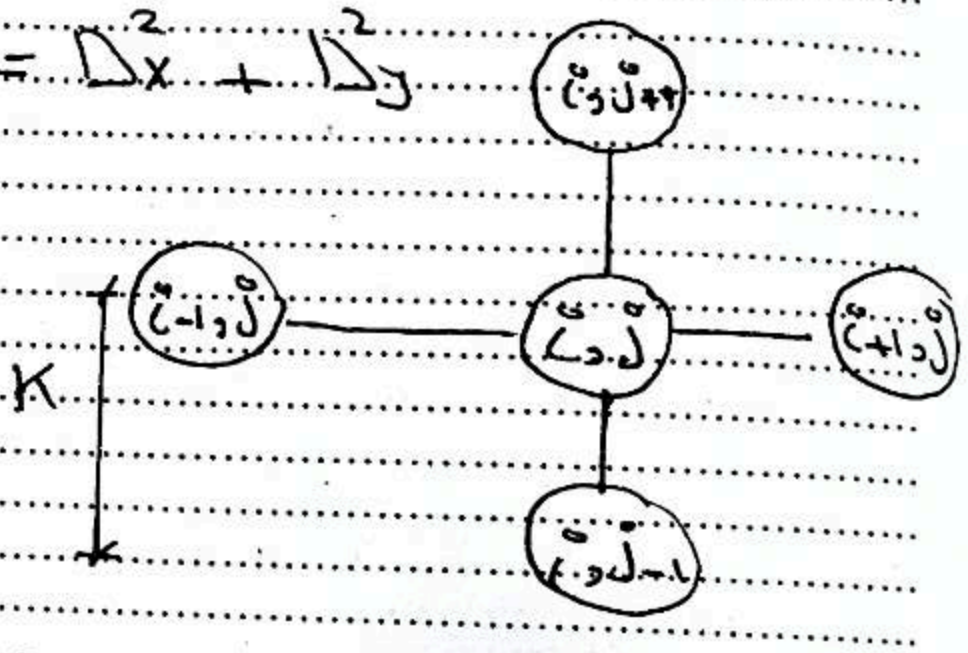
Engineering Analysis & Numerical Methods

The Numerical Solution of Partial Differential Equations

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta_x^2 + \Delta_y^2$$

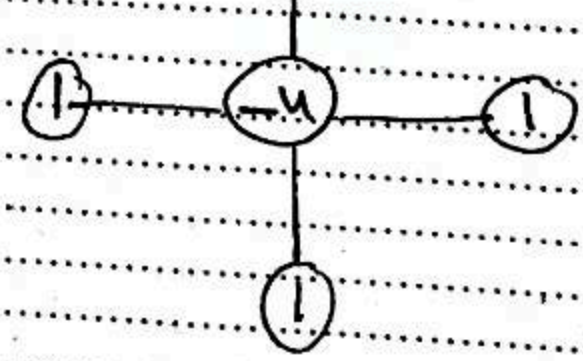


$$\nabla^2 z_i = \frac{z_{i-1,j} - 2z_{i,j} + z_{i+1,j}}{h^2} + \frac{z_{i,j-1} - 2z_{i,j} + z_{i,j+1}}{K^2}$$

