Partial Differential Equation (P.D.E.) :-

P.D.E. is diff. Eq. which involves partial derivatived of one or more dependent variables w.r.t. one or more independent variables.

Linear P.D.E. of 2^{nd} order :-

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$$

the above equation is non-homg. linear.
when $G=0$ then it will be homog. linear.

x, y are independent variables.

1. Hyperbolic if $B^2 - 4AC > 0$ (زائدة)
2. perbolic if $B^2 - 4AC = 0$ (مكافئة)
3. Elliptic if $B^2 - 4AC < 0$ (ناقصة)

to solve these equations using (method of separation) or (product method).

1: One - Dimensional Heat Flow

سريان الحرارة ذو البعد الواحد

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

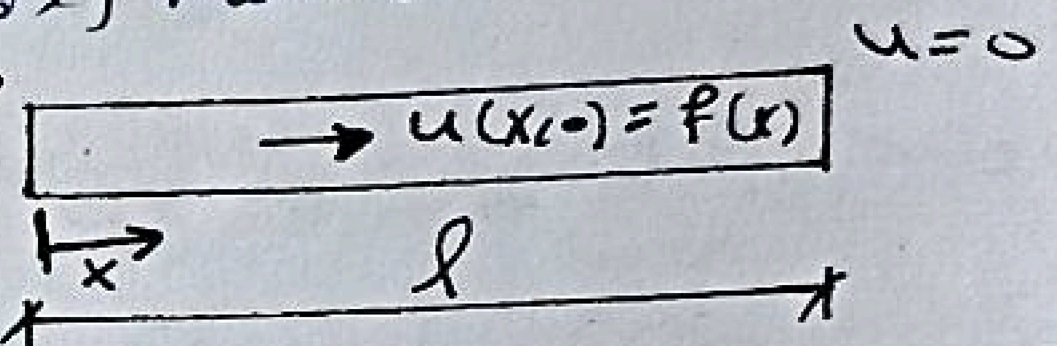
Example :- Solve the following P.D.E $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

use the following conditions.

$u(0,t) = u(l,t) = 0$

$u(x,0) = f(x)$

Sol. :- separation of variables method



u is dependent variable

x & t are indep. variables.

let $T = F(t)$ & $X = G(x)$

$$u(x,t) = F(t) \cdot G(x) = TX$$

$$\frac{\partial u}{\partial t} = X \cdot \dot{T} \quad , \quad \frac{\partial^2 u}{\partial x^2} = X'' \cdot T$$

بالتعويض $\Rightarrow X\dot{T} = c^2 X'' T \qquad \div (XT)$

$$\frac{\dot{T}}{T} = c^2 \frac{X''}{X} = -\lambda^2$$

part ① $c^2 \frac{X''}{X} = -\lambda^2 \Rightarrow c^2 X'' = -\lambda^2 X$

$$c^2 X'' + \lambda^2 X = 0 \Rightarrow c^2 m^2 + \lambda^2 = 0$$

$$m^2 = \frac{-\lambda^2}{c^2} \Rightarrow m_{1/2} = \mp \frac{\lambda}{c} i$$

$\therefore X = c_1 \cos \frac{\lambda}{c} x + c_2 \sin \frac{\lambda}{c} x$ $\left(\alpha = 0 ,\ \beta = \frac{\lambda}{c}\right)$

$$\int \frac{\dot{T}}{T} = \int -\lambda^2 \qquad \ln T = -\lambda^2 t + c_3$$

$$T = e^{-\lambda^2 t + c_3} = e^{-\lambda^2 t} * e^{c_3}$$

$$u(x,t) = XT$$

$$\therefore u(x,t) = \left(c_1 \cos \frac{\lambda}{c} x + c_2 \sin \frac{\lambda}{c} x\right) e^{-\lambda^2 t} * e^{c_3}$$

