2. Moment area method

It is a useful and simple method of determining sloper and deflections in beams. The method depends on bening moment diagram and geometry of the elastic curve. It involves two basic theorems that will illustrated.

To show the concept of this method, a default beam shown in figure is considered. If this beam is subjected to any loading, it will deform as shown exaggerated elastic curve. The bending moment diagram for it can be assume as shown exaggerated elastic curve. The bending moment diagram for it can be assume as shown. If we assume A and B as points at elastic curve, the tangent lines of the curve at these points can be drawn as shown. In this figure we can define the following symbols:

$\boldsymbol{\theta_{AB}}$: is the angle between the two tangent lines that are drawn from the points A and B.

$\boldsymbol{t_{A/B}}$: is the deviation of point A from a tangent line drawn at point B or the vertical distance from point A on the elastic curve to the point of intersection between the tangent line drawn from point B and vertical line that is perpendicular to the original position of point A on the beam.

$t_{B/A}$: is the same definition with replacing the symbols A and B with them

Theorm 1: the change in slope between tangent lines drawn to the elastic curve at points A and B is equal to area of bending moment diagram between these two points divided by EI.

$$\boldsymbol{\theta_{AB} = \frac{Area\ B.M.D\ (A-B)}{EI}}$$

Theorm 2: the distance $t_{A/B}$ is equal to moment of area of B.M.D between the points A and B about the point A, while the distance $t_{B/A}$ is equal to moment of the same area about the point B.

$$\boldsymbol{t_{A/B} = \frac{Area\ B.M.D\ (A-B) \times \bar{X}A}{EI}}$$

$$t_{B/A} = \frac{\boldsymbol{Area\ B.M.D\ (A-B) \times \bar{X}B}}{\boldsymbol{EI}}$$

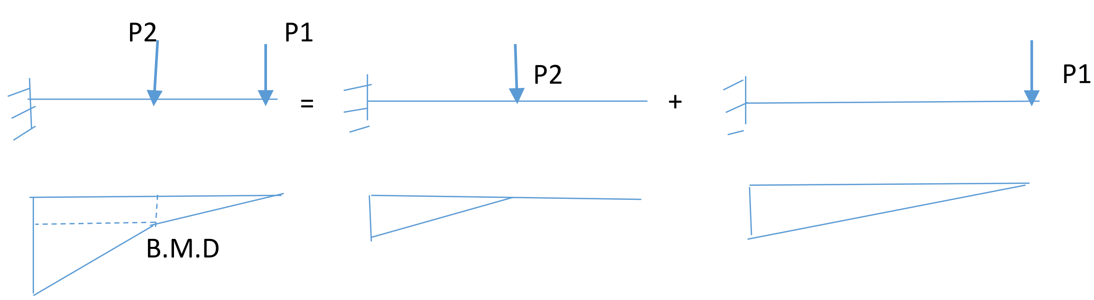
Steps of calculations

The following steps must be made for calculation of deflection and rotation values at any position in the beam:

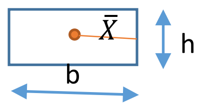
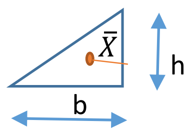
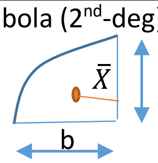
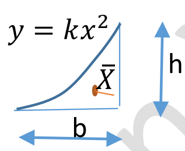
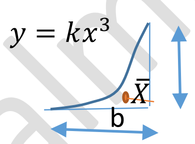
1. Find the reactions using equilibrium equations.
2. Draw bending moment diagram along the beam.
3. Draw the expected deformed shape (elastic curve) of the beam due to loads with exaggeration of the drawing.
4. Draw the tangent lines on the elastic curve at the end and supports of the beam and any other position that is required to solution of the problem.
5. Markes the deviations (t) and angles (θ) for the tangent lines at selected positions that ensure finding the required values of deflections (t) and angles (θ) at the selected positions using rules of triangles and angles.
6. Compute values of deviations (t) and angle (θ) at the selected positions using the two theorms.
7. Compute values of deflections and rotations at required positions using rules of triangles and angles.

Notes

- If the calculation of areas of B.M.D is complicated due to multi types of applied loads, the diagram must be divided into parts so that each part have areas that is easily calculated. Dividing of B.M.D. depends on dividing of applied loads so that each part of B.M.D. must be drawn according to one type of applied loads. Then, the total area of B.M.D. between two points and its moment about any point are the summations of areas of all parts of B.M.D. between these points and their moments about the point respectively.



