6. Dot Product (Scalar Product)

If $\bar{A} = a_1 i + a_2 j + a_3 k$

$\bar{B} = b_1 i + b_2 j + b_3 k$, then

$$\bar{A} \cdot \bar{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\bar{A}|\,|\bar{B}| \cos\theta$$

$$\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A} \quad \text{scalar}$$

Examples:

1. Consider the vectors $\bar{u} = 2i - j + k$, and $\bar{v} = i + j + 2k$

Find: (i) $\bar{u} \cdot \bar{v}$

(ii) the angle between $\bar{A}$ and $\bar{B}$.

solution:

$\bar{u} = 2i - j + k$

$\bar{v} = i + j + 2k$

$$\bar{u} \cdot \bar{v} = (2)(1) + (-1)(1) + (1)(2)$$
$$= 2 - 1 + 2$$
$$= 3$$

$$\bar{u} \cdot \bar{v} = |\bar{u}|\,|\bar{v}| \cos\theta$$

$$|\bar{u}| = \sqrt{4+1+1} = \sqrt{6}$$

$$|\bar{v}| = \sqrt{1+1+4} = \sqrt{6}$$

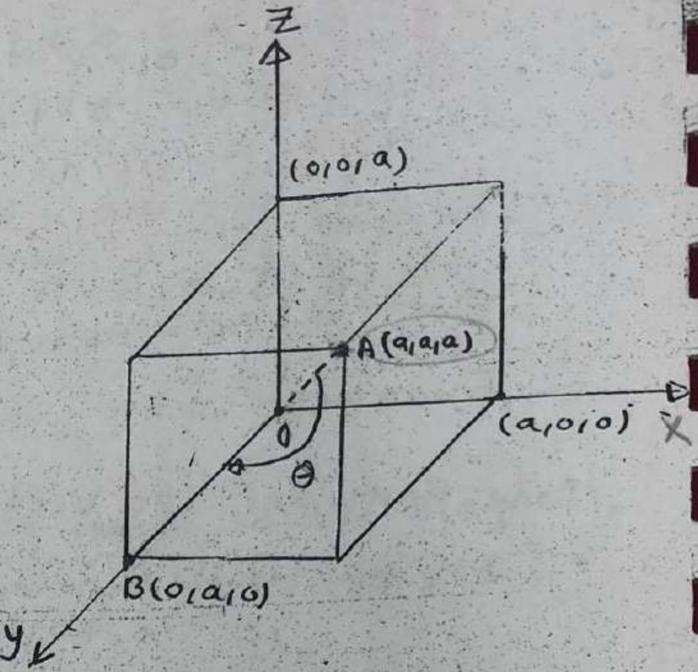
$$\bar{u} \cdot \bar{v} = \sqrt{6} \cdot \sqrt{6} \cos\theta$$

$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{\sqrt{6} \cdot \sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

$$\theta = \cos^{-1}(1/2)$$
$$= 60^\circ.$$

2. Find the angle between a diagonal of a cube and one of its sides.

Solution:



A (a, a, a)

O $(0, 0, 0)$

$\overline{OA} = ai + aj + ak$

B $(0, a, 0)$

$\overline{OB} = 0i + aj + 0k$

$$\cos\theta = \frac{\overline{OA} \cdot \overline{OB}}{|\overline{OA}|\, |\overline{OB}|}$$

$|\overline{OA}| = \sqrt{a^2 + a^2 + a^2}$

$= \sqrt{3a^2}$

$\overline{OB} = \sqrt{a^2}$

$$\cos\theta = \frac{a^2}{\sqrt{3a^2}\sqrt{a^2}} = \frac{a^2}{\sqrt{3a^4}} = \frac{a^2}{\sqrt{3}\,a^2} = \frac{1}{\sqrt{3}}$$

$\theta = \cos^{-1}(1/\sqrt{3})$

$= 54.7^\circ$

3. if $\bar{u} = i - 2j + 3k$, and $\bar{v} = 3i + 6j + 3k$, show that $\bar{u}$ is perpendicular to $\bar{v}$.