A B

$$\therefore u(x,t) = \left(A \cos \frac{\lambda}{c} x + B \sin \frac{\lambda}{c} x\right) e^{-\lambda^2 t}$$

$$B.C.S: \quad u(0,t) = 0 \Rightarrow 0 = A e^{-\lambda^2 t}$$

$$e^{-\lambda^2 t} \neq 0 \qquad \therefore A = 0$$

$$\therefore u(x,t) = \left(B \sin \frac{\lambda}{c} x\right) e^{-\lambda^2 t}$$

$$@ \; u(l,t) = 0 = B \sin \frac{\lambda l}{c} e^{-\lambda^2 t}$$

$$e^{-\lambda^2 t} \neq 0 \Rightarrow B \sin \frac{\lambda l}{c} = 0 \Rightarrow \frac{l\lambda}{c} = n\pi$$

$$\therefore \boxed{\lambda = \frac{nc\pi}{l}} \qquad \lambda l = nc\pi$$

هنا يتحول الحل الى Fourier half range بسبب وجود الـ

اي والان نبحث عن الزاوية التي جيبها = صفر ولكن نعرف ان π و $2\pi, 3\pi, n\pi$ الخ جيبهم = صفر

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x \cdot e^{-\left(\frac{n^2\pi^2 c^2}{l^2}\right)t}$$

Initial condition $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} B_n \cdot \sin \frac{n\pi}{l} x \cdot e^{0}$$

(دالة فردية)

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$$

2: Wave - Equation

اهتزاز السلك

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Example :- A uniform string is stretched between two points $(0,0)$ & $(l,0)$ is given by the intial displacement $f(x) = \begin{cases} x & 0 < x < l/2 \\ l - x & l/2 < x < l \end{cases}$ and released from rest. Find $u(x,t)$.

Sol. :- wave eqn. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

B.C.S :- $u(0,t) = 0$, $u(l,t) = 0$

$u(x,0) = f x$,

let $u(x,t) = X\,T$

$$X \ddot{T} = c^2 X'' T \qquad \div XT$$

$$\frac{\ddot{T}}{T} = c^2 \frac{X''}{X} = -\lambda^2$$

$$c^2 \frac{X''}{X} = -\lambda^2 \Rightarrow c^2 X'' + \lambda^2 X = 0$$

$$c^2X'' + \lambda^2 X = 0 \Rightarrow c^2m^2 + \lambda^2 = 0$$

$$m^2 = \frac{-\lambda^2}{c^2} \Rightarrow m_{1,2} = \mp \frac{\lambda}{c} i$$

$$\therefore X = c_1 \cos\frac{\lambda}{c}x + c_2 \sin\frac{\lambda}{c}x \quad \left(\alpha = 0,\ \beta = \frac{\lambda}{c}\right)$$

$$\frac{T''}{T} = -\lambda^2 \Rightarrow T'' + \lambda T = 0$$

$$m^2 + \lambda^2 = 0 \Rightarrow m^2 = -\lambda^2,\ m_{3,4} = \mp \lambda i$$

$$T = c_3 \cos\lambda t + c_4 \sin\lambda t$$

$$u(x,t) = \left(c_1 \cos\frac{\lambda}{c}x + c_2 \sin\frac{\lambda}{c}x\right)(c_3 \cos\lambda t + c_4 \sin\lambda t)$$

B.C1 $u(0,t) = 0 \Rightarrow \boxed{c_1 = 0}$

$$u(x,t) = c_2 \sin\frac{\lambda}{c}x \cdot (c_3 \cos\lambda t + c_4 \sin\lambda t)$$

$$u(x,t) = \sin\frac{\lambda}{c}x\,(A\cos\lambda t + B\sin\lambda t)$$

B.C2 $u(l,t) = 0 \Rightarrow 0 = \sin\frac{\lambda}{c}l\,(A\cos\lambda t + B\sin\lambda t)$

$$\sin\frac{\lambda l}{c} = 0 \Rightarrow \frac{\lambda l}{c} = n\pi \Rightarrow \lambda = \frac{n\pi c}{l}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(A_n \cos\frac{n\pi c}{l}t + B_n \sin\frac{n\pi c}{l}t\right)$$

