## *2.2.2. HEAT CONDUCTION IN CYLINDERS AND SPHERES*

Consider steady heat conduction through a hot-water pipe. Heat is continuously lost to the outdoors through the wall of the pipe, and we intuitively feel that heat transfer through the pipe is in the normal direction to the pipe surface and no significant heat transfer takes place in the pipe in other directions figure 2–11.

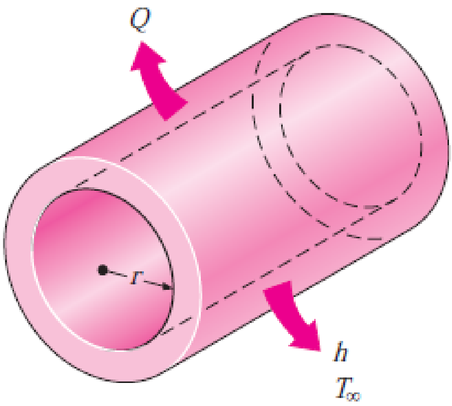



Figure 2-11 Heat is lost from a hot-water pipe to the air outside in the radial direction, and thus heat transfer from a long pipe is one-dimensional

The wall of the pipe, whose thickness is rather small, separates two fluids at different temperatures, and thus the temperature gradient in the radial direction will be relatively large. Further, if the fluid temperatures inside and outside the pipe remain constant, then heat transfer through the pipe is steady. Thus heat transfer through the pipe can be modeled as steady and one-dimensional. The temperature of the pipe in this case will depend on one direction only (the radial r-direction) and can be expressed as T= T(r).

Consider a long cylindrical layer (such as a circular pipe) of inner radius $r_1$, outer radius $r_2$, length L, and average thermal conductivity (k) figure 2–12. The two surfaces of the cylindrical layer are maintained at constant temperatures $T_1$ and $T_2$.

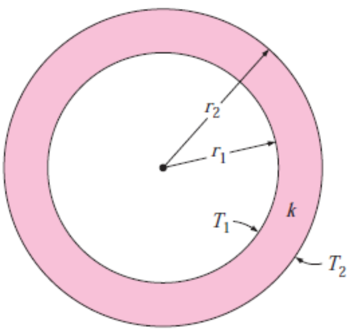



Figure 2-12 A long cylindrical pipe (or spherical shell) with specified inner and outer surface temperatures *T*1 and *T*2.

There is no heat generation in the layer and the thermal conductivity is constant. For one-dimensional heat conduction through the cylindrical layer, we have $T_{(r)}$. Then Fourier's law of heat conduction for heat transfer through the cylindrical layer can be expressed as

$$Q_{cond,cyl} = -kA\frac{dT}{dr} \quad \text{(W)} \qquad 2\text{-}24$$

where A=2$\pi$rL is the heat transfer area at location r. ***Note*** that A depends on r, and thus it varies in the direction of heat transfer. Separating the variables in the above equation and integrating from r = $r_1$, where $T_{(r1)}$= $T_1$, to r = $r_2$, where $T_{(r2)}$ = $T_2$, gives

$$\int_{r=r_1}^{r_2} \frac{Q_{cond,cyl}}{A} dr = \int_{T=T_1}^{T_2} kdT \qquad 2\text{-}25$$

Substituting A=2$\pi$rL and performing the integrations give

$$Q_{cond,cyl} = 2\pi L\mathrm{k}\frac{T_1 - T_2}{\ln\left({}^{r_2}/_{r_1}\right)} \quad \text{(W)} \qquad 2\text{-}26$$

since $Q_{cond,cyl}$ = constant. This equation can be rearranged as

$$Q_{cond,cyl} = \frac{T_1 - T_2}{R_{cyl}} \quad \text{(W)} \qquad 2\text{-}27$$

Where

$$R_{cyl} = \frac{\ln\left({}^{r_2}/_{r_1}\right)}{2\pi L\mathrm{k}} = \frac{\ln\left({}^{outer\ radius}/_{inner\ radius}\right)}{2\pi(length)(Thermal\ conductivity} \qquad 2\text{-}28$$

$R_{cyl}$ is the ***thermal resistance*** of the cylindrical layer against heat conduction, or simply the **conduction resistance** of the cylinder layer.