- The main rules that are detailed in the following table can be used for calculation of areas for different shapes of B.M.D. and their centroid.

shape	Area	Centroid ($\bar{X}$)
	A=bh	$\bar{X} = \frac{1}{2}$b
	$A = \frac{bh}{2}$	$\bar{X} = \frac{1}{3}$b
Parabola (2nd-deg) 	$A = \frac{2}{3}$bh	$\bar{X} = \frac{3}{8}$b
	$A = \frac{1}{3}$bh	$\bar{X} = \frac{1}{4}$b
	$A = \frac{1}{4}$bh	$\bar{X} = \frac{1}{5}$b

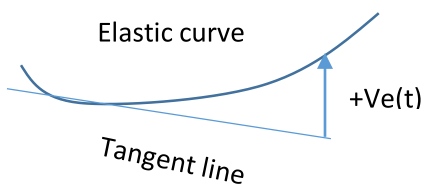
- if EI value is not constant within the beam and changes from part to the other, B.M.D must be drawn by dividing values of the diagram for each part (that has constant value of EI) by EI value for each part to get anew different diagram for each part to get anew different diagram ($\frac{M}{EI}$ diagram). The obtained diagram must be used to calculation of the areas and their moments for the required region without dividing the values of $\theta_{AB}, t_{A/B}$ and $t_{B/A}$ by EI value as follows:

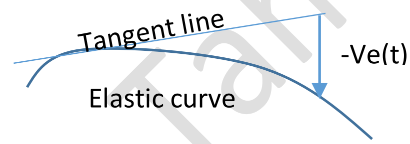
$$\theta_{AB} = Area\left(\frac{M}{EI}\right)_{A-B}$$

$$t_{A/B} = Area\left(\frac{M}{EI}\right)_{A-B} \times \bar{X}A,$$

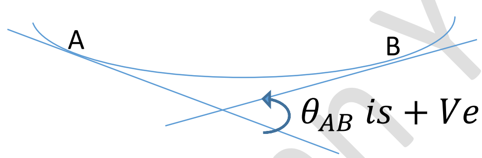
$$t_{B/A} = Area\left(\frac{M}{EI}\right)_{A-B} \times \bar{X}B$$

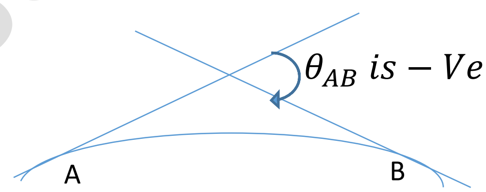
- The deviation $t_{A/B}$ or $t_{B/A}$ is positive when the point on elastic curve lies above tangent line while it is negative if the point lies below the tangent line as shown.





- The angle (θ_{AB}) is positive if its direction in c.c.w., while it is negative if its direction in c.w. as shown.



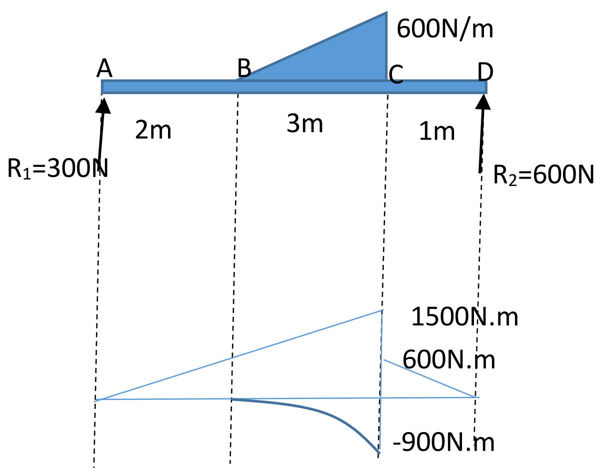


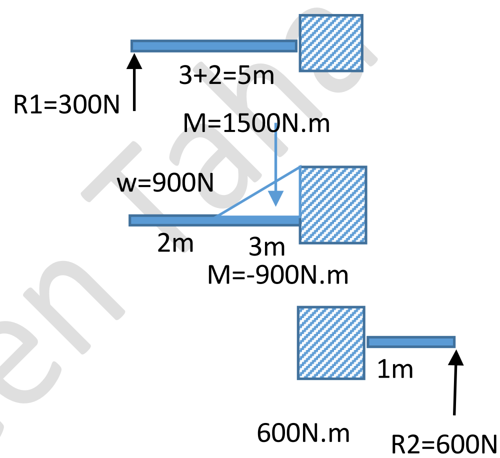
Notes: the measuring of angle is from left tangent line.

