Vibration

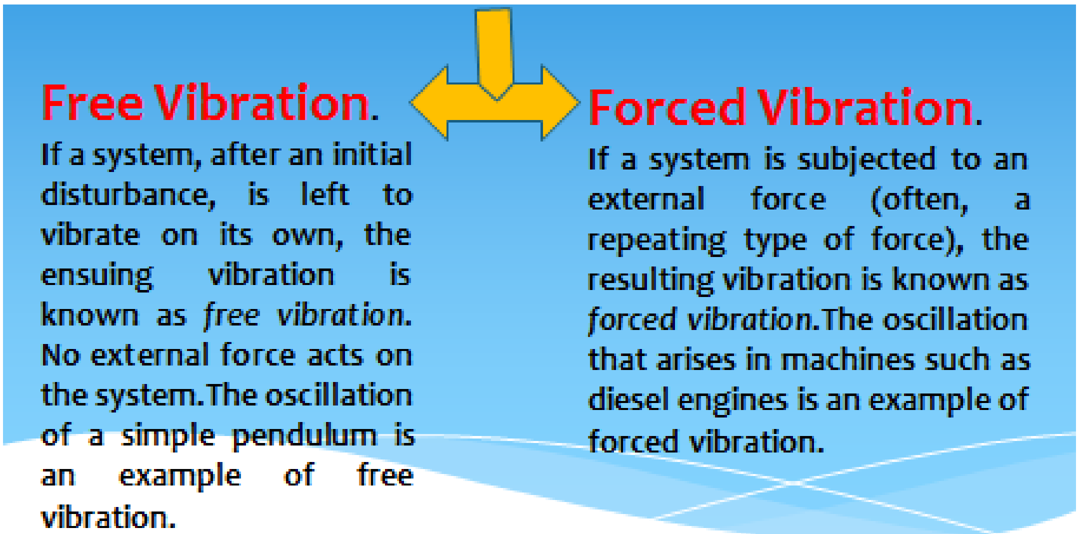
Any motion that repeats itself after an interval of time is called *vibration* or *oscillation*. The swinging of a pendulum and the motion of a plucked string are typical examples of vibration. The theory of vibration deals with the study of oscillatory motions of bodies and the forces associated with them.

A vibratory system, in general, includes a means for storing potential energy (spring or elasticity), a means for storing kinetic energy (mass or inertia), and a means by which energy is gradually lost (damper).

The vibration of a system involves the transfer of its potential energy to kinetic energy and of kinetic energy to potential energy, alternately. If the system is damped, some energy is dissipated in each cycle of vibration and must be replaced by an external source if a state of steady vibration is to be maintained.

Classification of Vibration

Vibration can be classified in several ways. Some of the important classifications are as follow:



Failures by Vibration

If the frequency of the external force coincides with one of the natural frequencies of the system, a condition known as *resonance* occurs, and the system undergoes dangerously large oscillations. Failures of such structures as buildings, bridges, turbines, and airplane wings have been associated with the occurrence of resonance.

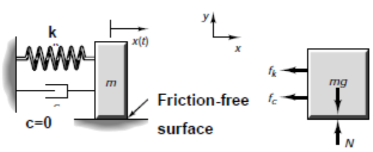
Case study

In April 1831, a brigade of soldiers marched in step across England's Broughton Suspension Bridge. According to accounts of the time, the bridge broke apart beneath the soldiers, throwing dozens of men into the water.

After this happened, the British Army reportedly sent new orders: Soldiers crossing a long bridge must "break stride," or not march in unison, to stop such a situation from occurring again.

Structures like bridges and buildings, although they appear to be solid and immovable, have a natural frequency of vibration within them. A force that's applied to an object at the same frequency as the object's natural frequency will amplify the vibration of the object in an occurrence called mechanical resonance.