We can repeat the analysis for a ***spherical layer*** by taking A=4$\pi r^2$ and performing the integrations in Eq. 2–25. The result can be expressed as

$$Q_{cond,sph} = \frac{T_1 - T_2}{R_{sph}} \qquad 2\text{-}29$$

Where

$$R_{sph} = \frac{r_2 - r_1}{4\pi r_1 r_2 \mathrm{k}} = \frac{outer\ raduis - inner\ raduis}{4\pi(inner\ raduis)(outer\ raduis)(Thermal\ conductivity} \qquad 2\text{-}30$$

$R_{sph}$ is the ***thermal resistance*** of the spherical layer against heat conduction, or simply the **conduction resistance** of the spherical layer.

Now consider steady one-dimensional heat flow through a cylindrical or spherical layer that is exposed to convection on both sides to fluids at temperatures $T_{\infty 1}$ and $T_{\infty 2}$ with heat transfer coefficients $h_1$ and $h_2$, respectively, as shown in figure 2–13.

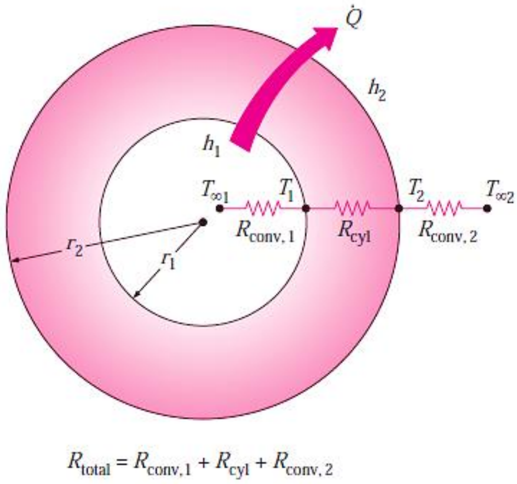



Figure 2-13 The thermal resistance network for a cylindrical (or spherical) shell subjected to convection from both the inner and the outer sides

The thermal resistance network in this case consists of one conduction and two convection resistances in series, just like the one for the plane wall, and the rate of heat transfer under steady conditions can be expressed as

$$Q = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}} \qquad 2\text{-}31$$

Where

$$R_{total} = R_{conv,1} + R_{cyl} + R_{conv,2}$$

$$= \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln\left({}^{r_2}/_{r_1}\right)}{2\pi L\mathrm{k}} + \frac{1}{(2\pi r_2 L)h_2} \qquad 2\text{-}32$$

Eq. (2-32) for a ***cylindrical*** layer, and

$$R_{total} = R_{conv,1} + R_{sph} + R_{conv,2}$$

$$= \frac{1}{(4\pi r_1{}^2)h_1} + \frac{(r_2 - r_1)}{4\pi r_1 r_2 \mathrm{k}} + \frac{1}{(4\pi r_2{}^2)h_2} \qquad 2\text{-}33$$

Eq. (2-33) for a ***spherical*** layer,

***Note*** that ***A*** in the convection resistance relation $R_{conv}$ = 1/*hA* is the ***surface area at which convection occurs***. It is equal to A=2πrL for a cylindrical surface and *A* = $4\pi r_1{}^2$ for a spherical surface of radius *r*. Also note that the thermal resistances are in series, and thus the total thermal resistance is determined by simply adding the individual resistances, just like the electrical resistances connected in series.