IF $h = K$

$$h^2 \nabla^2 z_i = z_{i-1,j} + z_{i+1,j} + z_{i,j-1} + z_{i,j+1} - 4z_{i,j}$$

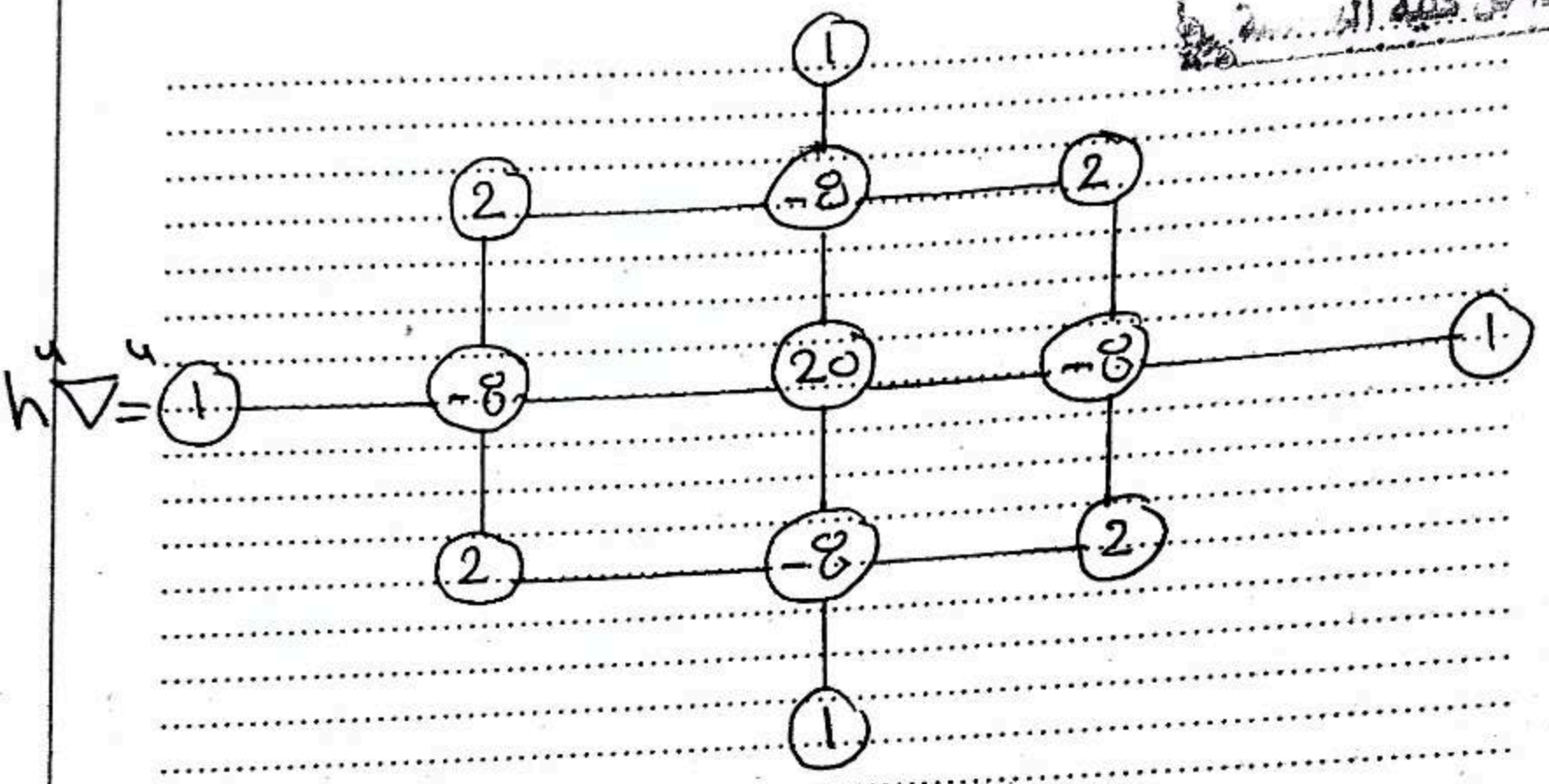
$h^2 \nabla^2 =$



$K = h$

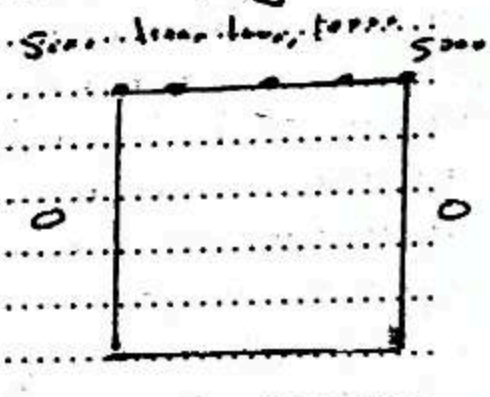
$$h^4 \nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