Free-body diagram and equations of motion



- Newton's Law:

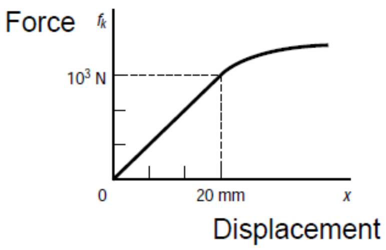
$$m\ddot{x}(t) = -kx(t)$$

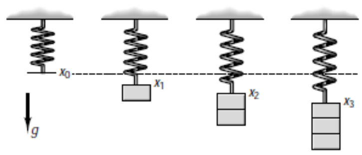
$$m\ddot{x}(t) + kx(t) = 0$$

$$x(0) = x_0, \dot{x}(0) = v_0$$

Stiffness

- From strength of materials (Solid Mech) recall:





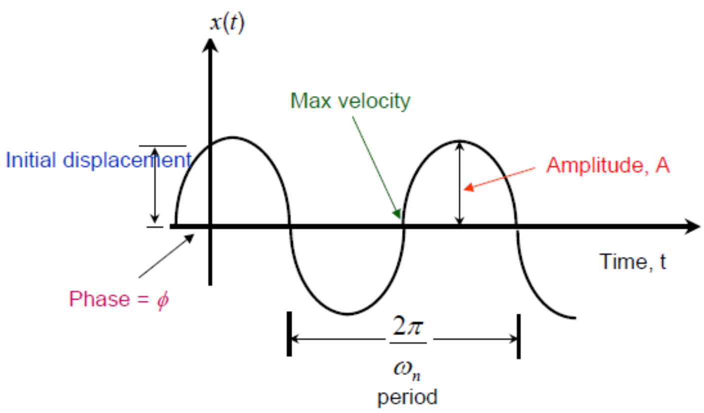
2nd Order Ordinary Differential Equation with Constant Coefficients

Divide by $m : \ddot{x}(t) + \omega_n^2 x(t) = 0$

$\omega_n = \sqrt{\dfrac{k}{m}}$: natural frequency in rad/s

$x(t) = A\sin(\omega_n t + \phi)$

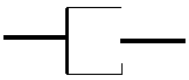
Periodic Motion



Example For m= 300 kg and ω_n =10 rad/s compute the stiffness:

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_n^2$$
$$= (300)10^2 \text{ kg/s}^2$$
$$= 3 \times 10^4 \text{ N/m}$$

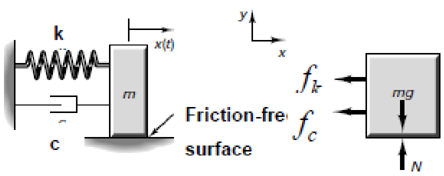
Linear Viscous Damping



$$f_c = c\dot{x}(t)$$

- A mathematical form
- Called a dashpot or viscous damper
- Somewhat like a shock absorber
- The constant c has units: Ns/m or kg/s

Spring-mass-damper systems



- From Newton's law:

$$m\ddot{x}(t) = -f_c - f_k$$
$$= -c\dot{x}(t) - kx(t)$$
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0$$
$$x(0) = x_0, \ \dot{x}(0) = v_0$$