## • Multilayered Cylinders and Spheres

Steady heat transfer through multilayered cylindrical or spherical shells can be handled just like multilayered plane walls discussed earlier by simply adding an additional resistance in series for each additional layer. For example, the steady heat transfer rate through the three-layered composite cylinder of length L shown in Fig. 2–14 with convection on both sides can be expressed as

$$Q = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}} \qquad 2\text{-}34$$

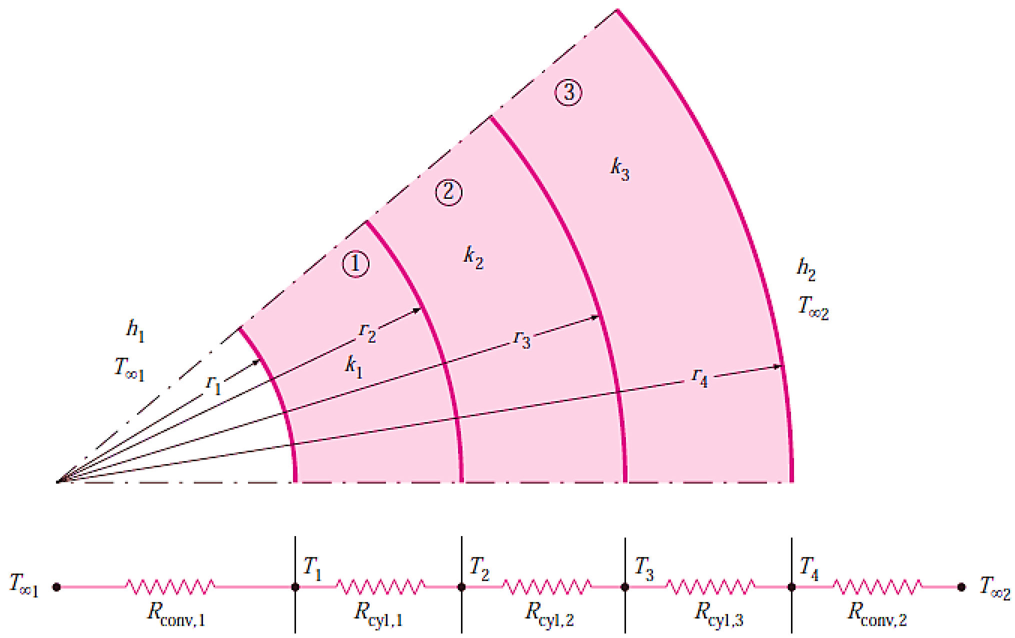



Figure 2-14 The thermal resistance network for heat transfer through a three-layered composite cylinder subjected to convection on both sides

where $R_{total}$ is the ***total thermal resistance***, expressed as

$$R_{total} = R_{conv,1} + R_{cyl,1} + R_{cyl,2} + R_{cyl,3} + R_{conv,2}$$

$$= \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi L k_1} + \frac{\ln(r_3/r_2)}{2\pi L k_2} + \frac{\ln(r_4/r_3)}{2\pi L k_3} + \frac{1}{h_2 A_4} \qquad \text{2-35}$$

where $A_1 = 2\pi rL$ and $A_4 = 4\pi r_1{}^2$

Equation 2–35 can also be used for a three-layered ***spherical shell*** by replacing the thermal resistances of cylindrical layers by the corresponding spherical ones.

**Example 3/**

A 17-m internal diameter spherical tank made of 2-cm-thick stainless steel ($k$ = 15 W/m · °C) is used to store iced water at $T_{\infty 1}$ = 0°C. The tank is located in a room whose temperature is $T_{\infty 2}$ = 22°C. The walls of the room are also at 22°C. The outer surface of the tank is black and heat transfer between the outer surface of the tank and the surroundings is by natural convection and radiation. The convection heat transfer coefficients at the inner and the outer surfaces of the tank are $h_1$ = 80 W/m² · °C and $h_2$ = 10 W/m² · °C, respectively. Determine the rate of heat transfer to the iced water in the tank