المشتقة الاولى $u_t(x,t) = \sum_{n=1}^{n=\infty} \sin\frac{n\pi x}{l}\left(-A_n \frac{n\pi c}{l}\sin\frac{n\pi c}{l}t + B_n\frac{n\pi c}{l}\cos\frac{n\pi c}{l}t\right)$

$$u_t(x,0) = 0 = \sum_{n=1}^{n=\infty} \sin\frac{n\pi x}{l} B_n \frac{n\pi c}{l} \Rightarrow B_n = 0$$

$$u(x,t) = \sum_{n=1}^{n=\infty} \sin\frac{n\pi x}{l} A_n \cos\frac{n\pi c}{l}t$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} A_n$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad p = l$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

x	$\sin\frac{n\pi x}{l}$
1 $(+)$	$-\frac{l}{n\pi}\cos\frac{n\pi}{l}x$
0 $(-)$	$\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}$

$$A_n = \frac{2}{l}\left[\frac{-xl}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{l/2}$$

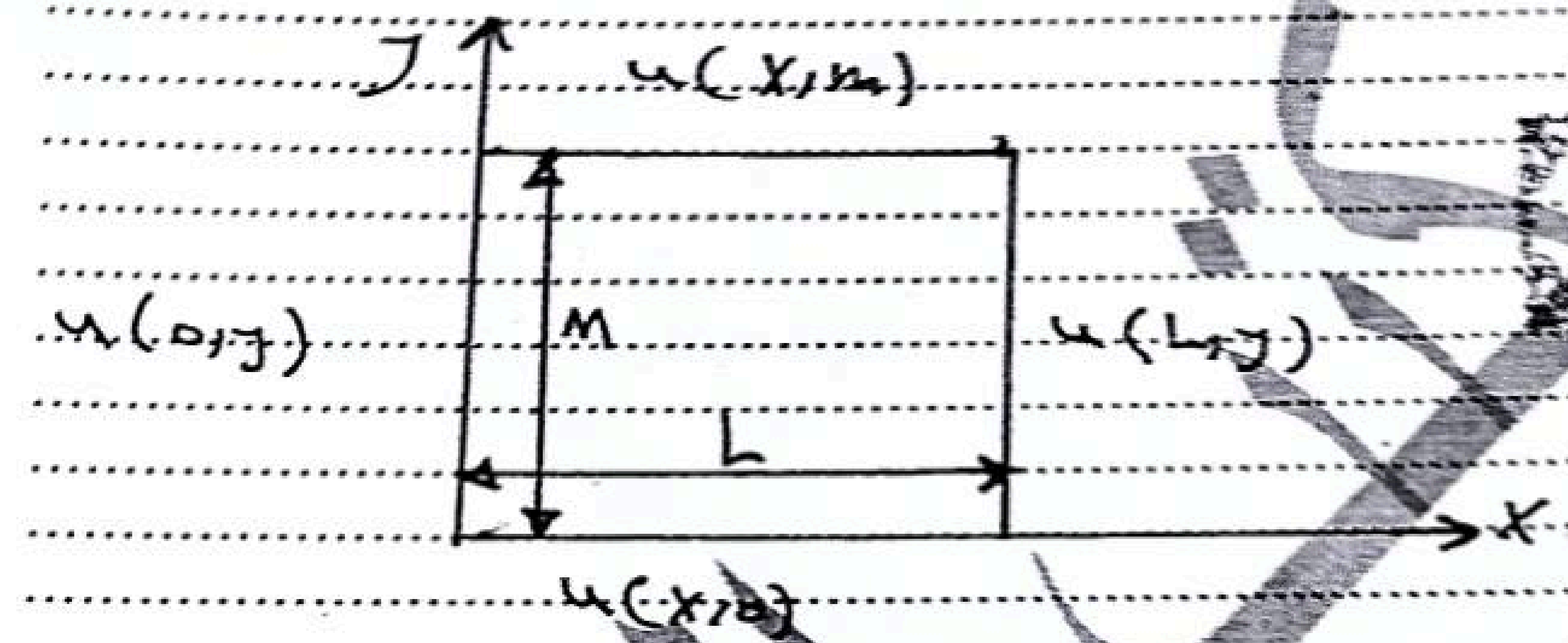
$$+ \frac{2}{l}\left[\frac{-l}{n\pi}(l-x)\cos\frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{l/2}^{l}$$

