Since $\zeta < 1$, the inverse transform yields

$$x(t) = \frac{1/m}{\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}t\right) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t \tag{E2.1.4}$$

2.2 VIBRATION OF MULTIDEGREE-OF-FREEDOM SYSTEMS

A typical n-degree-of-freedom system is shown in Fig. 2.7(a). For a multidegree-of-freedom system, it is more convenient to use matrix notation to express the equations of motion and describe the vibrational response. Let x_i denote the displacement of mass m_i measured from its static equilibrium position; $i = 1, 2, \ldots, n$. The equations of motion of the n-degree-of-freedom system shown in Fig. 2.7(a) can be derived from the free-body diagrams of the masses shown in Fig. 2.7(b) and can be expressed in matrix form as

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{f} \tag{2.60}$$

where $[m]$, $[c]$, and $[k]$ denote the mass, damping, and stiffness matrices, respectively:

$$[m] = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & m_n \end{bmatrix} \tag{2.61}$$

$$[c] = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & \cdots & 0 & 0 \\ -c_2 & c_2 + c_3 & -c_3 & \cdots & 0 & 0 \\ 0 & -c_3 & c_3 + c_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -c_{n-1} & c_n \end{bmatrix} \tag{2.62}$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & k_3 + k_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -k_{n-1} & k_n \end{bmatrix} \tag{2.63}$$

The vectors $\vec{x}$, $\dot{\vec{x}}$, and $\ddot{\vec{x}}$ indicate, respectively, the vectors of displacements, velocities, and accelerations of the various masses, and $\vec{f}$ represents the vector of forces acting on the masses:

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix}, \quad \dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{Bmatrix}, \quad \ddot{\vec{x}} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \vdots \\ \ddot{x}_n \end{Bmatrix}, \quad \vec{f} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} \tag{2.64}$$

where a dot over x_i represents a time derivative of x_i.

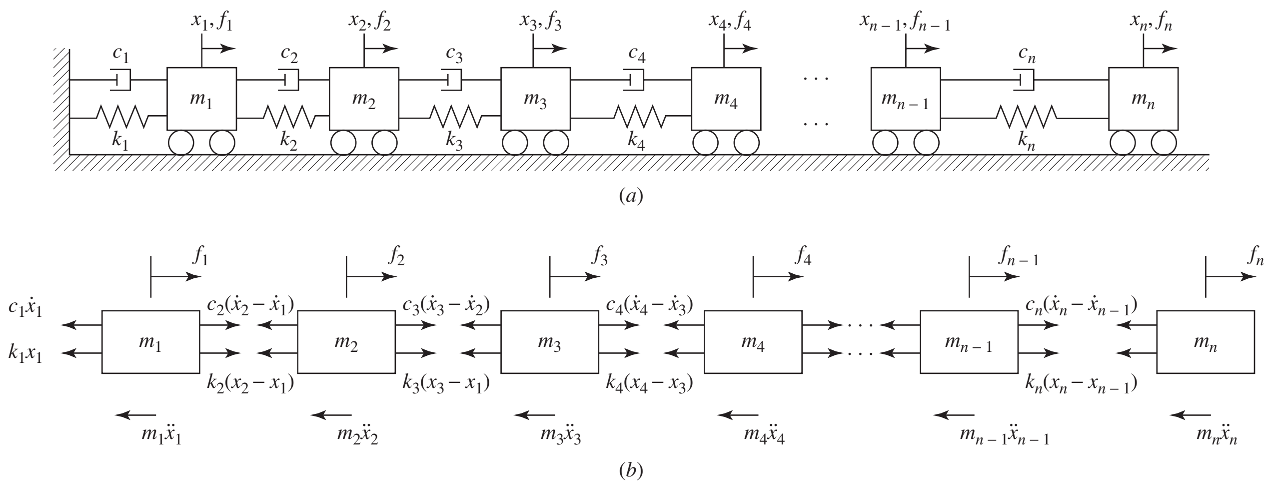


Figure 2.7 (*a*) An *n*-degree-of-freedom system; (*b*) free-body diagrams of the masses.

Note that the spring–mass–damper system shown in Fig. 2.7 is a particular case of a general n-degree-of-freedom system. In their most general form, the mass, damping, and stiffness matrices in Eq. (2.60) are fully populated and can be expressed as

$$[m] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{12} & m_{22} & m_{23} & \cdots & m_{2n} \\ \cdot & & & & \\ \cdot & & & & \\ m_{1n} & m_{2n} & m_{3n} & \cdots & m_{nn} \end{bmatrix} \tag{2.65}$$

$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{12} & c_{22} & c_{23} & \cdots & c_{2n} \\ \cdot & & & \cdots & \cdot \\ \cdot & & & \cdots & \cdot \\ \cdot & & & \cdots & \cdot \\ c_{1n} & c_{2n} & c_{3n} & \cdots & c_{nn} \end{bmatrix} \tag{2.66}$$

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{12} & k_{22} & k_{23} & \cdots & k_{2n} \\ \cdot & & & \cdots & \cdot \\ \cdot & & & \cdots & \cdot \\ \cdot & & & \cdots & \cdot \\ k_{1n} & k_{2n} & k_{3n} & \cdots & k_{nn} \end{bmatrix} \tag{2.67}$$

Equation (2.60) denotes a system of n coupled second-order ordinary differential equations. These equations can be decoupled using a procedure called *modal analysis*, which requires the natural frequencies and normal modes or natural modes of the system. To determine the natural frequencies and normal modes, the eigenvalue problem corresponding to the vibration of the undamped system is to be solved.

2.2.1 Eigenvalue Problem

The free vibration of the undamped system is governed by the equation

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{0} \tag{2.68}$$

The solution of Eq. (2.68) is assumed to be harmonic as

$$\vec{x} = \vec{X}\sin(\omega t + \phi) \tag{2.69}$$

so that

$$\ddot{\vec{x}} = -\omega^2\vec{X}\sin(\omega t + \phi) \tag{2.70}$$

where $\vec{X}$ is the vector of amplitudes of $\vec{x}(t)$, ϕ is the phase angle, and ω is the frequency of vibration. Substituting Eqs. (2.69) and (2.70) into Eq. (2.68), we obtain

$$[[k] - \omega^2[m]]\vec{X} = \vec{0} \tag{2.71}$$

Equation (2.71) represents a system of n algebraic homogeneous equations in unknown coefficients $X_1, X_2, \ldots, X_n$ (amplitudes of $x_1, x_2, \ldots, x_n$) with ω^2 playing the role of a parameter. For a nontrivial solution of the vector of coefficients $\vec{X}$, the determinant of the coefficient matrix must be equal to zero:

$$|[k] - \omega^2[m]| = 0 \tag{2.72}$$

Equation (2.72) is a polynomial equation of nth degree in ω^2 (ω^2 is called the *eigenvalue*) and is called the *characteristic equation* or *frequency equation*.

The roots of the polynomial give the n eigenvalues, $\omega_1^2, \omega_2^2, \ldots, \omega_n^2$. The positive square roots of the eigenvalues yield the natural frequencies of the system, $\omega_1, \omega_2, \ldots, \omega_n$. The natural frequencies are usually arranged in increasing order of magnitude, so that $\omega_1 \le \omega_2 \le \ldots \le \omega_n$. The lowest frequency ω_1 is referred to as the *fundamental frequency*. For each natural frequency ω_i, a corresponding nontrivial vector $\vec{X}^{(i)}$ can be obtained from Eq. (2.71):

$$[[k] - \omega_i^2[m]]\vec{X}^{(i)} = \vec{0} \tag{2.73}$$

The vector $\vec{X}^{(i)}$ is called the *eigenvector, characteristic vector, modal vector*, or *normal mode* corresponding to the natural frequency ω_i.

Of the n homogeneous equations represented by Eq. (2.73), any set of $n-1$ equations can be solved to express any $n-1$ quantities out of $X_1^{(i)}, X_2^{(i)}, \ldots, X_n^{(i)}$ in terms of the remaining $X^{(i)}$. Since Eq. (2.73) denotes a system of homogeneous equations, if $\vec{X}^{(i)}$ is a solution of Eq. (2.73), then $c_i\vec{X}^{(i)}$ is also a solution, where c_i is an arbitrary constant. This indicates that the shape of a natural mode is unique, but not its amplitude. Usually, a magnitude is assigned to the eigenvector $\vec{X}^{(i)}$ to make it unique using a process called *normalization*. A common normalization procedure, called *normalization with respect to the mass matrix*, consists of setting

$$\vec{X}^{(i)^{\mathrm{T}}}[m]\vec{X}^{(i)} = 1, \qquad i = 1, 2, \ldots, n \tag{2.74}$$

where the superscript T denotes the transpose.