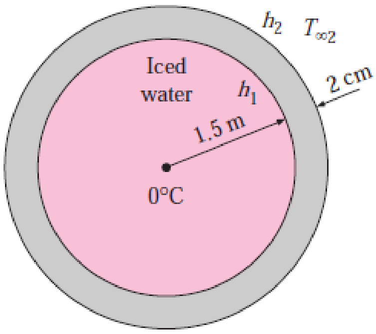
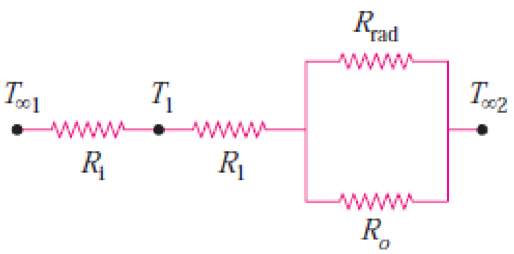



Figure 2-15 Schematic for Example 3

**Solution:**




The thermal resistance network for this problem is given in figure 2–15. Noting that the inner diameter of the tank is $D1$ = 3 m and the outer diameter is $D2$ = 3.04 m, the inner and the outer surface areas of the tank are

$$A_1 = \pi D_1^2 = \pi(3\text{ m})^2 = 28.3\text{ m}^2$$
$$A_2 = \pi D_2^2 = \pi(3.04\text{ m})^2 = 29.0\text{ m}^2$$

Also, the radiation heat transfer coefficient is given by

$$h_{rad} = \varepsilon\sigma(T_2^2 + T_{\infty 2}^2)(T_2 + T_{\infty 2})$$

But we do not know the outer surface temperature $T_2$ of the tank, and thus we cannot calculate $h_{rad}$. Therefore, we need to assume a $T_2$ value now and check the accuracy of this assumption later. We will repeat the calculations if necessary using a revised value for $T_2$.

We note that $T_2$ must be between 0°C and 22°C, but it must be closer to 0°C, since the heat transfer coefficient inside the tank is much larger. Taking $T_2 = 5°\text{C} = 278$ K, the radiation heat transfer coefficient is determined to be

$$h_{rad} = (1)(5.67 \times 10^{-8}\text{ W/m}^2\cdot\text{K}^4)[(295\text{ K})^2 + (278\text{ K})^2][(295 + 278)\text{ K}]$$
$$= 5.34\text{ W/m}^2\cdot\text{K} = 5.34\text{ W/m}^2\cdot°\text{C}$$

Then the individual thermal resistances become

$$R_i = R_{conv,\,1} = \frac{1}{h_1 A_1} = \frac{1}{(80\text{ W/m}^2\cdot°\text{C})(28.3\text{ m}^2)} = 0.000442°\text{C/W}$$
$$R_1 = R_{sphere} = \frac{r_2 - r_1}{4\pi k r_1 r_2} = \frac{(1.52 - 1.50)\text{ m}}{4\pi\ (15\text{ W/m}\cdot°\text{C})(1.52\text{ m})(1.50\text{ m})}$$
$$= 0.000047°\text{C/W}$$
$$R_o = R_{conv,\,2} = \frac{1}{h_2 A_2} = \frac{1}{(10\text{ W/m}^2\cdot°\text{C})(29.0\text{ m}^2)} = 0.00345°\text{C/W}$$
$$R_{rad} = \frac{1}{h_{rad} A_2} = \frac{1}{(5.34\text{ W/m}^2\cdot°\text{C})(29.0\text{ m}^2)} = 0.00646°\text{C/W}$$

The two parallel resistances $R_o$ and $R_{rad}$ can be replaced by an equivalent resistance $R_{equiv}$ determined from

$$\frac{1}{R_{equiv}} = \frac{1}{R_o} + \frac{1}{R_{rad}} = \frac{1}{0.00345} + \frac{1}{0.00646} = 444.7\text{ W/°C}$$

which gives

$$R_{equiv} = 0.00225°\text{C/W}$$

Now all the resistances are in series, and the total resistance is determined to be

$$R_{total} = R_i + R_1 + R_{equiv} = 0.000442 + 0.000047 + 0.00225 = 0.00274°\text{C/W}$$