$$\boxed{A_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\,\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\cos\frac{n\pi c}{l}t$$

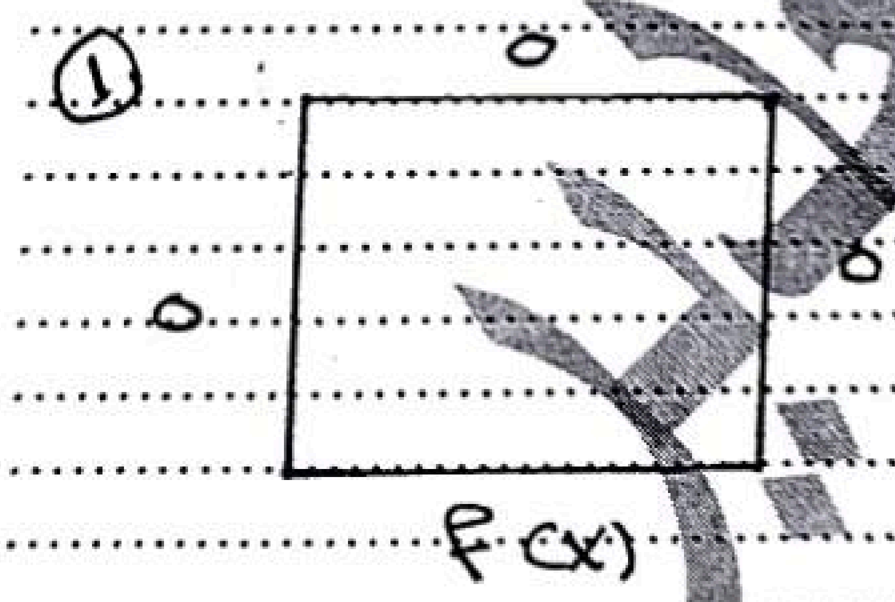
3: Two - Dimensional Laplace equation:

معادلة لابلاس لانتقال الحرارة ذات البعدين
في هذا التطبيق يكون سريان الحرارة ذو بعدين بالاتجاه (X) و(Y)
وهذا النوع من السريان شائع في الالواح المعدنية يعكس تطبيق سريان الحرارة ذو البعد الواحد الخاص بالاسلاك والقضبان المعدنية.

