7. Cross Product (Vector product)

If $\bar{A} = a_1 i + a_2 j + a_3 k$

$\bar{B} = b_1 i + b_2 j + b_3 k$

$$\bar{A} \times \bar{B} = |A|\,|B| \sin\theta\, \vec{n} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$\bar{A} \times \bar{B} \neq \bar{B} \times \bar{A}$

$\bar{A} \times \bar{B} = -(\bar{B} \times \bar{A})$

$i \times j = |i|\,|j| \sin 90\, \vec{n} = k$

$i \times j = k$

$j \times k = i$

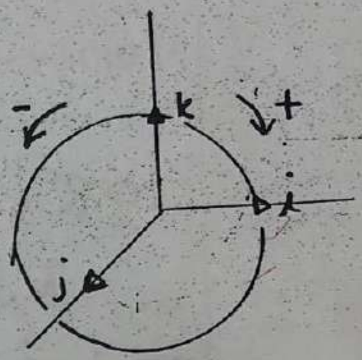
$k \times i = j$

$j \times i = -k$

$i \times k = -j$

$k \times j = -i$

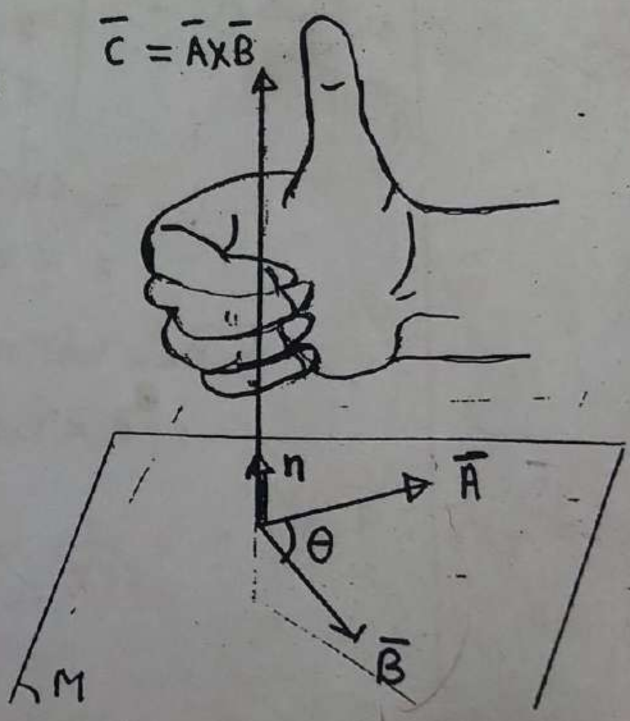
$i \times i = j \times j = k \times k = 0$



The cross product resuts a vector that is perpendicula to the vectors.

$\bar{A} \times \bar{B} = \vec{C}$, $\bar{C} \perp \bar{A}$ and $\bar{C} \perp \bar{B}$

$(\bar{A} \times \bar{B}) \perp$ plane contains $\bar{A}$ & $\bar{B}$

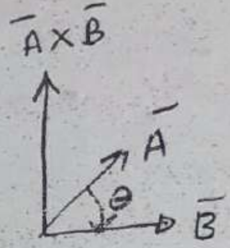


Example:

Given $\bar{A} = i + 2j - 2k$

$\bar{B} = 3i + 0j + k$

Find $\bar{A} \times \bar{B}$ and $\bar{B} \times \bar{A}$



Solution:

$$\bar{A} \times \bar{B} = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ 1 & 2 & -2 & 1 & 2 \\ 3 & 0 & 1 & 3 & 0 \end{array}\right.$$

$\ominus$ $\oplus$

$= 0k - 6j + 2i - j + 0i - 6k$

$= 2i - 7j - 6k$ (Vector)

OR:

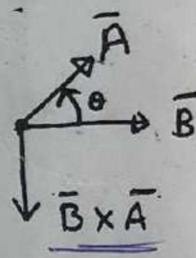
$$\bar{A} \times \bar{B} = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 3 & 0 & 1 \\ i & j & k \\ 1 & 2 & -2 \end{vmatrix} =$$

$\ominus$ $\oplus$

$= 2i + 0k - 6j - j + 0i - 6k$

$= 2i - 7j - 6k$

$$\bar{B} \times \bar{A} = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ 3 & 0 & 1 & 3 & 0 \\ 1 & 2 & -2 & 1 & 2 \end{array}\right.$$