Solution:

$\bar{u} \cdot \bar{v} = |\bar{u}|\, |\bar{v}| \cos\theta$

$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{|\bar{u}|\, |\bar{v}|}$$

$$= \frac{(1)(3) + (-2)(6) + (3)(3)}{\sqrt{14} \cdot \sqrt{54}} = \frac{0}{\sqrt{14 * 54}} = 0$$

$\cos\theta = 0$

$\theta = \cos^{-1}(0) = 90^\circ$

$\therefore \bar{u} \perp \bar{v}$.

Notes:

1- θ is acute (حاد) if and only if $u \cdot v > 0$

θ is obtuse if and only if $u \cdot v < 0$

$\theta = \pi/2$ " " " " $u \cdot v = 0$

2- The vector $u = u_1 i + u_2 j + u_3 k$, in space make the angles α, β and γ with i, j and k. These angles are called (direction angle) of the vector.

Thee ($\cos\alpha$, $\cos\beta$ and $\cos\gamma$) are called the (direction cosines) of the vector.

$$\cos\alpha = \frac{u_1}{|u|}, \quad \cos\beta = \frac{u_2}{|\bar{u}|}, \quad \cos\gamma = \frac{u_3}{|\bar{u}|}$$

For example:

$$\bar{u} = -i + 3j - 5k$$

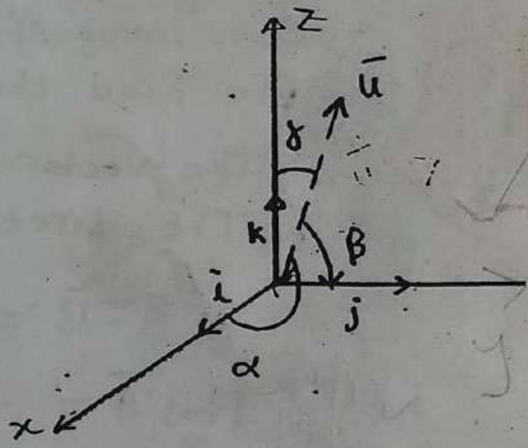
$$\cos\alpha = \frac{-1}{\sqrt{35}}, \quad \cos\beta = \frac{3}{\sqrt{35}}, \quad \cos\gamma = \frac{-5}{\sqrt{35}}$$

OR:

$$\cos\alpha = \frac{\bar{u} \cdot \bar{i}}{|u||i|} = \frac{-1}{\sqrt{35}}$$

$$\cos\beta = \frac{\bar{u} \cdot j}{|u||j|}$$

$$\cos\gamma = \frac{\bar{u} \cdot k}{|u||k|}$$



Example:

Given $\bar{V} = 2i - 4j + 4k$, Find the direction cosines, and the direction angles.

$$\cos\alpha = \frac{\bar{V} \cdot i}{|V||i|} = \frac{2}{\sqrt{36+1}} = \frac{1}{3}, \quad \alpha = \cos^{-1}(1/3) = 71^{\circ}$$

$$\cos\beta = \frac{\bar{V} \cdot j}{|V||j|} = \frac{-4}{6} = \frac{-2}{3}, \quad \beta = \cos^{-1}(-2/3) = 132^{\circ}$$

$$\cos\gamma = \frac{\bar{V} \cdot k}{|V||k|} = \frac{4}{6} = \frac{2}{3}, \quad \gamma = \cos^{-1}(2/3) = 48^{\circ}$$

Dot Product Applications:

