{0.64} = 0.325$$

PMF

$$P(X=k) = \binom{k+R-1}{k}(1-p)^k p^R = \frac{\Gamma(k+R)}{\Gamma(R)\Gamma(k+1)}(0.2)^k(0.8)^{1.04}$$

Numerical probabilities (rounded):

k	$P(X=k)$
0	0.793
1	0.165
2	0.0336
3	0.00682
4	0.00138
5	0.000278
6	0.0000559
7	0.0000112
8	0.00000226
9	0.000000454
10	0.0000000912

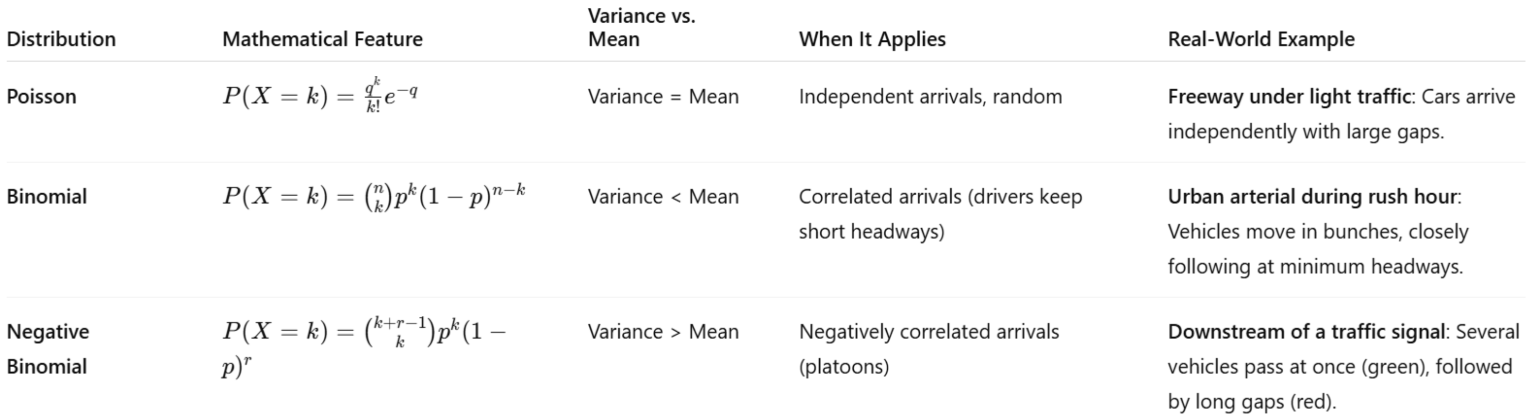
COMPARISON OF ARRIVAL DISTRIBUTIONS IN TRAFFIC FLOW

k	Case1: R=0.33, p=0.5	Case2: R=0.21, p=0.8	Case3: R=4.17, p=0.5	Case4: R=1.04, p=0.8
0	0.795536	0.954221	0.055553	0.792891
1	0.131264	0.040077	0.115827	0.164921
2	0.043645	0.004849	0.149707	0.033644
3	0.016949	0.000714	0.153948	0.006819
4	0.007055	0.000115	0.137976	0.001377
5	0.003055	1.93E-05	0.112727	0.000278
6	0.001357	3.35E-06	0.086142	5.59E-05
7	0.000613	5.95E-07	0.062576	1.12E-05
8	0.000281	1.07E-07	0.043686	2.26E-06
9	0.00013	1.96E-08	0.029536	4.54E-07
10	6.07E-05	3.60E-09	0.01945	9.12E-08
11	2.85E-05	6.69E-10	0.012527	1.83E-08
12	1.34E-05	1.25E-10	0.007918	3.67E-09
13	6.38E-06	2.35E-11	0.004925	7.37E-10
14	3.04E-06	4.43E-12	0.00302	1.48E-10

- **Case 1 (R=0.33, p=0.5):** Strongly peaked at 0 vehicles, with a small tail.
- **Case 2 (R=0.21, p=0.8):** Almost all probability at 0 vehicles (>95%).
- **Case 3 (R=4.17, p=0.5):** Broad distribution centered around 3–5 vehicles, representing heavy flows.
- **Case 4 (R=1.04, p=0.8):** Mostly 0–1 vehicles (~96%), but with a longer tiny tail.

Let’s put the three main **arrival distributions** into a clear **comparison table with real-world traffic examples**:

Table (2.2)