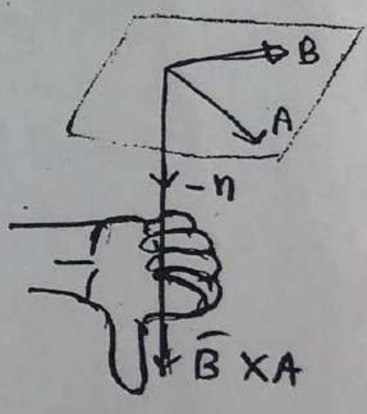


$= 6k + j - 0i + 6j - 2i - 0k$

$= -2i + 7j + 6k$ (Vector)

It can be noticed that

$\bar{A} \times \bar{B} = -(\bar{B} \times \bar{A})$



Example:

If $\bar{u} = 2i + j + k$, and $\bar{v} = -4i + 3j + k$, find $\bar{u} \times \bar{v}$.

Solution:

$$\bar{u} \times \bar{v} = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ 2 & 1 & 1 & 2 & 1 \\ -4 & 3 & 1 & -4 & 3 \end{array}\right|$$

$$= 6k - 4j + i - 2j - 3i + 4k$$

$$= -2i - 6j + 10k$$

$\bar{u} \times \bar{v}$ can be found by using determinants.

$$\bar{u} \times v = i \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + k \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}$$

$$= i(1-3) - j(2+4) + k(6+4)$$

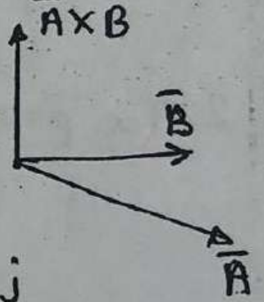
$$= -2i - 6j + 10k$$

Example:

Find a unit vector perpendicular to both $\bar{A} = 2i + j - k$, and $\bar{B} = i - j + 2k$.

Solution:

$(\bar{A} \times \bar{B})$ is a vector perpendicular to both $\bar{A}$ and $\bar{B}$, but it is not a unit vector.



$$\bar{A} \times \bar{B} = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ 2 & 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 1 & -1 \end{array}\right|$$

$$= -2k - j + 2i - 4j - i - k$$

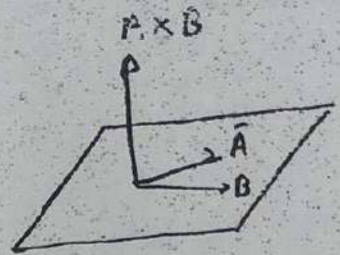
$$= i - 5j - 3k$$

$$u = \frac{i - 5j - 3k}{\sqrt{1 + 25 + 9}} = \frac{1}{\sqrt{35}} i - \frac{5}{\sqrt{35}} j - \frac{3}{\sqrt{35}} k$$

Cross Product Applications

1. To find a vector orthogonal to two vectors.

$\bar{A} \times \bar{B}$ produce a vector that is perpendicular to both $\bar{A}$ & $\bar{B}$, and normal to the plane contai[ning] them.

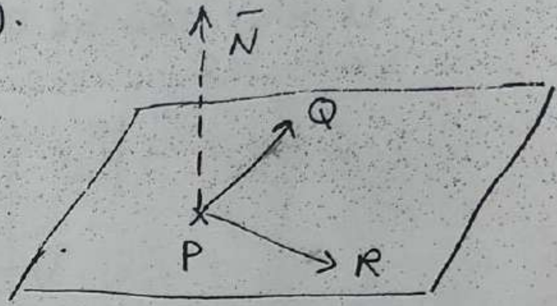


Example

Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Solution:

From P, write two vectors $\overline{PQ}$ and $\overline{PR}$.



$$\overline{PQ} = i + 2j - k$$

$$\overline{PR} = -2i + 2j + 2k$$

$$\bar{N} = \overline{PQ} \times \overline{PR}$$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = 6i + 6k$$

2. To prove that two vectors are parallel

if $\bar{A} \parallel \bar{B} \Rightarrow \bar{A} \times \bar{B} = 0$

$$\bar{A} \times \bar{B} = |\bar{A}| \cdot |\bar{B}| \sin 0 \, \vec{n} = 0$$

3- To find the (sine) of the angle between two vectors.

$$\bar{A} \times \bar{B} = |A|\,|B| \sin\theta \, \bar{n}$$

$$|\bar{A} \times \bar{B}| = |A|\,|B| \sin\theta$$

$$\therefore \sin\theta = \frac{|\bar{A} \times B|}{|\bar{A}|\,|\bar{B}|}$$

Example:

Show if $\bar{u} = 5i + k - j$, and $\bar{w} = -15i + 3j - 3k$ are parallel or not.