Ex1(622): for the beam shown in fig below , compute the moment of area of the moment diagram about the left end.

Sol:

1. R_1=300 N, R_2=600N.
2. The moment diagram by parts:





M=$(\sum M)_R$ will generally indicate the simplest manner of drawing the moment diagram by parts.

The moment at C in terms of the forces to the left of C is 1500 - 900 = 600N.m. the moment at C expressed in terms of the forces acting to the right of C.

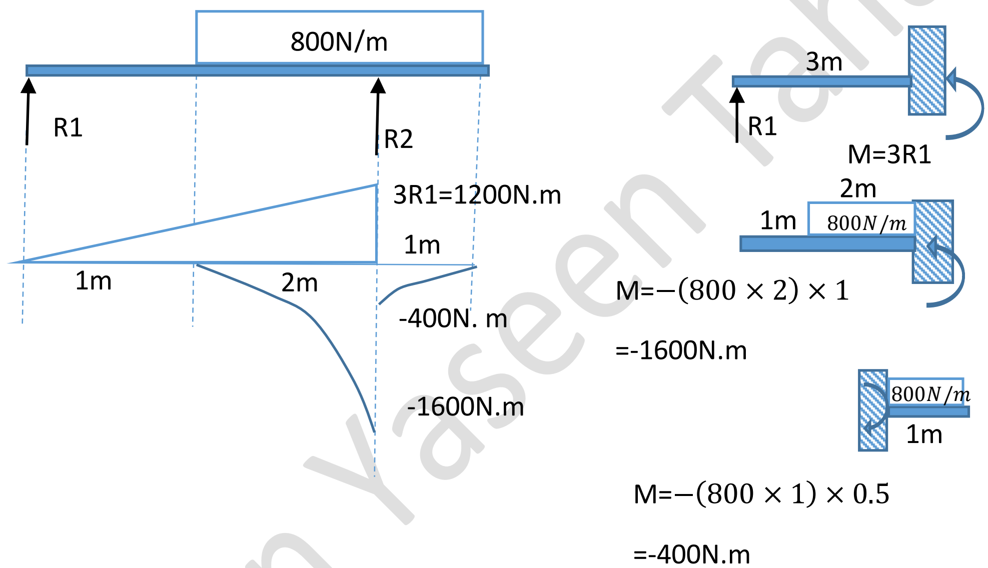
The moment of the area of M diagram about the left end A is now computed as equal to the sum of the moments of area of its parts

$$[(area)_{AD}.\bar{x}A] = \left(\frac{1500\times 5}{2}\right)\left(\frac{2}{3}\times 5\right) + \left(\frac{600\times 1}{2}\right)\left(5+\frac{1}{3}\times 1\right) - \left(\frac{900\times 3}{4}\right)\left(2+\frac{4}{5}\times 3\right)$$
$$= 11.13kN.m^3$$

Ex2(623) for the overhanging beam in fig. , compute the moment of area about C of the moment diagram included between the supports at A and C

Sol:

At any section between A and C , the conventional bending moment is computed more easily by applying M=$(\sum M)_L$ wheres between C and D it is simpler to apply M=$(\sum M)_R$,

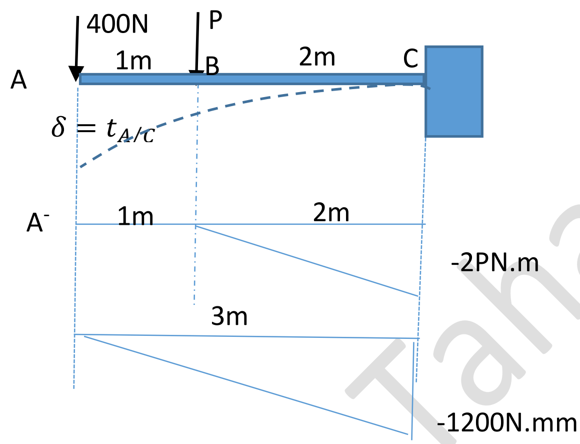


The value of the reactions need not be computed, the moment at C of all forces to the right of C, which is in accord with the fundamental definition M=$(\sum M)_L =$ M=$(\sum M)_R$, 3R1-1600=-400, 3R1=1200N.m