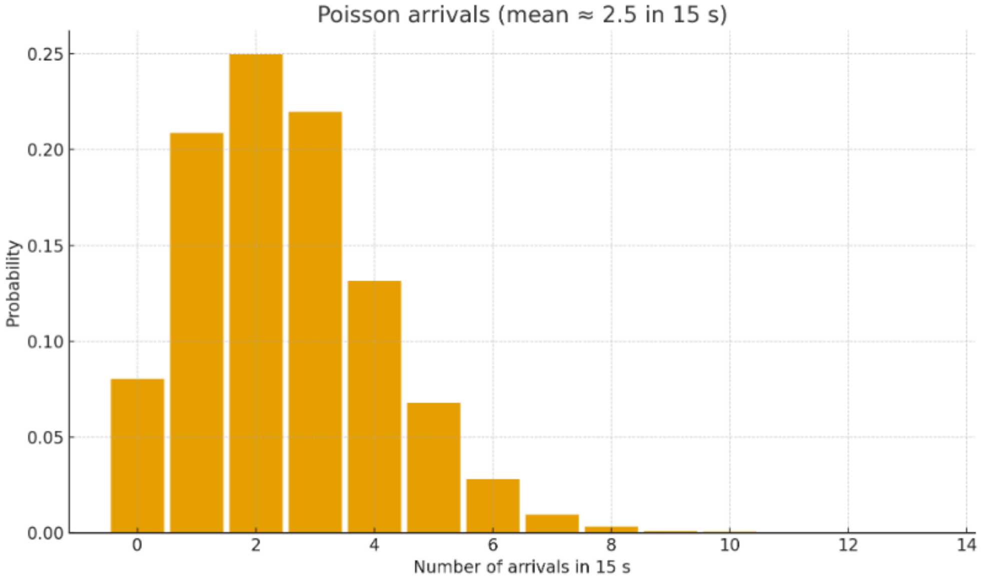


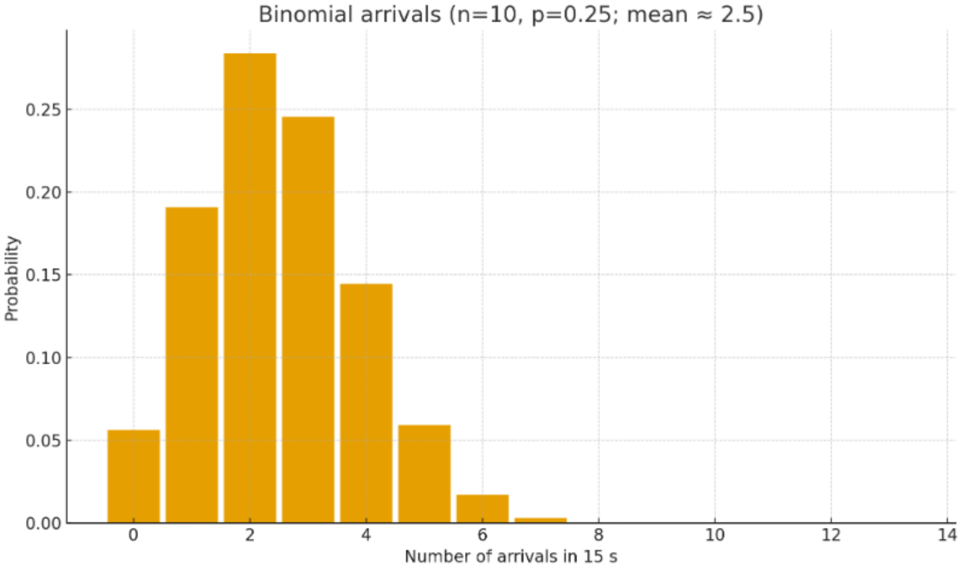
Distribution	Mathematical Feature	Variance vs. Mean	When It Applies	Real-World Example
Poisson	$P(X=k)=\frac{q^k}{k!}e^{-q}$	Variance = Mean	Independent arrivals, random	**Freeway under light traffic**: Cars arrive independently with large gaps.
Binomial	$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$	Variance < Mean	Correlated arrivals (drivers keep short headways)	**Urban arterial during rush hour**: Vehicles move in bunches, closely following at minimum headways.
Negative Binomial	$P(X=k)=\binom{k+r-1}{k}p^k(1-p)^r$	Variance > Mean	Negatively correlated arrivals (platoons)	**Downstream of a traffic signal**: Several vehicles pass at once (green), followed by long gaps (red).

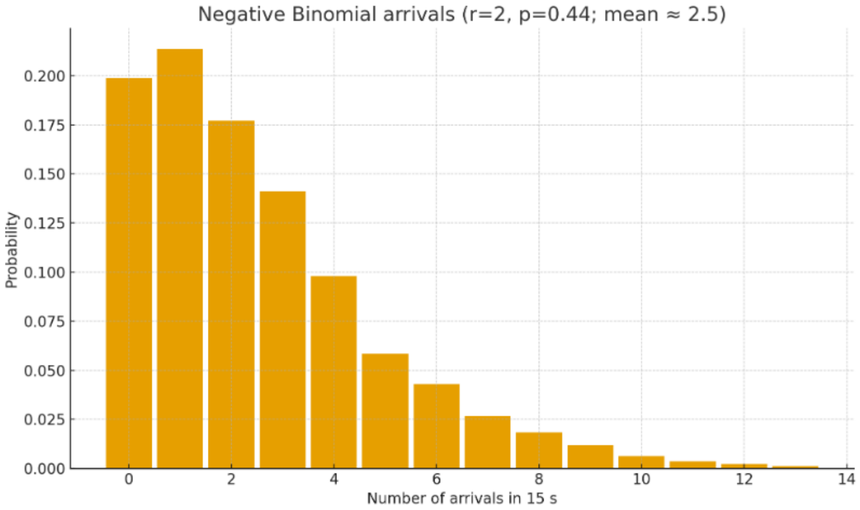
Interpretation:

- **Highway (light flow)** → arrivals are random → **Poisson fits best**.
- **Congested road** (drivers follow closely) → arrivals more predictable, less variation → **Binomial fits.**
- **Signalized intersection** (platoons released, then gap) → arrivals highly variable → **Negative Binomial fits**.