Engineering Analysis & Numerical Methods

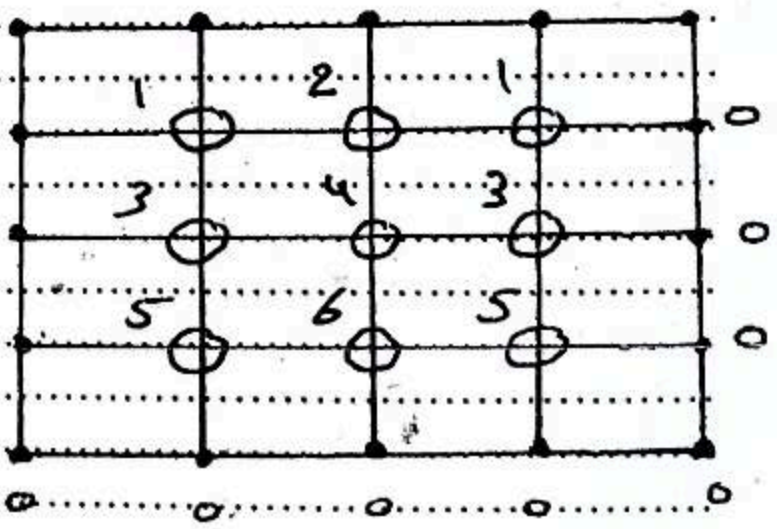


Example: Solve the partial differential equation for the plate shown in fig.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$\nabla^2 U = \begin{matrix} & (1) & \\ (1) & (-4) & (1) \\ & (1) & \end{matrix}$$



Engineering Analysis & Numerical Methods

$i=7$

$W_3 + W_7 + W_3 + W_7 + 2W_6 + 2W_6 - 8W_5$

$-8W_5 - 8W_8 + 20W_7 = \frac{10}{D}$ ..... (7)

$i=8$

$W_4 + W_4 + 2W_8 + 2W_8 + 2W_5 + 2W_5 - 8W_6$

$-8W_7 - 8W_6 - 8W_7 + 20W_8 = \frac{10}{D}$ ..... (8)

$$\begin{bmatrix} 23 & -8 & -7 & 2 & 1 & 0 & 0 & 0 \\ -16 & 21 & 4 & -8 & 0 & 1 & 0 & 0 \\ -8 & 2 & 22 & -8 & -8 & 2 & 1 & 0 \\ 4 & -8 & -16 & 20 & 4 & -8 & 0 & 1 \\ 1 & 0 & -8 & 2 & 23 & -8 & -8 & 2 \\ 0 & 1 & 4 & -8 & -16 & 21 & 4 & -8 \\ 0 & 0 & 2 & 0 & -16 & 4 & 22 & -8 \\ 0 & 0 & 0 & 2 & 8 & -16 & -16 & 20 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \\ W_7 \\ W_8 \end{Bmatrix} = \begin{Bmatrix} 1/D \\ 1/D \\ 1/D \\ 1/D \\ 1/D \\ 1/D \\ 1/D \\ 1/D \end{Bmatrix}$$

Engineering Analysis & Numerical Methods

i=1

$W_4 + W_1 + W_5 + W_1 + 2W_4 - 8W_3 - 8W_6 + 20W_1 = \frac{10}{D}$ ---- (1)

i=2

$W_2 + W_6 + 2W_3 + 2W_3 - 8W_1 - 8W_4 - 8W_1 + 20W_2 = \frac{10}{D}$ ---- (2)

i=3

$W_3 + W_7 + W_3 - 8W_4 - 8W_3 - 8W_4 + 2W_2 + 2W_6 + 20W_3 = \frac{10}{D}$ ---- (3)

i=4

$W_8 + 2W_1 + 2W_4 + 2W_5 + 2W_5 - 8W_2 - 8W_3 - 8W_6 - 8W_3 + 20W_4 = \frac{10}{D}$ ---- (4)

i=5

$W_1 + W_5 + W_5 + W_5 + 2W_4 + 2W_8 - 8W_3 - 8W_7 - 8W_6 + 20W_5 = \frac{10}{D}$ ---- (5)

