b. Variation of Parameters

$y'' + f(x)\,y' + g(x)\,y = R(x)$ — * اذا لم تكن $R(x)$ من ضمن الدوال السابقة في الجدول ، نحل بهذه الطريقة.

$y = y_h + y_p$

$y_p = u_1 * y_1 + u_2 * y_2$

$y_p' = (u_1 y_1' + u_1' y_1) + (u_2 y_2' + u_2' y_2)$;

cond. $u_1' y_1 + u_2' y_2 = 0$ ----- (1)

$y_p' = u_1 y_1' + u_2 y_2'$

$y_p'' = (u_1 y_1'' + u_1' y_1') + (u_2 y_2'' + u_2' y_2')$

sub. in D.E.

$\therefore R(x) = (u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2') + f(x)(u_1 y_1' + u_2 y_2') + g(x)(u_1 y_1 + u_2 y_2)$

$\therefore u_1 (y_1'' + f(x)\,y_1' + g(x)\,y_1) + u_2 (y_2'' + f(x)\,y_2' + g(x)\,y_2) + u_1' y_1' + u_2' y_2' = R(x)$ ---- (2)

Solve (1) & (2)

$$u_1' = \frac{-y_2}{y_1 y_2' - y_1' y_2} R(x)$$

$$u_2' = \frac{y_1}{y_1 y_2' - y_1' y_2} R(x)$$

$y_p' = -u_1 e^{-x} - u_2 x e^{-x} + u_2 e^{-x}$

Example:- Solve the D.E. $y'' + 2y' + y = e^{x}$

Sol. :- $m^2 + 2m + 1 = 0 \Rightarrow m_1 = m_2 = -1$

$\therefore y_h = c_1 e^{-x} + c_2 x e^{-x}$

$y_1 = e^{-x}$, $y_2 = x e^{-x}$

$$w(y_1, y_2) = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix} = e^{-2x} \begin{vmatrix} 1 & x \\ -1 & (1-x) \end{vmatrix}$$

$$= e^{-2x} (1 * (1-x) - (-1 * x)) = e^{-2x} (1) = e^{-2x}$$

$$u_1 = \int \frac{-y_2 R(x)}{w} dx = \int \frac{-x e^{-x} \cdot e^{x}}{e^{-2x}} dx = -\int x e^{2x} dx$$

integration by parts

$$= -\left[x \cdot \frac{e^{2x}}{2} - \int \frac{1}{2} e^{2x} (1) dx\right]$$

$$= -\left[\frac{x}{2} e^{2x} - \frac{1}{4} e^{2x}\right] = \frac{e^{2x}}{4} (1 - 2x)$$

$$u_2 = \int \frac{y_1 \cdot R(x)}{w} dx = \int \frac{e^{-x} \cdot e^{x}}{e^{-2x}} dx = \int e^{2x} dx$$

$$= \frac{1}{2} e^{2x}$$

$y_t = y_h + y_p$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{e^{2x}}{4} (1 - 2x) \cdot e^{-x} + \frac{e^{2x}}{2} \cdot x e^{-x}$$

$y_p = u_1 y_1 + u_2 y_2 = u_1 e^{-x} + u_2 x e^{-x}$

$y_p' = (-u_1 e^{-x} + \cancel{u_1' e^{-x}}) + (u_2(-x e^{-x} + e^{-x}) + \cancel{u_2' x e^{-x}})$

$u_1' y_1 + u_2' y_2 = 0 \Rightarrow u_1' e^{-x} + u_2' x e^{-x} = 0$

$= \frac{e^x}{2}\left(\frac{1}{2}(1-2x)+x\right)$

$= \frac{e^x}{2}\left(\frac{1}{2}-x+x\right) = \frac{e^x}{4}$

$y = y_h + y_p = c_1 e^{-x} + c_2 x e^{-x} + \frac{e^x}{4}$

غير وارد

Example:- Solve the D.E. $y''' - 3y'' + 3y' - y = \frac{e^x}{x}$

$D^3 - 3D^2 + 3D - 1 = 0$

$D = 1, \quad 1 - 3 + 3 - 1 = 0 \quad (o.k)$

$\therefore D_1 = 1$

$(D-1)(D-1)$

$D_2 = D_3 = 1$

$$\begin{array}{r} D^2 - 2D + 1 \\ D-1 \overline{\big) D^3 - 3D^2 + 3D - 1} \\ \mp D^3 \pm D^2 \\ \hline -2D^2 + 3D \\ \pm 2D^2 \mp 2D \\ \hline D - 1 \\ \cancel{D-1} \\ \hline 0 \end{array}$$

$\therefore y_h = c_1 e^x + c_2 e^x x + c_3 e^x x^2$

$\therefore y_1 = e^x, \quad y_2 = x e^x, \quad y_3 = x^2 e^x$

$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$

$$w(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$\frac{P}{q} = \frac{\pm 1}{\pm 1}$