2.2.2 Orthogonality of Modal Vectors

The modal vectors possess an important property known as *orthogonality* with respect to the mass matrix $[m]$ as well as the stiffness matrix $[k]$ of the system. To see this property, consider two distinct eigenvalues ω_i^2 and ω_j^2 and the corresponding eigenvectors $\vec{X}^{(i)}$ and $\vec{X}^{(j)}$. These solutions satisfy Eq. (2.71), so that

$$[k]\vec{X}^{(i)} = \omega_i^2[m]\vec{X}^{(i)} \tag{2.75}$$

$$[k]\vec{X}^{(j)} = \omega_j^2[m]\vec{X}^{(j)} \tag{2.76}$$

Premultiplication of both sides of Eq. (2.75) by $\vec{X}^{(j)^{\mathrm{T}}}$ and Eq. (2.76) by $\vec{X}^{(i)^{\mathrm{T}}}$ leads to

$$\vec{X}^{(j)^{\mathrm{T}}}[k]\vec{X}^{(i)} = \omega_i^2\vec{X}^{(j)^{\mathrm{T}}}[m]\vec{X}^{(i)} \tag{2.77}$$

$$\vec{X}^{(i)^{\mathrm{T}}}[k]\vec{X}^{(j)} = \omega_j^2\vec{X}^{(i)^{\mathrm{T}}}[m]\vec{X}^{(j)} \tag{2.78}$$

Noting that the matrices $[k]$ and $[m]$ are symmetric, we transpose Eq. (2.78) and subtract the result from Eq. (2.77), to obtain

$$(\omega_i^2 - \omega_j^2)\vec{X}^{(j)^{\mathrm{T}}}[m]\vec{X}^{(i)} = 0 \tag{2.79}$$

Since the eigenvalues are distinct, $\omega_i^2 \neq \omega_j^2$ and Eq. (2.79) leads to

$$\vec{X}^{(j)^{\mathrm{T}}}[m]\vec{X}^{(i)} = 0, \qquad i \neq j \tag{2.80}$$

Substitution of Eq. (2.80) in Eq. (2.77) results in

$$\vec{X}^{(j)^{\mathrm{T}}}[k]\vec{X}^{(i)} = 0, \qquad i \neq j \tag{2.81}$$

Equations (2.80) and (2.81) denote the orthogonality property of the eigenvectors with respect to the mass and stiffness matrices, respectively. When $j = i$, Eqs. (2.77) and (2.78) become

$$\vec{X}^{(i)^{\mathrm{T}}}[k]\vec{X}^{(i)} = \omega_i^2 \vec{X}^{(i)^{\mathrm{T}}}[m]\vec{X}^{(i)} \tag{2.82}$$

If the eigenvectors are normalized according to Eq. (2.74), Eq. (2.82) gives

$$\vec{X}^{(i)^{\mathrm{T}}}[k]\vec{X}^{(i)} = \omega_i^2 \tag{2.83}$$

By considering all the eigenvectors, Eqs. (2.74) and (2.83) can be written in matrix form as

$$[X]^{\mathrm{T}}[m][X] = [I] = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \tag{2.84}$$

$$[X]^{\mathrm{T}}[k][X] = [\omega_i^2] = \begin{bmatrix} \omega_1^2 & & & 0 \\ & \omega_2^2 & & \\ & & \ddots & \\ 0 & & & \omega_n^2 \end{bmatrix} \tag{2.85}$$

where the $n \times n$ matrix $[X]$, called the *modal matrix*, contains the eigenvectors $\vec{X}^{(1)}$, $\vec{X}^{(2)}, \ldots, \vec{X}^{(n)}$ as columns:

$$[X] = \left[\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \cdots \quad \vec{X}^{(n)}\right] \tag{2.86}$$

2.2.3 Free Vibration Analysis of an Undamped System Using Modal Analysis

The free vibration of an undamped n-degree-of-freedom system is governed by the equations

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{0} \tag{2.87}$$

The n coupled second-order homogeneous differential equations represented by Eq. (2.87) can be uncoupled using modal analysis. In the analysis the solution, $\vec{x}(t)$, is expressed as a superposition of the normal modes $\vec{X}^{(i)}$, $i = 1, 2, \ldots, n$:

$$\vec{x}(t) = \sum_{i=1}^{n} \eta_i(t)\vec{X}^{(i)} = [X]\vec{\eta}(t) \tag{2.88}$$

where $[X]$ is the modal matrix, $\eta_i(t)$ are unknown functions of time, known as *modal coordinates* (or *generalized coordinates*), and $\vec{\eta}(t)$ is the vector of modal coordinates:

$$\vec{\eta}(t) = \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \\ \vdots \\ \eta_n(t) \end{Bmatrix} \tag{2.89}$$

Equation (2.88) represents the *expansion theorem* and is based on the fact that eigenvectors are orthogonal and form a basis in n-dimensional space. This implies that any vector, such as $\vec{x}(t)$, in n-dimensional space can be generated by a linear combination of a set of linearly independent vectors, such as the eigenvectors $\vec{X}^{(i)}$, $i = 1, 2, \ldots, n$. Substitution of Eq. (2.88) into Eq. (2.87) gives

$$[m][X]\ddot{\vec{\eta}} + [k][X]\vec{\eta} = \vec{0} \tag{2.90}$$

Premultiplication of Eq. (2.90) by $[X]^{\mathrm{T}}$ leads to

$$[X]^{\mathrm{T}}[m][X]\ddot{\vec{\eta}} + [X]^{\mathrm{T}}[k][X]\vec{\eta} = \vec{0} \tag{2.91}$$

In view of Eqs. (2.84) and (2.85), Eq. (2.91) reduces to

$$\ddot{\vec{\eta}} + [\omega_i^2]\vec{\eta} = \vec{0} \tag{2.92}$$

which denotes a set of n uncoupled second-order differential equations:

$$\frac{d^2\eta_i(t)}{dt^2} + \omega_i^2\eta_i(t) = 0, \quad i = 1, 2, \ldots, n \tag{2.93}$$

If the initial conditions of the system are given by

$$\vec{x}(t=0) = \vec{x}_0 = \begin{Bmatrix} x_{1,0} \\ x_{2,0} \\ \vdots \\ x_{n,0} \end{Bmatrix} \tag{2.94}$$

$$\dot{\vec{x}}(t=0) = \dot{\vec{x}}_0 = \begin{Bmatrix} \dot{x}_{1,0} \\ \dot{x}_{2,0} \\ \vdots \\ \dot{x}_{n,0} \end{Bmatrix} \tag{2.95}$$

the corresponding initial conditions on $\vec{\eta}(t)$ can be determined as follows.

Premultiply Eq. (2.88) by $[X]^{\mathrm{T}}[m]$ and use Eq. (2.84) to obtain

$$\vec{\eta}(t) = [X]^{\mathrm{T}}[m]\vec{x}(t) \tag{2.96}$$

Thus,

$$\begin{Bmatrix} \eta_1(0) \\ \eta_2(0) \\ \vdots \\ \eta_n(0) \end{Bmatrix} = \vec{\eta}(0) = [X]^{\mathrm{T}}[m]\vec{x}_0 \tag{2.97}$$

$$\begin{Bmatrix} \dot{\eta}_1(0) \\ \dot{\eta}_2(0) \\ \vdots \\ \dot{\eta}_n(0) \end{Bmatrix} = \dot{\vec{\eta}}(0) = [X]^{\mathrm{T}}[m]\dot{\vec{x}}_0 \tag{2.98}$$

The solution of Eq. (2.93) can be expressed as [see Eq. (2.3)]

$$\eta_i(t) = \eta_i(0)\cos\omega_i t + \frac{\dot{\eta}_i(0)}{\omega_i}\sin\omega_i t, \qquad i = 1, 2, \ldots, n \tag{2.99}$$

where $\eta_i(0)$ and $\dot{\eta}_i(0)$ are given by Eqs. (2.97) and (2.98) as

$$\eta_i(0) = \vec{X}^{(i)^{\mathrm{T}}}[m]\vec{x}_0 \tag{2.100}$$

$$\dot{\eta}_i(0) = \vec{X}^{(i)^{\mathrm{T}}}[m]\dot{\vec{x}}_0 \tag{2.101}$$

Once $\eta_i(t)$ are determined, the free vibration solution, $\vec{x}(t)$, can be found using Eq. (2.88).

Example 2.2 Find the free vibration response of the two-degree-of-freedom system shown in Fig. 2.8 using modal analysis for the following data: $m_1 = 2$ kg, $m_2 = 5$ kg, $k_1 = 10$N/m, $k_2 = 20$N/m, $k_3 = 5$N/m, $x_1(0) = 0.1$ m, $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 5$ m/s.

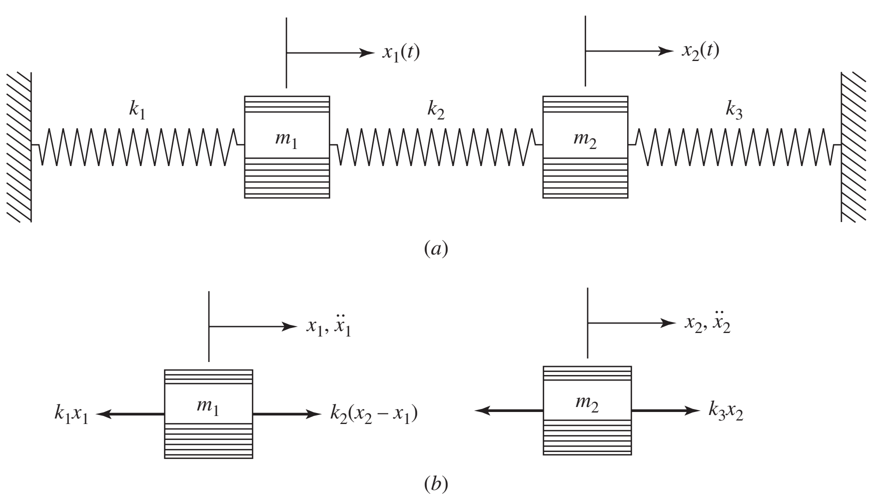


Figure 2.8 Two-degree-of-freedom system: (*a*) system in equilibrium; (*b*) free-body diagrams.