Divide equation of motion by m

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = 0$$

where $\omega_n = \sqrt{k/m}$ and

$$\zeta = \frac{c}{2\sqrt{km}} = \text{damping ratio (dimensionless)}$$

$$\zeta = \frac{c}{c_{cr}};$$

Fundamentals of Linear Vibrations

1. Single Degree-of-Freedom Systems
2. Two Degree-of-Freedom Systems
3. Multi-DOF Systems
4. Continuous

Single Degree-of-Freedom Systems

A spring-mass system

1- Newton law

- General solution for any simple oscillator

2. Equivalent springs

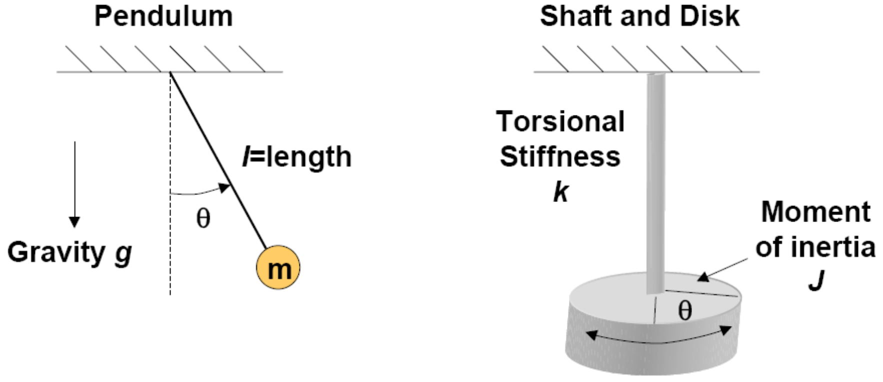
- Spring in series and in parallel

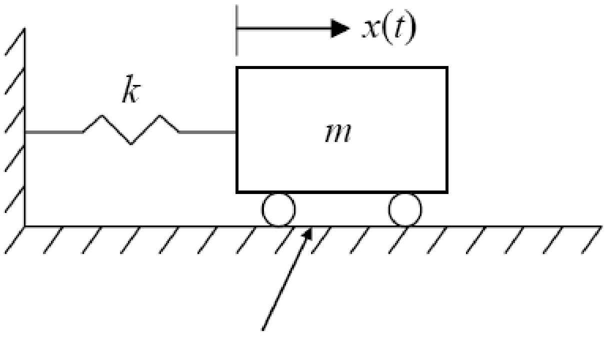
3. Energy Methods

- Strain energy & kinetic energy (Work-energy statement)

Undamped Free Vibrations of Single Degree of Freedom Systems :

