5/ Gauss - Jordan Elimination

هذه الطريقة مشابهة لطريقة حذف كاوس الاعتيادية غير اننا نحصل في هذه الطريقة على مصفوفة قطرية واحدة بدل من حصولنا على مصفوفة مثلثية عليا حيث يتم في هذه الطريقة حذف جميع العناصر الواقعة فوق وتحت القطر الرئيسي (اي نجعلها اصفار) بينما تتحول عناصر القطر الرئيسي الى واحد.

Example:- Solve the following set of linear algebraic equations using the Gauss-Jordan method.

$$2x_1 - 4x_2 + 6x_3 = 5$$
$$x_1 + 3x_2 - 7x_3 = 2$$
$$7x_1 + 5x_2 + 9x_3 = 4$$

Solution

للحصول على ①

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 5 \\ 1 & 3 & -7 & 2 \\ 7 & 5 & 9 & 4 \end{array}\right] \xrightarrow{r_1/2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5/2 \\ 1 & 3 & -7 & 2 \\ 7 & 5 & 9 & 4 \end{array}\right] \xrightarrow[r_1*-7+r_3]{r_1*-1+r_2}$$

للحصول على ①

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 5/2 \\ 0 & 5 & -10 & -1/2 \\ 0 & 19 & -12 & -27/2 \end{array}\right] \xrightarrow{r_2/5} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5/2 \\ 0 & 1 & -2 & -1/10 \\ 0 & 17 & -12 & -27/2 \end{array}\right] \xrightarrow[r_2*-17+r_3]{r_2*2+r_1}$$

$-2*-7+5=19$

$-2*2+3=-1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2.3 \\ 0 & 1 & -2 & -0.1 \\ 0 & 0 & 26 & -11.6 \end{array}\right] \xrightarrow{r_3/26} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2.3 \\ 0 & 1 & -2 & -0.1 \\ 0 & 0 & 1 & -0.44 \end{array}\right]$$

$$\xrightarrow[r_3 * 1 + r_1]{r_3 * 2 + r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.85 \\ 0 & 1 & 0 & -0.99 \\ 0 & 0 & 1 & -0.45 \end{array}\right]$$

$\therefore X_1 = 1.85 \quad X_2 = -0.99 \quad X_3 = -0.45$

Eigen Values & Eigen Vectors :-

A variety of practical problems having to do with alternating currents & voltage and other oscillatory phenomena lead to linearly algebric system of the type:

$$a_{11} X_1 + a_{12} X_2 + \text{---} + a_{1n} X_n = \lambda X_1$$

$$a_{21} X_1 + a_{22} X_2 + \text{---} + a_{2n} X_n = \lambda X_2$$

$$\vdots$$

$$a_{n1} X_1 + a_{n2} X_2 + \text{---} + a_{nn} X_n = \lambda X_n$$

$$A X = \lambda X$$

wher λ = Eigen Values , X = Eigen Vectors

Example :- Find Eigen Values & Vectors

if $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0$$

$$\lambda_1 = 3 \quad , \quad \lambda_2 = 1$$

$$\boxed{\lambda^2 - 4\lambda + 3 = 0}$$

for $\lambda = 3$

$-X_1 + X_2 = 0$

$X_1 - X_2 = 0$

$\therefore X_1 = X_2$

$$\begin{Bmatrix} X_1 \\ X_1 \end{Bmatrix}, \quad \lambda = 3$$

for $\lambda = 1$

$X_1 + X_2 = 0$

$X_1 + X_2 = 0$

$X_2 = -X_1$

$$\begin{Bmatrix} X_1 \\ -X_1 \end{Bmatrix}, \quad \lambda = 1$$

Example:- Find Eigen Values & Vectors

if $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = 0$$

$(\lambda - 2)(\lambda - 3)(\lambda - 5)$

$\lambda = 2, 3, 5$ Eigen values

For $\lambda = 2$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

$3X_3 = 0 \Rightarrow X_3 = 0$

$X_1 + X_2 + 4X_3 = 0$

$X_1 = -X_2$

$$\begin{Bmatrix} X_1 \\ -X_1 \\ 0 \end{Bmatrix}, \lambda = 2$$

For $\lambda = 3$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = 0$$

$2X_3 = 0, \quad X_3 = 0$

$-X_2 + 6X_3 = 0, \quad X_2 = 0$

$X_1 =$ arbitrary

$$\begin{Bmatrix} X_1 \\ 0 \\ 0 \end{Bmatrix}, \lambda = 3$$

For $\lambda = 5$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

$-3X_2 + 6X_3 = 0$

$\therefore X_2 = 2X_3$

$-2X_1 + X_2 + 4X_3 = 0$

$\therefore X_1 = 3X_3$

$$\begin{bmatrix} 3X_3 \\ 2X_3 \\ X_3 \end{bmatrix} \quad \lambda = 5$$

Example: Determine the eigen values and eigen vectors for matrix A

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

solution:

$$A = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} = 0$$

$$0 = (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 2-\lambda \\ 2 & 2 \end{vmatrix}$$

$0 = (1-\lambda)\left[(2-\lambda)(3-\lambda)-(2)(1)\right] - 1\left[(1)(2) - (2)(2-\lambda)\right]$

$0 = \lambda^3 - 6\lambda^2 + 11\lambda - 6$

let $\lambda = 1$

$(1)^3 - 6(1)^2 + 11(1) - 6 = 0$ o.k

$$\begin{array}{r|l} & \lambda^2 - 5\lambda + 6 \\ \hline \lambda - 1 & \lambda^3 - 6\lambda^2 + 11\lambda - 6 \\ & \lambda^3 - \lambda^2 \\ \hline & -5\lambda^2 + 11\lambda \\ & -5\lambda^2 + 5\lambda \\ \hline & 6\lambda - 6 \\ & 6\lambda - 6 \\ \hline & 0 \quad 0 \end{array}$$

$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$

For $\lambda = 1$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = 0$$

$0\,X_1 + 0\,X_2 - 1(X_3) = 0 \Rightarrow X_3 = 0$

$1(X_1) + 1(X_2) + 1(X_3) = 0$ but $X_3 = 0$

$X_1 + X_2 = 0 \Rightarrow X_1 = -X_2$

$$\begin{Bmatrix} X_1 \\ -X_1 \\ 0 \end{Bmatrix}, \ \lambda = 1 \quad \text{or} \quad \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$$