Functions and their Graphs

Functions

A function from a set D to a set Y is a rule that assigns a unique (single) element

$$f(x) \in Y \quad to\ each\ element\ x \in D$$

The set D of all possible input values is called the **domain** of the function. The set of all values of f(x) as x varies throughout D is called the **range** of the function. The range may not include every element in the set Y. The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers.

Example:

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

- The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$ .The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = \left(\sqrt{y}\right)^2$ for $y \geq 0$.
- The formula y= 1/x gives a real y-value for every x except x=0. *We cannot divide any number by zero*. The range of y= 1/x the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since y=1/(1/y)
- The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$ .The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).
- In $y = \sqrt{4 - x}$ the quantity (4 – x) cannot be negative. That is, $4 - x \geq 0$ or $x \leq 4$ .The formula gives real y-values for all $x \leq 4$ .The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

- The formula $y = \sqrt{1-x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1-x^2$ is negative and its square root is not a real number. The values of $1-x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $y = \sqrt{1-x^2}$ is [0, 1].

Example

Find the Domain of the given functions

1. $f(x) = \frac{1}{x-3}$
2. $f(x) = \sqrt{x^2-3}$
3. $f(x) = \sqrt{x-3x^2}$
4. $f(x) = \frac{x}{|x|+1}$
5. $f(x) = \sqrt{8-x} + \sqrt{x-5}$
6. $f(x) = \frac{1}{x^2}$
7. $f(x) = \sqrt[3]{x}$

Sol.

1. $f(x) = \frac{1}{x-3}$
   The domain will be all real numbers except the numbers that make the denominator equals zero i.e. , $x - 3 \neq 0 \ \ or \ x \neq 3$. Then the domain is [D= R/{x=3}].
2. $f(x) = \sqrt{x^2-3}$
   The domain will be all real numbers except the numbers that make the expression under the square root negative i.e.
   $x^2 - 3 \geq 0 \ or \ x^2 \geq 3 \ or \ x \geq \pm\sqrt{3}$
   Then the domain will be
   $$D = \left(-\infty, -\sqrt{3}\right] \cup \left[\sqrt{3}, \infty\right) \ or \ R/\{\left(-\sqrt{3}, \sqrt{3}\right)\}$$
3. $f(x) = \sqrt{x-3x^2}$
   The domain will be all real numbers except the numbers that make the expression under the square root negative i.e.
   $$x - 3x^2 \geq 0 = x(1-3x) \geq 0$$
   $$x = 0 \ or \ x = \frac{1}{3}$$
   Then the number under the square root will be negative if
   $$x < 0 \ or \ x > \frac{1}{3}$$

And the domain will be only the closed interval [0,1/3]

4. $f(x) = \frac{x}{|x|+1}$

   The domain will be all real numbers except the numbers that make the denominator equals zero i.e., $|x| + 1 \neq 0 \; or \; |x| \neq -1$, and since there is no number that makes this possible then the domain is all real numbers. D=R

5. $f(x) = \sqrt{8-x} + \sqrt{x-5}$

The domain will be all real numbers except the numbers that make the expression under the square root negative i.e.

$$(8-x) \geq 0 \; and \; (x-5) \geq 0$$
$$8 \geq x \; and \; x \geq 5$$

Then the number under the square root will be negative if

$$x < 5 \; or \; x > 8$$

And the domain will be only the closed interval [5,8].

6. $f(x) = \frac{1}{x^2}$

The domain will be all real numbers except the numbers that make the denominator equals zero i.e., $x^2 \neq 0 \; or \; x \neq 0$, then the domain is all real numbers except *x*=0. *D=R/{x=0}*

7. $f(x) = \sqrt[3]{x}$

   Since the number under the cube root can be negative or positive, then the domain will be all real numbers. $D = (-\infty, \infty)$.

