Laplace Transformation

Chapter 1

The Laplace Transform

DEFINITION OF THE LAPLACE TRANSFORM

Let $F(t)$ be a function of t specified for $t > 0$. Then the *Laplace transform* of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$, is defined by

$$\mathcal{L}\{F(t)\} \;=\; f(s) \;=\; \int_0^{\infty} e^{-st}\, F(t)\, dt \qquad (1)$$

where we assume at present that the parameter s is real. Later it will be found useful to consider s complex.

The Laplace transform of $F(t)$ is said *to exist* if the integral (1) *converges* for some value of s; otherwise it does not exist. For sufficient conditions under which the Laplace transform does exist, see Page 2.

NOTATION

If a function of t is indicated in terms of a capital letter, such as $F(t)$, $G(t)$, $Y(t)$, etc., the Laplace transform of the function is denoted by the corresponding lower case letter, i.e. $f(s)$, $g(s)$, $y(s)$, etc. In other cases, a tilde (~) can be used to denote the Laplace transform. Thus, for example, the Laplace transform of $u(t)$ is $\tilde{u}(s)$.

LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

The adjacent table shows Laplace transforms of various elementary functions. For details of evaluation using definition (1), see Problems 1 and 2. For a more extensive table see Appendix B, Pages 245 to 254.

	$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$
1.	1	$\frac{1}{s}$ $\quad s > 0$
2.	t	$\frac{1}{s^2}$ $\quad s > 0$
3.	t^n $n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$ $\quad s > 0$ *Note.* Factorial $n = n! = 1 \cdot 2 \cdots n$ Also, by definition $0! = 1$.
4.	e^{at}	$\frac{1}{s-a}$ $\quad s > a$
5.	$\sin at$	$\frac{a}{s^2+a^2}$ $\quad s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2}$ $\quad s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}$ $\quad s > \lvert a \rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2}$ $\quad s > \lvert a \rvert$

SOME IMPORTANT PROPERTIES OF LAPLACE TRANSFORMS

In the following list of theorems we assume, unless otherwise stated, that all functions satisfy the conditions of *Theorem 1-1* so that their Laplace transforms exist.

1. Linearity property.

Theorem 1-2. If c_1 and c_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transforms $f_1(s)$ and $f_2(s)$ respectively, then

$$\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\} = c_1f_1(s) + c_2f_2(s) \qquad (2)$$

The result is easily extended to more than two functions.

Example.
$$\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} = 4\mathcal{L}\{t^2\} - 3\mathcal{L}\{\cos 2t\} + 5\mathcal{L}\{e^{-t}\}$$
$$= 4\left(\frac{2!}{s^3}\right) - 3\left(\frac{s}{s^2+4}\right) + 5\left(\frac{1}{s+1}\right)$$
$$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$$

The symbol $\mathcal{L}$, which transforms $F(t)$ into $f(s)$, is often called the *Laplace transformation operator*. Because of the property of $\mathcal{L}$ expressed in this theorem, we say that $\mathcal{L}$ is a *linear operator* or that it has the *linearity property*.

2. First translation or shifting property.

Theorem 1-3. If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a) \qquad (3)$$

Example. Since $\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+4}$, we have

$$\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

3. Second translation or shifting property.

Theorem 1-4. If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$, then

$$\mathcal{L}\{G(t)\} = e^{-as}f(s) \qquad (4)$$

Example. Since $\mathcal{L}\{t^3\} = \dfrac{3!}{s^4} = \dfrac{6}{s^4}$, the Laplace transform of the function

$$G(t) = \begin{cases} (t-2)^3 & t > 2 \\ 0 & t < 2 \end{cases}$$

is $6e^{-2s}/s^4$.

