Numerical Methods

System of Linear equation :-

$$a_{11} X_1 + a_{12} X_2 + \cdots + a_{1n} X_n = b_1$$
$$a_{21} X_1 + a_{22} X_2 + \cdots + a_{2n} X_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{n1} X_1 + a_{n2} X_2 + \cdots + a_{nn} X_n = b_n$$

In matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

or $[A]\{X\} = \{b\}$

$\boxed{I}$ Direct Methods :- (using Inverses)

$\{AX = B\} \longrightarrow (X = A^{-1} \cdot B)$

$x_1, x_2, \cdots$ قيم المتغيرات المطلوبة = X

$b_1, b_2, \cdots$ قيم ثوابت المعادلات = B

معكوس المصفوفة = A^{-1}

Example :- Use Inverse method to solve the following system of linear equations.

$$3X_1 + 2X_2 - X_3 = 4$$
$$5X_1 \qquad + 2X_3 = 17$$
$$-X_2 + 3X_3 = 5$$

solution:

step 1 $\quad A = \begin{bmatrix} 3 & 2 & -1 \\ 5 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$

$\det A = +3\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} - 2\begin{bmatrix} 5 & 2 \\ 0 & 3 \end{bmatrix} + (-1)\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A = 3\,[(0) - (-1)(2)] - 2\,[(5)(3) - (0)(2)] - 1\,[(5)(-1) - (0)]$

$\det A = 6 - 30 + 5 = -19$

Step 2 $\quad$ Find A^c

$A_{11} = +\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} = +[0 - (-1)(2)] = 2$

$A_{13} = +\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = +[(5)(-1) - 0] = -5$

$A_{21} = +\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = -[(2)(3) - (-1)(-1)] = -5$

$\vdots$

$\therefore A^c = \begin{bmatrix} 2 & -15 & -5 \\ -5 & 9 & 3 \\ 4 & -11 & -10 \end{bmatrix}$

step 3 $\quad$ Find $[A^c]^T$

$[A^c]^T = \begin{bmatrix} 2 & -5 & 4 \\ -15 & 9 & -11 \\ -5 & 3 & -10 \end{bmatrix}$

Step 4

$A^{-1} = \dfrac{[A^c]^T}{\text{Det} A} \Rightarrow A^{-1} = \dfrac{\begin{bmatrix} 2 & -5 & 4 \\ -15 & 9 & -11 \\ -5 & 3 & -10 \end{bmatrix}}{-19}$

$A^c = \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$$X_3 = \frac{|D_3|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & -2 \\ 2 & 3 & 0 \\ 3 & 3 & -1 \end{vmatrix}}{-7} = -1$$

Example:- use Inverse method to solve the above example.

$$X_1 + 2X_2 + 3X_3 = -2$$

$$2X_1 + 3X_2 + 2X_3 = 0$$

$$3X_1 + 3X_2 + 4X_3 = -1$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$$

$$AX = B \Rightarrow X = A^{-1} B$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{-1}{7} \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

where $|A| = -7$

2 Cramer's Rule :-

If the coefficient matrix A of a system $AX=B$ of n linear equations in n unknowns is nonsingular, the system has the unique soln.

$$X_1 = \frac{|D_1|}{|A|}, \quad X_2 = \frac{|D_2|}{|A|}, \quad \dots, \quad X_n = \frac{|D_n|}{|A|}$$

Where D_i is the matrix obtained from A by replacing the ith column of A by the column vector B.

Example : Solve the linear equations by Cramer's rule.

$$X_1 + 2X_2 + 3X_3 = -2$$
$$2X_1 + 3X_2 + 2X_3 = 0$$
$$3X_1 + 3X_2 + 4X_3 = -1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 0 \\ -1 \end{Bmatrix}$$

$$X_1 = \frac{|D_1|}{|A|} = \frac{\begin{vmatrix} -2 & 2 & 3 \\ 0 & 3 & 2 \\ -1 & 3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix}} = \frac{-7}{-7} = 1$$

$$X_2 = \frac{|D_2|}{|A|} = \frac{\begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & 2 \\ 3 & -1 & 4 \end{vmatrix}}{-7} = 0$$

$D \quad X_3$