When we define a function with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real *x-values* for which the formula gives real *y-values*, the so-called ***natural domain***. If we want to restrict the domain in some way, we must say so. The domain of a function may be restricted by context. For example, the domain of

$$y = x^2$$

is the entire set of real numbers. To restrict the function to, say, positive values of *x*, we would write($y = x^2, x > 0$). Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$ The range of($y = x^2, x \geq 2$) is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation, the range is $\{y | y \geq 4\} \; or \; [4, \infty)$.

When the range of a function is a set of real numbers, the function is said to be ***real-valued***. The domains and ranges of many ***real-valued*** functions of a real variable are intervals or combinations of intervals. The intervals may be open, closed, or half open, and may be finite or infinite

Example

Find the range of the given functions

1. $y = \frac{1}{x^2}$

   The range can be calculated as the set of all values of f(x) as x varies throughout the domain. Then first we have to find the domain of the function $y = \frac{1}{x^2}$ which has been calculated previously as *D=R/{x=0}*. Then the range will be restricted with the x values. Now y cannot be negative since y is a fraction of two positive numbers (1 and x^2), then the range will be $y > 0 \ or \ (0, \infty)$.

2. $y = \sqrt{1 - x^2}$

   The domain will be all real numbers that make $(1 - x^2) \geq 0 \ or \ x^2 \leq 1$ then the domain is [-1,1] while the range can be calculated as x varies throughout the closed interval [-1,1] which makes y values varies through the closed interval [0,1].

3. $y = |x|$

   The domain of this function is $(-\infty, \infty)$ and as x varies throughout this domain the values of y are positive values only since $|x|$ is always positive even if x values are negative. Then the range will be $[0, \infty)$.

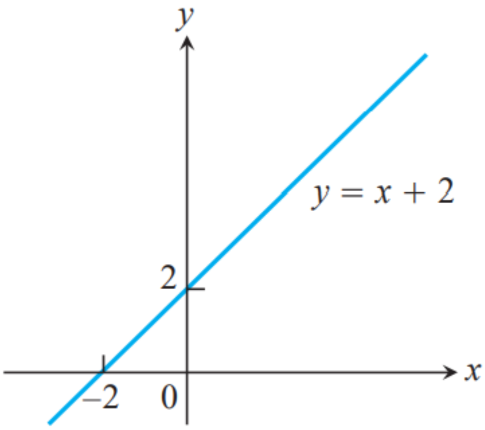
Graphs of Functions

Another way to visualize a function is its graph. If f is a function with domain D, its graph consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f.

Example

The graph of the function $f(x) = x + 2$ is the set of points with coordinates

(x, y) for which $y = x + 2$. Its graph is sketched below



The graph of a function *f* is a useful picture of its behaviour. If (x, y) is a point on the graph, then *y=f(x)* is the height of the graph above the point *x*. The height may be positive or negative, depending on the sign of *f(x)*.

Graph of some Frequently encountered Functions

There are a number of important types of functions frequently encountered in calculus.

- **Linear Functions**
  A function of the form $\boldsymbol{y = mx + b}$ for constants $\boldsymbol{m}$ and $\boldsymbol{b}$, is called a linear function. Examples of some linear functions with different slopes ($\boldsymbol{m}$) are shown in the figure below