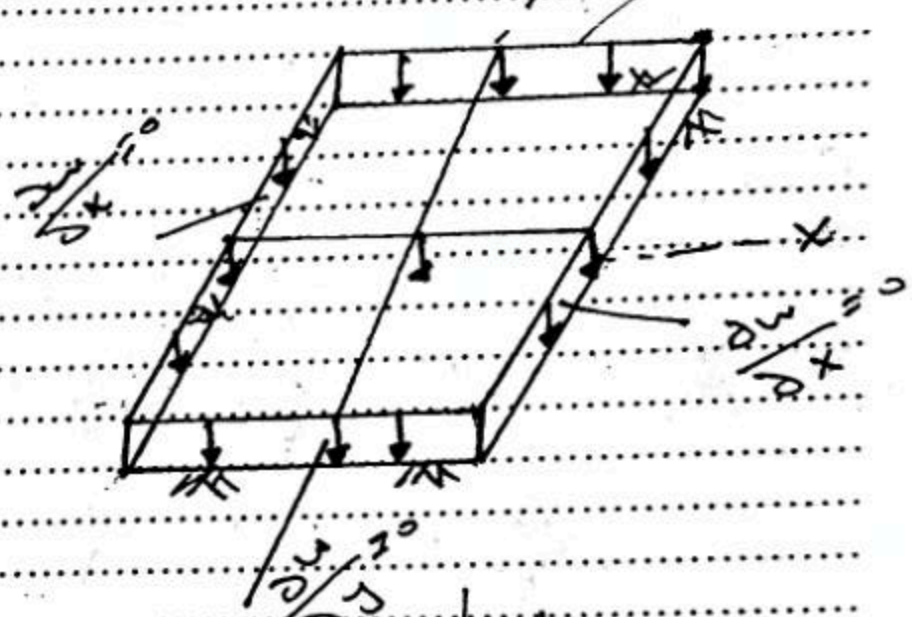
i=6

$W_2 + W_6 + 2W_3 + 2W_3 + 2W_7 + 2W_7 - 8W_4 - 8W_5 - 8W_8 - 80W_5 + 20W_6 = \frac{10}{D}$ ---- (6)