Examples of Single-Degree-of-Freedom Systems





Equation of Motion: $m\ddot{x} + kx = 0$

Or, in another form: $\ddot{x} + \omega_n^2 x = 0$

Any simple oscillator

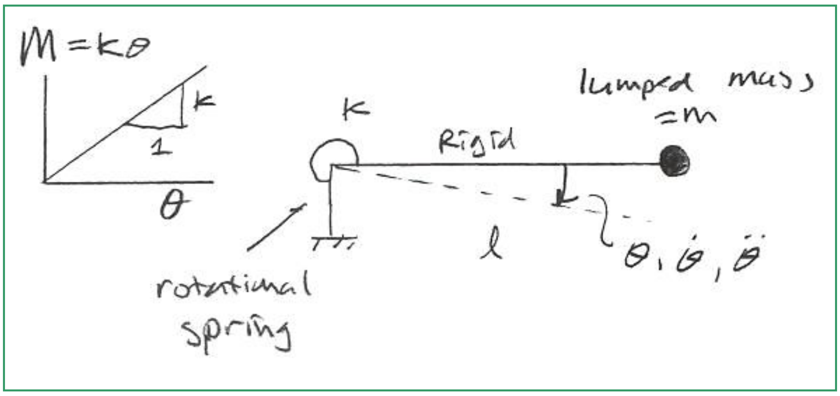
<u>General approach:</u>

1. Select coordinate system
2. Apply small displacement
3. Draw FBD
4. Apply Newton's Laws:

$$\Sigma F = \frac{d}{dt}(m\dot{x})$$

$$\Sigma M = \frac{d}{dt}(I\dot{\theta})$$

Simple oscillator – Example 1



$$I = mass\ moment\ of\ inertia = ml^2$$

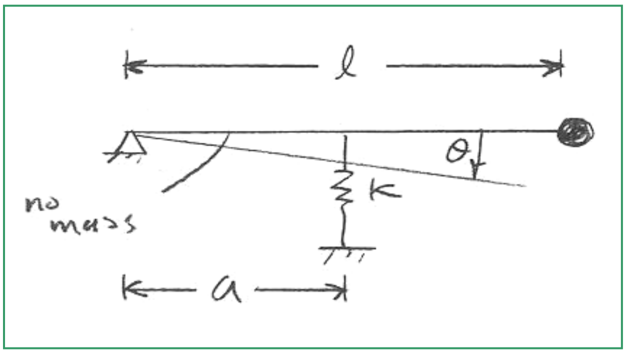
$$\Sigma M = I\ddot{\theta}$$

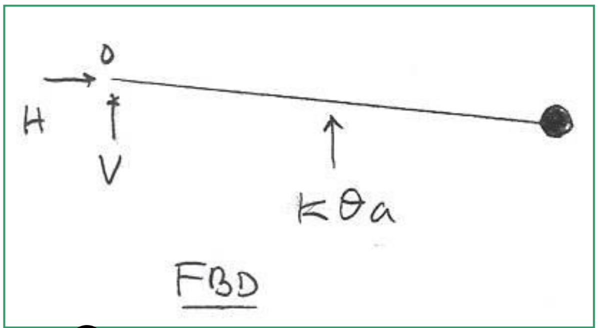
$$-K\theta = I\ddot{\theta}$$

$$ml^2\ddot{\theta} + K\theta = 0$$

$$\omega_n = \sqrt{\frac{K}{ml^2}}$$

Simple oscillator – Example 2





$$\Sigma M_o = I_o\ddot{\theta}$$

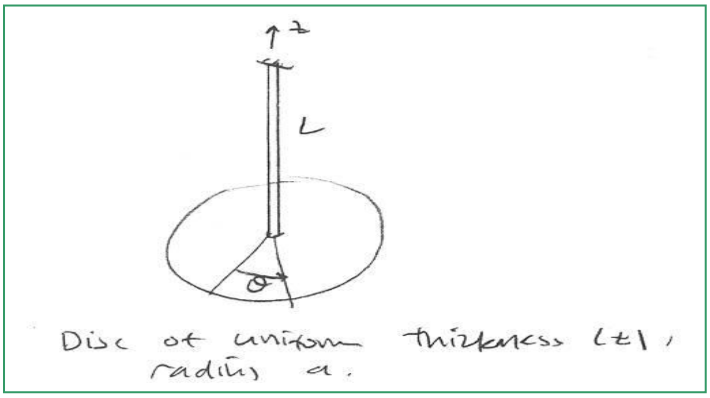
$$-(k\theta a)a = ml^2\ddot{\theta}$$

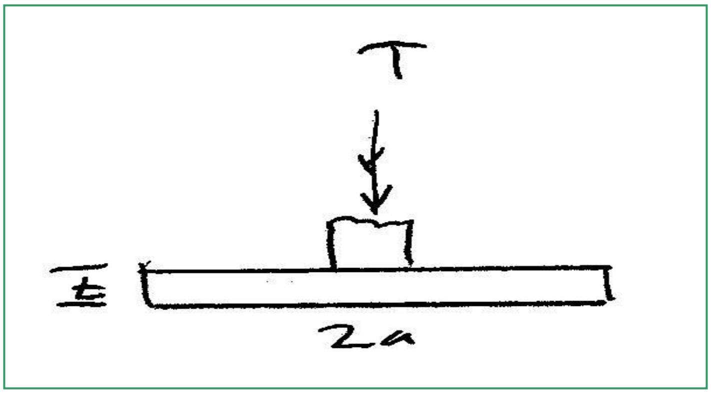