Solution:

if $\bar{u} \parallel \bar{w}$, then $\bar{u} \times \bar{w} = 0$

$$\bar{u} \times \bar{w} = \begin{vmatrix} i & j & k \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix} \begin{matrix} i & j \\ 5 & -1 \\ -15 & 3 \end{matrix}$$

$$= 15k - 15j + 3i + 15j - 3i - 15k$$

$$= 0$$

$\therefore \bar{u} \parallel \bar{w}$.

Example:

Use cross product to find the sine of the angle between the vectors $\bar{A} = 2i + 3j - 6k$, and $\bar{B} = 2i + 3j + 6k$.

Solution:

$$\sin\theta = \frac{|\bar{A} \times \bar{B}|}{|A|\,|B|} \quad \text{as} \quad \bar{A} \times \bar{B} = |A|\,|B| \sin\theta \, \vec{n}$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} i & j & k \\ 2 & 3 & -6 \\ 2 & 3 & +6 \end{vmatrix} \begin{matrix} i & j \\ 2 & 3 \\ 2 & 3 \end{matrix} = \cancel{6k} - 12j + 18i - 12j + 18i - \cancel{6k}$$

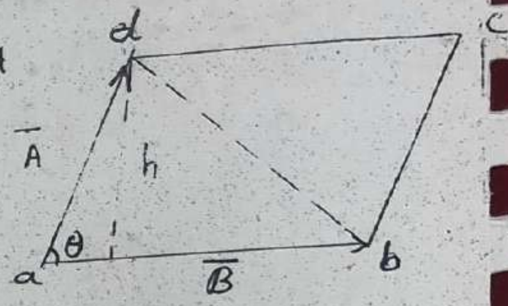
$$= 36i - 24j$$

$$|A| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

$$|B| = 7$$

$$\sin\theta = \frac{|\bar{A} \times \bar{B}|}{|7|\,|7|} = \frac{\sqrt{(36)^2 + (24)^2}}{49} \approx 0.883$$

4. To find the area of a parallelogram determined by two vectors.



Parallelogram Area = Base X height

$\sin\theta = \frac{h}{|A|}$

$\therefore h = |\bar{A}| \sin\theta = \text{height}$

$|\bar{B}| = \text{Base}$

$\therefore$ Area = Base * height

$|\bar{B}||h|$

$= |B|\,|A| \sin\theta$

$= |A|\,|B| \sin\theta$

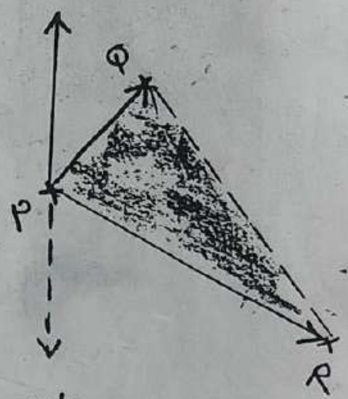
$= |\bar{A} \times \bar{B}|$

Triangle area $= \frac{1}{2}(|\bar{A} \times \bar{B}|)$

Example:

- Find the area of the triangle determined by the point $P(1,1,1)$, $Q(2,1,3)$ and $R(3,-1,1)$.
- Find a unit vector perpendicular to the plane PQR.

Solution:



ΔPQR Area $= \frac{1}{2}(|\bar{PQ} \times \bar{PR}|)$

$\bar{PQ} = i + 0j + 2k$

$\bar{PR} = 2i - 2j + 0k$

$$\bar{PQ} \times \bar{PR} = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ 1 & 0 & 2 & 1 & 0 \\ 2 & -2 & 0 & 2 & -2 \end{array}\right.$$