1. To prove that two vectors are perpendicular to each other.

$\bar{A} \perp \bar{B}$ if $\bar{A} \cdot \bar{B} = 0$

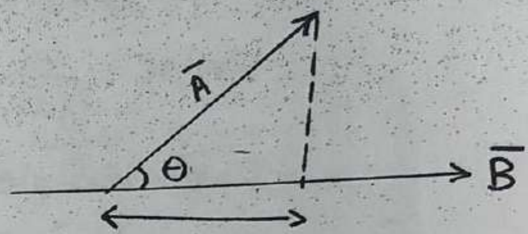
$\bar{A} \cdot \bar{B} = |\bar{A}|\,|\bar{B}| \cos\theta$

if $A \perp B$, $\theta = 90$, then $\bar{A} \cdot \bar{B} = 0$

2. To find the angles between any two vectors.

$$\theta = \cos^{-1}\left(\frac{\bar{A} \cdot \bar{B}}{|A|\,|B|}\right)$$

3. To find the scalar projection (projection length) of $\bar{A}$ on $\bar{B}$.



$$\bar{A} \cdot \bar{B} = |A|\,|B| \cos\theta$$

$$|A| \cos\theta = \mathrm{Proj}_B A = \frac{\bar{A} \cdot \bar{B}}{|\bar{B}|}$$

$$\bar{A} \cdot \bar{B} = \underbrace{|A| \cos\theta}\,|B|$$

$$\therefore \mathrm{Proj}_{\bar{B}} \bar{A} = |\bar{A}| \cos\theta = \frac{\bar{A} \cdot \bar{B}}{|\bar{B}|}$$

تركيه اخصه

4. To find the $\boxed{\text{vector projection}}$ of $\bar{A}$ on $\bar{B}$.

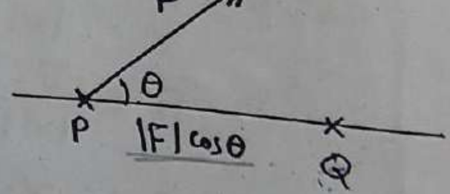
The vector projection is a magnitude and direction.
The direction is same as $U_{\bar{B}}$ which is $\dfrac{\bar{B}}{|\bar{B}|}$

$$\therefore \bar{A} \cdot \bar{B} = |A| \cos\theta\, |B|$$

$$\mathrm{Proj}_{\bar{B}} \bar{A} = \frac{\bar{A} \cdot \bar{B}}{|B|} \cdot \frac{\bar{B}}{|B|} = \frac{(\bar{A} \cdot \bar{B})}{|B|^2}\,\bar{B}$$

5. To find the work (W) done by a constant force $(\bar{F})$.

The quantity $(|F| \cos\theta)$ is the component of the force $\bar{F}$ in the direction of the motion, and $|\bar{PQ}|$ is the distance travelled by the object.

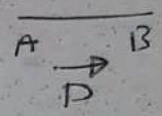


$|F|\,|F| \cos\theta$

$$\text{work} = \underline{W} = \bar{F} \cdot \overline{PQ} = |\bar{F}|\,|\overline{PQ}| \cos\theta$$

قوة إزاحة

$$= (|\bar{F}| \cos\theta)\,|PQ|$$



Examples:

$\vec{F} \cdot \vec{D}$

4. Find the angle between $\bar{A} = i - 2j - 2k$, and $\bar{B} = 6i + 3j + 2k$, and find the component of $\bar{B}$ in the direction of $\bar{A}$.

Solution:

$$\bar{A} \cdot \bar{B} = |A|\,|B| \cos\theta$$

$$\cos\theta = \frac{\bar{A} \cdot \bar{B}}{|\bar{A}|\,|\bar{B}|} = \frac{(6) + (-6) + (-4)}{|3|\,|7|} = \frac{-4}{21}$$

$$\theta = \cos^{-1}(-4/21) = 101^\circ.$$

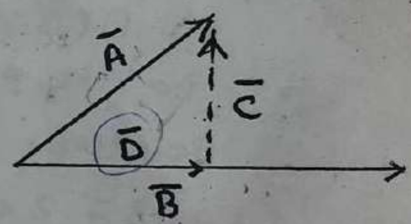
$$\underset{A}{\text{Proj}}\, \bar{B} = \frac{\bar{A} \cdot \bar{B}}{|A|} = \frac{-4}{3}$$

5. Let $\bar{A} = 2i - j + 3k$, and $\bar{B} = 4i - j + 2k$, Find

a) The Vector component of $\bar{A}$ along $\bar{B}$.

b) The vector component of $\bar{A}$ orthogonal to $\bar{B}$.