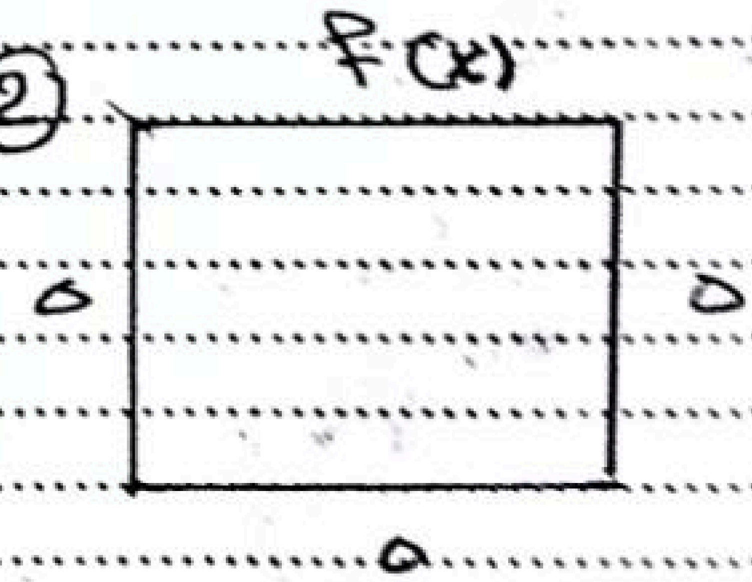


$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

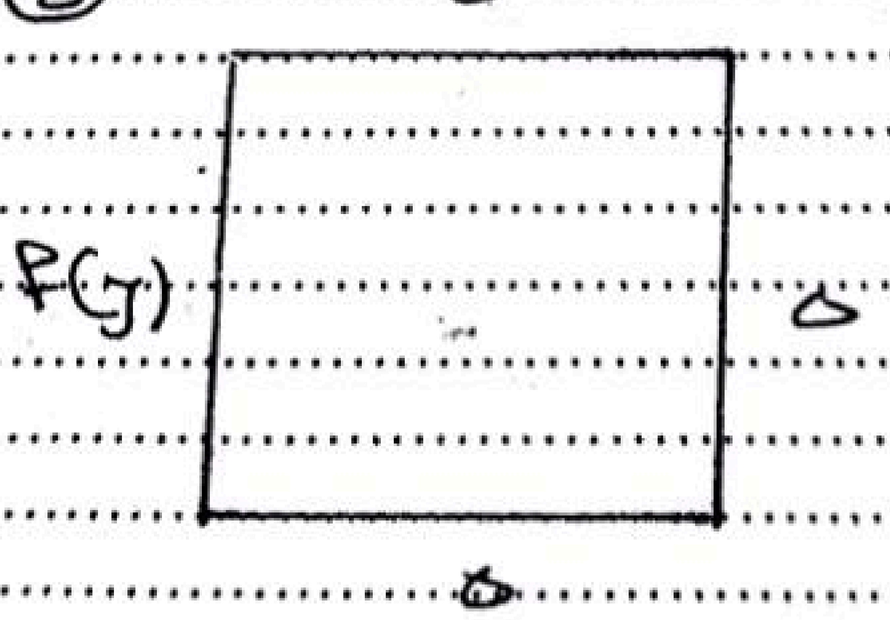
(1)



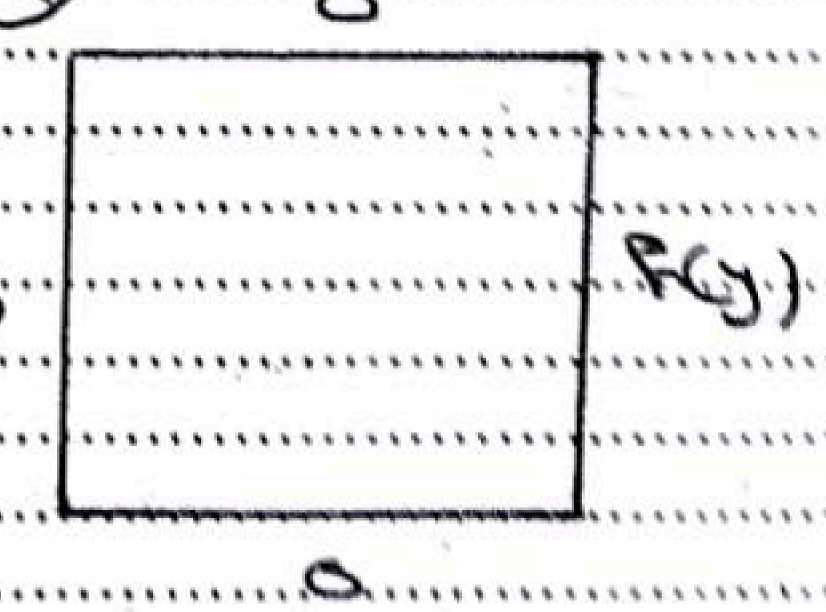
(2)