$= -2k + 4j + 0i - 0j + 4i - 0k$

$= 4i + 4j - 2k$

$|\bar{PQ} \times \bar{PR}| = \sqrt{16+16+4} = \sqrt{36} = 6$

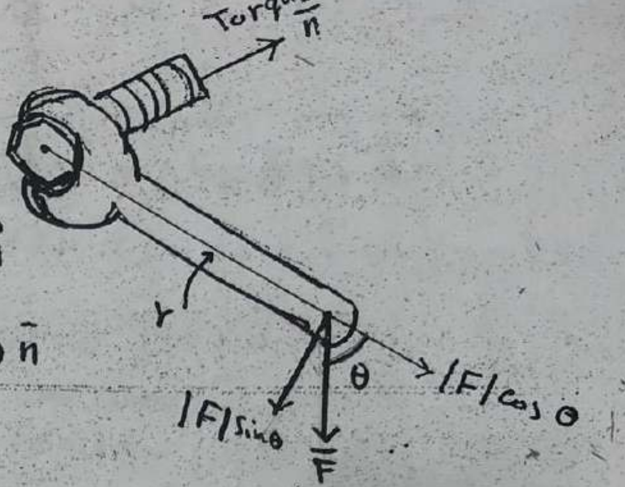
Area $= \frac{1}{2}|\bar{PQ} \times \bar{PR}| = \frac{1}{2} \times 6 = 3$ sq. units

$\bar{U}_N = \frac{\bar{PQ} \times \bar{PR}}{|\bar{PQ} \times \bar{PR}|} = \frac{4i+4j-2k}{6} = \frac{2}{3}i + \frac{2}{3}j - \frac{2}{3}k$

or $-\frac{2}{3}i - \frac{2}{3}j + \frac{2}{3}k$

5. To Calculate the torque vector and magnitude.

The torque magnitude is the product of the length of the lever arm (r) and the component of the force (F) perpendicular to (r).



Torque magnitude = $|T|$

$|T| = |\bar{F}| \sin\theta |\bar{r}|$

Torque vector $= |F| \sin\theta |\bar{r}| \vec{n}$

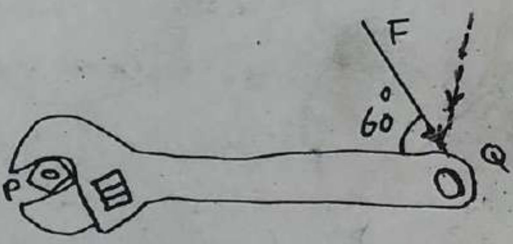
$= |\bar{F}||\bar{r}| \sin\theta \bar{n}$

$= \bar{F} \times \bar{r}$

Example:

Find the magnitude of the torque exerted by the force F on the bolt at P if $|PQ| = 8''$, and $|\bar{F}| = 30$ lb. (See the Fig).

Solution:



$|T| = \text{torque} = |F| \sin\theta |\bar{r}|$

$|F| = 30$ lb , $|r| = 8'' = \frac{8}{12}$ ft

$|T| = |F| \sin\theta |r|$

$= 30 * \sin 60 * \frac{8}{12}$

$= 10\sqrt{3}$ lb.ft

8. Triple Scalar product

$\bar{A} \cdot \bar{B} \cdot \bar{C}$ (undefined)

$\bar{A} \cdot (\bar{B} + \bar{C}) = \bar{A} \cdot \bar{B} + \bar{A} \cdot \bar{C}$ (Distribution Law)

$(\bar{A} + \bar{B}) \cdot (\bar{C} + \bar{D}) = \bar{A} \cdot \bar{C} + \bar{A} \cdot \bar{D} + \bar{B} \cdot \bar{C} + \bar{B} \cdot \bar{D}$

$\bar{A} \cdot (\bar{B} \times \bar{C})$ = Triple scalar product (scalar result)

or Box product

$\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{B} \times \bar{C}) \cdot \bar{A}$

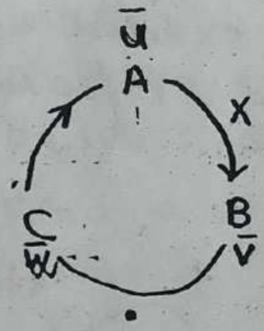
$= \underbrace{|(\bar{B} \times \bar{C})|}_{\text{مساحة القاعدة}} \underbrace{|A| \, |\cos\theta|}_{\text{الارتفاع}}$

