Vectors

1- Definition

Some physical quantities, like length and mass, are
determined when their magnitudes are given in terms of
specific units. Such quantities are called (scalars).
Other quantities, like forces and velocities, in which
the direction as well as the magnitude is important,
are called (Vectors).

2- Vector Representation

i- From two points (Initial and Terminal)

$$\overline{AB} = (x_2 - x_1)\,i + (y_2 - y_1)\,j + (z_2 - z_1)\,k$$

where

$A(x_1, y_1, z_1) \equiv$ initial point

$B(x_2, y_2, z_2) \equiv$ terminal point

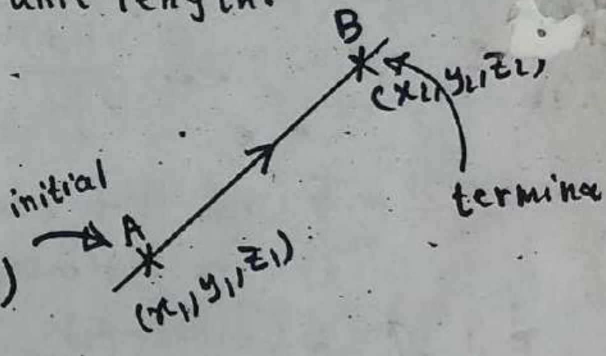
i is a vector along x-axis from (0,0,0) to (1,0,0)

j is a vector along y-axis from (0,0,0) to (0,1,0)

k is a vector along z-axis from (0,0,0) to (0,0,1)

i, j and k are vectors of unit length.

ii- Vector in plane can be represented as



$\bar{V}$ = Vector length $(\cos\theta\, i + \sin\theta\, j)$
where θ is the inclination angle

OR : From Line equation

$$ax + by + c = 0 \qquad \text{(Line Equation)}$$

$$\bar{V} = bi - aj \quad \text{or} \quad \bar{V} = -bi + aj$$

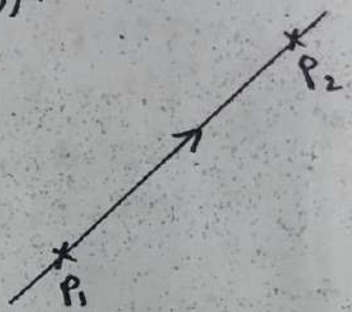
3- Vector Length (Magnitude)

$$|\overline{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Examples:

1- Find the vector $\overline{P_1P_2}$ if $P_1(2,4,-7)$, and $P_2(0,5,2)$

$$\overline{P_1P_2} = (0-2)i + (5-4)j + (2-(-7))k$$
$$= -2i + j + 9k$$
$$\overrightarrow{P_1P_2} = \langle -2, 1, 9 \rangle$$
$$\overline{P_2P_1} = (2-0)i + (4-5)j + (-7-2)k$$
$$= 2i - j - 9k$$



Note that $\overline{P_1P_2} = -2i + j + 9k$, and

$P_2P_1 = 2i - j - 9k$

$\therefore \overline{P_1P_2} = -(\overline{P_2P_1})$

$\therefore$ Multiplying by (-1) will reverse the direction.

$$|\overline{P_1P_2}| = \text{Vector length}$$
$$= \sqrt{(-2)^2 + (1)^2 + (9)^2}$$
$$= \sqrt{4+1+81}$$
$$= \sqrt{86} \text{ unit length}$$

2 - Find the vector $\overline{OP}$, if O is the origin, and P is the midpoint of the vector $\overline{AB}$ joining $A(2,1,-2)$ and $B(6,0,-4)$.

$$O(0,0,0), \quad P\left(\frac{2+6}{2}, \frac{1+0}{2}, \frac{-2+(-4)}{2}\right) = \left(4, \frac{1}{2}, -3\right)$$

$$\overline{OP} = (4-0)i + \left(\tfrac{1}{2}-0\right)j + (-3-0)k$$
$$= 4i + \tfrac{1}{2}j - 3k$$
$$= \left\langle 4, \tfrac{1}{2}, -3 \right\rangle$$

$$|\overline{OP}| = \sqrt{16 + \tfrac{1}{4} + 9} = \sqrt{\frac{64+1+36}{4}}$$
$$= \frac{\sqrt{101}}{2} \text{ units}$$

3- Find a vector of length (two) and making an angle of 30° with the positive x-axis.