SOLUTION The equations of motion can be expressed as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E2.2.1}$$

For free vibration, we assume harmonic motion as

$$x_i(t) = X_i \cos(\omega t + \phi), \qquad i = 1, 2 \tag{E2.2.2}$$

where X_i is the amplitude of $x_i(t)$, ω is the frequency, and ϕ is the phase angle. Substitution of Eq. (E2.2.2) into Eq. (E2.2.1) leads to the eigenvalue problem

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E2.2.3}$$

Using the known data, Eq. (E2.2.3) can be written as

$$\begin{bmatrix} -2\omega^2 + 30 & -20 \\ -20 & -5\omega^2 + 25 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E2.2.4}$$

For a nontrivial solution of X_1 and X_2, the determinant of the coefficient matrix in Eq. (E2.2.4) is set equal to zero to obtain the frequency equation:

$$\begin{vmatrix} -2\omega^2 + 30 & -20 \\ -20 & -5\omega^2 + 25 \end{vmatrix} = 0$$

or

$$\omega^4 - 20\omega^2 + 35 = 0 \tag{E2.2.5}$$

The roots of Eq. (E2.2.5) give the natural frequencies of vibration of the system as

$$\omega_1 = 1.392028 \text{ rad/s}, \qquad \omega_2 = 4.249971 \text{ rad/s} \tag{E2.2.6}$$

Substitution of $\omega = \omega_1 = 1.392028$ in Eq. (E2.2.4) leads to $X_2^{(1)} = 1.306226 X_1^{(1)}$, while $\omega = \omega_2 = 4.249971$ in Eq. (E2.2.4) yields $X_2^{(2)} = -0.306226 X_1^{(2)}$. Thus, the mode shapes or eigenvectors of the system are given by

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.306226 \end{Bmatrix} X_1^{(1)} \tag{E2.2.7}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.306226 \end{Bmatrix} X_1^{(2)} \tag{E2.2.8}$$

where $X_1^{(1)}$ and $X_1^{(2)}$ are arbitrary constants. By normalizing the mode shapes with respect to the mass matrix, we can find the values of $X_1^{(1)}$ and $X_1^{(2)}$ as

$$\vec{X}^{(1)^{\mathrm{T}}}[m]\vec{X}^{(1)} = (X_1^{(1)})^2 \{\, 1 \quad 1.306226 \,\} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 1.306226 \end{Bmatrix} = 1$$

or $X_1^{(1)} = 0.30815$, and

$$\vec{X}^{(2)^{\mathrm{T}}}[m]\vec{X}^{(2)} = (X_1^{(2)})^2\{\,1 \quad -0.306226\,\}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{Bmatrix} 1 \\ -0.306226 \end{Bmatrix} = 1$$

or $X_1^{(2)} = 0.63643$. Thus, the modal matrix becomes

$$[X] = \begin{bmatrix} \vec{X}^{(1)} & \vec{X}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.30815 & 0.63643 \\ 0.402513 & -0.19489 \end{bmatrix} \tag{E2.2.9}$$

Using

$$\vec{x}(t) = [X]\vec{\eta}(t) \tag{E2.2.10}$$

Eq. (E2.2.1) can be expressed in scalar form as

$$\frac{d^2\eta_i(t)}{dt^2} + \omega_i^2\eta_i(t) = 0, \qquad i = 1, 2 \tag{E2.2.11}$$

The initial conditions of $\eta_i(t)$ can be determined using Eqs. (2.100) and (2.101) as

$$\eta_i(0) = \vec{X}^{(i)^{\mathrm{T}}}[m]\vec{x}(0) \quad \text{or} \quad \vec{\eta}(0) = [X]^{\mathrm{T}}[m]\vec{x}(0) \tag{E2.2.12}$$

$$\dot{\eta}_i(0) = \vec{X}^{(i)^{\mathrm{T}}}[m]\dot{\vec{x}}(0) \quad \text{or} \quad \dot{\vec{\eta}}(0) = [X]^{\mathrm{T}}[m]\dot{\vec{x}}(0) \tag{E2.2.13}$$

$$\vec{\eta}(0) = \begin{bmatrix} 0.30815 & 0.63643 \\ 0.402513 & -0.19489 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{Bmatrix} 0.1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.61630 \\ 1.27286 \end{Bmatrix} \tag{E2.2.14}$$

$$\dot{\vec{\eta}}(0) = \begin{bmatrix} 0.30815 & 0.63643 \\ 0.402513 & -0.19489 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{Bmatrix} 0 \\ 5 \end{Bmatrix} = \begin{Bmatrix} 10.06282 \\ -4.87225 \end{Bmatrix} \tag{E2.2.15}$$

The solution of Eq. (E2.2.11) is given by Eq. (2.99):

$$\eta_i(t) = \eta_i(0)\cos\omega_i t + \frac{\dot{\eta}_i(0)}{\omega_i}\sin\omega_i t, \qquad i = 1, 2 \tag{E2.2.16}$$

Using the initial conditions of Eqs. (E2.2.14) and (E2.2.15), we find that

$$\eta_1(t) = 0.061630\cos 1.392028t + 7.22889\sin 1.392028t \tag{E2.2.17}$$

$$\eta_2(t) = 0.127286\cos 4.249971t - 1.14642\sin 4.24997t \tag{E2.2.18}$$

The displacements of the masses m_1 and m_2, in meters, can be determined from Eq. (E2.2.10) as

$$\begin{aligned} \vec{x}(t) &= \begin{bmatrix} 0.30815 & 0.63643 \\ 0.402513 & -0.19489 \end{bmatrix} \begin{Bmatrix} 0.061630\cos 1.392028t + 7.22889\sin 1.392028t \\ 0.127286\cos 4.249971t - 1.14642\sin 4.24997t \end{Bmatrix} \\ &= \begin{Bmatrix} 0.018991\cos 1.392028t + 2.22758\sin 1.392028t + 0.081009\cos 4.24997t \\ \quad - 0.72962\sin 4.24997t \\ 0.024807\cos 1.392028t + 2.909722\sin 1.392028t - 0.024807\cos 4.24997t \\ \quad + 0.223426\sin 4.24997t \end{Bmatrix} \end{aligned} \tag{E2.2.19}$$

2.2.4 Forced Vibration Analysis of an Undamped System Using Modal Analysis

The equations of motion can be expressed as

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{f}(t) \tag{2.102}$$

The eigenvalues ω_i^2 and the corresponding eigenvectors $\vec{X}^{(i)}$, $i = 1, 2, \ldots, n$, of the system are assumed to be known. The solution of Eq. (2.102) is assumed to be given by a linear combination of the eigenvectors as

$$\vec{x}(t) = \sum_{i=1}^{n} \eta_i(t)\vec{X}^{(i)} = [X]\vec{\eta}(t) \tag{2.103}$$

where $\eta_i(t)$ denote modal coordinates and $[X]$ represents the modal matrix. Substituting Eq. (2.103) into Eq. (2.102) and premultiplying the result by $[X]^{\mathrm{T}}$ results in

$$[X]^{\mathrm{T}}[m][X]\ddot{\vec{\eta}} + [X]^{\mathrm{T}}[k][X]\vec{\eta} = [X]^{\mathrm{T}}\vec{f} \tag{2.104}$$

Using Eqs. (2.84) and (2.85), Eq. (2.104) can be written as

$$\ddot{\vec{\eta}} + [\omega_i^2]\vec{\eta} = \vec{Q} \tag{2.105}$$

where $\vec{Q}$ is called the *vector of modal forces* (or *generalized forces*) given by

$$\vec{Q}(t) = [X]^{\mathrm{T}}\vec{f}(t) \tag{2.106}$$

The n uncoupled differential equations indicated by Eq. (2.105) can be expressed in scalar form as

$$\frac{d^2\eta_i(t)}{dt^2} + \omega_i^2\eta_i(t) = Q_i(t), \qquad i = 1, 2, \ldots, n \tag{2.107}$$

where

$$Q_i(t) = \vec{X}^{(i)^{\mathrm{T}}}\vec{f}(t), \qquad i = 1, 2, \ldots, n \tag{2.108}$$

Each of the equations in (2.107) can be considered as the equation of motion of an undamped single-degree-of-freedom system subjected to a forcing function. Hence, the solution of Eq. (2.107) can be expressed, using $\eta_i(t)$, $Q_i(t)$, $\eta_{i,0}$, and $\dot{\eta}_{i,0}$ in place of $x(t)$, $F(t)$, x_0, and $\dot{x}_0$, respectively, and setting $\omega_d = \omega_i$ and $\zeta = 0$ in Eqs. (2.57)–(2.59), as

$$\eta_i(t) = \int_0^t Q_i(\tau)h(t-\tau)\,d\tau + g(t)\eta_{i,0} + h(t)\dot{\eta}_{i,0} \tag{2.109}$$

with

$$h(t) = \frac{1}{\omega_i}\sin\omega_i t \tag{2.110}$$

$$g(t) = \cos\omega_i t \tag{2.111}$$

The initial values $\eta_{i,0}$ and $\dot{\eta}_{i,0}$ can be determined from the known initial conditions $\vec{x}_0$ and $\dot{\vec{x}}_0$, using Eqs. (2.97) and (2.98).