PLOT EXAMPLE ARRIVAL HISTOGRAMS (POISSON VS. BINOMIAL VS. NEGATIVE BINOMIAL) WITH THE SAME MEAN FLOW (SAY 600 VEH/HR)


Poisson arrivals (mean ≈ 2.5 in 15 s)
Probability
0.00
0.05
0.10
0.15
0.20
0.25
0
2
4
6
8
10
12
14
Number of arrivals in 15 s


Binomial arrivals (n=10, p=0.25; mean ≈ 2.5)
Probability
0.00
0.05
0.10
0.15
0.20
0.25
0
2
4
6
8
10
12
14
Number of arrivals in 15 s

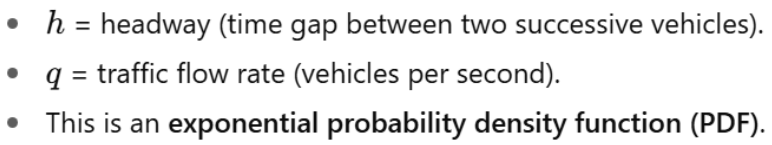

Negative Binomial arrivals (r=2, p=0.44; mean ≈ 2.5)
Probability
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
0
2
4
6
8
10
12
14
Number of arrivals in 15 s

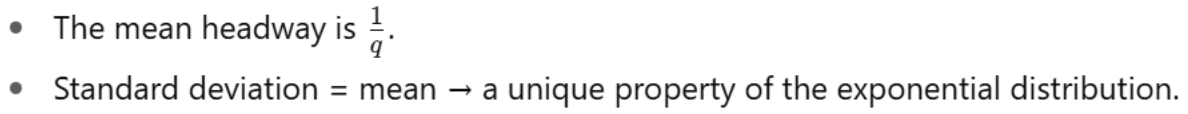
2.3 HEADWAY DISTRIBUTIONS

EXPONENTIAL

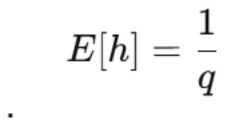
The first distribution is the exponential distribution, shown in figure 2.6. This is defined by:

$$P(h|q) = q\, e^{-qh} \tag{2.6}$$


- h = headway (time gap between two successive vehicles).
- q = traffic flow rate (vehicles per second).
- This is an **exponential probability density function (PDF)**.


- The mean headway is $\frac{1}{q}$.
- Standard deviation = mean → a unique property of the exponential distribution.

- Key Properties: This equation has **a single parameter**, being the **flow q**. That is the inverse of the average headway.

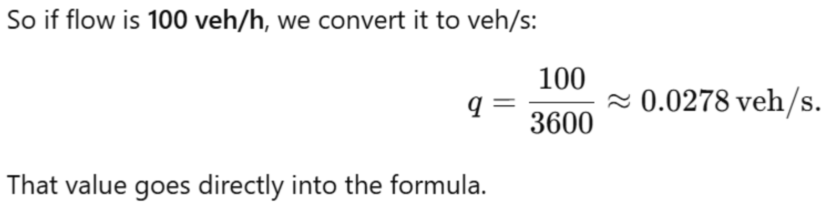


$$E[h] = \frac{1}{q}$$

.

It assumes **independence of vehicle arrivals**:

- Each driver decides independently when to depart.
- Equivalent to a **Poisson process** of arrivals.



So if flow is **100 veh/h**, we convert it to veh/s:

$$q = \frac{100}{3600} \approx 0.0278\,\text{veh/s}.$$

That value goes directly into the formula.

- The underlying assumption:
  - all drivers can choose their moment of arrival independently. Consider that each time step a driver considers to leave with a fixed probability.
  - This is a very good assumption on quiet roads, when there are no interactions between the vehicles. The interactions occur once vehicles are limited in choosing their headway, mostly indicated by the minimum headway.

- Limitations

At higher flows, drivers cannot choose freely:

  - **Interactions occur**, and vehicles must maintain at least a **minimum headway**.
  - The exponential model becomes unrealistic, since it allows arbitrarily small headways.

- illustration how this works out for a real-life case is shown in figure 2.6

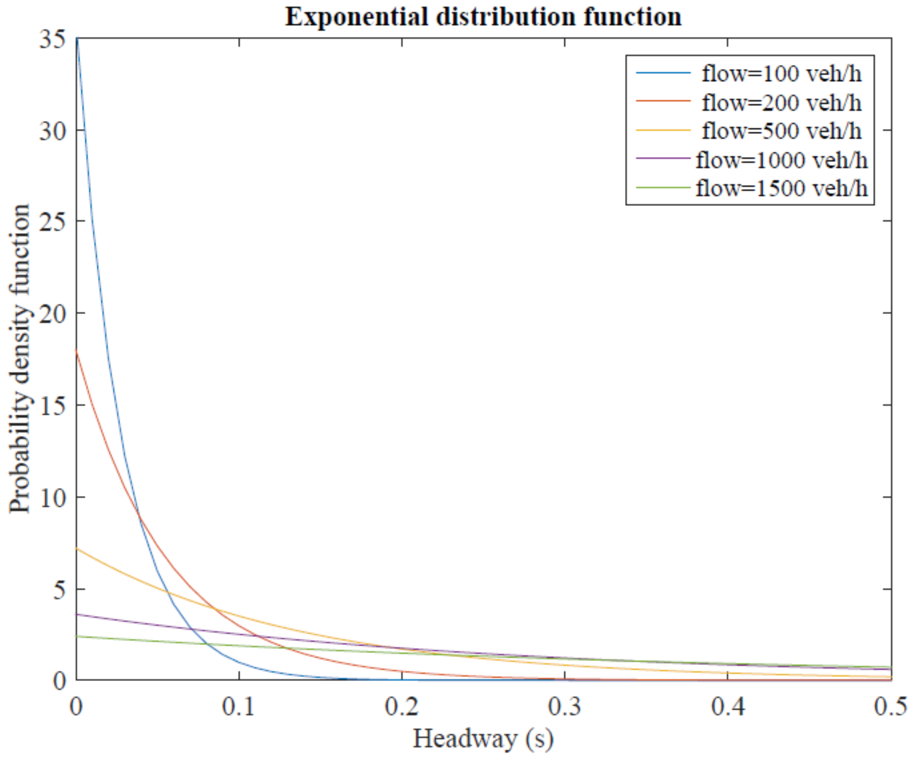


Figure 2.6: The probability density function for headways according to the exponential distribution for different flow values

- Each curve in the figure corresponds to a different flow (q).
- **Higher flow (q)** = smaller average headway (vehicles are closer together).
- **Lower flow** = larger headway (vehicles are spaced further apart).