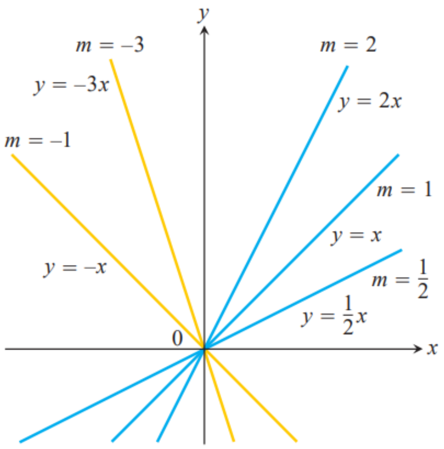


Figure (a) linear Functions with

different slopes

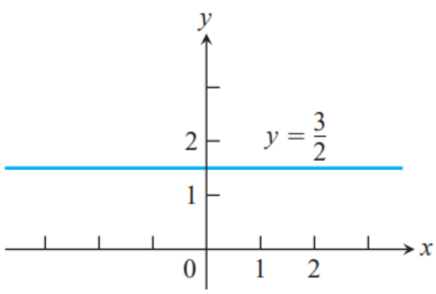


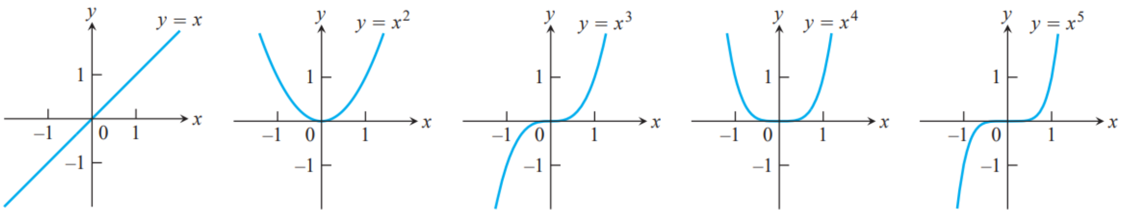
Figure (b) linear function with slope =0

- **Power Functions**

A function of the form $f(x) = x^a$ where a is a constant, is called a power function. There are several important cases to consider.

(i) when $a=n$, where n is positive.

The graphs of $f(x) = x^n$ for n=1,2, 3, 4, 5, are displayed in figure below. These functions are defined for all real values of x. Notice that as the power n gets larger, the curves tend to flatten toward the *x-axis* on the interval (-1,1) and also rise more steeply for $|x|>1$. Each curve passes through the point (1, 1) and through the origin.



Figure

Graphs of $f(x) = x^n \; for \; n = 1,2,3,4,5 \; defined \; for -\infty < x < \infty$

(ii) when a = -1 or a = -2.

The graphs of the functions $f(x) = x^{-1} and \; f(x) = x^{-2}$ are shown in figure below. Both functions are defined for all $x \neq 0$ (you can never divide by zero).

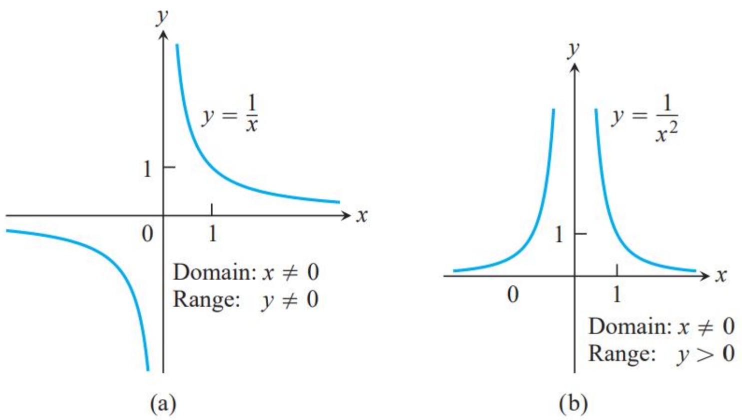
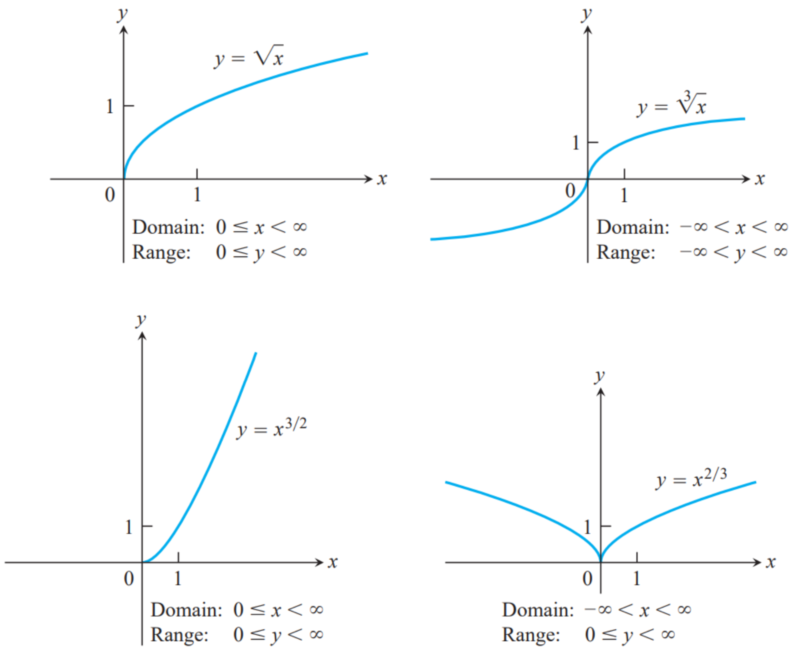