2.2.5 Forced Vibration Analysis of a System with Proportional Damping

In proportional damping, the damping matrix $[c]$ in Eq. (2.60) can be expressed as a linear combination of the mass and stiffness matrices as

$$[c] = \alpha[m] + \beta[k] \tag{2.112}$$

where α and β are known constants. Substitution of Eq. (2.112) into Eq. (2.60) yields

$$[m]\ddot{\vec{x}} + (\alpha[m] + \beta[k])\dot{\vec{x}} + [k]\vec{x} = \vec{f} \tag{2.113}$$

As indicated earlier, in modal analysis, the solution of Eq. (2.113) is assumed to be of the form

$$\vec{x}(t) = [X]\vec{\eta}(t) \tag{2.114}$$

Substituting Eq. (2.114) into Eq. (2.113) and premultiplying the result by $[X]^{\mathrm{T}}$ leads to

$$[X]^{\mathrm{T}}[m][X]\ddot{\vec{\eta}} + (\alpha[X]^{\mathrm{T}}[m][X]\dot{\vec{\eta}} + \beta[X]^{\mathrm{T}}[k][X]\dot{\vec{\eta}}) + [X]^{\mathrm{T}}[k][X]\vec{\eta} = [X]^{\mathrm{T}}\vec{f} \tag{2.115}$$

When Eqs. (2.84) and (2.85) are used, Eq. (2.115) reduces to

$$\ddot{\vec{\eta}} + (\alpha[I] + \beta[\omega_i^2])\dot{\vec{\eta}} + [\omega_i^2]\vec{\eta} = \vec{Q} \tag{2.116}$$

where

$$\vec{\mathrm{Q}} = [X]^{\mathrm{T}}\vec{f} \tag{2.117}$$

By defining

$$\alpha + \beta\omega_i^2 = 2\zeta_i\omega_i, \qquad i = 1, 2, \ldots, n \tag{2.118}$$

where ζ_i is called the *modal viscous damping factor* in the ith mode, Eq. (2.116) can be rewritten in scalar form as

$$\frac{d^2\eta_i(t)}{dt^2} + 2\zeta_i\omega_i\frac{d\eta_i(t)}{dt} + \omega_i^2\eta_i(t) = Q_i(t), \qquad i = 1, 2, \ldots, n \tag{2.119}$$

Each of the equations in (2.119) can be considered as the equation of motion of a viscously damped single-degree-of-freedom system whose solution is given by Eqs. (2.57)–(2.59). Thus, the solution of Eq. (2.119) is given by

$$\eta_i(t) = \int_0^t Q_i(\tau)h(t-\tau)\,d\tau + g(t)\eta_{i,0} + h(t)\dot{\eta}_{i,0} \tag{2.120}$$

where

$$h(t) = \frac{1}{\omega_{di}}e^{-\zeta_i\omega_i t}\sin\omega_{di}t \tag{2.121}$$

$$g(t) = e^{-\zeta_i\omega_i t}\left(\cos\omega_{di}t + \frac{\zeta_i\omega_i}{\omega_{di}}\sin\omega_{di}t\right) \tag{2.122}$$

and ω_{di} is the ith frequency of damped vibration:

$$\omega_{di} = \sqrt{1-\zeta_i^2}\,\omega_i \tag{2.123}$$

2.2.6 Forced Vibration Analysis of a System with General Viscous Damping

The equations of motion of an n-degree-of-freedom system with arbitrary viscous damping can be expressed in the form of Eq. (2.60):

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{f} \tag{2.124}$$

In this case, the modal matrix will not diagonalize the damping matrix, and an analytical solution is not possible in the configuration space. However, it is possible to find an analytical solution in the state space if Eq. (2.124) is expressed in state-space form. For this, we add the identity $\dot{\vec{x}}(t) = \dot{\vec{x}}(t)$ to an equivalent form of Eq. (2.124) as

$$\dot{\vec{x}}(t) = \dot{\vec{x}}(t) \tag{2.125}$$

$$\ddot{\vec{x}}(t) = -[m]^{-1}[c]\dot{\vec{x}}(t) - [m]^{-1}[k]\vec{x}(t) + [m]^{-1}\vec{f} \tag{2.126}$$

By defining a $2n$-dimensional state vector $\vec{y}(t)$ as

$$\vec{y}(t) = \begin{Bmatrix} \vec{x}(t) \\ \dot{\vec{x}}(t) \end{Bmatrix} \tag{2.127}$$

Eqs. (2.125) and (2.126) can be expressed in state form as

$$\dot{\vec{y}}(t) = [A]\vec{y}(t) + [B]\vec{f}(t) \tag{2.128}$$

where the coefficient matrices $[A]$ and $[B]$, of order $2n \times 2n$ and $2n \times n$, respectively, are given by

$$[A] = \begin{bmatrix} [0] & [I] \\ -[m]^{-1}[k] & -[m]^{-1}[c] \end{bmatrix} \tag{2.129}$$

$$[B] = \begin{bmatrix} [0] \\ [m]^{-1} \end{bmatrix} \tag{2.130}$$

Modal Analysis in State Space For the modal analysis, first we consider the free vibration problem with $\vec{f} = \vec{0}$ so that Eq. (2.128) reduces to

$$\dot{\vec{y}}(t) = [A]\vec{y}(t) \tag{2.131}$$

This equation denotes a set of $2n$ first-order ordinary differential equations with constant coefficients. The solution of Eq. (2.131) is assumed to be of the form

$$\vec{y}(t) = \vec{Y}e^{\lambda t} \tag{2.132}$$

where $\vec{Y}$ is a constant vector and λ is a constant scalar. By substituting Eq. (2.132) into Eq. (2.131), we obtain, by canceling the term $e^{\lambda t}$ on both sides,

$$[A]\vec{Y} = \lambda\vec{Y} \tag{2.133}$$

Equation (2.133) can be seen to be a standard algebraic eigenvalue problem with a nonsymmetric real matrix, $[A]$. The solution of Eq. (2.133) gives the eigenvalues λ_i and the corresponding eigenvectors $\vec{Y}^{(i)}$, $i = 1, 2, \ldots, 2n$. These eigenvalues and eigenvectors can be real or complex. If λ_i is a complex eigenvalue, it can be shown that its complex conjugate ($\overline{\lambda}_i$) will also be an eigenvalue. Also, the eigenvectors $\vec{Y}^{(i)}$ and $\overline{\vec{Y}}^{(i)}$, corresponding to λ_i and $\overline{\lambda}_i$, will also be complex conjugates to one another. The eigenvectors $\vec{Y}^{(i)}$ corresponding to the eigenvalue problem, Eq. (2.133), are called the *right eigenvectors* of the matrix $[A]$. The eigenvectors corresponding to the transpose of the matrix are called the *left eigenvectors* of $[A]$. Thus, the left eigenvectors, corresponding to the eigenvalues λ_i, are obtained by solving the eigenvalue problem

$$[A]^{\mathrm{T}}\vec{Z} = \lambda\vec{Z} \tag{2.134}$$

Since the determinants of the matrices $[A]$ and $[A]^{\mathrm{T}}$ are equal, the characteristic equations corresponding to Eqs. (2.133) and (2.134) will be identical:

$$|[A] - \lambda[I]| \equiv |[A]^{\mathrm{T}} - \lambda[I]| = 0 \tag{2.135}$$

Thus, the eigenvalues of Eqs. (2.133) and (2.134) will be identical. However, the eigenvectors of $[A]$ and $[A]^{\mathrm{T}}$ will be different. To find the relationship between $\vec{Y}^{(i)}$, $i = 1, 2, \ldots, 2n$ and $\vec{Z}^{(j)}$, $j = 1, 2, \ldots, 2n$, the eigenvalue problems corresponding to $\vec{Y}^{(i)}$ and $\vec{Z}^{(j)}$ are written as

$$[A]\vec{Y}^{(i)} = \lambda_i\vec{Y}^{(i)} \quad \text{and} \quad [A]^{\mathrm{T}}\vec{Z}^{(j)} = \lambda_j\vec{Z}^{(j)} \tag{2.136}$$

or

$$\vec{Z}^{(j)^{\mathrm{T}}}[A] = \lambda_j\vec{Z}^{(j)^{\mathrm{T}}} \tag{2.137}$$

Premultiplying the first of Eq. (2.136) by $\vec{Z}^{(j)^{\mathrm{T}}}$ and postmultiplying Eq. (2.137) by $\vec{Y}^{(i)}$, we obtain

$$\vec{Z}^{(j)^{\mathrm{T}}}[A]\vec{Y}^{(i)} = \lambda_i\vec{Z}^{(j)^{\mathrm{T}}}\vec{Y}^{(i)} \tag{2.138}$$

$$\vec{Z}^{(j)^{\mathrm{T}}}[A]\vec{Y}^{(i)} = \lambda_j\vec{Z}^{(j)^{\mathrm{T}}}\vec{Y}^{(i)} \tag{2.139}$$

Subtracting Eq. (2.139) from Eq. (2.138) gives

$$(\lambda_i - \lambda_j)\vec{Z}^{(j)^{\mathrm{T}}}\vec{Y}^{(i)} = 0 \tag{2.140}$$

Assuming that $\lambda_i \neq \lambda_j$, Eq. (2.140) yields

$$\vec{Z}^{(j)^{\mathrm{T}}}\vec{Y}^{(i)} = 0, \qquad i,\ j = 1, 2, \ldots, 2n \tag{2.141}$$

which show that the ith right eigenvector of $[A]$ is orthogonal to the jth left eigenvector of $[A]$, provided that the corresponding eigenvalues λ_i and λ_j are distinct. By substituting Eq. (2.141) into Eq. (2.138) or Eq. (2.139), we find that

$$\vec{Z}^{(j)^{\mathrm{T}}}[A]\,\vec{Y}^{(i)} = 0, \qquad i,\ j = 1,\ 2,\ \ldots,\ 2n \tag{2.142}$$