Vector = Vector length $(\cos\theta\, i + \sin\theta\, j)$

$$\bar{V} = 2(\cos 30\, i + \sin 30\, j)$$

$$= 2\left(\frac{\sqrt{3}}{2} i + \frac{1}{2} j\right)$$

$$= \sqrt{3}\, i + j$$

4. Find the vector tangent to the curve $y = \frac{x^3}{2} + \frac{1}{2}$ at the point $(1,1)$.

We have to find the tangent equation.

$$m = \frac{dy}{dx} = \frac{1}{2}\cdot 3x^2 + 0 = \frac{3}{2}x^2$$

$$m\Big|_{(1,1)} = \frac{3}{2}$$

$$m = \frac{y - y_1}{x - x_1} \Longrightarrow \frac{3}{2} = \frac{y-1}{x-1}$$

$$3x - 3 = 2y - 2$$

$$3x - 2y - 1 = 0 \quad \text{(Equation)}$$

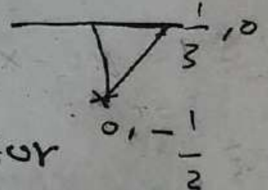
$x = 0 \quad y = -\frac{1}{2}$ $\qquad 0, -\frac{1}{2}$

$y = 0 \quad x = \frac{1}{3}$ $\qquad \frac{1}{3}, 0$

$\left(0 - \frac{1}{3}\right) i + -\frac{1}{2} j$

$\bar{L}$: $-2i - 3j$ (or)

$2i + 3j$



5. Find the length, and the angle that the vector $\bar{V} = 5i + 12j$ makes with the positive x-axis.

$\bar{V}$ = Vector length $(\cos\theta\, i + \sin\theta\, j)$ $\qquad \frac{1}{9} + \frac{1}{4}$

$\therefore$ Vector length $*$ $\cos\theta = 5$ $\qquad \frac{13}{36} = \frac{\sqrt{13}}{6}$

Vector length $*$ $\sin\theta = 12$

Vector length $= \sqrt{5^2 + 12^2} = \sqrt{169} = 13$

$$\bar{V} = 13(\cos\theta\, i + \sin\theta\, j)$$

$$\left.\begin{array}{l} 13\cos\theta = 5 \\ 13\sin\theta = 12 \end{array}\right\} \theta = \tan^{-1}(5/12).$$

4. Unit Vector

$$U_{\bar{v}} = \frac{\text{Vector}}{\text{length}} = \frac{\bar{v}}{|\bar{v}|} = \text{Direction}$$

Examples

6. Find the direction of the vector $\bar{v} = 3i + 2j - k$.

$$\text{Direction} = \text{unit vector} = \frac{\bar{v}}{|\bar{v}|} = \frac{3i+2j-k}{\sqrt{9+4+1}}$$

$$= \frac{3i+2j-k}{\sqrt{14}}$$

$$= \frac{3}{\sqrt{14}} i + \frac{2}{\sqrt{14}} j - \frac{1}{\sqrt{14}} k$$

Check the length:

$$|U_{\bar{v}}| = \sqrt{\left(\frac{3}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(-\frac{1}{\sqrt{14}}\right)^2} = \sqrt{\frac{9}{14} + \frac{4}{14} + \frac{1}{14}}$$

$$= \sqrt{\frac{14}{14}} = \sqrt{1} = 1 \implies \therefore \text{unit vector.}$$