Then the steady rate of heat transfer to the iced water becomes

$$Q = \frac{T_{\infty 2} - T_{\infty 1}}{R_{total}} = \frac{(22 - 0)°\text{C}}{0.00274°\text{C/W}} = \mathbf{8029\ W} \qquad (\text{or } Q = 8.027\text{ kJ/s})$$

To check the validity of our original assumption, we now determine the outer surface temperature from

$$Q = \frac{T_{\infty 2} - T_2}{R_{equiv}} \longrightarrow T_2 = T_{\infty 2} - QR_{equiv}$$
$$= 22°\text{C} - (8029\text{ W})(0.00225°\text{C/W}) = 4°\text{C}$$

## 2.2.3. CRITICAL RADIUS OF INSULATION

We know that adding more insulation to a wall or to the attic always decreases heat transfer. The thicker the insulation, the lower the heat transfer rate. This is expected, since the heat transfer area A is constant, and adding insulation always increases the thermal resistance of the wall without increasing the convection resistance.

Adding insulation to a cylindrical pipe or a spherical shell, however, is a different matter. The additional insulation increases the conduction resistance of the insulation layer but decreases the convection resistance of the surface because of the increase in the outer surface area for convection. The heat transfer from the pipe may increase or decrease, depending on which effect dominates.

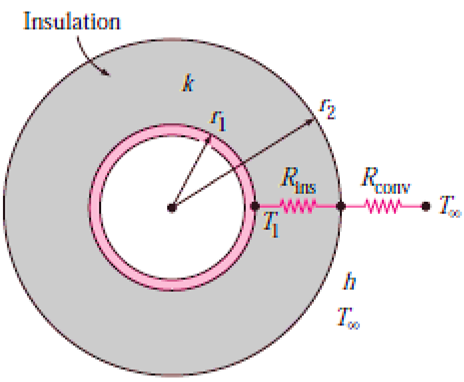



Figure 2-16 An insulated cylindrical pipe exposed to convection from the outer surface and the thermal resistance network associated with it.

Consider a cylindrical pipe of outer radius $\boldsymbol{r_1}$ whose outer surface temperature $\boldsymbol{T_1}$ is maintained constant figure 2–16. The pipe is now insulated with a material whose thermal conductivity is $\boldsymbol{k}$ and outer radius is $\boldsymbol{r_2}$. Heat is lost from the pipe to the surrounding medium at temperature $\boldsymbol{T}_\infty$, with a convection heat transfer coefficient *h*. The rate of heat transfer from the insulated pipe to the surrounding air can be expressed as figure 2–17.

$$Q = \frac{T_1 - T_\infty}{R_{ins} + R_{conv}} = \frac{T_1 - T_\infty}{\frac{\ln\left({r_2}/{r_1}\right)}{2\pi L k} + \frac{1}{h(2\pi r_2 L)}} \quad \text{(W)} \qquad 2\text{-}36$$

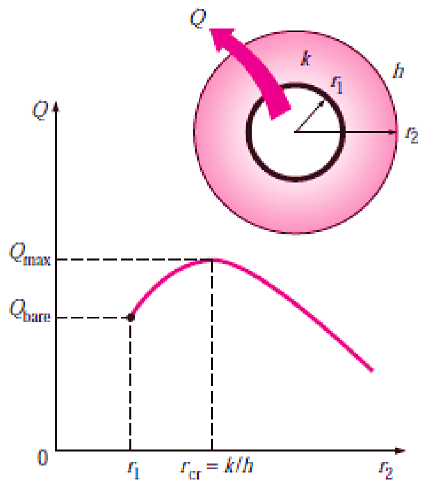



Figure 2-17

The variation of with the outer radius of the insulation $\boldsymbol{r_2}$ is plotted in Fig. 2–17. The value of $\boldsymbol{r_2}$ at which reaches a maximum is determined from the requirement that *d*/*dr*2=0 (zero slope). Performing the differentiation and solving for $\boldsymbol{r_2}$ yields the **critical radius of insulation** for a cylindrical body to be