Engineering Analysis & Numerical Methods

Example: Find the deflection of the plate shown in fig. q = 10 kN/m²

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{10}{D}$$



Top view

Engineering Analysis & Numerical Methods

Solution of Higher-Order Partial Differential Equation

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$$

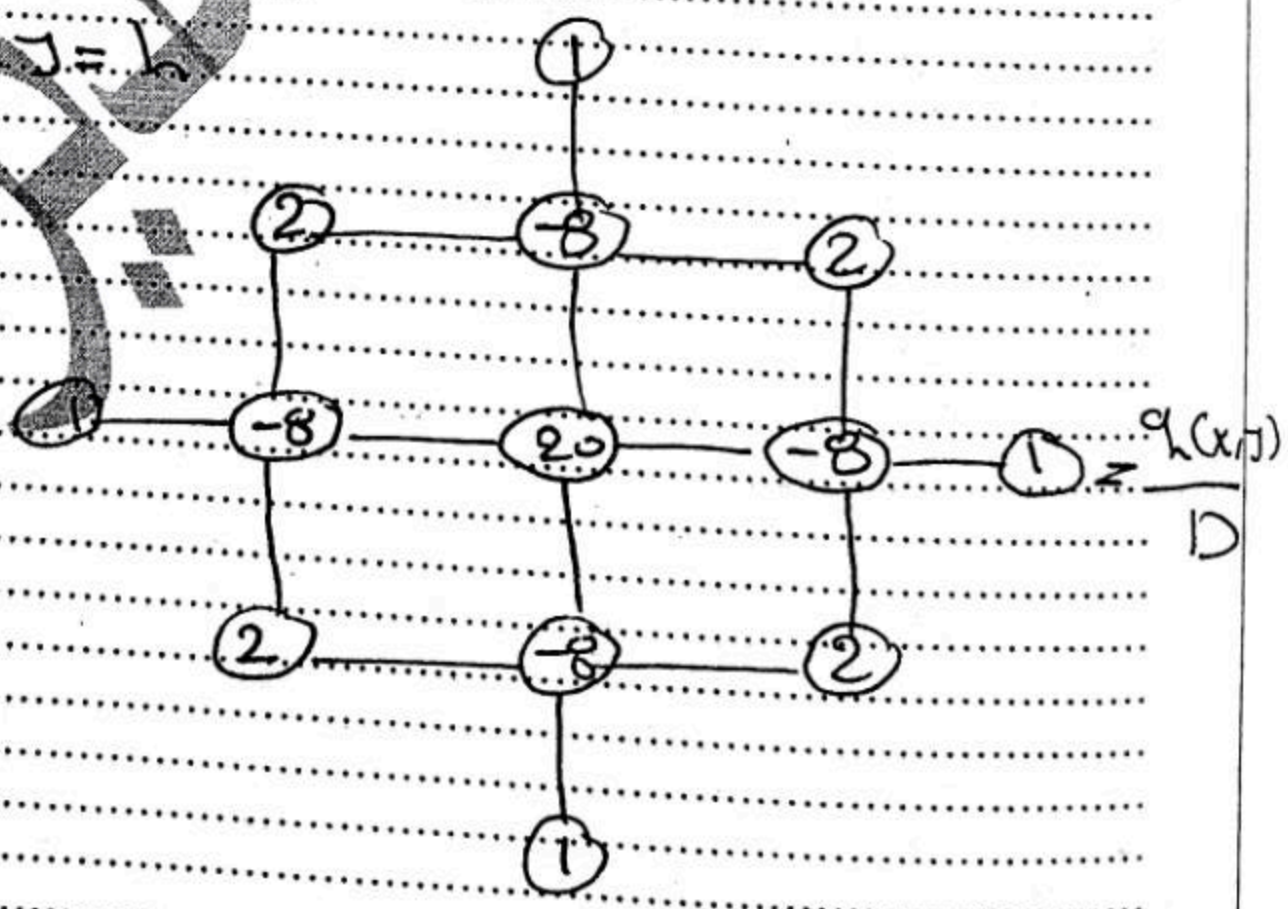
or $\nabla^4 w = q(x,y)/D$

$$D = \frac{Et^3}{12(1-\nu^2)}$$

Where: w = deflection
E = modulus of elasticity
ν = Poisson's ratio
t = thickness of the slab
q = uniform load.

$\Delta x = \Delta y = h$

$$h^4 \nabla^4 w = \frac{q(x,y)}{D}$$



Engineering Analysis & Numerical Methods

$i=12$

$h_{11}+h_8+h_{11}+h_{15}-4h_{12}=0$ ---(12)

$i=13$

$h_{14}+h_{10}+h_{14}+5-4h_{13}=0$ ---(13)

$i=14$

$h_{13}+h_{14}+h_{15}+5-4h_{14}=0$ ---(14)

$i=15$

$h_{14}+h_{12}+h_{14}+5-4h_{15}=0$ ---(15)

$$\begin{bmatrix}
-4 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -4 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -4
\end{bmatrix}
\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{13} \\ h_{14} \\ h_{15} \end{bmatrix}
=
\begin{bmatrix} -10 \\ 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 0 \\ 0 \\ -10 \\ 0 \\ 0 \\ 0 \\ -5 \\ -5 \\ -15 \end{bmatrix}$$

Engineering Analysis & Numerical Methods

$i=1$

$10 + h_5 + h_2 + h_5 - 4h_1 = 0$ ——— (1)

$i=2$

$h_1 + h_6 + h_3 + h_6 - 4h_2 = 0$ ——— (2)

$i=3$

$h_2 + h_7 + h_4 + h_7 - 4h_3 = 0$ ——— (3)

$i=4$

$h_3 + h_8 + h_3 + h_8 - 4h_4 = 0$ ——— (4)

$i=5$

$10 + h_1 + h_6 + h_9 - 4h_5 = 0$ ——— (5)

$i=6$

$h_5 + h_2 + h_7 + h_{10} - 4h_6 = 0$ ——— (6)

$i=7$

$h_6 + h_3 + h_8 + h_{11} - 4h_7 = 0$ ——— (7)

$i=8$

$h_7 + h_4 + h_7 + h_{12} - 4h_8 = 0$ ——— (8)

$i=9$

$10 + h_5 + h_{10} + h_5 - 4h_9 = 0$ ——— (9)