For example:

- At **flow = 100 veh/h**, the curve starts very high at small h, then decays quickly. This reflects long expected gaps between vehicles (low density).
- At **flow = 1500 veh/h**, the curve is much flatter → vehicles arrive more frequently and closer together.

- In summary: The exponential headway distribution is mathematically simple and realistic for **light traffic (free flow)**, but fails in **moderate to heavy traffic**, which is why **composite headway models** are introduced.

Testing for Exponential Distribution

- The **best method**: perform a **formal statistical test**, e.g.,
    - **Kolmogorov–Smirnov (K-S) test** – compares the observed distribution with the theoretical exponential distribution.
- **Rule of thumb**:
    - For the exponential distribution, the **mean = standard deviation**.
    - If measured headway data satisfy this property (and arrivals appear independent), it strongly suggests that the arrivals are **exponentially distributed**.

Relationship to Poisson Arrivals

If headways follow an exponential distribution, then:

- The **arrival process** follows a **Poisson distribution**.
- This means the probability of observing *n* vehicles in a time interval *t* is:

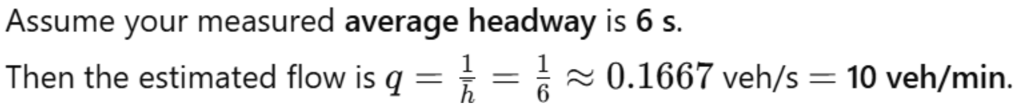
$$P(N(t) = n) = \frac{(qt)^n}{n!} e^{-qt}$$

where *q* is the flow rate.

- Interpretation:

- The exponential headway distribution describes the **time between arrivals**.
- The Poisson distribution describes the **number of arrivals in a given period**.

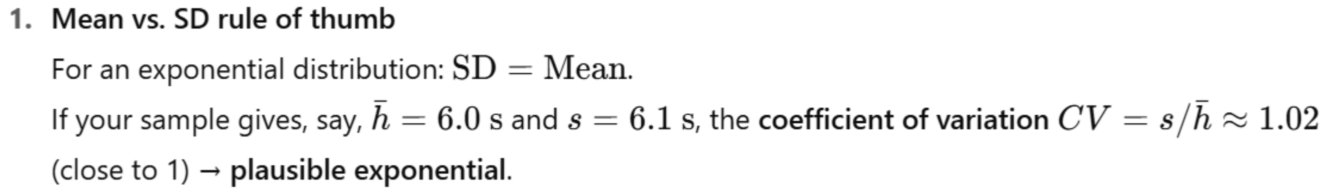
Example



Assume your measured **average headway** is **6 s**.

Then the estimated flow is $q = \frac{1}{\bar{h}} = \frac{1}{6} \approx 0.1667$ veh/s = **10 veh/min**.

- **Quick checks for “exponential headways”**



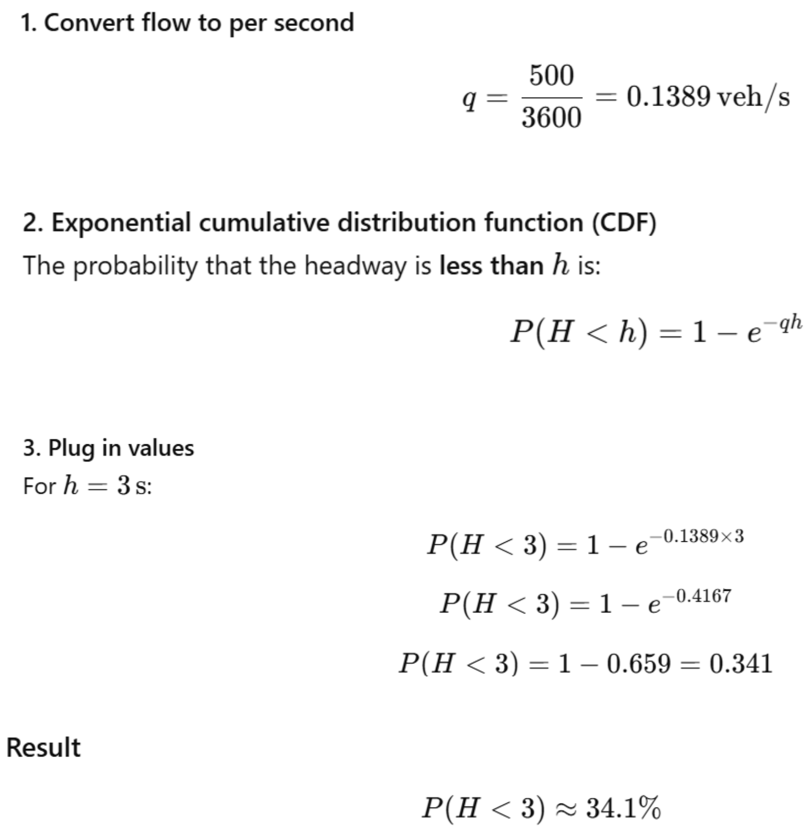
1. **Mean vs. SD rule of thumb**

   For an exponential distribution: $\mathrm{SD} = \mathrm{Mean}$.