$$ml^2\ddot{\theta} + ka^2\theta = 0$$

$$I = I_{cg} + md^2 = ml^2$$

$$\omega_n = \sqrt{\frac{k}{m}\left(\frac{a}{l}\right)}$$

Simple oscillator – Example 3





$$I = \frac{ma^2}{2}$$

$$\theta = \frac{TL}{JG} \Rightarrow \left(\frac{JG}{L}\right)\theta = T$$

$$Equivalent\ stiffness: K = \frac{JG}{L}$$

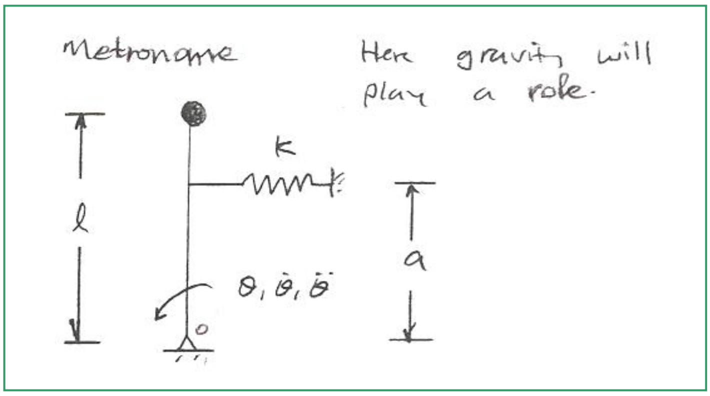
$$\omega_n^2 = \frac{2GJ}{ma^2 L}$$

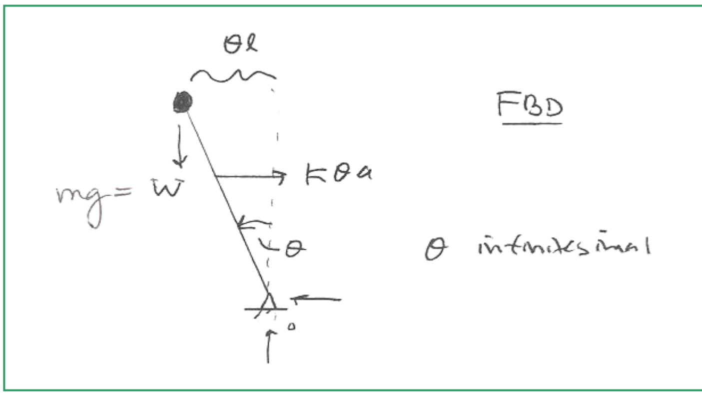
$$\Sigma M_z = I\ddot{\theta}$$

$$-T = I\ddot{\theta}$$

$$\frac{ma^2}{2}\ddot{\theta} + \frac{GJ}{L}\theta = 0$$

Example 4 :





$$\Sigma M_o = I_o \ddot{\theta}$$

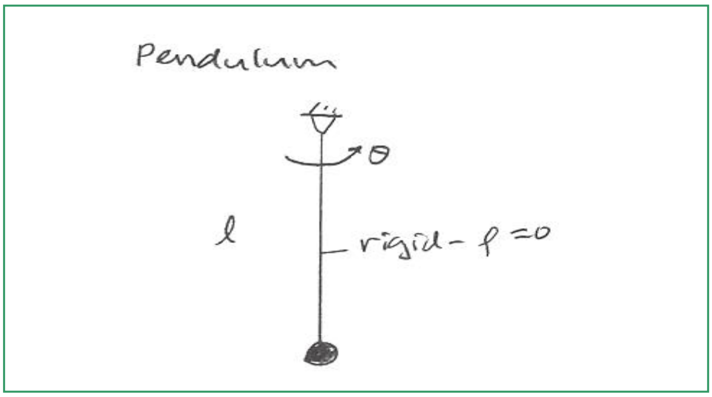
$$-(k\theta a)a + W\theta l = ml^2\ddot{\theta}$$

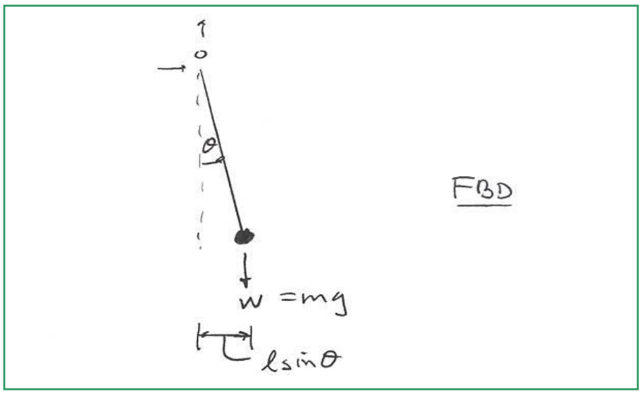
$$ml^2\ddot{\theta} + \theta(ka^2 - Wl) = 0$$

$$\omega_n^2 = \frac{ka^2 - Wl}{ml^2}$$

$$\omega_n = \omega_n(a)$$

Example 5 :