Figure (a) when a = -1
(b) when a = -2

(iii) when $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, and \frac{2}{3}$

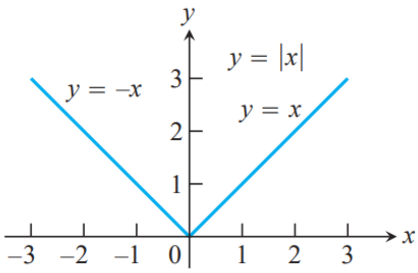
The functions $f(x) = x^{\frac{1}{2}} = \sqrt{x}$ and $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$ are the square root and cube root functions, respectively. The domain of the square root function is $[0, \infty)$ but the cube root function is defined for all real x. Their graphs are displayed in figure below along with the graphs of

$$f(x) = x^{\frac{3}{2}} \; and \; x^{\frac{2}{3}}$$



- **Piecewise defined Function**

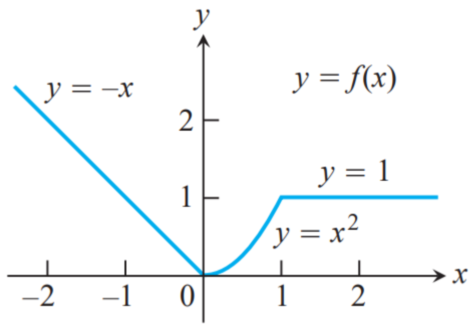
Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**