(3)



(4)



① $u(0,y)=0$, $u(L,y)=0$, $u(x,h)=0$
$u(x,0)=f(x)$

② $u(0,y)=0$, $u(L,y)=0$, $u(x,0)=0$
$u(x,h)=f(x)$

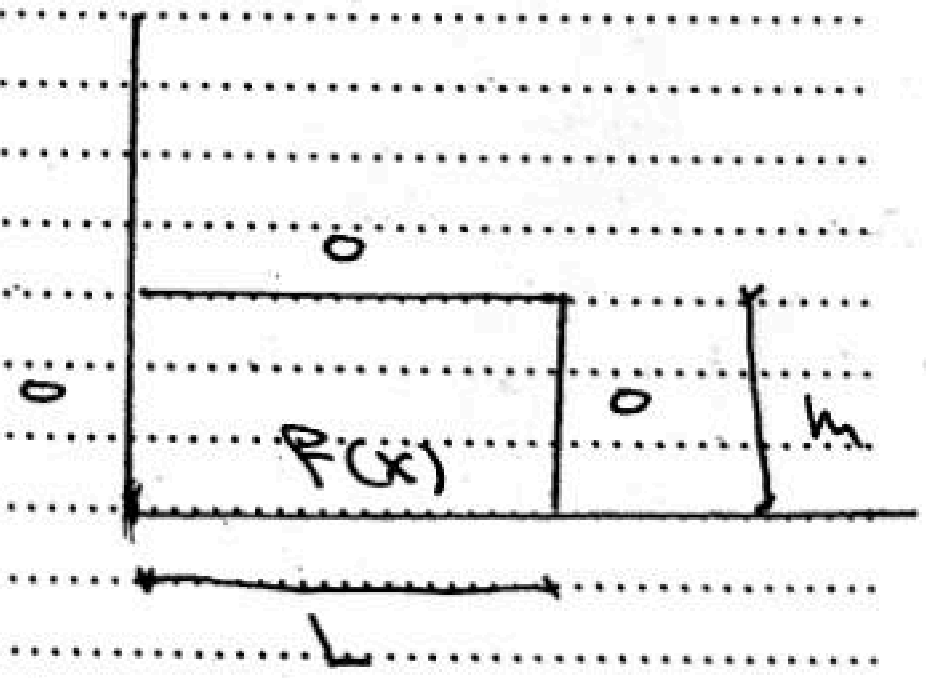
③ $u(x,0)=0$, $u(x,h)=0$, $u(L,y)=0$
$u(0,y)=f(y)$

④ $u(x,0)=0$, $u(x,h)=0$, $u(0,y)=0$
$u(L,y)=f(y)$

* (يراجع تسلسل التحويلية)

Example: Solve the following P.D.E

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$



Sol.

$$u_{(x,y)}=X\,Y$$

$$\frac{\partial^2 u}{\partial x^2}=X''\,Y$$

$$\frac{\partial^2 u}{\partial y^2}=X\,Y''$$

$$X''Y+X\,Y''=0 \Rightarrow X''\,Y=-X\,Y'' \div(XY)$$

$$\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda^2$$

$$\frac{X''}{X}=-\lambda^2 \Rightarrow X''=-\lambda^2 X$$

$$m_{1,2}=\mp\lambda i \Rightarrow \alpha=0 \quad , \beta=\cancel{\lambda}$$

$\therefore X = c_1 \sin \lambda x + c_2 \cos \lambda x$

$\frac{Y''}{Y} = \lambda^2 \Rightarrow Y'' = \lambda^2 Y \Rightarrow m_{3,4} = \mp \lambda$