We cannot define ω_n

since we have $sin\theta$ term

If $\theta << 1, sin\theta \approx \theta$:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$\Sigma M_o = I_o \ddot{\theta}$$

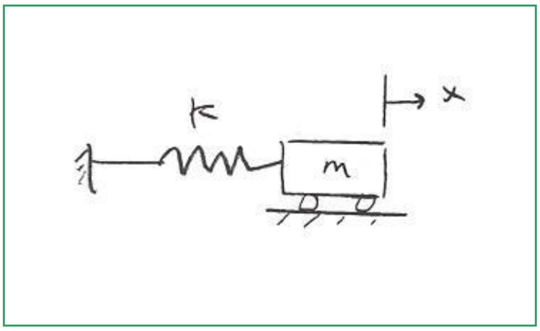
$$-Wl\,sin\theta = ml^2\ddot{\theta}$$

$$ml^2\ddot{\theta} + mgl\,sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}sin\theta = 0$$

Energy methods

Work-Energy principles



$$U = \frac{1}{2}kx^2 \text{ ------ } (strain - energy)$$

$$T = \frac{1}{2}m\dot{x}^2 \text{ ------ } (Kinetic - energy)$$

$$E = U + T = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$$

Work-energy principles have many uses, but one of the most useful is to derive the equations of motion.

Conservation of energy: E = const.

$$\frac{d}{dt}(E) = 0$$

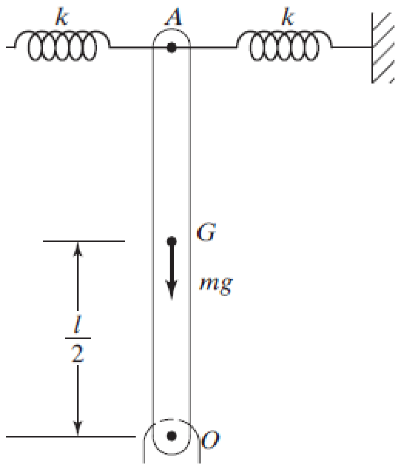
$$kx\dot{x} + m\dot{x}\ddot{x} = 0$$

$$\boxed{m\ddot{x} + kx = 0}$$

Consider a uniform rigid bar, of mass m and length l, pivoted at one end and connected symmetrically by two springs at the other end, as shown in Fig. below. Assuming that the springs are unstretched when the bar is vertical, derive the equation of motion of the system for small angular displacements (θ) of the bar about the pivot point, and investigate the stability behavior of the system.

Sol:

$$\frac{ml^2}{3}\ddot{\theta} + (2kl\sin\theta)l\cos\theta - W\frac{l}{2}\sin\theta = 0$$



For small oscillations, Eq. (E.1) reduces to

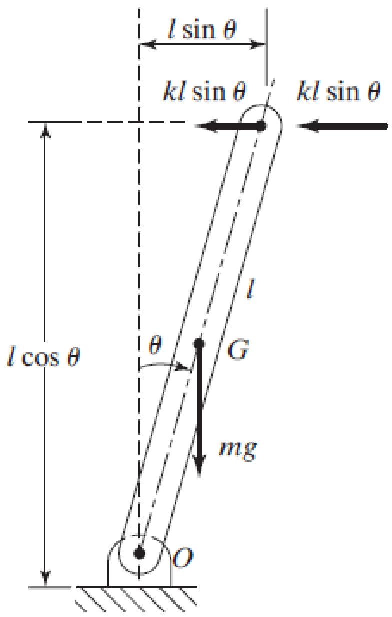
$$\frac{ml^2}{3}\ddot{\theta} + 2kl^2\theta - \frac{Wl}{2}\theta = 0$$

or

$$\ddot{\theta} + \alpha^2\theta = 0$$

where

$$\alpha^2 = \left(\frac{12kl^2 - 3Wl}{2ml^2}\right)$$



$$\omega_n = \left(\frac{12kl^2 - 3Wl}{2ml^2}\right)^{1/2}$$