By setting $i = j$ in Eq. (2.138) or Eq. (2.139), we obtain

$$\vec{Z}^{(i)^{\mathrm{T}}}[A]\vec{Y}^{(i)} = \lambda_i\vec{Z}^{(i)^{\mathrm{T}}}\vec{Y}^{(i)}, \qquad i = 1,\ 2,\ \ldots,\ 2n \tag{2.143}$$

When the right and left eigenvectors of $[A]$ are normalized as

$$\vec{Z}^{(i)^{\mathrm{T}}}\vec{Y}^{(i)} = 1, \qquad i = 1,\ 2,\ \ldots,\ 2n \tag{2.144}$$

Eq. (2.143) gives

$$\vec{Z}^{(i)^{\mathrm{T}}}[A]\vec{Y}^{(i)} = \lambda_i, \qquad i = 1,\ 2,\ \ldots,\ 2n \tag{2.145}$$

Equations (2.144) and (2.145) can be expressed in matrix form as

$$[Z]^{\mathrm{T}}[Y] = [I] \tag{2.146}$$

$$[Z]^{\mathrm{T}}[A][Y] = [\lambda_i] \tag{2.147}$$

where the matrices of right and left eigenvectors are defined as

$$[Y] \equiv \left[\vec{Y}^{(1)} \ \ \vec{Y}^{(2)} \ \ \cdots \ \ \vec{Y}^{(2n)}\right] \tag{2.148}$$

$$[Z] \equiv \left[\vec{Z}^{(1)} \ \ \vec{Z}^{(2)} \ \ \cdots \ \ \vec{Z}^{(2n)}\right] \tag{2.149}$$

and the diagonal matrix of eigenvalues is given by

$$[\lambda_i] = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_{2n} \end{bmatrix} \tag{2.150}$$

In the modal analysis, the solution of the state equation, Eq. (2.128), is assumed to be a linear combination of the right eigenvectors as

$$\vec{y}(t) = \sum_{i=1}^{2n} \eta_i(t)\vec{Y}^{(i)} = [Y]\,\vec{\eta}(t) \tag{2.151}$$

where $\eta_i(t), i = 1, 2, \ldots, 2n$, are modal coordinates and $\vec{\eta}(t)$ is the vector of modal coordinates:

$$\vec{\eta}(t) = \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \\ \vdots \\ \eta_{2n}(t) \end{Bmatrix} \tag{2.152}$$

Substituting Eq. (2.151) into Eq. (2.128) and premultiplying the result by $[Z]^{\mathrm{T}}$, we obtain

$$[Z]^{\mathrm{T}}[Y]\dot{\vec{\eta}}(t) = [Z]^{\mathrm{T}}[A]\,[Y]\,\vec{\eta}(t) + [Z]^{\mathrm{T}}[B]\,\vec{f}(t) \tag{2.153}$$

In view of Eqs. (2.146) and (2.147), Eq. (2.153) reduces to

$$\dot{\vec{\eta}}(t) = [\lambda_i]\,\vec{\eta}(t) + \vec{Q}(t) \tag{2.154}$$

which can be written in scalar form as

$$\frac{d\eta_i(t)}{dt} = \lambda_i\eta_i(t) + Q_i(t), \qquad i = 1,\ 2,\ \ldots,\ 2n \tag{2.155}$$

where the vector of modal forces is given by

$$\vec{Q}(t) = [Z]^{\mathrm{T}}[B]\ \vec{f}(t) \tag{2.156}$$

and the ith modal force by

$$Q_i(t) = \vec{Z}^{(i)^{\mathrm{T}}}[B]\vec{f}(t), \qquad i = 1,\ 2,\ \ldots,\ 2n \tag{2.157}$$

The solutions of the first-order ordinary differential equations, Eq. (2.155), can be expressed as

$$\eta_i(t) = \int_0^t e^{\lambda_i(t-\tau)} Q_i(\tau)\,d\tau + e^{\lambda_i t}\eta_i(0), \qquad i = 1,\ 2,\ \ldots,\ 2n \tag{2.158}$$

which can be written in matrix form as

$$\vec{\eta}(t) = \int_0^t e^{[\lambda_i](t-\tau)}\vec{Q}(\tau)\,d\tau + e^{[\lambda_i]t}\vec{\eta}(0) \tag{2.159}$$

where $\vec{\eta}(0)$ denotes the initial value of $\vec{\eta}(t)$. To determine $\vec{\eta}(0)$, we premultiply Eq. (2.151) by $\vec{Z}^{(i)^{\mathrm{T}}}$ to obtain

$$\vec{Z}^{(i)^{\mathrm{T}}}\vec{y}(t) = \vec{Z}^{(i)^{\mathrm{T}}}[Y]\ \vec{\eta}(t) \tag{2.160}$$

In view of the orthogonality relations, Eq. (2.141), Eq. (2.160) gives

$$\eta_i(t) = \vec{Z}^{(i)^{\mathrm{T}}}\vec{y}(t), \qquad i = 1,\ 2,\ \ldots,\ 2n \tag{2.161}$$

By setting $t = 0$ in Eq. (2.161), the initial value of $\eta_i(t)$ can be found as

$$\eta_i(0) = \vec{Z}^{(i)^{\mathrm{T}}}\vec{y}(0), \qquad i = 1,\ 2,\ \ldots,\ 2n \tag{2.162}$$

Finally, the solution of Eq. (2.128) can be expressed, using Eqs. (2.151) and (2.159), as

$$\vec{y}(t) = \int_0^t [Y]e^{[\lambda_i](t-\tau)}\vec{Q}(\tau)\,d\tau + [Y]e^{[\lambda_i]t}\vec{\eta}(0) \tag{2.163}$$

Example 2.3 Find the forced response of the viscously damped two-degree-of-freedom system shown in Fig. 2.9 using modal analysis for the following data: $m_1 =$ 2 kg, $m_2 = 5$ kg, $k_1 = 10$ N/m, $k_2 = 20$ N/m, $k_3 = 5$ N/m, $c_1 = 2$ N · s/m, $c_2 = 3$ N · s/m, $c_3 = 1.0$ N · s/m, $f_1(t) = 0$, $f_2(t) = 5$ N, and $t \geq 0$. Assume the initial conditions to be zero.

SOLUTION The equations of motion of the system are given by

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{f} \tag{E2.3.1}$$

where

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \tag{E2.3.2}$$

$$[c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \tag{E2.3.3}$$

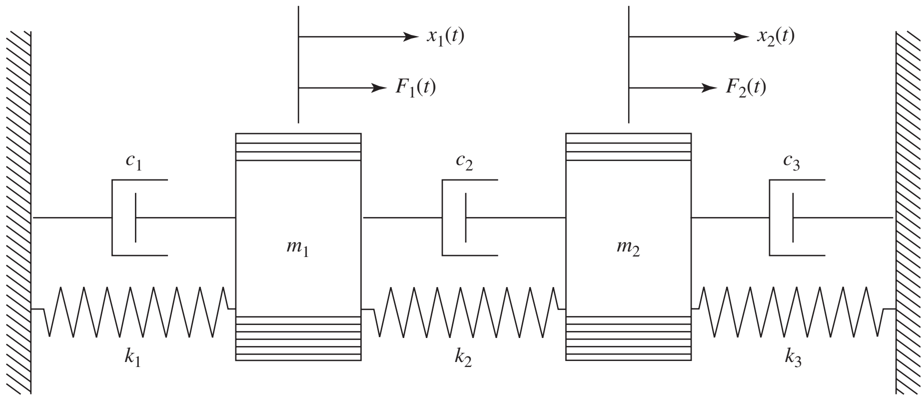


Figure 2.9 Viscously damped two-degree-of-freedom system.

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 30 & -20 \\ -20 & 25 \end{bmatrix} \tag{E2.3.4}$$

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}, \quad \ddot{\vec{x}} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}, \quad \vec{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{E2.3.5}$$

The equations of motion can be stated in state form as

$$\dot{\vec{y}} = [A]\vec{y} + [B]\vec{f} \tag{E2.3.6}$$

where

$$[A] = \begin{bmatrix} [0] & [I] \\ -[m]^{-1}[k] & -[m]^{-1}[c] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 10 & -\frac{5}{2} & \frac{3}{2} \\ 4 & -5 & \frac{3}{5} & -\frac{4}{5} \end{bmatrix} \tag{E2.3.7}$$

$$[B] = \begin{bmatrix} [0] \\ [m]^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \tag{E2.3.8}$$

$$\vec{y} = \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \tag{E2.3.9}$$

The solution of the eigenvalue problem

$$[A]\vec{Y} = \lambda\vec{Y}$$

or

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 10 & -\frac{5}{2} & \frac{3}{2} \\ 4 & -5 & \frac{3}{5} & -\frac{4}{5} \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{Bmatrix} = \lambda \begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{Bmatrix} \tag{E2.3.10}$$

is given by

$$\begin{aligned} \lambda_1 &= -1.4607 + 3.9902i \\ \lambda_2 &= -1.4607 - 3.9902i \\ \lambda_3 &= -0.1893 + 1.3794i \\ \lambda_4 &= -0.1893 - 1.3794i \end{aligned} \tag{E2.3.11}$$

$$[Y] \equiv \left[\vec{Y}^{(1)} \quad \vec{Y}^{(2)} \quad \vec{Y}^{(3)} \quad \vec{Y}^{(4)}\right]$$

$$= \begin{bmatrix} -0.0754 - 0.2060i & -0.0754 + 0.2060i & -0.0543 - 0.3501i & -0.0543 + 0.3501i \\ 0.0258 + 0.0608i & 0.0258 - 0.0608i & -0.0630 - 0.4591i & -0.0630 + 0.4591i \\ 0.9321 & 0.9321 & 0.4932 - 0.0085i & 0.4932 + 0.0085i \\ -0.2803 + 0.0142i & -0.2803 - 0.0142i & 0.6452 & 0.6452 \end{bmatrix} \tag{E2.3.12}$$

