Infinite Sequences and Infinite Series

1- Infinite Sequences

A sequence is a list of numbers

$a_1, a_2, a_3, a_4, ---, a_n$

in a given order, each of a_1, a_2, a_3 and so on represents a number. Thes are the TERMS of the sequence.

For example, the sequence

$2, 4, 6, 8, 10 ---, 2n$

has a first term $a_1 = 2$, second term $a_2 = 4$ and nth term $a_n = 2n$. The integer n is called the index of a_n.

Note
order is important. The sequence $2, 4, 6, 8 ----$ is not the same as the sequence $4, 2, 6, 8$.

sequences can be described by writting rules that specify their terms, such as

$a_n = \sqrt{n}$

$a_n = (-1)^{n+1} \frac{1}{n}$

$a_n = \frac{n-1}{n}$

or by listing terms,

$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots \sqrt{n}\}$

$\{a_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4} \ldots (-1)^{n+1}\frac{1}{n}\}$

$\{a_n\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots, \frac{n-1}{n}\}$

we also somtimes writ

$$\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$$

Difinition: The infinite sequence of numbers is a function whose domain is the set of positive integers.

For example, the general behaviour of the following sequences are:

(1) $a_n = 2n = 2, 4, 6, 8 - - - -$

(2) $a_n = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} - - - -$

(3) $a_n = \frac{n+1}{n} = 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4} - - -$

Example: write the first five terms of the following sequences:

1- $\left[\frac{2n-1}{3n+2}\right]$ or $a_n = \frac{2n-1}{3n+2}$ or $\left\{\frac{2n-1}{3n+2}\right\}$

$n = 1, 2, 3, 4, 5$ (five terms)

$a_1 = \frac{1}{5}, a_2 = \frac{3}{8}, a_3 = \frac{5}{11}, a_4 = \frac{7}{14}, a_5 = \frac{9}{17}$

The five terms are $\left[\frac{1}{5}, \frac{3}{8}, \frac{5}{11}, \frac{7}{14}, \frac{9}{17}\right]$

2- $a_n = \frac{1+(-1)^n}{n}$

$n = 1, 2, 3, 4, 5$

$a_1 = 0, a_2 = 1, a_3 = 0, a_4 = \frac{1}{2}, a_5 = 0$

the five terms are $[0, 1, 0, \frac{1}{2}, 0]$

Convergence and Divergence

Somtimes the number in a sequce approach a single value as the index "n" increases.

For example

(1) $\left[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} ----, \frac{1}{n}\right]$

the terms approach zero as n gets large.

(2) $\left[0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ----, 1-\frac{1}{n}\right]$

the terms approach 1.

in this case, it can be said that the sequence Converges to the specific number.

Definition

If an converges L, we write

$\lim\limits_{n\to\infty} a_n = L$, or simply $a_n \longrightarrow L$

and call L the unit of the sequence

on the other hand, if the sequence never converging to simple value or whose terms get larger than any number as n increases, then it can be said that the sequence Diverges to infinity or negative infinity.

For example,

(1) $\{1, -1, 1, -1, 1, -1 \text{ -----}, (-1)^{n+1}\}$

(2) $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4} \text{ ---}, \sqrt{n}\}$

Calculating Limits of Sequences

Theorm 1:

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B real numbers. The following rules hold if

$$\lim_{n\to\infty} a_n = A \quad \text{and} \quad \lim_{n\to\infty} b_n = B$$

1- Sum Rule: $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$

2- Difference Rule: $\lim\limits_{n\to\infty} (a_n - b_n) = A - B$

3- Constant multiple Rul: $\lim\limits_{n\to\infty} (K b_n) = K \cdot B$

4- product Rule: $\lim\limits_{n\to\infty} (a_n \cdot b_n) = A \cdot B$

5- Quotient Rule: $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$

Nots:

$$\frac{\infty}{\text{كمية}} = \infty, \quad \frac{\text{كمية}}{\infty} = 0, \quad \infty + \infty = \infty, \quad \infty \mp \text{كمية} = \infty$$

Examples:

a. $\lim_{n\to\infty} \left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty} \left(\frac{1}{n}\right) = -1 \cdot \frac{1}{\infty} = -1 \cdot 0 = 0$

(constant multiple Rule)

b- $\lim_{n\to\infty} \left(\frac{n-1}{n}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)$

$= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1$

(Difference Rule)

c- $\lim_{n\to\infty} \frac{5}{n^2} = 5 \lim_{n\to\infty} \frac{1}{n^2} = 5 \cdot 0 = 0$

(constant multiple Rule)

d- $\lim_{n\to\infty} \frac{4 - 7n^6}{n^6 + 3} = \lim_{n\to\infty} \frac{\left(\frac{4}{n^6} - 7\right)}{\left(1 + \frac{3}{n^6}\right)}$