   If your sample gives, say, $\bar{h} = 6.0$ s and $s = 6.1$ s, the **coefficient of variation** $CV = s/\bar{h} \approx 1.02$ (close to 1) → **plausible exponential**.

Example:

Probability that headway h<3 , Take **flow = 500 veh/h**.



1. Convert flow to per second

$$q = \frac{500}{3600} = 0.1389\,\text{veh/s}$$

2. Exponential cumulative distribution function (CDF)

The probability that the headway is **less than** h is:

$$P(H < h) = 1 - e^{-qh}$$

3. Plug in values

For $h = 3\,\text{s}$:

$$P(H < 3) = 1 - e^{-0.1389\times3}$$

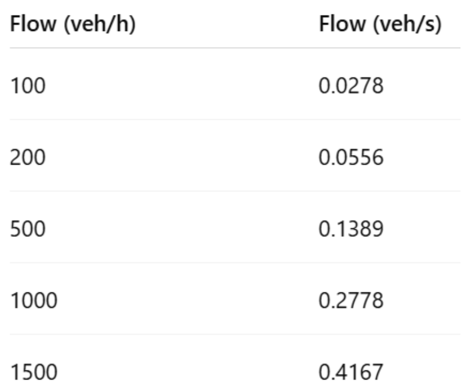
$$P(H < 3) = 1 - e^{-0.4167}$$

$$P(H < 3) = 1 - 0.659 = 0.341$$

Result

$$P(H < 3) \approx 34.1\%$$

So, when the flow is **500 veh/h**, about **one-third of headways are expected to be less than 3 seconds**.



Flow (veh/h)	Flow (veh/s)
100	0.0278
200	0.0556
500	0.1389
1000	0.2778
1500	0.4167

- At **low flows (100 veh/h)**, vehicles are widely spaced (avg headway ~36 s).
- At **high flows (1500 veh/h)**, vehicles arrive more frequently (avg headway ~2.4 s).

COMPOSITE HEADWAY MODELS

- Whereas the exponential distribution function works well for low flows, for higher flows the distribution function is not very good. For these situations, so called composite headway distributions are being used

- The basic idea is that At any moment, a fraction Φ of vehicles are free (not closely following), and the rest 1−Φ are constrained (following a leader) and have a headway distribution function $P_{constraint}(h)$.

- In a composite headway distribution, these two distribution functions are combined. The combined distribution function can hence be expressed as:

$$P(h) = \Phi P_{\text{free}}(h) + (1-\Phi) P_{\text{constraint}}(h) \tag{2.7}$$

Figure (2.7) shows an **example of the composite headway distribution** and how it is estimated from real-life traffic data.

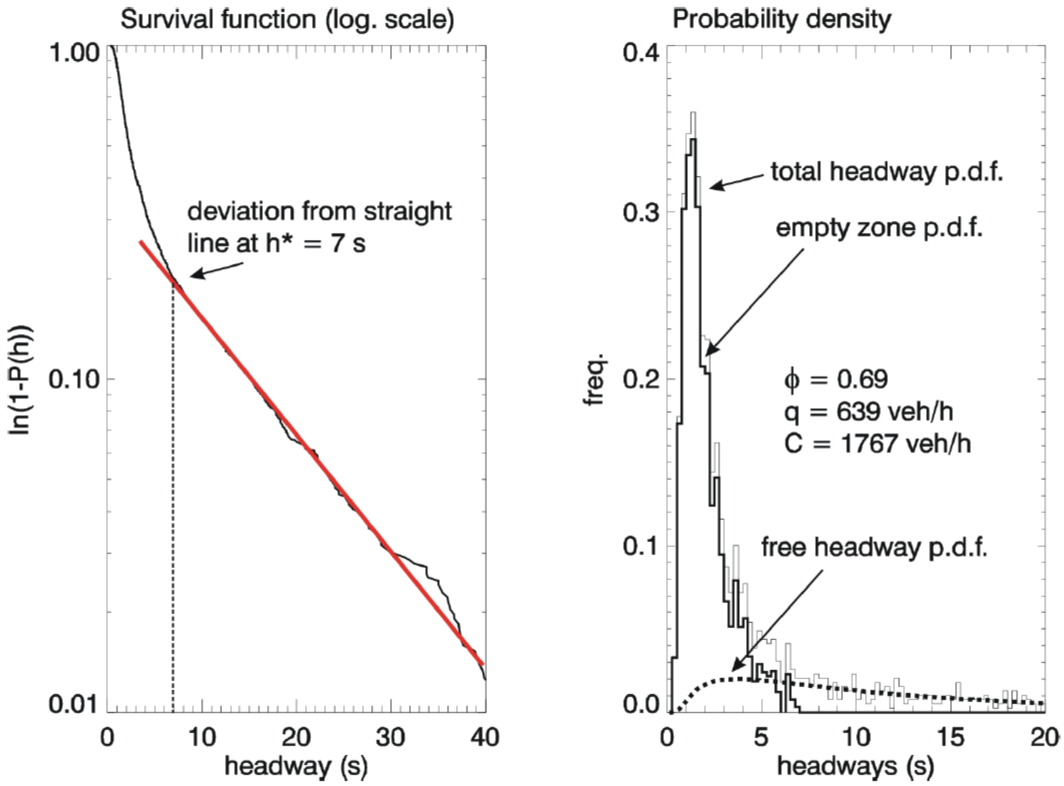


Figure 2.7: Example of the composite headway distribution and its estimation for real life data

Left Plot: Survival Function (log scale)

- The vertical axis is ln(1−P(h)), i.e., the logarithm of the Survival Function (the probability that a headway exceeds a certain value).
- In theory, if headways followed a **pure exponential distribution**, the plot would form a **straight line** on this log-scale graph.
- For **large headways (> 7 s)**, the points indeed lie approximately along a straight line → supporting the exponential model in this range.
- For **small headways (< 7 s)**, the line deviates downward → indicating that short headways don’t fit the exponential assumption. This deviation is due to **driver following behavior and minimum safe distances** (vehicles can’t follow too closely in reality).

