Ex: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+13}\right\}$

In this example we have $F(s)=\frac{1}{s^2+6s+13}$ we will compare with

$$F(s)=\frac{b}{(s+a)^2+b^2}$$

$$F(s)=\frac{1}{s^2+6s+13}=\frac{1}{s^2+6s+9-9+13}=\frac{1}{2}\frac{2}{(s+3)^2+4} \cancel{\frac{1}{2}e^{-3t}\sin 2t}$$

$$\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+4}\right\}=\frac{1}{2}e^{-3t}\sin 2t=f(t)$$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

$$\frac{1}{s(s^2+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}=\frac{As^2+A+Bs^2+Cs}{s(s^2+1)}$$

$A+B=0 \Rightarrow B=-1$

$A=1$

$C=0$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{-s}{s^2+1}\right\}$$

$$=1-\cos t$$

L.T. Solution of Linear Equations With Constant Coefficient:

Consider the problem of using L.T. to solve non-homogenous n order ordinary diff eq. with Constant Coefficient.

$$a_n y^{n}+a_{n-1}y^{n-1}+\cdots+a_1 y'+a_0 y=f(t)$$

with the conditions

$y(0)=C_0$

$y'(0)=C_1, \;\cdots\;, y^{(n-1)}(0)=C_{n-1}$

The procedure of solving this eq. by L.T. is as follows:

(1) Take L.T. to both side of diff. eq. and use the initial eq. to get algebric eq. unknown $Y(s)$.

(2) Solve the algebric eq. to determine $Y(s)$.

(3) Use the inverse L.T. of $Y(s)$ to find $y(t)$.

Ex:- Solve the O.D.E.

$$y' - 2y = e^{5t} \quad , y(0) = 3$$

using L.T.

(1) $\mathcal{L}\{y' - 2y\} = \mathcal{L}\{e^{5t}\}$

$\mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{e^{5t}\}$

(2) $\mathcal{L}\{y\} = Y(s)$ $\qquad \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 3$

$\mathcal{L}\{e^{5t}\} = \frac{1}{s-5}$

$sY(s) - 3 - 2Y(s) = \frac{1}{s-5}$

$Y(s)(s-2) = \frac{1}{s-5} + 3 = \frac{1+3s-15}{(s-5)}$

$Y(s) = \frac{3s-14}{(s-2)(s-5)}$

$\frac{3s-14}{(s-2)(s-5)} = \frac{A}{s-2} + \frac{B}{s-5}$

$= \frac{As - 5A + Bs - 2B}{(s-2)(s-5)}$

$3s - 14 = (A+B)s - 5A - 2B$

$A + B = 3 \quad] \times 2$

$-5A - 2B = -14$

$-3A = -8 \rightarrow A = \frac{8}{3}$

$B = 3 - \frac{8}{3} = \frac{1}{3}$

$\mathcal{L}^{-1}\{Y(s)\} = y(t)$

$y(t) = \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}$

$y(t) = \frac{8}{3}e^{2t} + \frac{1}{3}e^{5t}$

Ex Solve diff. eq. by L.T.

$y'' - 2y' - 8y = 0 \qquad y(0) = 3 \quad , y'(0) = 6$

① $\mathcal{L}\{y'' - 2y' - 8y\} = \mathcal{L}\{0\}$ $\qquad \mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0)$

$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} - 8\mathcal{L}\{y\} = 0 \qquad \mathcal{L}\{y'\} = sY(s) - y(0)$

$s^2 Y(s) - 3s \cancel{-6} - 2sY(s) \cancel{+6} - 8Y(s) = 0$

$(s^2 - 2s - 8)Y(s) = 3s \Rightarrow Y(s) = \frac{3s}{s^2 - 2s - 8} = \frac{As + 2A + Bs - 4B}{(s-4)(s+2)}$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{3s}{s^2 - 2s - 8}\right\} \qquad \frac{3s}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$

$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \qquad A = \frac{12}{6} = 2, \quad B = 1 \frac{5}{4}$

$y(t) = 2e^{4t} + e^{-2t}$

$\frac{5}{2}$

$A + B = 3$ ①

$2A - 4B = 0$ ②

$B = \frac{1}{2}A$

$A = 3 - \frac{1}{2}$

$= \frac{6-1}{2}$

$= \frac{5}{2}$

$B = \frac{5}{2}$

H.w

① $y'' + y = e^{-2t} \sin t \qquad y(0) = y'(0) = 0$

② $y''' + 4y'' + 5y' + 2y = 10\cos t \qquad y(0) = y'(0) = 0, \quad y''(0) = 3$

③ $y'' + 2y' + 5y = f(t) \qquad y(0) = y'(0) = 0$

$f(t) = \begin{cases} 1 & 0 < t < \pi \\ 0 & t > \pi \end{cases}$

④ $y''' - 5y'' + 7y' - 3y = 20\sin t$

$y(0) = y'(0) = 0 \qquad y''(0) = -2$

$(A+B)s + 2(A - 2B)$

$A + B = 3$ ---- ①

$A - 2B = 0$ → ② $\qquad \cancel{A = 2B}$

$\boxed{B = \frac{1}{2}A}$

$\boxed{= 1}$

$A + \frac{1}{2}A = 3$

$\frac{\cancel{3}}{2} A = 3$

$A = \frac{6}{\cancel{3}}$ 2