$f(1) = 1 - 3 + 3 - 1 = 0$

$\boxed{m_1 = 1}$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & +3 & -1 \\ & & 1 & -2 & +1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

$= D^2 - 2D + 1 = 0 \quad (D-1)(D-1) = D_2 = D_3 = 1$

$$\omega(y_1, y_2, y_3) = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x + e^x & x^2e^x + 2xe^x \\ e^x & xe^x + 2e^x & x^2e^x + 2xe^x + 2(xe^x + \dots \end{vmatrix}$$

$$= e^{3x} \begin{vmatrix} 1 & x & x^2 \\ 1 & x+1 & x^2+2x \\ 1 & x+2 & x^2+4x+1 \end{vmatrix}$$

$$= e^{3x}\left[1\left[(x^3+4x^2+x+x^2+4x+1) - (x^3+2x^2+2x^2+4x)\right] - x\left[(x^2+4x+1) - (x^2+2x)\right] + x^2\left[(x+2)-(x+1)\right]\right]$$

$$\omega(y_1, y_2, y_3) = e^{3x}$$

$$u_1 = \int \frac{\begin{vmatrix} 0 & y_2 & y_3 & \dots & y_n \\ 0 & \bar{y}_2 & \bar{y}_3 & & \vdots \\ R(x) & dy_2^n/dx^n & dy_3^n/dx^n & \dots & dy_n^n/dx^n \end{vmatrix}}{\omega(y_1, y_2, \dots, y_n)}$$

$$\bar{u}_1 = \frac{\begin{vmatrix} 0 & xe^x & x^2e^x \\ 0 & xe^x + e^x & x^2e^x + 2xe^x \\ e^x/x & xe^x + 2e^x & x^2e^x + 2xe^x + 2(xe^x + e^x) \end{vmatrix}}{\omega(y_1, y_2, y_3)}$$

w $\quad$ rhs

$$= \frac{1}{e^{3x}} \cdot \frac{e^{x}}{x} \cdot e^{2x} \begin{vmatrix} x & x^2 \\ x+1 & x^2+2x \end{vmatrix}$$

$$= \frac{e^{x}}{xe^{3x}} \cdot e^{2x} \left[(x^3+2x^2) - (x^3+x^2) \right]$$

$$(2x^2 - x)$$

$$\overline{U_1} = \frac{e^{x}}{xe^{3x}} \cdot x^2 e^{2x} = x$$

$$U_1 = \int x \, dx = \frac{x^2}{2}$$

$$\overline{U_2} = \frac{-1}{\omega(y_1, y_2, y_3)} \cdot \begin{vmatrix} e^{x} & 0 & x^2e^{x} \\ e^{x} & 0 & x^2e^{x}+2xe^{x} \\ e^{x} & \frac{e^{x}}{x} & x^2e^{x}+2xe^{x}+2(xe^{x}+e^{x}) \end{vmatrix}$$

$$= -e^{-3x} \cdot \frac{e^{x}}{x} \begin{vmatrix} e^{x} & x^2e^{x} \\ e^{x} & x^2e^{x}+2xe^{x} \end{vmatrix}$$

$$= -e^{-3x} \cdot \frac{e^{x}}{x} \cdot e^{2x} \begin{vmatrix} 1 & x^2 \\ 1 & x^2+2x \end{vmatrix}$$

$$\overline{U_2} = -e^{-3x} \cdot \frac{e^{x}}{x} (+2xe^{2x}) = -2$$

$$U_2 = \int u_2 \, dx$$

$$U_2 = -2x$$

$$\bar{U}_3 = \frac{1}{W(y_1, y_2, y_3)} \begin{vmatrix} e^x & xe^x & 0 \\ e^x & xe^x + e^x & 0 \\ e^x & xe^x + 2e^x & \frac{e^x}{x} \end{vmatrix}$$

$$= \frac{1}{e^{3x}} \cdot \frac{e^x}{x} \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$

$$= \frac{1}{e^{3x}} \cdot \frac{e^x}{x} \cdot e^{2x} \begin{vmatrix} 1 & x \\ 1 & x+1 \end{vmatrix}$$

$$\bar{U}_3 = \frac{1}{x}(1) = \frac{1}{x} \Rightarrow U_3 = \ln x$$

$$\therefore y_p = \frac{x^2}{2} e^x - 2x^2 e^x + x^2 e^x \ln x$$

$u_1 \; y_1 \qquad u_2 \; y_2 \qquad y_3 \; u_3$

$$y = y_h + y_p$$

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + \frac{x^2}{2} e^x - 2x^2 e^x + x^2 e^x \ln x$$