$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

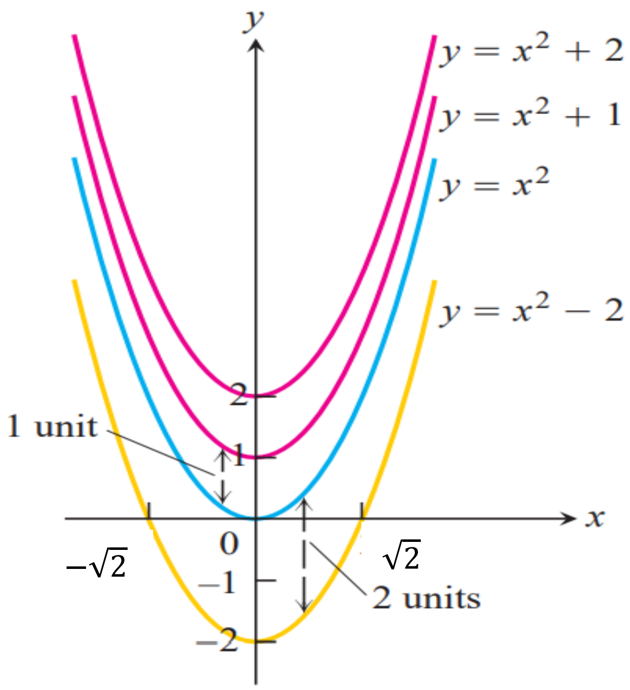
Example

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

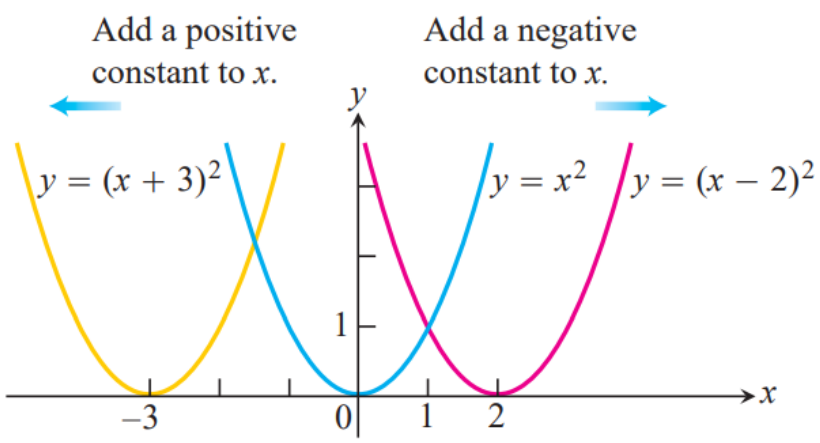


Shifting a Graph of a Function

1. To shift the graph of a function **straight up**, add a positive constant to the righthand side of the formula **(*f(x)*+*c*)**
2. To shift the graph of a function **straight down**, add a negative constant to the right-hand side of the formula **(*f(x)*-*c*)**
3. To shift the graph of to the **left**, add a positive constant to *x*. **(y= *f(x*+*c*))**
4. To shift the graph of to the **right**, add a negative constant to *x*.(**y= *f(x*-*c*))**



To shift the graph of $f(x) = x^2$ up we add positive constants to the function and to shift the graph down we add negative constants to the function.



To shift the graph of $f(x) = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x.

Scaling and Reflecting a Graph of a Function

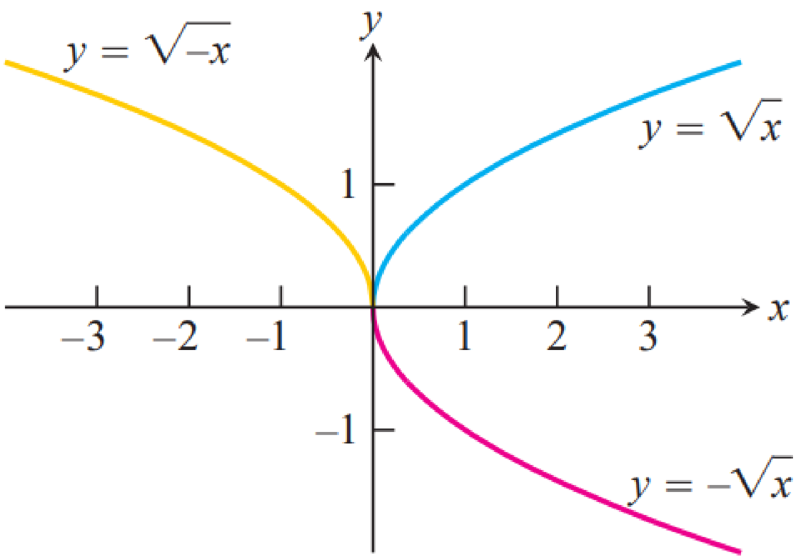
Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$,

