$$= \left[\frac{x^2}{2} + xy\right]_2^x + \left[xy + \frac{2y^2}{2}\right]_3^y$$

$$= \frac{x^2}{2} + xy - 2 - 2y + 2y + y^2 - 6 - 9$$

$$\therefore \frac{x^2}{2} + xy + y^2 = 17$$

4 Linear First-Order Eq.s:-

The genral form $\boxed{\frac{dy}{dx} + P(x)\, y = Q(x)}$

i- $I.F. = e^{\int P(x)\, dx}$

ii- $I.F. * y = \int I.F.\; Q(x)\, dx$

$$\frac{d}{dx}\left(y\, e^{\int P(x)\, dx}\right) = \frac{dy}{dx}\, e^{\int P(x)\, dx} + y\, P(x)\, e^{\int P(x)\, dx}$$

$$= e^{\int P(x)\, dx}\left(\frac{dy}{dx} + y\, P(x)\right)$$

$e^{\int P(x)\, dx}$ is an integrating factor

$$\therefore \boxed{y\, e^{\int P(x)\, dx} = \int Q(x) \cdot e^{\int P(x)\, dx}\, dx + c}$$

Example:- Solve the D.E. $\frac{dy}{dx} + 2xy = 4x$

$\int P(x)\, dx = 2x\, dx = x^2 \Rightarrow e^{\int P(x)\, dx} = e^{x^2}$

$y e^{x^2} = \int 4x\, e^{x^2}\, dx \Rightarrow x^2 2\, e^{x^2} + c$

$\cancel{y = 2x^2 + c e^{-x^2}}$ X

$u = x^2 \Rightarrow du = 2x\, dx$

$\int 4x e^{x^2} dx = \int 2 \cdot (2x\, dx) e^{x^2} = \int 2 e^u du$

$2e^u = 2e^{x^2}$

$y = 2 + c e^{-x^2}$

$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$

Example:- Solve the D.E. $y' + \cos x \, y = \cos 2x$ $\div (\sin x)$

$$\frac{dy}{dx} + \frac{\cos x}{\sin x} y = \frac{\cos 2x}{\sin x}$$

$\therefore$ $P(x) = \frac{\cos x}{\sin x}$ $\qquad Q(x) = \frac{\cos 2x}{\sin x}$

$$I.F. = e^{\int P(x)\,dx} = e^{\int \frac{\cos x}{\sin x}dx} = e^{\ln \sin x} = \sin x$$

$u = 2x$, $du = 2dx$, $dx = \frac{du}{2}$

$$I.F. * y = \int I.F. * Q(x)\, dx$$

$$\sin x \, y = \int \sin x * \frac{\cos 2x}{\sin x}\, dx \quad * \cancel{\frac{2}{2}}$$

$$y \sin x = \frac{1}{2} \sin 2x + c$$

[5] Special First_Order Bernoulli's Eq.:-

A non_L.D.E which can be transformed into L.D.E by change of the dependent variables.

$$\frac{dy}{dx} + y\,P(x) = y^n\,Q(x)$$

or

$$y^{-n}\frac{dy}{dx} + y^{-n+1}\,P(x) = Q(x)$$

Note:- n = any real no. $\quad n > 0$

$z = y^{1-n}$

$\frac{dz}{dy} = (1-n)y^{-n}$; $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx}\,\frac{y^n}{1-n}$ subs. $\left(\frac{dz}{dx}\,\frac{y^n}{1-n}\right) + P(x)y = Q(x)y^n$

$\frac{dz}{dx} + P(x)(1-n)y^{1-n} = Q(x)(1-n)$

$z' + (1-n)P(x)\,z = (1-n)\,Q(x)$ L.D.E of z

i_ let $z = y^{1-n}$

ii_ $\frac{dz}{dx} = (1-n)\,y^{-n}\,\frac{dy}{dx}$

iii_ نعوض المعادلة مثل الـ linear وتحل

Example 8: Solve the D.E. $3xy' + y + x^2y^4 = 0$

$y' + \frac{1}{3x}y = -\frac{x}{3}y^4 \quad *(-3)$

$\therefore n = 4$

$z = y^{1-n} = y^{-3}$, $\frac{dz}{dx} = (1-n)y^{-4}\frac{dy}{dx}$

$\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx} \Longrightarrow \frac{dz}{dx} - \frac{1}{x}z = x$

$P(x) = \frac{-1}{x}$, $Q(x) = x$

$I.F = e^{-\int\frac{dx}{x}} = \frac{1}{x}$

$$\frac{z}{x} = \int x \cdot \frac{1}{x} \, dx + c$$

$$\Rightarrow \frac{z}{x} = x + c \Longrightarrow z = x^2 + cx$$

$$y^{-3} = z \Longrightarrow y^3 = \frac{1}{x^2 + cx}$$

H.W. :- Solve i- $y \ln y \, dx + (x - \ln y) \, dy = 0$

ii- $\frac{dy}{dx} + \frac{1}{3} y = \frac{1}{3}(1-2x) y^4$

Examples

Ex 1:- Solve the D.E. $(2y + x^2) \, dx = x \, dy$

$M = 2y + x^2$, $N = -x$ ∴ non-homog.

$\frac{\partial M}{\partial y} = 2$, $\frac{\partial N}{\partial x} = -1$ ∴ not exact.

$$\frac{2y + x^2}{x} = \frac{dy}{dx} \quad \div (x \, dx)$$

$$\frac{dy}{dx} - \frac{2}{x} y = x \qquad \therefore \text{linear}$$

$$P(x) = \frac{-2}{x} \qquad Q(x) = x$$

Ex. 4:- Solve the D.E. $(\cos 2x)\, y' + (2\sin 2x)\, y = 2$

$\cos 2x \dfrac{dy}{dx} + 2\sin 2x\, y - 2 = 0$

$\cos 2x\, dy + (2\sin 2x\, y - 2)\, dx = 0$ ∴ non-homog.

$M = 2\sin 2x\, y - 2 \quad , N = \cos 2x$

$\dfrac{\partial M}{\partial y} = 2\sin 2x \quad , \dfrac{\partial N}{\partial x} = -2\sin 2x$

∴ not exact

$\dfrac{dy}{dx} + 2\tan 2x\, y = \dfrac{2}{\cos 2x}$

∴ Linear with $p(x) = 2\tan 2x$; $Q(x) = \dfrac{2}{\cos 2x}$

$I.F. = e^{\int 2\tan 2x\, dx} = e^{2\int \frac{\sin 2x}{\cos 2x} dx}$

$= e^{-\int \frac{-2\sin 2x}{\cos 2x} dx} = e^{-\ln \cos 2x} = \dfrac{1}{\cos 2x}$

$\dfrac{y}{\cos 2x} = \displaystyle\int \dfrac{2}{\cos 2x \cdot \cos 2x}\, dx + C$

$= 2\displaystyle\int \sec^2 2x\, dx + C$

$\dfrac{y}{\cos 2x} = 2\tan 2x + C$

$y = \cos 2x\, (2\tan 2x) + C$