$$r_{cr,clinder} = \frac{k}{h} \quad \text{(m)} \qquad 2\text{-}37$$

Note that the critical radius of insulation depends on the thermal conductivity of the insulation ***k*** and the external convection heat transfer coefficient ***h***.

The discussions above can be repeated for a sphere, and it can be shown in a similar manner that the ***critical radius of insulation*** for a ***spherical shell*** is

$$r_{cr,clinder} = \frac{2k}{h} \quad \text{(m)} \qquad 2\text{-}38$$

where *k* is the thermal conductivity of the insulation and *h* is the convection heat transfer coefficient on the outer surface.

**Example 4/**

A 17-mm-diameter and 5-m-long electric wire is tightly wrapped with a 2-mm-thick plastic cover whose thermal conductivity is $k = 0.15$ W/m · °C. Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at $T_\infty = 30$°C with a heat transfer coefficient of $h = 12$ W/m² · °C, determine the temperature at the interface of the wire and the plastic cover in steady operation. Also determine whether doubling the thickness of the plastic cover will increase or decrease this interface temperature.

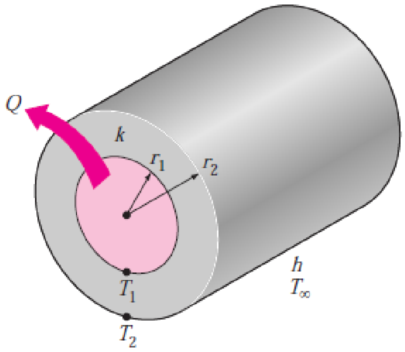
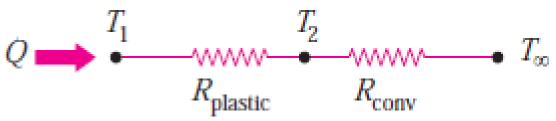



Figure 2-18 Schematic for Example 4

## Solution:




$$\dot{Q} = \dot{W}_e = VI = (8\ \text{V})(10\ \text{A}) = 80\ \text{W}$$

$$A_2 = (2\pi r_2)L = 2\pi(0.0035\ \text{m})(5\ \text{m}) = 0.110\ \text{m}^2$$

$$R_{\text{conv}} = \frac{1}{hA_2} = \frac{1}{(12\ \text{W/m}^2\cdot{}^\circ\text{C})(0.110\ \text{m}^2)} = 0.76^\circ\text{C/W}$$

$$R_{\text{plastic}} = \frac{\ln(r_2/r_1)}{2\pi kL} = \frac{\ln(3.5/1.5)}{2\pi(0.15\ \text{W/m}\cdot{}^\circ\text{C})(5\ \text{m})} = 0.18^\circ\text{C/W}$$

and therefore

$$R_{\text{total}} = R_{\text{plastic}} + R_{\text{conv}} = 0.76 + 0.18 = 0.94^\circ\text{C/W}$$

Then the interface temperature can be determined from

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{\text{total}}} \longrightarrow T_1 = T_\infty + \dot{Q}R_{\text{total}}$$

$$= 30^\circ\text{C} + (80\ \text{W})(0.94^\circ\text{C/W}) = 105^\circ\text{C}$$

$$r_{cr} = \frac{k}{h} = \frac{0.15\ \text{W/m}\cdot{}^\circ\text{C}}{12\ \text{W/m}^2\cdot{}^\circ\text{C}} = 0.0125\ \text{m} = 12.5\ \text{mm}$$