4. Change of scale property.

Theorem 1-5. If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F(at)\} = \frac{1}{a}f\left(\frac{s}{a}\right) \qquad (5)$$

Example. Since $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$, we have

$$\mathcal{L}\{\sin 3t\} = \frac{1}{3}\,\frac{1}{(s/3)^2+1} = \frac{3}{s^2+9}$$

Theorm (6)

Let $f(t)$ be a continous $(n-1)$ derivitives, $f'(t), f''(t), \dots,$ $f^{(n-1)}(t)$ for all $t \geq 0$ and assume that $f, f', \dots, f^{(n-1)}$ have exponitional order α. If f^{n} is continous in every interval $[0, t]$, then $\mathcal{L}\{f^{(n)}\}$ exist for $s > \alpha$ and

$$\mathcal{L}\{f^{(n)}(t)\} = s^{n}\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

Ex: Apply the last theorm with $n = 2$ to find $\mathcal{L}\{\sin bt\}$

$$\mathcal{L}\{f''(t)\} = s^{2}\mathcal{L}\{f(t)\} - s f(0) - f'(0)$$

$f(t) = \sin bt \qquad f'(t) = b\cos bt$

$f''(t) = -b^{2}\sin bt \qquad f(0) = 0, \quad f'(0) = b$

$$\mathcal{L}\{-b^{2}\sin bt\} = s^{2}\mathcal{L}\{\sin bt\} - s(0) - b$$

$$-b^{2}\mathcal{L}\{\sin bt\} - s^{2}\mathcal{L}\{\sin bt\} = -b \quad \} * -1$$

$$\mathcal{L}\{\sin bt\}(b^{2} + s^{2}) = b$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^{2} + b^{2}}$$

H.w Find Laplace for the following functions:

1. $f(t) = \sin at \sin bt$
2. $f(t) = t^{2}\cos bt$
3. $f(t) = t^{3}e^{at}$
4. $f(t) = e^{at}t^{2}$

Other Method:-

~~$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$~~

Theorem (7): suppose that $f(t)$ is a function which continuous and of exponitional order which has the Laplace transform (6)

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Ex:

find L.T. of $(t^2 \sin bt)$

since

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

there fore

$$\mathcal{L}\{t^2 \sin bt\} = (-1)^2 \frac{d^2}{ds^2} F(s)$$ where $F(s)$ is the L.T. of

$f(t) = \sin bt$

$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$, $\frac{d}{ds} F(s) = \frac{-2bs}{(s^2+b^2)^2}$, $\frac{d^2}{ds^2} F(s) = \frac{-2b(s^2+b^2)^2 + 8bs^2(s^2+b^2)}{(s^2+b^2)^4}$

$$= \frac{(s^2+b^2)[-2b(s^2+b^2)+8bs^2]}{(s^2+b^2)^{4\,3}} = \frac{6bs^2-2b^3}{(s^2+b^2)^3}$$

$$\therefore \mathcal{L}\{t^2 \sin bt\} = \frac{6bs^2-2b^3}{(s^2+b^2)^3}$$

The Inverse of Laplace Transform:-

Here we consider the inverse problems of L.T., if we have L.T. $F(s)$ of a certain function $f(t)$. Then the problem is to find function $f(t)$.

We will use the notation $L^{-1}\{F(s)\}$ to denoties the inverse L.T. of $f(t)$ and we write;

$$f(t) = L^{-1}\{F(s)\}$$

in order to find the L^{-1}. T., we need the following table:-

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$
1	$1/s$
e^{at}	$1/(s-a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s^2/(s^2-b^2)$
t^n	$n!/s^{n+1}$
$t \sin bt$	$2bs/(s^2+b^2)^2$
$t \cos bt$	$(s^2-b^2)/(s^2+b^2)^2$
$t^n e^{at}$	$n!/(s-a)^{n+1}$
$e^{-at} \sin bt$	$b/[(s+a)^2+b^2]$
$e^{-at} \cos bt$	$(s+a)/[(s+a)^2+b^2]$
$\dfrac{\sin bt - bt\cos bt}{2b^3}$	$1/(s^2+b^2)^2$
$\dfrac{t \sin bt}{2b}$	$s/(s^2+b^2)^2$