Second Order Differential Equations With Constant Coefficients

$ay'' + by' + cy = R(x)$ standard Form

i- IF $R(x) = 0$ called homogenous

ii- IF $R(x) \neq 0$ called non-homogenous

i- Homogenous Form

$ay'' + by' + cy = 0$

auxiliary equation

D = operator

$Dy = \frac{dy}{dx}$, $D(Dy) = D^2y$, $D(D^2y) = D^3y$

$y = e^{mx}$

or $(aD^2 + bD + c)y = 0$

$y' = me^{mx}$

let $y = e^{mx} \Rightarrow e^{mx}(am^2 + bm + c) = 0$

$y'' = m^2e^{mx}$

Since e^{mx} can never be zero, it is thus necessary that

$am^2 + bm + c = 0$, $m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$

اقرب الترتيب

$\boxed{a}$ IF $m_1 \neq m_2$

∴ $y = c_1e^{m_1x} + c_2e^{m_2x}$ (جذران حقيقيان غير متساويان)

$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$

$e^{mx}(am^2 + bm + c) = 0$

b. If $m_1 = m_2 = m$

$\therefore y = c_1 e^{mx} + c_2 x e^{mx}$ (جذران حقيقيان متساويان)

c. If $\alpha \mp i\beta$

$\therefore y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ (جذران خياليان)

Example 1:- Solve $y'' + 3y' + 2y = 0$

$m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0$

$m_1 = -1 \quad , \quad m_2 = -2 \Rightarrow y = c_1 e^{-x} + c_2 e^{-2x}$

Example 2:- Solve $y'' + 5y' + 10y = 0$

$m^2 + 5m + 10 = 0 \Rightarrow m_{1,2} = \dfrac{-5 \mp \sqrt{25 - 4(1)(10)}}{2(1)}$

$= \dfrac{-5 \mp \sqrt{-15}}{2} = \dfrac{-5}{2} \mp \dfrac{\sqrt{15}}{2} i$

$\alpha = \dfrac{-5}{2} \quad , \quad \beta = \dfrac{\sqrt{15}}{2}$

$\therefore y = e^{-5/2 x}\left(c_1 \cos \dfrac{\sqrt{15}}{2} x + c_2 \sin \dfrac{\sqrt{15}}{2} x\right)$

H.W. (1) $4y'' - 5y' - 6y = 0$ (4) $9y'' + 24y' + 16 = 0$

(2) $y'' - 5y' + 6y = 0$ (5) $y'' + 8y' + 25 = 0$

(3) $y'' - 6y' + 9 = 0$

ii- Non-Homogenous 2nd Order D.E.
when $R(x) \neq 0$

1. Assume $R(x) = 0$ and find y_h
2. Find y_p by → method of undetermined coef. / → variation of parameters.
3. Let $y_t = y_h + y_p$

$\boxed{a}$ Method of undetermined coefficients:-

$$ay'' + by' + cy = f(x)$$

i- assuming y_p
ii- substituting y_p into the given diff. eq.
iii- determining the arbitrary constants in y_p

$R(x)$	y_p	$R(x)$	y_p (example)
K	A	2	A
Kx^n	$Ax^n + Bx^{n-1} + \dots + C$	$3x^2$	$Ax^2 + Bx + C$
Ke^{rx}	Ae^{rx}	$3e^{2x}$	Ae^{rx}
$K\cos rx$ / $K\sin rx$	$A\cos rx + B\sin rx$	$2\cos 2x$	$A\cos 2x + B\sin 2x$
		$3x^2 + \cos 2x$	$Ax^2 + Bx + C + D\cos 2x + E\sin 2x$
		$3x^2 e^{2x}$	$(Ax^2 + Bx + C)e^{2x}$

Example 1:- Solve the D.E. $y'' - y' - 6y = e^{x}$

Sol.:- $m^2 - m - 6 = 0 \implies (m+2)(m-3) = 0$

$m_1 = -2 \quad , \quad m_2 = 3$

$\therefore y_h = c_1 e^{-2x} + c_2 e^{3x}$

$R(x) = e^{x} \implies y_p = Ae^{x} \implies y_p' = Ae^{x} \implies y_p'' = Ae^{x}$

Sub. $Ae^{x} - Ae^{x} - 6Ae^{x} = e^{x}$

$\therefore -6A = 1 \implies A = -\frac{1}{6}$

$\therefore y_p = -\frac{1}{6} e^{x}$

$\therefore y = y_h + y_p \implies y = c_1 e^{-2x} + c_2 e^{3x} - \frac{1}{6} e^{x}$

Example 2:- Solve the D.E. $y'' + 4y' + 4y = \cos x$

Sol. $m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0 \implies m_1 = m_2 = -2$

$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$

$R(x) = \cos x$

$\therefore y_p = A \cos x + B \sin x$

$y_p' = -A \sin x + B \cos x$

$y_p'' = -A \cos x - B \sin x$

Sub. $-A\cos x - B\sin x - 4A\sin x + 4B\cos x$
$+ 4A\cos x + 4B\sin x = \cos x$

$(3A+4B)\cos x + (3B-4A)\sin x = \cos x$

$3A+4B=1 \quad , \quad 3B-4A=0$

$\Rightarrow B = \frac{4}{3}A \Rightarrow 3A + 4\left(\frac{4}{3}\right)A = 1$

$\therefore \frac{25}{3}A = 1 \Rightarrow A = \frac{3}{25}$

$3B = 4 * \frac{3}{25} \Rightarrow B = \frac{4}{25}$

$\therefore y_p = \frac{3}{25}\cos x + \frac{4}{25}\sin x$

$y = y_h + y_p$

H.W (3): Solve the D.E. $y'' + y' - 6y = x^2 + 1$
$y'' + 2y' - 3y = \sin x + \cos 2x$