$i=10$

$h_9 + h_6 + h_{11} + h_{13} - 4h_{10} = 0$ ——— (10)

$i=11$

$10 + h_7 + h_{12} + h_{14} - 4h_{11} = 0$ ——— (11)

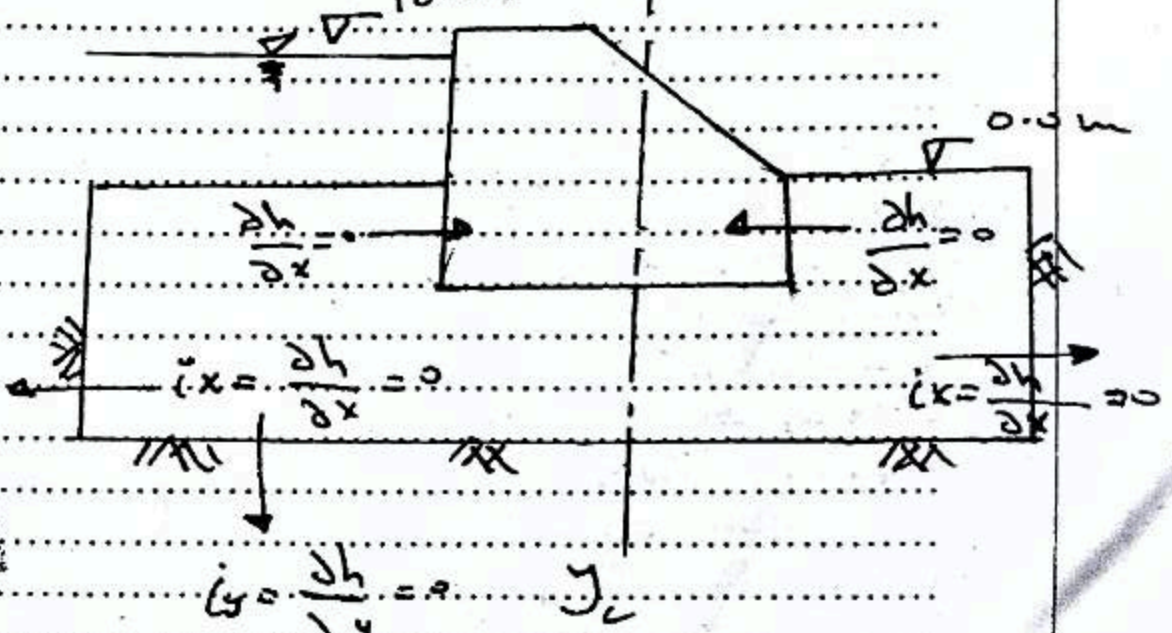
Engineering Analysis & Numerical Methods