Thus, h∗≈7h seconds is the Cutoff Headway where exponential behavior starts to hold.

Right Plot: Probability Density Function (PDF)

- This shows the **empirical distribution of headways**.
- Three main components are visible:
    1. **Empty Zone PDF** (near zero): Headways shorter than the minimum safe distance are practically impossible.
    2. **Free Headway PDF** (longer headways): Follows an exponential decline, reflecting random arrivals when vehicles are not constrained.
    3. **Total Headway PDF**: Combination of constrained short headways and free long headways → the COMPOSITE DISTRIBUTION.
- The figure also provides traffic parameters:
    - ϕ=0.69: Proportion of constrained (platooned) vehicles.
    - q=639 veh/h: Observed traffic flow.
    - C=1767 veh/h: Estimated roadway capacity.

The figure shows how **observed headways deviate from an exponential distribution at short times due to car-following constraints**. For headways larger than about 7 seconds, the exponential model holds. The composite distribution is built by combining the constrained (platoon) headways with the exponential free-flow headways.

Key Idea of Composite Headway Distribution

- Real traffic does not follow a **single distribution** (like pure exponential).
- Instead, it is **composite**:
    - For small headways, limited by car-following → constrained headways.
    - For large headways, arrivals behave more randomly → exponential tail.

This combined model reflects **real-life driver behavior** much better than assuming a single exponential distribution.

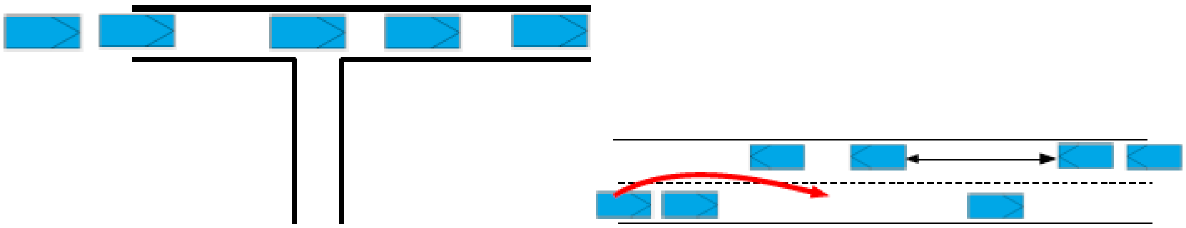
CRITICAL GAP

- **Definition**: The smallest time gap between vehicles that a driver is willing to accept to complete a maneuver (like turning, merging, or overtaking).
- It’s driver- and situation-dependent: some drivers need longer gaps; others accept shorter ones.

SITUATIONS

Critical gap matters in cases such as:

- **Turning** into a priority road.
- **Merging** into a traffic stream.
- **Lane changing**.
- **Overtaking**.



(a) Turning onto a road which has priority

(b) Overtaking

This figure illustrates **two typical situations where the critical gap concept is applied**

(a) Turning onto a road which has priority

- A vehicle on a minor road wants to turn onto a **major road** with continuous traffic flow.
- The driver must wait until there is a **sufficient time gap** between vehicles on the main road to safely merge.
- The **critical gap** here is the **minimum headway** (time between two consecutive vehicles on the main road) that the turning driver considers safe to enter.
- If gaps are smaller than the critical gap, the driver keeps waiting; once a gap ≥ critical gap occurs, the maneuver happens.

(b) Overtaking

- A vehicle in one lane wants to **overtake** another vehicle by moving into the opposite lane.
- The red curved arrow shows the **maneuver** (moving to the opposite lane and back).
- The driver must consider:
  - The **time needed to complete the overtaking maneuver**.
  - The **speed and distance of oncoming vehicles** in the opposite lane.
  - A **safety margin** (extra time/space for uncertainty).
- The **critical gap** is the **minimum safe headway** in the oncoming traffic that allows the overtaking vehicle to finish the maneuver without conflict.

Example: Overtaking

- Suppose a driver needs **3 seconds** to pass on the opposite lane.
- During those 3 seconds, opposing traffic also advances forward.
- To calculate the critical gap:
  1. Compute the **space needed** for the overtaking vehicle (time × speed).
  2. **Add** the distance covered by opposing vehicles during that time.
  3. Include a **safety margin**.
- Convert this total distance back into **time gap** by dividing by the speed of the opposing traffic.
- This gives the minimum gap needed in the opposing lane for the overtaking to be safe.

- The driver's decision depends on whether the available **headway ≥ critical gap**.
- This directly ties into inflow capacity: the larger the critical gap, the fewer vehicles can merge or overtake per unit time.

INFLOW CAPACITY

This links critical gap to **road capacity**.

- Imagine the **main traffic stream** has a headway distribution P(h).
- There's a continuous queue of vehicles wanting to merge, each needing at least the critical gap gc .
- **Step 1**: Count how many acceptable gaps (≥ gc) occur per unit of time.
- **Step 2**: That number equals the maximum vehicles that can enter (inflow capacity).
- **Key Point**: Flow (vehicles per unit time) = inverse of headway. So the inflow capacity depends on the headway distribution.

- **Critical gap** = the minimum safe gap a driver accepts.
- **Inflow capacity** = how many vehicles can merge given the distribution of available gaps and the critical gap requirement.