7. Find a unit vector having the same direction as the vector $3i - 4j$.

$$U_{\bar{v}} = \frac{3i-4j}{\sqrt{9+16}} = \frac{3}{5} i - \frac{4}{5} j$$

check: $|U_{\bar{v}}| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$

8. Find a unit vector tangent to the curve $y = x^2$ at the point $(2,4)$.

$$m = \frac{dy}{dx} = 2x \implies m\Big|_{(2,4)} = 2(2) = 4$$

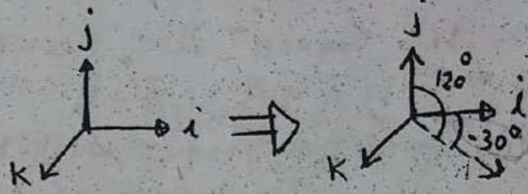
$$4 = \frac{y-4}{x-2} \implies \begin{array}{l} 4x - 8 = y - 4 \\ 4x - y - 4 = 0 \end{array}$$

$$\bar{v} = -i - 4j \quad \text{(or)} \quad \bar{v} = i + 4j$$

$$U_v = \frac{i+4j}{\sqrt{17}} = \frac{1}{\sqrt{17}} i + \frac{4}{\sqrt{17}} j$$

9. Find the unit vector obtained by rotating (j) through 120° in the clockwise direction.

$|j| = 1$



$\therefore$ The required vector is

$1\,(\cos(-30^\circ)\,i + \sin(30^\circ)\,j)$

$= \cos(-30)\,i + \sin(30)\,j$

$= \frac{\sqrt{3}}{2}\,i + \frac{1}{2}\,j$

sq(3)/2 i - 1/2 j

5. Vectors operations

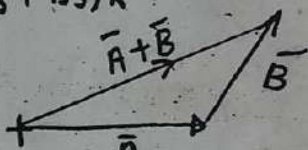
if $\bar{A} = a_1 i + a_2 j + a_3 k$

$\bar{B} = b_1 i + b_2 j + b_3 k$

c = scalar quantity, then

(i) $\bar{A} + \bar{B} = (a_1 i + a_2 j + a_3 k) + (b_1 i + b_2 j + b_3 k)$

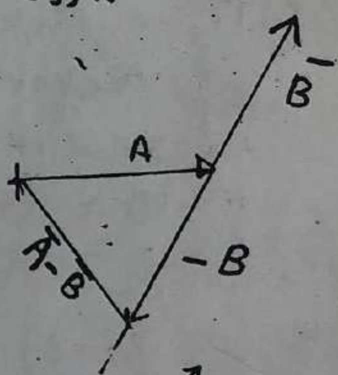
$= (a_1 + b_1)\,i + (a_2 + b_2)\,j + (a_3 + b_3)\,k$



(ii) $\bar{A} - \bar{B} = \bar{A} + (-\bar{B})$

$= (a_1 i + a_2 j + a_3 k) + (-b_1 i - b_2 j - b_3 k)$

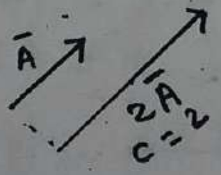
$= (a_1 - b_1)\,i + (a_2 - b_2)\,j + (a_3 - b_3)\,k$

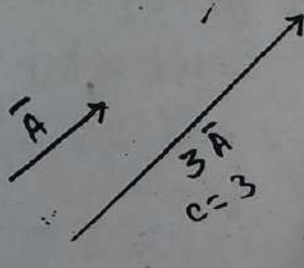


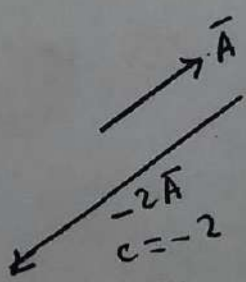
(iii) $c\bar{A} = c a_1 i + c a_2 j + c a_3 k$

$c\bar{A}$ is a vector parallel to $\bar{A}$, and have a length (c) times as $\bar{A}$.