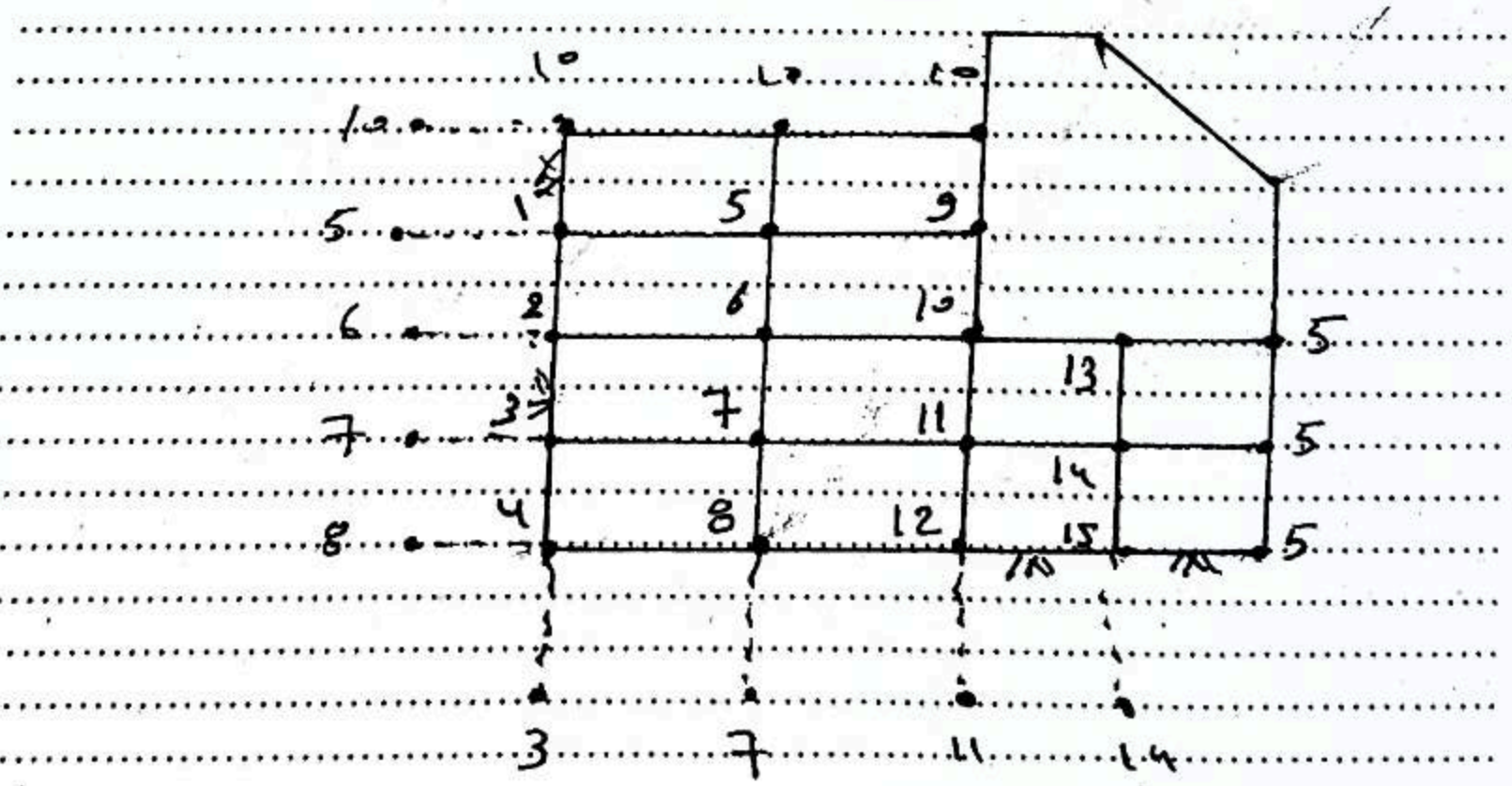
$U_1 = 4.286$, $U_2 = 5.268$, $U_3 = 1.875$, $U_4 = 2.500$

$U_5 = 7.14$, $U_6 = 7.82$

Example:- Solve the partial differential equation for the dam shown in fig.

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$





Engineering Analysis & Numerical Methods

$i = 1$

$100 + 0 + U_3 + U_2 - 4U_1 = 0$ ---------- (1)

$i = 2$

$100 + U_1 + U_4 + U_1 - 4U_2 = 0$ ---------- (2)

$i = 3$

$U_1 + 0 + U_5 + U_5 - 4U_3 = 0$ ---------- (3)

$i = 4$

$U_2 + U_3 + U_6 + U_3 - 4U_4 = 0$ ---------- (4)