The solution of the eigenvalue problem

$$[A]^{\mathrm{T}}\vec{Z} = \lambda\vec{Z}$$

or

$$\begin{bmatrix} 0 & 0 & -15 & 4 \\ 0 & 0 & 10 & -5 \\ 1 & 0 & -\frac{5}{2} & \frac{3}{5} \\ 0 & 1 & \frac{3}{2} & -\frac{4}{5} \end{bmatrix} \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{Bmatrix} = \lambda \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{Bmatrix} \tag{E2.3.13}$$

gives λ_i as indicated in Eq. (E2.3.11) and $\vec{Z}^{(i)}$ as

$$[Z] \equiv [\vec{Z}^{(1)} \quad \vec{Z}^{(2)} \quad \vec{Z}^{(3)} \quad \vec{Z}^{(4)}]$$

$$= \begin{bmatrix} 0.7736 & 0.7736 & 0.2337 - 0.0382i & 0.2337 + 0.0382i \\ -0.5911 + 0.0032i & -0.5911 - 0.0032i & 0.7775 & 0.7775 \\ 0.0642 - 0.1709i & 0.0642 + 0.1709i & 0.0156 - 0.1697i & 0.0156 + 0.1697i \\ -0.0418 + 0.1309i & -0.0418 - 0.1309i & 0.0607 - 0.5538i & 0.0607 + 0.5538i \end{bmatrix} \tag{E2.3.14}$$

The vector of modal forces is given by

$$\vec{Q}(t) = [Z]^{\mathrm{T}}[B]\vec{f}(t)$$

$$= \begin{bmatrix} 0.7736 & -0.5911 + 0.0032i & 0.0642 - 0.1709i & -0.0418 + 0.1309i \\ 0.7736 & -0.5911 - 0.0032i & 0.0642 + 0.1709i & -0.0418 - 0.1309i \\ 0.2337 - 0.0382i & 0.7775 & 0.0156 - 0.1697i & 0.0607 - 0.5538i \\ 0.2337 + 0.0382i & 0.7775 & 0.0156 + 0.1697i & 0.0607 + 0.5538i \end{bmatrix}$$

$$\cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.5 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{Bmatrix} 0 \\ 5 \end{Bmatrix} = \begin{Bmatrix} -0.0418 + 0.1309i \\ -0.0418 - 0.1309i \\ 0.0607 - 0.5538i \\ 0.0607 + 0.5538i \end{Bmatrix} \tag{E2.3.15}$$

Since the initial values, $x_1(0), x_2(0), \dot{x}_1(0)$, and $\dot{x}_2(0)$, are zero, all $\eta_i(0) = 0, i = 1, 2, 3, 4$, from Eq. (2.162). Thus, the values of $\eta_i(t)$ are given by

$$\eta_i(t) = \int_0^t e^{\lambda_i(t-\tau)} Q_i(\tau)\, d\tau, \qquad i = 1, 2, 3, 4 \tag{E2.3.16}$$

since $Q_i(\tau)$ is a constant (complex quantity), Eq. (E2.3.16) gives

$$\eta_i(t) = \frac{Q_i}{\lambda_i}(e^{\lambda_i t} - 1), \qquad i = 1, 2, 3, 4 \tag{E2.3.17}$$

Using the values of Q_i and λ_i from Eqs. (E2.3.15) and (E2.3.11), $\eta_i(t)$ can be expressed as

$$\begin{aligned} \eta_1(t) &= (0.0323 - 0.0014i)\, [e^{(-1.4607+3.9902i)t} - 1] \\ \eta_2(t) &= (0.0323 + 0.0014i)\, [e^{(-1.4607-3.9902i)t} - 1] \\ \eta_3(t) &= (-0.4 + 0.0109i)\, [e^{(-0.1893+1.3794i)t} - 1] \\ \eta_4(t) &= (-0.4 - 0.0109i)\, [e^{(-0.1893-1.3794i)t} - 1] \end{aligned} \tag{E2.3.18}$$

Finally, the state variables can be found from Eq. (2.151) as

$$\vec{y}(t) = [Y]\vec{\eta}(t) \tag{E2.3.19}$$

In view of Eqs. (E2.3.12) and (E2.3.18), Eq. (E2.3.19) gives

$$\begin{aligned} y_1(t) &= 0.0456\left[e^{(-1.4607+3.9902i)t} - 1\right] \quad \text{m} \\ y_2(t) &= 0.0623\left[e^{(-1.4607-3.9902i)t} - 1\right] \quad \text{m} \\ y_3(t) &= -0.3342\left[e^{(-0.1893+1.3794i)t} - 1\right] \quad \text{m/s} \\ y_4(t) &= -0.5343\left[e^{(-0.1893-1.3794i)t} - 1\right] \quad \text{m/s} \end{aligned} \tag{E2.3.20}$$

2.3 RECENT CONTRIBUTIONS

Single-Degree-of-Freedom Systems Anderson and Ferri [5] investigated the properties of a single-degree-of-freedom system damped with generalized friction laws. The system was studied first by using an exact time-domain method and then by using first-order harmonic balance. It was observed that the response amplitude can be increased or decreased by the addition of amplitude-dependent friction. These results suggest that in situations where viscous damping augmentation is difficult or impractical, as in the case of space structures and turbomachinery bladed disks, beneficial damping properties can be achieved through the redesign of frictional interfaces. Bishop et al. [6] gave an elementary explanation of the Duhamel integral as well as Fourier and Laplace transform techniques in linear vibration analysis. The authors described three types of receptances and explained the relationships between them.

Multidegree-of-Freedom Systems The dynamic absorbers play a major role in reducing vibrations of machinery. Soom and Lee [7] studied the optimal parameter design of linear and nonlinear dynamic vibration absorbers for damped primary systems. Shaw et al. [8] showed that the presence of nonlinearities can introduce dangerous instabilities, which in some cases may result in multiplication rather than reduction of the vibration amplitudes. For systems involving a large number of degrees of freedom, the size of the eigenvalue problem is often reduced using a model reduction or dynamic condensation process to find an approximate solution rapidly. Guyan reduction is a popular technique used for model reduction [9]. Lim and Xia [10] presented a technique for dynamic condensation based on iterated condensation. The quantification of the extent of nonproportional viscous damping in discrete vibratory systems was investigated by Prater and Singh [11]. Lauden and Akesson derived an exact complex dynamic member stiffness matrix for a damped second-order Rayleigh–Timoshenko beam vibrating in space [12].

The existence of classical real normal modes in damped linear vibrating systems was investigated by Caughey and O'Kelly [13]. They showed that the necessary and sufficient condition for a damped system governed by the equation of motion

$$[I]\ddot{\vec{x}}(t) + [A]\dot{\vec{x}}(t) + [B]\vec{x}(t) = \vec{f}(t) \tag{2.164}$$

to possess classical normal modes is that matrices $[A]$ and $[B]$ be commutative; that is, $[A][B] = [B][A]$. The scope of this criterion was reexamined and an alternative form of the condition was investigated by other researchers [14]. The settling time of a system can be defined as the time for the envelope of the transient part of the system response to move from its initial value to some fraction of the initial value. An expression for the settling time of an underdamped linear multidegree-of-freedom system was derived by Ross and Inman [15].

REFERENCES

1. S. S. Rao, *Mechanical Vibrations*, 4th ed., Prentice Hall, Upper Saddle River, NJ, 2004.
2. L. Meirovitch, *Fundamentals of Vibrations*, McGraw-Hill, New York, 2001.
3. D. J. Inman, *Engineering Vibration*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2001.
4. A. K. Chopra, *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, Prentice Hall, Englewood Cliffs, NJ, 1995.
5. J. R. Anderson and A. A. Ferri, Behavior of a single-degree-of-freedom system with a generalized friction law, *Journal of Sound and Vibration*, Vol. 140, No. 2, pp. 287–304, 1990.

6. R. E. D. Bishop, A. G. Parkinson, and J. W. Pendered, Linear analysis of transient vibration, *Journal of Sound and Vibration*, Vol. 9, No. 2, pp. 313–337, 1969.
7. A. Soom and M. Lee, Optimal design of linear and nonlinear vibration absorbers for damped system, *Journal of Vibrations, Acoustics, Stress and Reliability in Design*, Vol. 105, No. 1, pp. 112–119, 1983.
8. J. Shah, S. W. Shah, and A. G. Haddow, On the response of the nonlinear vibration absorber, *Journal of Non-linear Mechanics*, Vol. 24, pp. 281–293, 1989.
9. R. J. Guyan, Reduction of stiffness and mass matrices, *AIAA Journal*, Vol. 3, No. 2, p. 380, 1965.
10. R. Lim and Y. Xia, A new eigensolution of structures via dynamic condensation, *Journal of Sound and Vibration*, Vol. 266, No. 1, pp. 93–106, 2003.
11. G. Prater and R. Singh, Quantification of the extent of non-proportional viscous damping in discrete vibratory systems, *Journal of Sound and Vibration*, Vol. 104, No. 1, pp. 109–125, 1986.
12. R. Lauden and B. Akesson, Damped second-order Rayleigh–Timoshenko beam vibration in space: an exact complex dynamic member stiffness matrix, *International Journal for Numerical Methods in Engineering*, Vol. 19, No. 3, pp. 431–449, 1983.
13. T. K. Caughey and M. E. J. O'Kelly, Classical normal modes in damped systems, *Journal of Applied Mechanics*, Vol. 27, pp. 269–271, 1960.
14. A. S. Phani, On the necessary and sufficient conditions for the existence of classical normal modes in damped linear dynamic systems, *Journal of Sound and Vibration*, Vol. 264, No. 3, pp. 741–745, 2002.
15. A. D. S. Ross and D. J. Inman, Settling time of underdamped linear lumped parameter systems, *Journal of Sound and Vibration*, Vol. 140, No. 1, pp. 117–127, 1990.