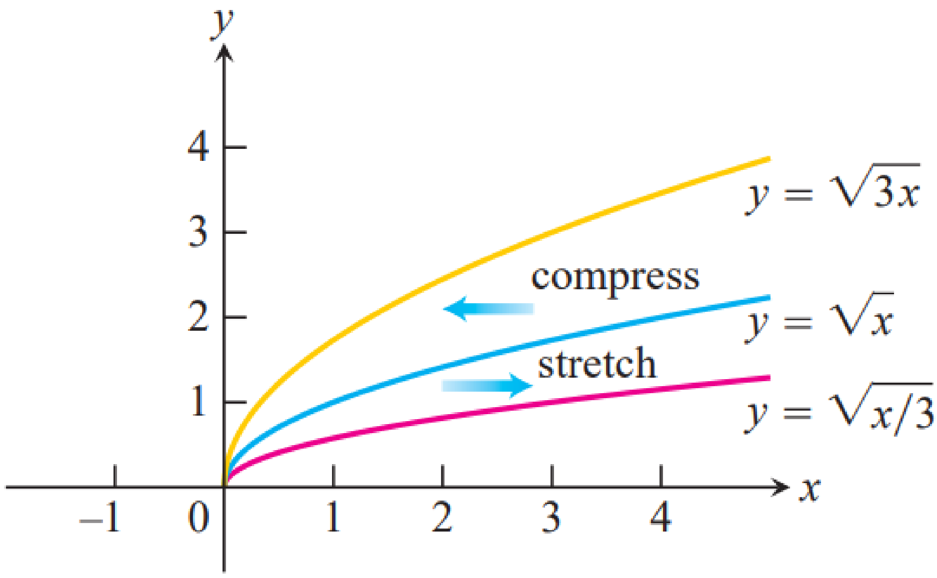
$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

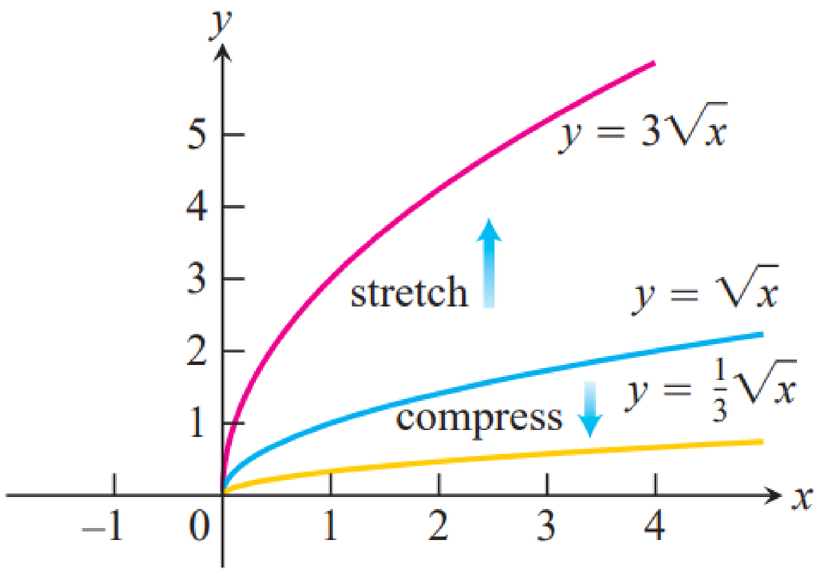
$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.



$y = \sqrt{-x}$
y
$y = \sqrt{x}$
1
x
–3
–2
–1
1
2
3
–1
$y = -\sqrt{x}$



y
4
3
2
1
$y = \sqrt{3x}$
compress
$y = \sqrt{x}$
stretch
$y = \sqrt{x/3}$
x
–1
0
1
2
3
4



y
5
4
3
2
1
$y = 3\sqrt{x}$
stretch
$y = \sqrt{x}$
compress
$y = \frac{1}{3}\sqrt{x}$
x
–1
0
1
2
3
4

Example

If $f(x) = \sin x \quad sketch\ f(2x) and\ f(\frac{x}{2})$

Sol.