Example:







(iv) $\bar{A} = \bar{B}$, if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$

Examples:

10. Given $\bar{A} = 3i - 4j - k$, $\bar{B} = i - j + 7k$, Find

i) $\bar{A} + \bar{B}$; $\bar{A} - \bar{B}$

ii) a vector with the same direction as $\bar{A}$ but with half the length of $\bar{A}$.

iii) a vector oppositely directed to $\bar{B}$ but with twice the length of $\bar{B}$.

solution:

$\bar{A} = 3i - 4j - k$

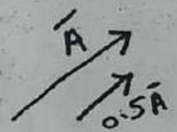
$\bar{B} = i - j + 7k$

* $\bar{A} + \bar{B} = 4i - 5j + 6k$

* $\bar{A} - \bar{B} = 2i - 3j - 8k$

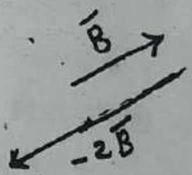
* $0.5\,\bar{A} = 0.5\,(3i - 4j - k)$

$= 1.5i - 2j - 0.5k$



* $-2\bar{B} = -2(i - j + 7k)$

$= -2i + 2j - 14k$



11. Let $\bar{u} = \langle 1,3 \rangle$, $\bar{V} = \langle 2,1 \rangle$, $\bar{W} = \langle 4,-1 \rangle$. Find the vector x that satisfies $2\bar{u} - \bar{V} + x = 7x + \bar{W}$.

Solution:

$\bar{u} = i + 3j$, $\bar{V} = 2i + j$, $\bar{W} = 4i - j$

$2\bar{u} - \bar{V} + x = 7x + \bar{W}$

$\cancel{2i} + 6j - \cancel{2i} - \cancel{j} + x = 7x + 4i - \cancel{j}$

$-4i + 6j = 7x - x = 6x$

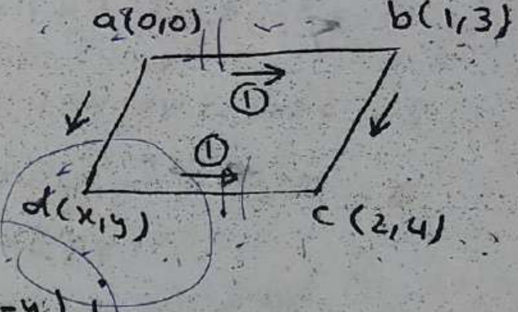
$$x = \frac{-4i + 6j}{6}$$

$= -\frac{2}{3}i + j$

12. Use vectors to find the fourth vertex of a parallelogram, three of whose vertices are (0,0), (1,3), and (2,4).

Solution:

Consider the figure.



$\overline{ab} = i + 3j$

$\overline{dc} = (2-x)i + (4-y)j$

$\overline{ab} = \overline{dc}$

$\therefore\ i + 3j = (2-x)i + (4-y)j$

$1 = 2 - x \Longrightarrow x = 1$

$3 = 4 - y \Longrightarrow y = 1$

$\therefore\ d(x,y) \equiv d(1,1)$

The other solution:

$\overline{ad} = \overline{bc}$

$\left.\begin{array}{l} xi + yj = i + j \\ x = 1,\ y = 1 \end{array}\right\}\ d = (1,1)$

13. Find two unit vectors parallel to the line $y = 3x + 2$

Solution:

The line equation is $3x - y + 2 = 0$

$\overline{L} = i + 3j$

$\therefore\ \overline{u}_L = \dfrac{i+3j}{\sqrt{10}} = \dfrac{1}{\sqrt{10}}i + \dfrac{3}{\sqrt{10}}j$

OR

$\overline{L} = -i - 3j$

$\therefore\ \overline{u}_L = \dfrac{-i-3j}{\sqrt{10}} = -\dfrac{1}{\sqrt{10}}i - \dfrac{3}{\sqrt{10}}j$