PROBLEMS

2.1 A building frame with four identical columns that have an effective stiffness of k and a rigid floor of mass m is shown in Fig. 2.10. The natural period of vibration of the frame in the horizontal direction is found to be 0.45 s. When a heavy machine of mass 500 kg is mounted (clamped) on the floor, its natural period of vibration in the horizontal direction is found to be 0.55 s. Determine the effective stiffness k and mass m of the building frame.

2.2 The propeller of a wind turbine with four blades is shown in Fig. 2.11. The aluminum shaft AB on which the blades are mounted is a uniform hollow shaft of outer diameter 2 in., inner diameter 1 in., and length 10 in. If

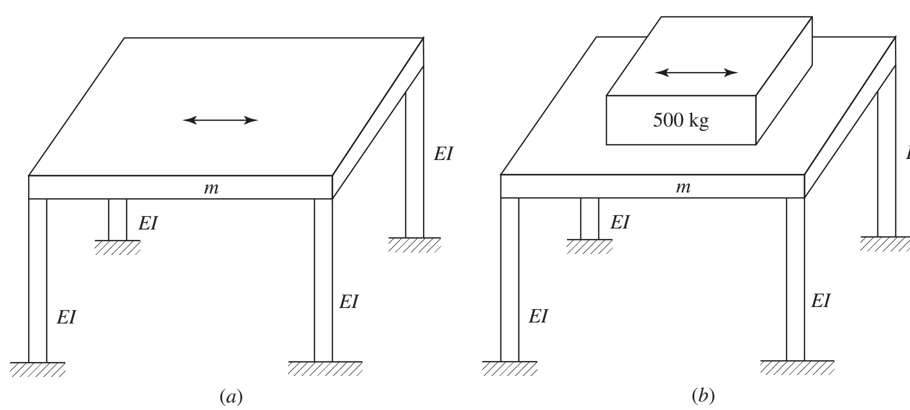


Figure 2.10

each blade has a mass moment of inertia of 0.5 lb-in.-sec^2, determine the natural frequency of vibration of the blades about the y-axis. [*Hint*: The torsional stiffness k_t of a shaft of length l is given by $k_t = GI_0/l$, where G is the shear modulus ($G = 3.8 \times 10^6$ psi for aluminum) and I_0 is the polar moment of inertia of the cross section of the shaft.]

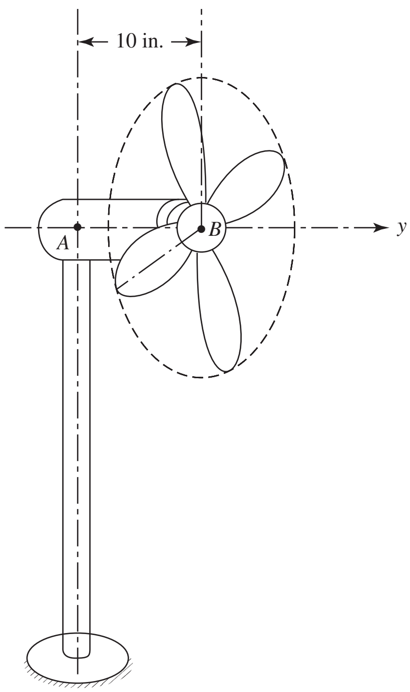


Figure 2.11

2.3 What is the difference between the damped and undamped natural frequencies and natural time periods for a damping ratio of 0.5?

2.4 A spring–mass system with mass 1 kg is found to vibrate with a natural frequency of 10 Hz. The same system when immersed in an oil is observed to vibrate with a natural frequency of 9 Hz. Find the damping constant of the oil.

2.5 Find the response of an undamped spring–mass system subjected to a constant force F_0 applied during $0 \le t \le \tau$ using a Laplace transform approach. Assume zero initial conditions.

2.6 A spring–mass system with mass 10 kg and stiffness 20,000 N/m is subjected to the force shown in Fig. 2.12. Determine the response of the mass using the convolution integral.

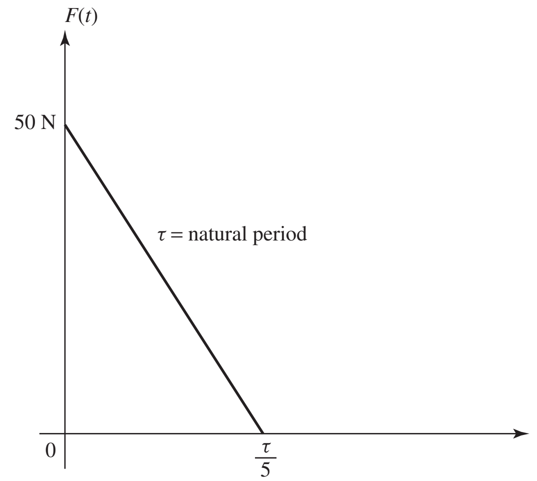


Figure 2.12

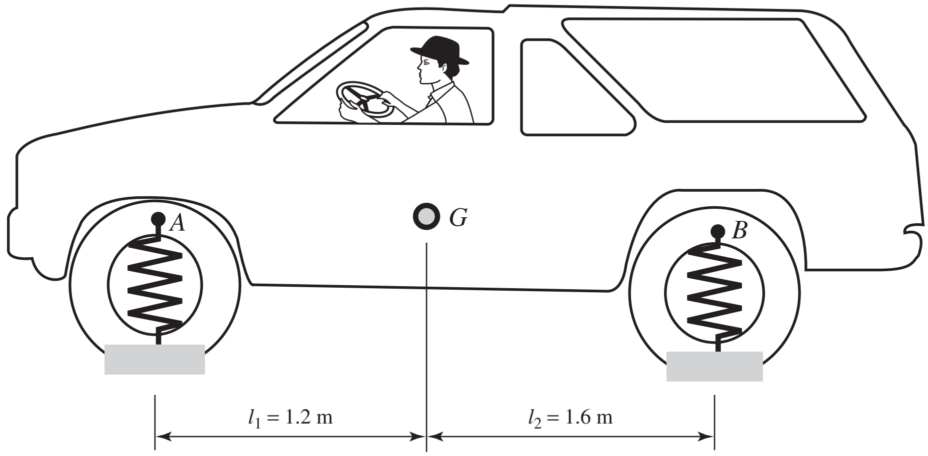
2.7 Find the response of a spring–mass system subjected to the force $F(t) = F_0 e^{i\omega t}$ using the method of Laplace transforms. Assume the initial conditions to be zero.

2.8 Consider a spring–mass system with $m = 10$ kg and $k = 5000$ N/m subjected to a harmonic force $F(t) = 400 \cos 10t$ N. Find the total system response with the initial conditions $x_0 = 0.1$ m and $\dot{x}_0 = 5$ m/s.

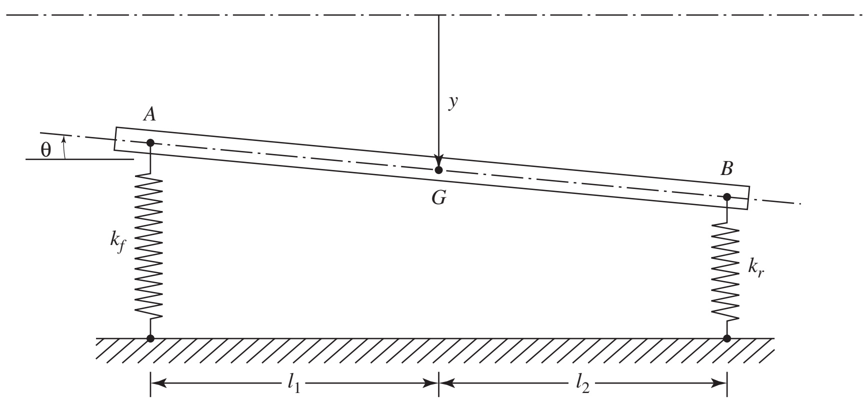
2.9 Consider a spring–mass–damper system with $m = 10$ kg, $k = 5000$ N/m, and $c = 200$ N·s/m subjected to a harmonic force $F(t) = 400 \cos 10t$ N. Find the steady-state and total system response with the initial conditions $x_0 = 0.1$ m and $\dot{x}_0 = 5$ m/s.

2.10 A simplified model of an automobile and its suspension system is shown in Fig. 2.13 with the following data: mass $m = 1000$ kg, radius r of gyration about the center of mass $G = 1.0$ m, spring constant of front suspension $k_f = 20$ kN/m, and spring constant of rear suspension $k_r = 15$ kN/m.

(a) Derive the equations of motion of an automobile by considering the vertical displacement of the center of mass y and rotation of the body about the center of mass θ as the generalized coordinates.



(*a*)



(*b*)

Figure 2.13

(b) Determine the natural frequencies and mode shapes of the automobile in bounce (up-and-down motion) and pitch (angular motion) modes.

2.11 Find the natural frequencies and the m-orthogonal mode shapes of the system shown in Fig. 2.9(a) for the following data: $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$.