- Vehicles from the minor road can only enter when there is a gap h in the major stream.
- Depending on how **large** that gap is, **more than one vehicle** can enter:
  - If gc≤h<2gc: at most **1 vehicle** enters.
  - If 2gc≤h<3 : at most **2 vehicles** enter.
  - If 3gc≤h<4gc: at most **3 vehicles** enter.
  - … and so on.

This leads to a summation over all possible numbers of vehicles that could enter per gap.

One vehicle case (Eq. 2.8):

$$q_{\text{in,1 vehicle}} = q \int_{g_c}^{2g_c} P(h)\, dh$$

• q= flow rate of major stream (gap frequency = 1/E[h]).

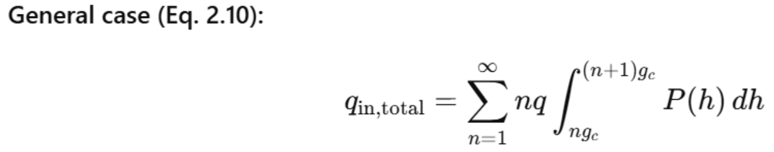
• Probability that a headway lies in [gc,2gc].

Two vehicle case (Eq. 2.9):

$$q_{\text{in,2 vehicles}} = 2q \int_{2g_c}^{3g_c} P(h)\, dh$$

• Probability of gap in $[2g_c,3g_c]$.

• Two vehicles can enter, hence the factor **2**.

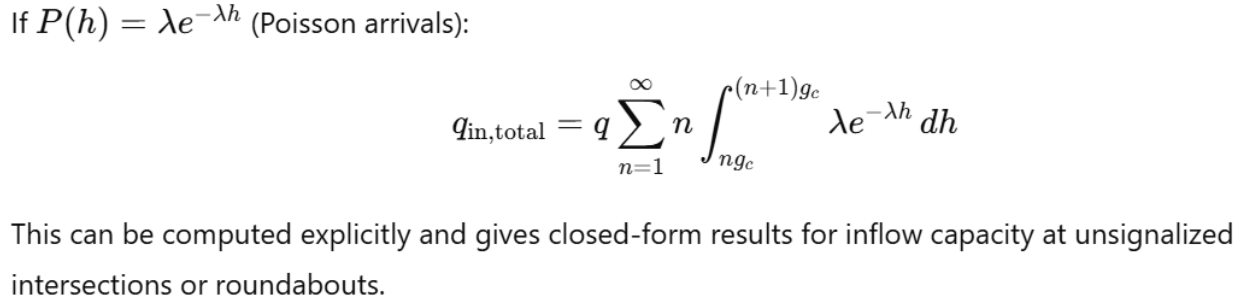


General case (Eq. 2.10):

$$q_{\text{in,total}} = \sum_{n=1}^{\infty} nq \int_{ng_c}^{(n+1)g_c} P(h)\,dh$$

- n = number of vehicles entering in that gap.
- Each term:
  - nq = rate of gaps multiplied by number of vehicles per gap.
  - Integral = probability that the gap length lies between ng_c and $(n+1)g_c$.
- This summation accounts for the fact that **larger gaps allow multiple vehicles** to pass.

Special Case: Exponential Distribution

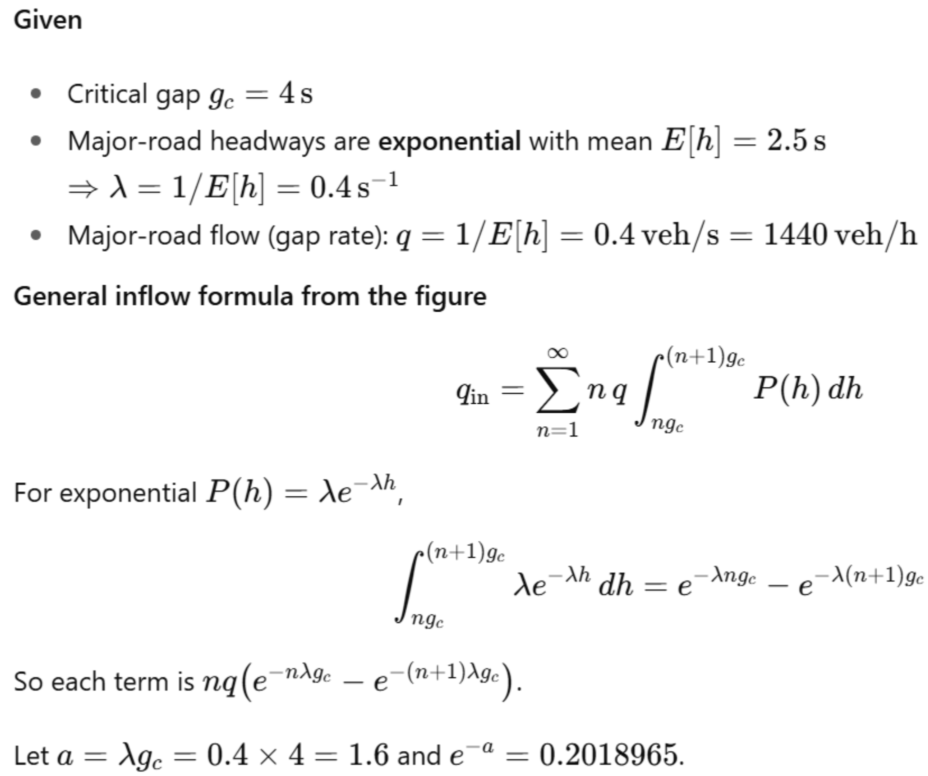


If $P(h) = \lambda e^{-\lambda h}$ (Poisson arrivals):

$$q_{\text{in,total}} = q \sum_{n=1}^{\infty} n \int_{ng_c}^{(n+1)g_c} \lambda e^{-\lambda h}\,dh$$

This can be computed explicitly and gives closed-form results for inflow capacity at unsignalized intersections or roundabouts.