$= \frac{0 - 7}{1 + 0} = -7$

(Sum + quotient Rule)

Theorm 2:

The following six sequences converge to limits listed blow:

1- $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

2- $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3- $\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \quad (x > 0)$

4- $\lim_{n\to\infty} x^{n} = 0 \quad (|x| < 1)$

$= \infty \quad (|x| \geqslant 1)$

5- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{n} = e^{x} \quad (\text{any } x)$

6- $\lim_{n\to\infty} \frac{x^{n}}{n!} = 0 \quad (\text{any } x)$

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim \frac{f'(n)}{g'(n)}$ (L'-Hopitals Rule)

Examples:

These are examples of the limits:

a. $\left[\frac{\ln(n^2)}{n}\right]$

$$\lim_{n\to\infty}\frac{\ln(n^2)}{n}=\lim_{n\to\infty}\frac{2\ln(n)}{n}=2\cdot\lim_{n\to\infty}\frac{\ln(n)}{n}$$

$= 2\cdot 0 = 0$ formula 1 (converges)

b. $\left[\sqrt[n]{n^2}\right]$

$$\lim_{n\to\infty}\left[\sqrt[n]{n^2}\right]=\lim_{n\to\infty}(n^2)^{\frac{1}{n}}=\lim_{n\to\infty}(n)^{2\cdot\frac{1}{n}}$$

$$=\lim_{n\to\infty}\left(n^{\frac{1}{n}}\right)^2=\lim\left(\sqrt[n]{n}\right)^2=(1)^2$$

formula 2 (converges)

c. $\left[\sqrt[n]{3n}\right]$

$$\lim_{n\to\infty}\left(\sqrt[n]{3n}\right)=\lim_{n\to\infty}3^{\frac{1}{n}}\cdot(n)^{\frac{1}{n}}=1\cdot 1=1$$

formula 3 with $x=3$ and formula 2 (converges) d- $\left[\left(-\frac{1}{2}\right)^n \right]$

$$\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$$

formula 4 with $|x|<1$
(converges)

e- $\left[2^n \right]$

$$\lim_{n\to\infty} (2)^n = \infty$$

formula 4 with $|x|\geq 1$
(diverges)

Note:

اذا كانت a_n كسرية

اذا كانت درجة البسط اقل من درجة المقام	اذا كانت درجة البسط اكبر من درجة المقام	اذا كانت درجة البسط = درجة المقام
$\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n = \infty$	$\lim_{n\to\infty} a_n = \dfrac{\text{معامل اكبر اس بالبسط}}{\text{معامل اكبر اس بالمقام}}$

حيث يتم ايجاد قيمة $\lim_{n\to\infty} a_n$ اما بطريقة القسمة على اكبر اس او بقاعدة (L'-Hopital)

Examples :

a. $\left[\dfrac{3n^2-5n}{5n^2+2n-6}\right]$

the order of the numerator is the same as the order of the denominator. the result is $\frac{3}{5}$. (معامل اكبر اس بالبسط/معامل اكبر اس بالمقام)

$$\lim_{n\to\infty}\frac{3n^2-5n}{5n^2+2n-6} = \lim \frac{3-\frac{5}{n}}{5+\frac{2}{n}} = \frac{3-0}{5+0-0}$$

$$= \frac{3}{5} \qquad \text{(converges)}$$

b. $\left[\left(\dfrac{2n+3}{3n-7}\right)^4\right]$

$$\lim_{n\to\infty}\left(\frac{2n+3}{3n-7}\right)^4 = \lim_{n\to\infty}\left(\frac{2+\frac{3}{n}}{3-\frac{7}{n}}\right)^4$$

$$= \left(\frac{2+0}{3-0}\right)^4 = \left(\frac{2}{3}\right)^4 \qquad \text{(converges)}$$

c. $\left[\dfrac{n^3+1}{n^2+3}\right]$

The order of the numerator > the order of the denominator
$\therefore$ the result is ∞

$$\lim_{n\to\infty}\left(\frac{n^3+1}{n^2+3}\right) = \lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{f'(n)}{g'(n)}$$

$$= \lim_{n\to\infty} \frac{3n^2}{2n} = \lim_{n\to\infty} \frac{6n}{2} = \infty \text{ (diverges)}$$

d- $\left[\frac{n}{n^2+3}\right]$

the order of numerator < the order of denominator.
the results is zero.

$$\lim_{n\to\infty} \left(\frac{n}{n^2+3}\right) = \lim_{n\to\infty} \frac{1}{2n+0} = \frac{1}{\infty} = 0$$

(diverges)

e- $\left[\frac{\ln(n)}{n}\right]$

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 \quad \text{(converges)}$$