2.12 Determine the natural frequencies and the m-orthogonal mode shapes of the system shown in Fig. 2.14.

2.13 Find the free vibration response of the system shown in Fig. 2.8(a) using modal analysis. The data are as follows: $m_1 = m_2 = 10$ kg, $k_1 = k_2 = k_3 = 500$ N/m, $x_1(0) = 0.05$ m, $x_2(0) = 0.10$ m, and $\dot{x}_1(0) = \dot{x}_2(0) = 0$.

2.14 Consider the following data for the two-degree-of-freedom system shown in Fig. 2.9: $m_1 = 1$ kg, $m_2 = 2$ kg, $k_1 = 500$ N/m, $k_2 = 100$ N/m, $k_3 = 300$ N/m, $c_1 = 3$ N·s/m, $c_2 = 1$ N·s/m, and $c_3 = 2$ N·s/m.

(a) Derive the equations of motion.

(b) Discuss the nature of error involved if the off-diagonal terms of the damping matrix are neglected in the equations derived in part (a).

(c) Find the responses of the masses resulting from the initial conditions $x_1(0) = 5$ mm, $x_2(0) = 0$, $\dot{x}_1(0) = 1$ m/s, and $\dot{x}_2(0) = 0$.

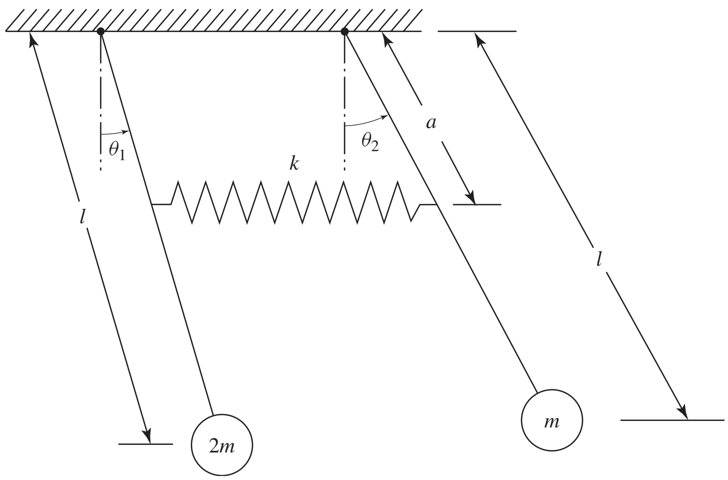


Figure 2.14

2.15 Determine the natural frequencies and m-orthogonal mode shapes of the three-degree-of-freedom system shown in Fig. 2.15 for the following data: $m_1 = m_3 = m$, $m_2 = 2\ m$, $k_1 = k_4 = k$, and $k_2 = k_3 = 2\ k$.

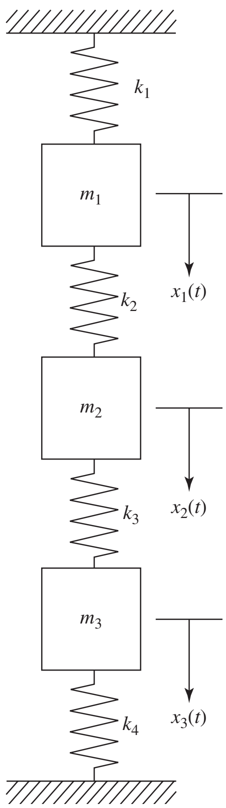


Figure 2.15

2.16 Find the free vibration response of the system described in Problem 2.14 using modal analysis

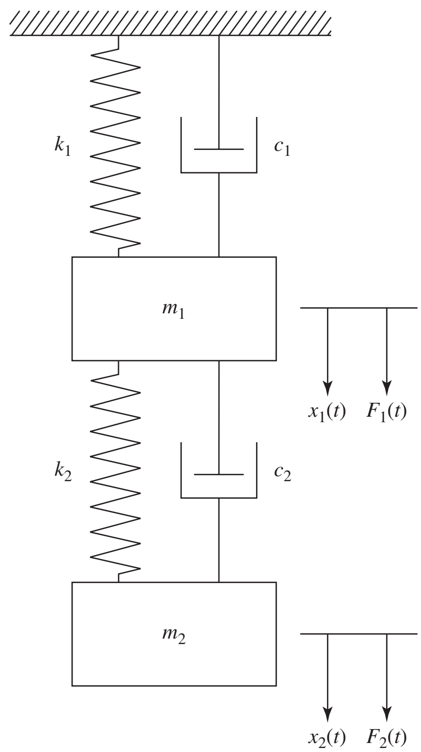


Figure 2.16

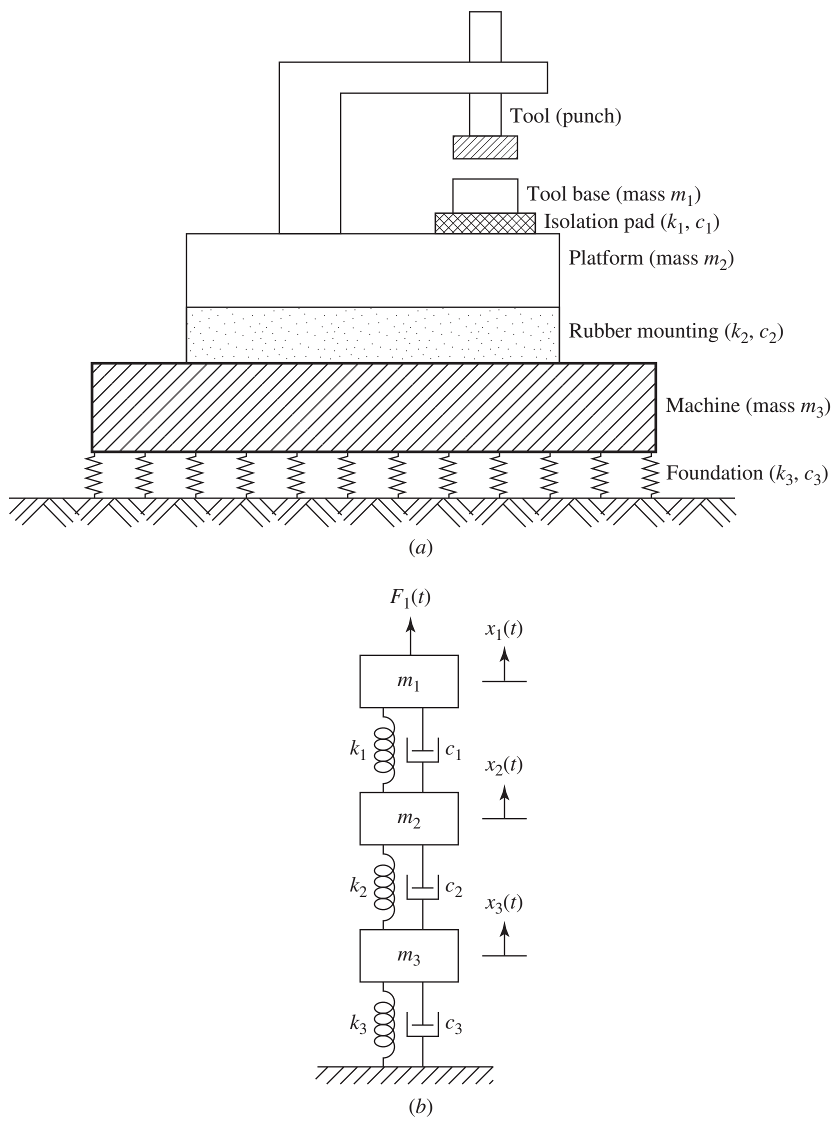


Figure 2.17

with the following data: $m = 2$ kg, $k = 100$ N/m, $x_1(0) = 0.1$ m, and $x_2(0) = x_3(0) = \dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0) = 0$.

2.17 Consider the two-degree-of-freedom system shown in Fig. 2.16 with the following data: $m_1 = 10$ kg, $m_2 = 1$ kg, $k_1 = 100$ N/m, $k_2 = 10$ N/m, and dampers c_1 and c_2 corresponding to proportional damping with $\alpha = 0.1$ and $\beta = 0.2$. Find the steady-state response of the system.

2.18 A punch press mounted on a foundation as shown in Fig. 2.17(a) has been modeled as a three-degree-of-freedom system as indicated in Fig. 2.17(b). The data are as follows: $m_1 = 200$ kg, $m_2 = 2000$ kg, $m_3 = 5000$ kg, $k_1 = 2 \times 10^5$ N/m, $k_2 = 1 \times 10^5$ N/m, and $k_3 = 5 \times 10^5$ N/m. The damping constants c_1, c_2, and c_3 correspond to modal damping ratios of $\zeta_1 = 0.02$, $\zeta_2 = 0.04$, and $\zeta_3 = 0.06$ in the first, second, and third modes of the system, respectively.

Find the response of the system using modal analysis when the tool base m_1 is subjected to an impact force $F_1(t) = 500\delta(t)$ N.

2.19 A spring–mass–damper system with $m = 0.05$ lb-sec^2/in., k = 50 lb/in., and $c = 1$ lb-sec/in., is subjected to a harmonic force of magnitude 20 lb. Find the resonant amplitude and the maximum amplitude of the steady-state motion.

2.20 A machine weighing 25 lb is subjected to a harmonic force of amplitude 10 lb and frequency 10 Hz. If the maximum displacement of the machine is to be restricted to 1 in., determine the necessary spring constant of the foundation for the machine. Assume the damping constant of the foundation to be 0.5 lb-sec/in.