Solution:



(a): $\underset{\bar{B}}{\text{Proj}}\, \bar{A} = \frac{\bar{A} \cdot \bar{B}}{|\bar{B}|}$

$$= \frac{15}{\sqrt{21}}$$

Vector Component $= \frac{15}{\sqrt{21}} \frac{\bar{B}}{|\bar{B}|}$

$$= \frac{15}{\sqrt{21}\,\sqrt{21}} \bar{B}$$

$$= \frac{15}{21}(4i - j + 2k)$$

$$= \frac{5}{7}(4i - j + 2k)$$

$$= \frac{20}{7} i - \frac{5}{7} j + \frac{10}{7} k$$

$$= \bar{D}$$

(b)

$$\bar{D} + \bar{C} = \bar{A}$$

$$\bar{C} = \bar{A} - \bar{D} = \text{orthogonal component of } \bar{A}$$

$$= (2i - j + 3k) - \left(\frac{20}{7} i - \frac{5}{7} j + \frac{10}{7} k\right)$$

$$= -\frac{6}{7} i - \frac{2}{7} j + \frac{11}{7} k$$

6. Show that $A(2,-1,1)$, $B(3,2,-1)$ and $C(7,0,-2)$ are vertices of a right triangle.

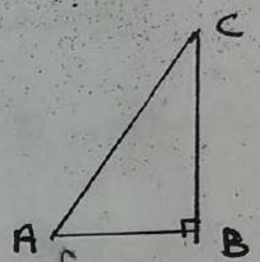
solution:

Three sides enable us to write three vectors:

$\bar{AB} = i + 3j - 2k$, $|\bar{AB}| = \sqrt{14}$

$\bar{AC} = 5i + j - 3k$, $|\bar{AC}| = \sqrt{35}$

$\bar{BC} = 4i - 2j - k$, $|\bar{BC}| = \sqrt{21}$



$\because |\bar{AC}|^2 = |AB|^2 + |BC|^2$

$\therefore$ The right angle is at the vertex B.

7. Find the value of x, so that the vector from $A(1,-1,3)$ to $B(3,0,5)$ is perpendicular to the vector from A to the point $p(x,x,x)$.

Solution:

$\bar{AB} = 2i + j + 2k$

$\bar{AP} = (x-1)i + (x+1)j + (x-3)k$

$\bar{AB} \perp \bar{AP} \Rightarrow \therefore \bar{AB} \cdot \bar{AP} = 0$

$2(x-1) + (x+1) + 2(x-3) = 0$

$2x - \cancel{2} + x + \cancel{1} + 2x - 6 = 0$

$5x = 7$

$x = 7/5$

$\therefore p(7/5, 7/5, 7/5)$.

8. Let $\bar{v} = ai + j$, and $\bar{u} = 4i + 3j$, find a so that

i- $\bar{v}$ and $\bar{u}$ are othogonal. $\vec{v} \perp \vec{u}$ dot product

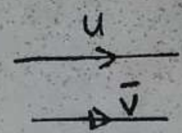
ii - $\bar{v}$ and $\bar{u}$ are parallel.

iii - The angle between $\bar{u}$ and $\bar{v}$ is 30°.

solution:

(i) if $\bar{u} \perp \bar{v}$, $\bar{v} \cdot \bar{u} = 0 \Rightarrow 4a + 3 = 0 \Rightarrow a = -3/4$.