Example:



Given

- Critical gap $g_c = 4\,\mathrm{s}$
- Major-road headways are **exponential** with mean $E[h] = 2.5\,\mathrm{s}$
  $\Rightarrow \lambda = 1/E[h] = 0.4\,\mathrm{s}^{-1}$
- Major-road flow (gap rate): $q = 1/E[h] = 0.4\,\mathrm{veh/s} = 1440\,\mathrm{veh/h}$

General inflow formula from the figure

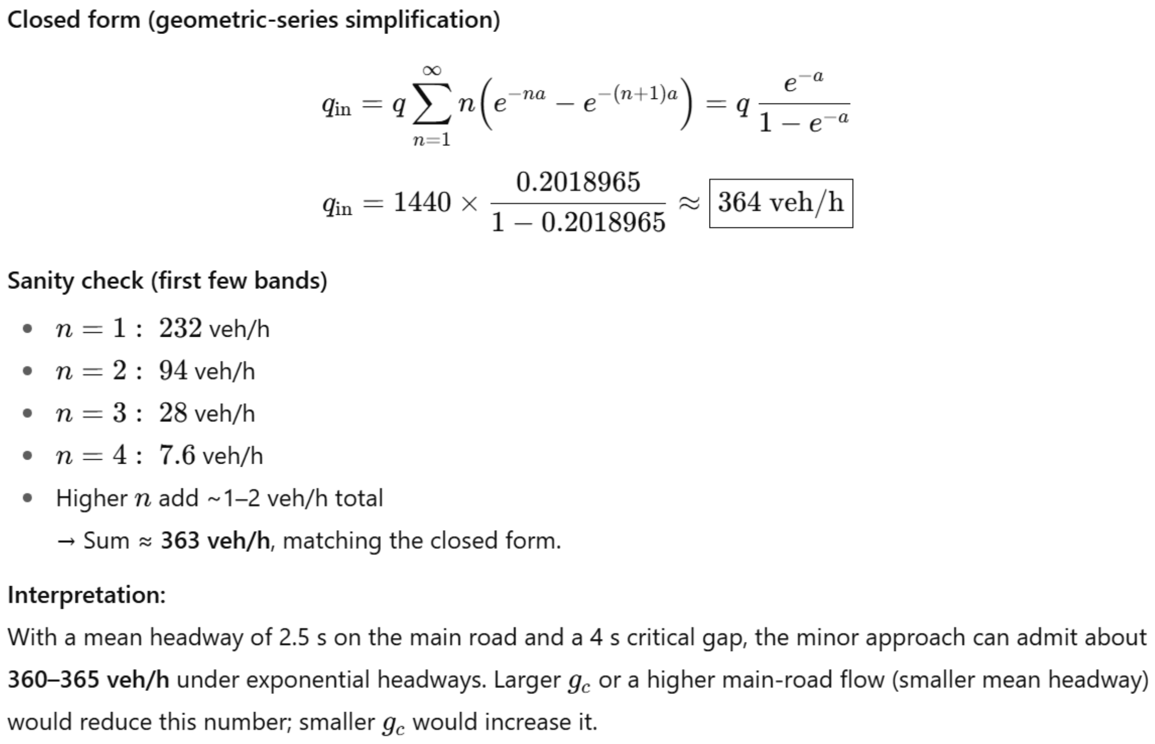
$$q_{\text{in}} = \sum_{n=1}^{\infty} n\,q \int_{ng_c}^{(n+1)g_c} P(h)\,dh$$

For exponential $P(h) = \lambda e^{-\lambda h}$,

$$\int_{ng_c}^{(n+1)g_c} \lambda e^{-\lambda h}\,dh = e^{-\lambda n g_c} - e^{-\lambda (n+1) g_c}$$

So each term is $nq\big(e^{-n\lambda g_c} - e^{-(n+1)\lambda g_c}\big)$.

Let $a = \lambda g_c = 0.4 \times 4 = 1.6$ and $e^{-a} = 0.2018965$.



Closed form (geometric-series simplification)

$$q_{\text{in}} = q\sum_{n=1}^{\infty} n\Big(e^{-na} - e^{-(n+1)a}\Big) = q\,\frac{e^{-a}}{1-e^{-a}}$$

$$q_{\text{in}} = 1440 \times \frac{0.2018965}{1-0.2018965} \approx \boxed{364\ \text{veh/h}}$$

Sanity check (first few bands)

- $n = 1:\ 232$ veh/h
- $n = 2:\ 94$ veh/h
- $n = 3:\ 28$ veh/h
- $n = 4:\ 7.6$ veh/h
- Higher n add ~1–2 veh/h total

→ Sum ≈ **363 veh/h**, matching the closed form.

Interpretation:

With a mean headway of 2.5 s on the main road and a 4 s critical gap, the minor approach can admit about **360–365 veh/h** under exponential headways. Larger g_c or a higher main-road flow (smaller mean headway) would reduce this number; smaller g_c would increase it.

References

1. Knoop, V. L. (2017). Introduction to traffic flow theory: An introduction with exercises. *Delft University of Technology: Delft, The Netherlands*.
2. Manual, H. C. (2000). Highway capacity manual. *Washington, DC*, *2*(1), 1.
3. Traffic engineering / Roger P. Roess, Elena S. Prassas, William R. McShane. -4th ed.