$$y = \sin x \quad (red)$$

$$y = sin(2x) \quad (blue)$$

$$y = \sin\left(\frac{x}{2}\right) \quad (green)$$

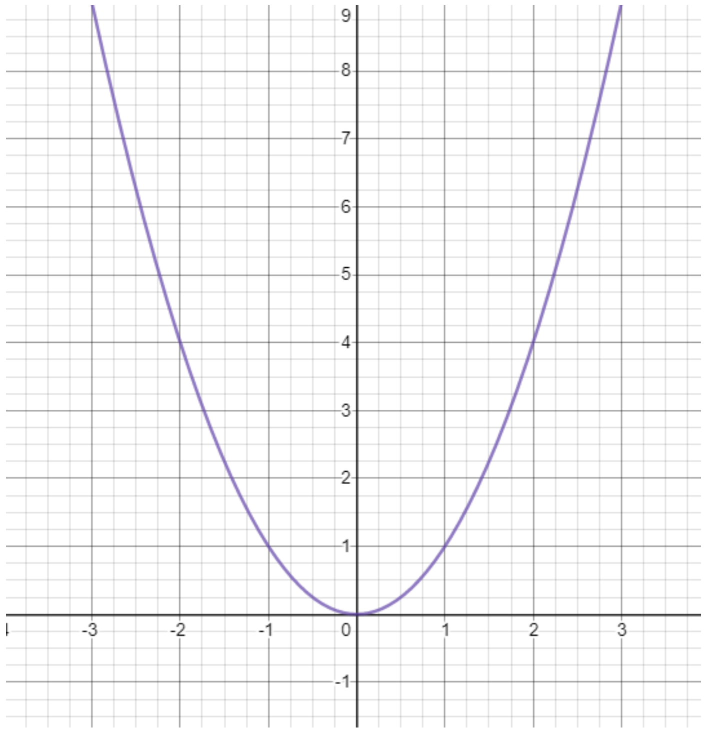
Example

If $f(x) = x^2 \ \ sketch \ \ f(x) = (x-2)^2 + 1$

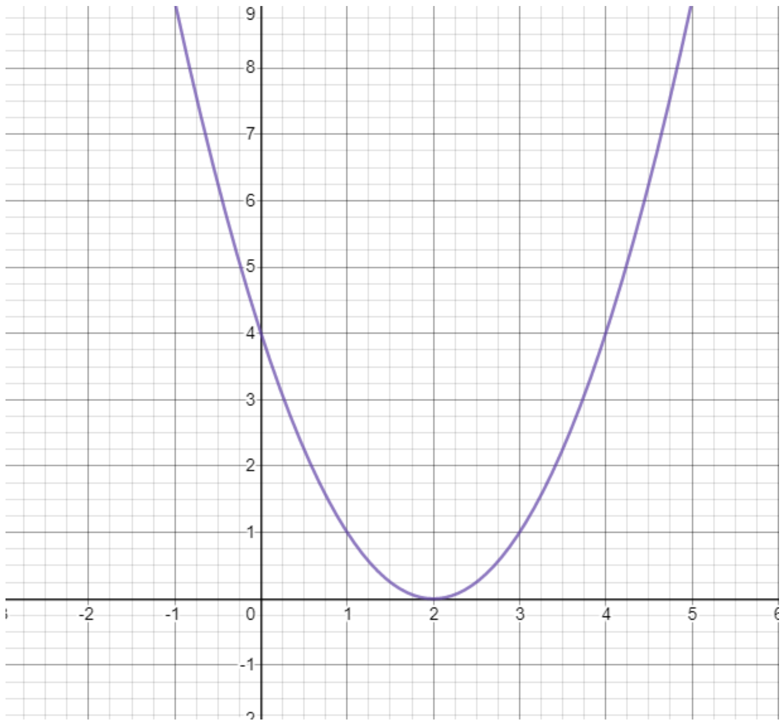
Sol.

First sketch $f(x) = x^2$ then $f(x) = (x-2)^2$ by shifting $f(x) = x^2$ right by 2 units then shift $f(x) = (x-2)^2$ up by one unit as in the figures below

1. $f(x) = x^2$



2. $f(x) = (x-2)^2$



3. $f(x) = (x-2)^2 + 1$