(ii) if $\bar{u} \parallel \bar{v}$, the angle between them will be zero.



$$\bar{u} \cdot \bar{v} = |\bar{u}|\,|v| \cos 0$$

$$4a + 3 = \sqrt{a^2+1} \cdot \sqrt{16+9}$$

$$4a + 3 = \sqrt{25(a^2+1)}$$

$$(4a+3)^2 = 25a^2 + 25$$

$$16a^2 + 24a + 9 = 25a^2 + 25$$

$$9a^2 - 24a + 16 = 0$$

$$(3a - 4)^2 = 0 \Rightarrow a = \frac{4}{3}$$

20
400
20

(iii) $\bar{u} \cdot \bar{v} = \sqrt{a^2+1} * 5 * \cos 30$

$$(ai + j) \cdot (4i + 3j) = \frac{5\sqrt{3(a^2+1)}}{2}$$

$$(4a + 3)^2 = \frac{25[3(a^2+1)]}{4}$$

$$16a^2 + 24a + 9 = \frac{75a^2 + 25}{4}$$

$$64a^2 + 96a + 36 = 75a^2 + 25$$

$$11a^2 - 96a + 11 = 0$$

$$a = \frac{96 \mp \sqrt{(96)^2 - 4(11)(11)}}{22}$$

9. Find the work done by a force $\overline{F} = 3\dot{j}$ (Newton) applied to a point that moves on a line from (1,3) to (4,7). Assume distance is measured in meters.

1,3 4,7

solution:

$W = \overline{F} \cdot \overline{d}$

$\overline{F} = 3j$

$\overline{d} = (4-1)i + (7-3)j = 3i + 4j$

$W = (3j) \cdot (3i + 4j) = (0)(3) + (3)(4)$

$= 12$ Joule.

---

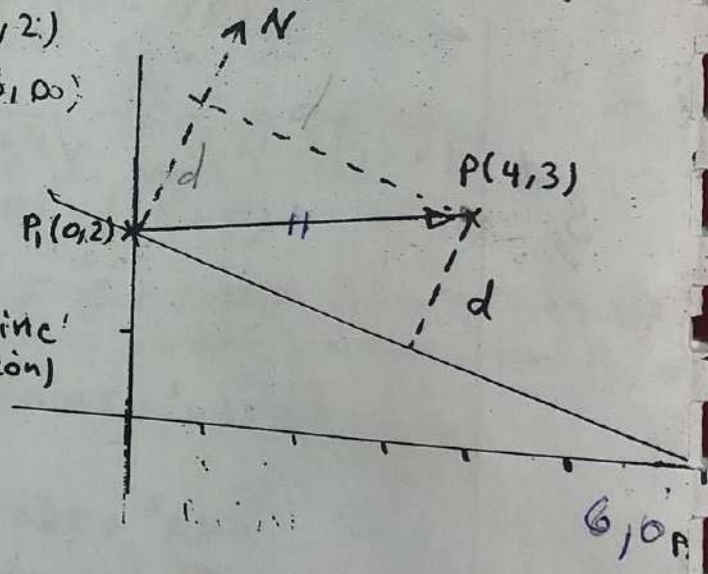
10. Calculate the distance between the point (4,3) and the line $x + 3y = 6$.

Solution:

The Line equation $X + 3y - 6 = 0$ can be plotted by

$x = 0, \; y = 2 \Rightarrow P_1 \; (0, 2)$

$y = 0, \; x = 6 \Rightarrow P_2 \; (6, 0)$



From P_1 draw the normal vector $\overline{N}$.

$N = i + 3j$ (From Line equation)

Find the vector $\overline{P_1P}$

$\overline{P_1P} = (4-0)i + (3-2)j$

$= 4i + j$

$$d = \underset{\overline{N}}{\text{Proj}} \; \overline{P_1P} = \frac{\overline{P_1P} \cdot \overline{N}}{|\overline{N}|}$$

$$= \frac{(4)(1) + (1)(3)}{\sqrt{1+9}} = \frac{7}{\sqrt{10}}$$