

5) Gauss - Jordan Elimination

هذه الطريقة مشابهة لطريقة حذف كاويا التقليدية غير أننا نعمل في هذه الطريقة على تصفئة كل صف في واحدة بدل من جعلنا كل صف صفه صليته عليا حيث يتم في هذه الطريقة حذف جميع العناصر الواقعة فوق و تحت القطر الرئيسي (أي جعلها صفرًا) بينما تتحول عناصر القطر الرئيسي إلى واحد.

Example: Solve the following set of linear algebraic equations using the Gauss-Jordan method.

$$2x_1 - 4x_2 + 6x_3 = 5$$

$$x_1 + 3x_2 - 7x_3 = 2$$

$$7x_1 + 5x_2 + 9x_3 = 4$$

Solution

$$\begin{bmatrix} 2 & -4 & 6 & | & 5 \\ 1 & 3 & -7 & | & 2 \\ 7 & 5 & 9 & | & 4 \end{bmatrix} \xrightarrow{r_1/2} \begin{bmatrix} 1 & -2 & 3 & | & 5/2 \\ 1 & 3 & -7 & | & 2 \\ 7 & 5 & 9 & | & 4 \end{bmatrix} \begin{array}{l} r_1 \times -1 + r_2 \\ r_1 \times -7 + r_3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 3 & | & 5/2 \\ 0 & 5 & -10 & | & -1/2 \\ 0 & 19 & -12 & | & -27/2 \end{bmatrix} \xrightarrow{r_2/5} \begin{bmatrix} 1 & -2 & 3 & | & 5/2 \\ 0 & 1 & -2 & | & -1/10 \\ 0 & 19 & -12 & | & -27/2 \end{bmatrix} \begin{array}{l} r_2 \times 2 + r_1 \\ r_2 \times -19 + r_3 \end{array}$$

$-2 \times 2 + 3 = -1$
 $-2 \times -7 + 5 = 19$

$$\begin{bmatrix} 1 & 0 & -1 & | & 2/3 \\ 0 & 1 & -2 & | & -0.1 \\ 0 & 0 & 26 & | & -11.6 \end{bmatrix} \xrightarrow{r_3/26} \begin{bmatrix} 1 & 0 & -1 & | & 2/3 \\ 0 & 1 & -2 & | & -0.1 \\ 0 & 0 & 1 & | & -0.44 \end{bmatrix}$$

$$\begin{array}{l} r_3 \times 2 + r_2 \\ \longrightarrow \\ r_3 \times 1 + r_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.85 \\ 0 & 1 & 0 & -0.99 \\ 0 & 0 & 1 & -0.45 \end{array} \right]$$

$$\therefore x_1 = 1.85$$

$$x_2 = -0.99$$

$$x_3 = -0.45$$

Engineering Analysis & Numerical Methods

Eigen Values & Eigen Vectors

A variety of practical problems having to do with alternating currents & voltage and other oscillatory phenomena lead to linear algebraic system of the type:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = \lambda x_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = \lambda x_2$$

$$\vdots$$
$$a_{nr}x_1 + a_{nr}x_2 + \dots + a_{nr}x_n = \lambda x_n$$

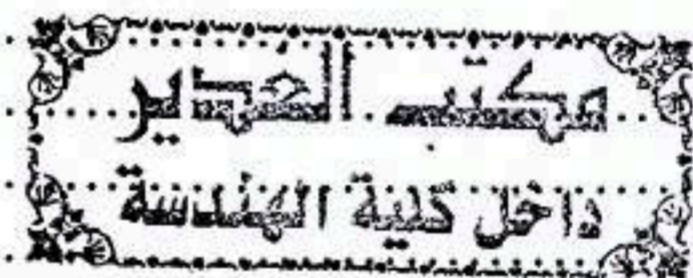
$$A \cdot X = \lambda X$$

where $\lambda = \text{Eigen Values}$, $X = \text{Eigen Vectors}$

Example: Find Eigen Values & Vectors

$$\text{if } A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$



$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0$$

$$\lambda_1 = 3 \quad \lambda_2 = 1$$

$$\lambda^2 - 4\lambda + 3 = 0$$

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For $\lambda = 3$

$$-x_1 + x_2 = 0$$

$$x_1 - x_2 = 0$$

$$\therefore x_1 = x_2$$

$$\begin{Bmatrix} x_1 \\ x_1 \end{Bmatrix}, \lambda = 3$$

For $\lambda = 1$

$$x_1 + x_2 = 0$$

$$x_1 + x_2 = 0$$

$$x_2 = -x_1$$

$$\begin{Bmatrix} x_1 \\ -x_1 \end{Bmatrix}, \lambda = 1$$

Example: Find Eigen Values & Vectors.

$$\text{if } A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

$$(\lambda - 3)(\lambda - 2)(\lambda - 5)$$

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$\lambda = 2, 3, 5$ Eigen values.

For $\lambda = 2$

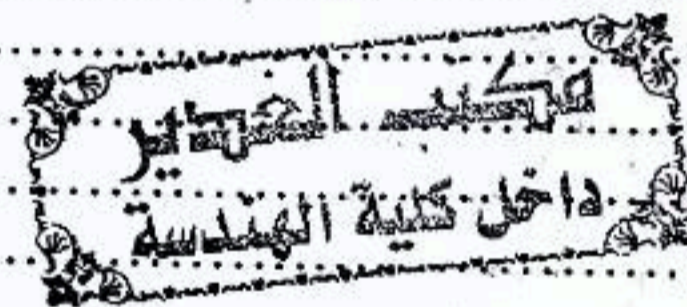
$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$3x_3 = 0 \Rightarrow x_3 = 0$$

$$x_1 + x_2 + 4x_3 = 0$$

$$x_1 = -x_2$$

$$\begin{cases} x_1 \\ -x_1 \\ 0 \end{cases}, \lambda = 2$$



For $\lambda = 3$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{cases} x_1 \\ x_2 \\ x_3 \end{cases} = 0$$

$$2x_3 = 0, x_3 = 0$$

$$-x_2 + 6x_3 = 0, x_2 = 0$$

$x_1 = \text{arbitrary}$

$$\begin{cases} x_1 \\ 0 \\ 0 \end{cases}, \lambda = 3$$

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For $\lambda = 5$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-3x_2 + 6x_3 = 0$$

$$\therefore x_2 = 2x_3$$

$$-2x_1 + x_2 + 4x_3 = 0$$

$$\therefore x_1 = 3x_3$$

$$\begin{bmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{bmatrix} \quad \lambda = 5$$

Example: Determine the eigen values and eigen vectors for matrix A

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} = 0$$

$$= (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 2-\lambda \\ 2 & 2 \end{vmatrix}$$

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$$0 = (1-\lambda) [(2-\lambda)(3-\lambda) - (2)(1)] - 1 [(1)(2) - (2)(2-\lambda)]$$

$$0 = \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

let $\lambda = 1$

$$(1)^3 - 6(1)^2 + 11(1) - 6 = 0 \quad 0.12$$

$$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

For $\lambda = 1$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

$$0 \cdot x_1 + 0 \cdot x_2 - 1(x_3) = 0 \Rightarrow x_3 = 0$$

$$1(x_1) + 1(x_2) + 1(x_3) = 0 \quad \text{but } x_3 = 0$$

$$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$$

$$\begin{Bmatrix} x_1 \\ -x_1 \\ 0 \end{Bmatrix}, \lambda = 1 \quad \text{or} \quad \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$$