We obtain the moment of area of the M diagram between A and C about C by applying

$$\left[(area)_{Ac}.\bar{x}c = \sum ax\right](area)_{AC}.\bar{x}c = \left(\frac{3 \times 1200}{2}\right)\left(\frac{3}{3}\right) - \left(\frac{2 \times 1600}{3}\right)\left(\frac{2}{4}\right) = 1270N.m^3$$

Ex3(633) for the cantilever beam in fig , it assumed that E=12GN/m^2 I=10×10^6 mm^4. What value of P will cause a 20mm deflection at the free end?



$$t_{A/C} = \frac{1}{EI}(area)_{AC}.\bar{x}_A$$

$$-\delta = \frac{1}{EI}\left[-\left(\frac{2\times 2P}{2}\right)\left(1+\frac{2}{3}\times 2\right)-\left(\frac{3\times 1200}{2}\right)\left(\frac{2}{3}\times 3\right)\right]$$

$$EI\delta = (4.667P + 3600)N.m^3$$

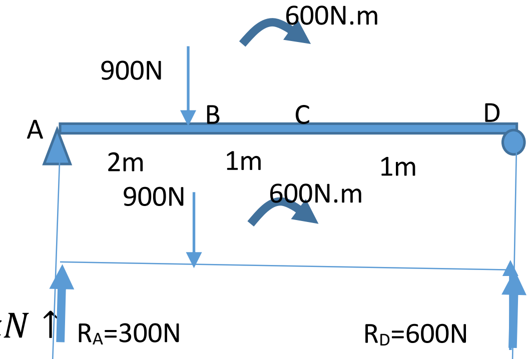
$$(12\times 10^9)(10\times 10^{-6})(20\times 10^{-3}) = 4.667P + 3600,$$

P=-257N

The minus sign of P indicates that the direction of P must be opposite to that originally assumed that is P must act upward.

Ex 4for the beam shown in figure, find:

1- Deflection valuesat B and C
2- Rotation values at support.



Sol:

$\overset{+}{\curvearrowleft} M_A = 0$

$900 \times 2 + 600 - R_D \times 4 = 0 \Longrightarrow R_D = 600kN \uparrow$

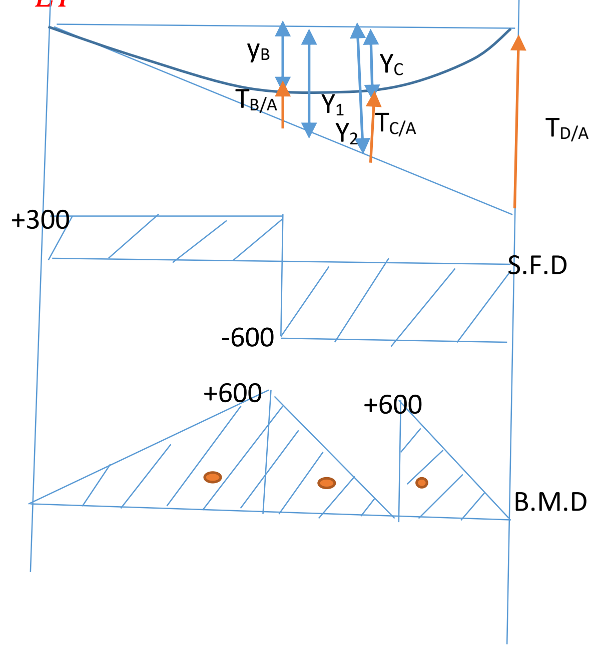
$\overset{+}{\uparrow} F_y = 0 \Longrightarrow R_A + 600 - 900 = 0$