$\therefore Y = c_3 e^{\lambda y} + c_4 e^{-\lambda y}$

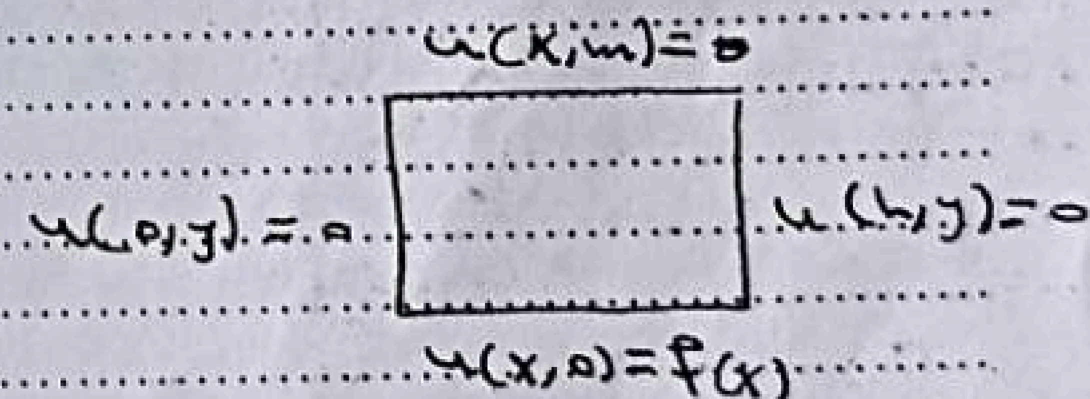
sub. $\sinh \lambda y$ & $\cosh \lambda y$ instead of $e^{\lambda y}$ & $e^{-\lambda y}$

$\therefore Y = c_3 \sinh \lambda y + c_4 \cosh \lambda y$

$u = X\,Y$

$\therefore u(x,y) = (c_1 \sin \lambda x + c_2 \cos \lambda x)(c_3 \sinh \lambda y + c_4 \cosh \lambda y)$

B.C.S:-



① $u(0,y) = 0$

② $u(L,y) = 0$

③ $u(x,m) = 0$

④ $u(x,0) = f(x)$

① $u(0,y) = 0$

$0 = (c_1 \sin 0 + c_2 \cos 0)(c_3 \sinh \lambda y + c_4 \cosh \lambda y)$

$0 = c_2 (c_3 \sinh \lambda y + c_4 \cosh \lambda y)$

$\therefore c_2 = 0$ القوس $\neq 0$ لأن (Y) متغير مستمر

$\therefore u(x,y) = c_1 \sin \lambda x \,(c_3 \sinh \lambda y + c_4 \cosh \lambda y)$

let $c_2 * c_3 = A$ & $c_1 * c_4 = B$

$u(x,y) = \sin \lambda x (A \sinh \lambda y + B \cosh \lambda y)$

(2) $u(L,y) = 0$

$0 = \sin \lambda L\, (A \sinh \lambda y + B \cosh \lambda y)$

$\sin \lambda L = 0$

$\lambda L = n\pi$ $\qquad n = 1, 2, 3, \ldots, \infty$

$\therefore \lambda = \frac{n\pi}{L}$

$u(x,y) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left(A \sinh \frac{n\pi}{L} y + B \cosh \frac{n\pi}{L} y\right)$

(3) $u(x,m) = 0$

$0 = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left(A \sinh \frac{n\pi}{L} m + B \cosh \frac{n\pi}{L} m\right)$

$\sin \frac{n\pi}{L} x \neq 0$ لأن قيمته متغير (X)

$\therefore \left(A \sinh \frac{n\pi}{L} m + B \cosh \frac{n\pi}{L} m\right) = 0$

$$B = \frac{A \cdot \sinh \frac{n\pi}{L} m}{\cosh \frac{n\pi}{L} m}$$

(4) $u(x,0) = f(x)$

$f(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left(A \cancel{\sinh 0}^{\,0} + B \cancel{\cosh 0}^{\,1}\right)$

$f(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \cdot A_n$

$$\therefore A_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x \cdot dx$$