$i = 5$

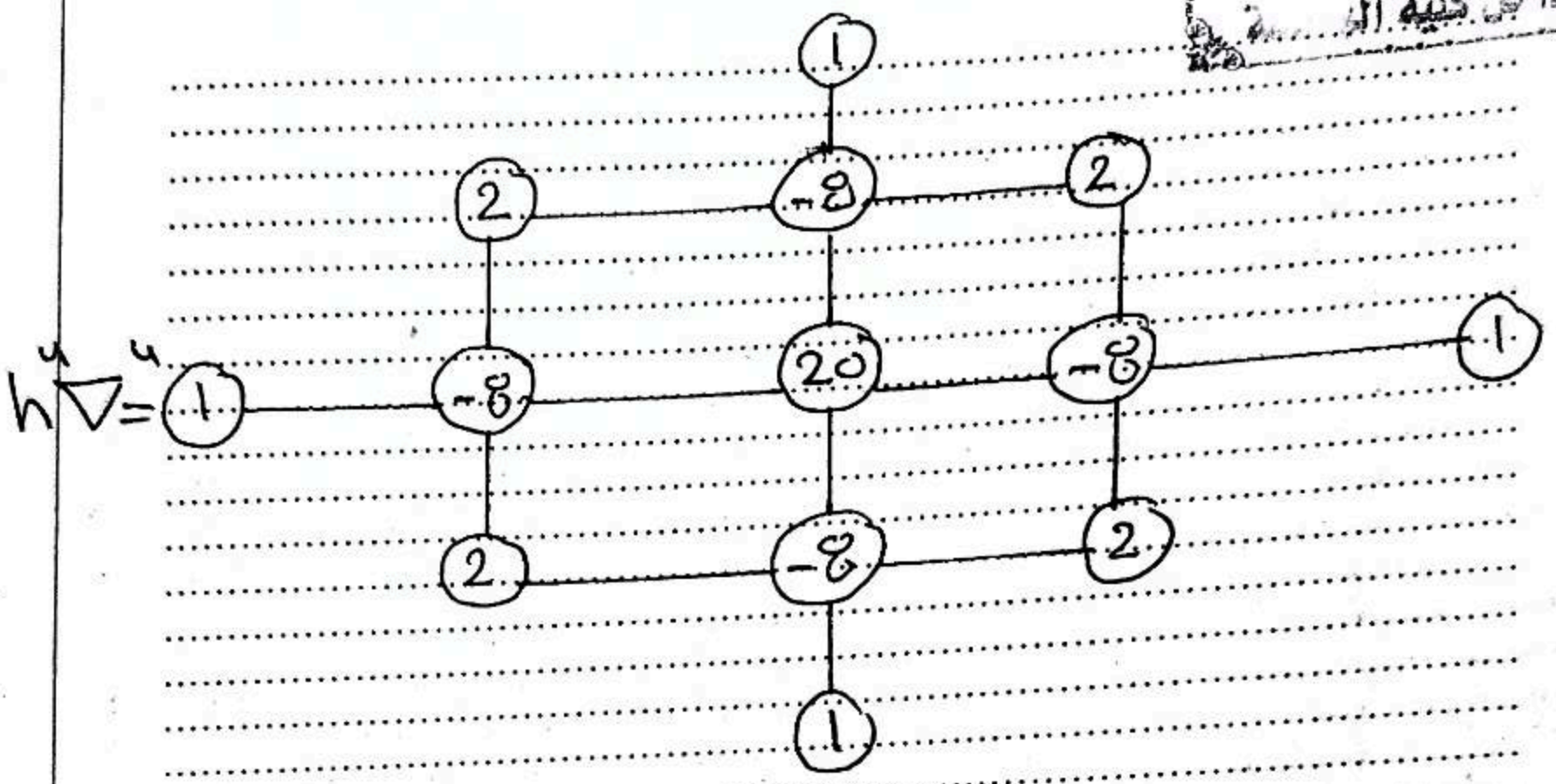
$U_3 + 0 + 0 + U_6 - 4U_5 = 0$ ---------- (5)

$i = 6$

$U_4 + U_5 + 0 + U_5 - 4U_6 = 0$ ---------- (6)

$$\begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 2 & -4 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} -100 \\ -100 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

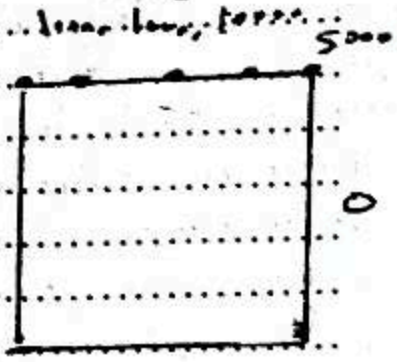
Engineering Analysis & Numerical Methods



$h^4\nabla^4 =$

Example:- Solve the partial differential equation for the plate shown in fig.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$\nabla^2 U = \begin{matrix} & (1) & \\ (1) & (-4) & (1) \\ & (1) & \end{matrix}$$