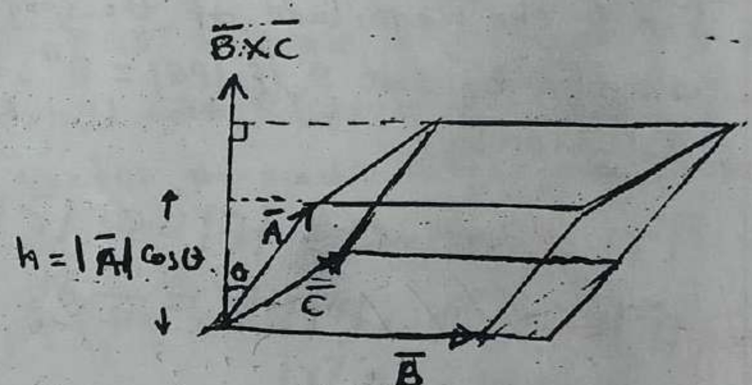
هنا عملنا Dot product بين A و (B×C)

$|\bar{A} \cdot (\bar{B} \times \bar{C})|$ = The volume of parallelpiped.

$\bar{B}$ and $\bar{C}$ are the base and $|(\bar{B} \times \bar{C})|$ is the base are

$|A| \cos\theta$ = height





$|\bar{B} \times \bar{C}|$ = Base area

$|\bar{A}| \cos\theta$ = height = $\text{Proj}_{\bar{A}} (\bar{B} \times \bar{C})$

V = Base area * height

Example:

Given $\bar{u} = 2i + j$, $\bar{v} = 2i - j + k$, and $\bar{w} = i + 2k$

- Verify that $(\bar{u} \times \bar{v}) \cdot \bar{w} = (\bar{v} \times \bar{w}) \cdot \bar{u} = (\bar{w} \times \bar{u}) \cdot v$
- Find the volume of the box determined by $\bar{u}, \bar{v}$ and $\bar{w}$

solution:

$$\bar{u}\times\bar{v} = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ 2 & 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 2 & -1 \end{array}\right| = -2k + i - 2j - 2k$$

$$= i - 2j - 4k$$

$$(\bar{u}\times\bar{v})\cdot\bar{w} = (i - 2j - 4k)\cdot(i + 2k) = 1 - 8 = -7$$

$$\bar{v}\times\bar{w} = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ 2 & -1 & 1 & 2 & -1 \\ 1 & 0 & 2 & 1 & 0 \end{array}\right| = j - 2i - 4j + k$$

$$= -2i - 3j + k$$

$$(\bar{v}\times\bar{w})\cdot\bar{u} = (-2i - 3j + k)\cdot(2i + j) = -4 - 3 = -7$$

parallelepiped volume $= |(\bar{u}\times\bar{v})\cdot W|$

$= 7.0$ units cubed

Triple Vector product

$$\bar{A}\times\bar{B}\times\bar{C} = (\bar{A}\times\bar{B})\times\bar{C}$$

$$(\bar{A}\times\bar{B})\times\bar{C} = (\bar{A}\cdot\bar{C})\,\bar{B} - (\bar{B}\cdot\bar{C})\,\bar{A}$$

$$(\bar{B}\times\bar{C})\times\bar{A} = (\bar{B}\cdot\bar{A})\,\bar{C} - (\bar{C}\cdot\bar{A})\,\bar{B}$$

$$A\times(B\times C) = (\bar{A}\cdot\bar{C})\,\bar{B} - (\bar{A}\cdot\bar{B})\,\bar{C}$$

Example:

Use two methods to find $(\bar{A}\times\bar{B})\times\bar{C}$, if

$\bar{A} = i - j + 2k$, $\bar{B} = 2i + j + k$ and $\bar{C} = i + 2j - k$.

Solution:

$$\bar{A}\times\bar{B} = -3i + 3j + 3k$$

$$(\bar{A}\times\bar{B})\times\bar{C} = (-3i + 3j + 3k)\times(i + 2j - k)$$

$$= -9i - 9k$$

Alternative Method:

$$\overbrace{(A\times B)\times C}^{①\,\cdot} = (A\cdot C)\,B - (B\cdot C)\,A \quad \text{②}\,\cdot$$

$\bar{A} \cdot \bar{C} = 1 - 2 - 2 = -3$

$\bar{B} \cdot \bar{C} = 2 + 2 - 1 = 3$

$$
\begin{aligned}
(A \times B) \times C &= (A \cdot C)B - (B \cdot C)A \\
&= (-3)(2i + j + k) - 3(i - j + 2k) \\
&= -6i - 3j - 3k - 3i + 3j - 6k \\
&= -9i - 9k
\end{aligned}
$$