$R_A = 300kN \uparrow$

$$t_{D/A} = \frac{1}{EI} \times \left[\frac{1}{2} \times 600 \times 1 \times \left(\frac{2}{3} \times 1\right) + \frac{1}{2} \times 600 \times 1 \times \left(1 + \frac{2}{3} \times 1\right) + \frac{1}{2} 600 \times 2 \times \left(2 + \frac{2}{3}\right)\right] = \frac{1}{EI} \times [200 + 300 + 1600] \Longrightarrow t_{D/A} = \frac{2300}{EI}$$

$$t_{C/A} = \frac{1}{EI} \times = \frac{1}{EI} \times [200 + 1000] \Longrightarrow t_{C/A} = \frac{1200}{EI}$$

$$t_{B/A} = \frac{1}{EI} \times \left[\frac{1}{2} \times 600 \times 1 \times \left(\frac{2}{3}\right)\right] \Longrightarrow t_{B/A} = \frac{400}{EI}$$



$$\frac{y_2}{3} = \frac{t_{D/A}}{4} \Longrightarrow \frac{y_2}{3} = \frac{2300/EI}{4}$$

$$\therefore y_2 = \frac{1725}{EI}$$

$$\frac{y_1}{2} = \frac{t_{D/A}}{4} \Longrightarrow \frac{y_1}{2} = \frac{2300/EI}{4}$$

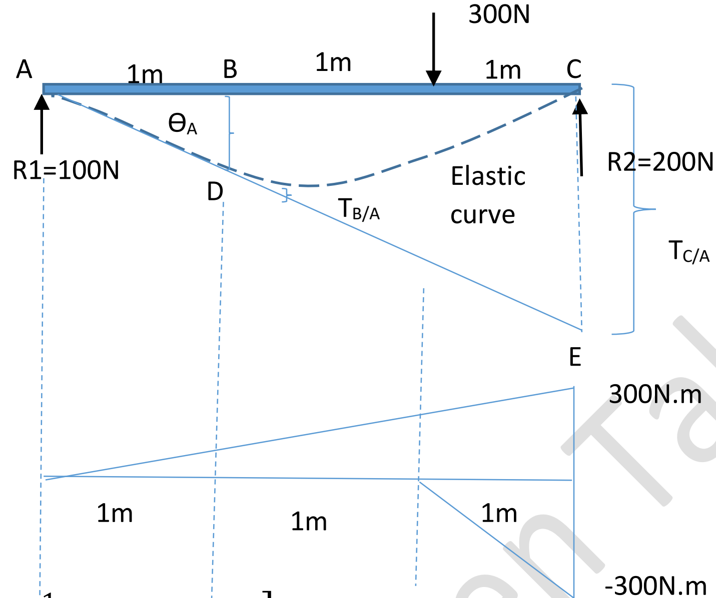
$$\therefore y_1 = \frac{1150}{EI}$$

$$y_B = y_1 - t_{B/A} = \frac{1150}{EI} - \frac{400}{EI} = \frac{750}{EI} \downarrow$$

$$y_C = y_2 - t_{C/A} = \frac{1725}{EI} - \frac{1200}{EI} = \frac{525}{EI} \downarrow$$

$$\theta_A = \frac{t_{D/A}}{4} = \frac{2300/EI}{4} = \frac{575}{EI} \curvearrowright c.w$$

Ex 5(649) the simple beam in fig supports a concentrated load of 300 N at 2m from the left support. Compute the value of EIS at B which is 1m from the left support.



$$\left[t_{c/A} = \frac{1}{EI}(area)_{cA}.\bar{x}_c\right]$$

$$t_{c/A} = \frac{1}{EI}\left[\left(\frac{2\times300}{2}\right)\left(\frac{1}{3}\times 3\right) - \left(\frac{1\times300}{2}\right)\left(\frac{1}{3}\times 1\right)\right] = \frac{400}{EI}$$

Triangle ABD is similar to triangle ACE

$$BD = \frac{1}{3}\times t_{c/A} = \frac{400}{3EI}$$

The deviation $t_{B/A}$ s next obtained from

$$\left[t_{B/A} = \frac{1}{EI}(area)_{BA}.\bar{x}_B\right]$$

$$t_{B/A} = \frac{1}{EI}\left[\left(\frac{1\times 100}{2}\right)\left(\frac{1}{3}\times 1\right)\right] = \frac{100}{6EI}$$

$$[\delta = BD = t_{B/A}],\ \delta = \frac{1}{EI}\left[\frac{400}{3} - \frac{100}{6}\right] \rightarrow EI\delta = 116.7N.m^3$$

The slope of elastic curve $\rightarrow \theta_A \approx tan\theta_A = \frac{CE}{AC} = \frac{t_{\frac{c}{A}}}{AC},\quad t_{\frac{c}{A}} = \frac{400}{EI}$

$$\theta_A = \frac{400/EI}{3} = \frac{400}{3EI}$$