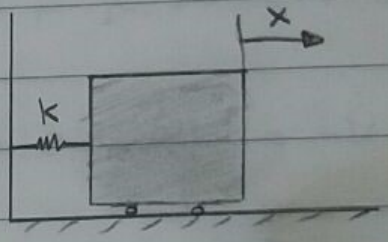
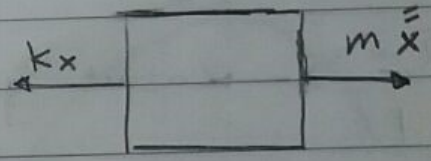
EX.1 Derive the displacement formula and natural frequancy of undamped free Vibration for Single degree of freedom.



Sol

$$-kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$



$$\ddot{x} + \frac{k}{m}x = 0$$

Putting $w^2 = \frac{k}{m}$ then

$$\ddot{x} + w^2 x = 0 \qquad \text{---} \quad (1)$$

assume $x = e^{mt}$

$\ddot{x} = m^2 e^{mt}$ Sub in (1)

$$m^2 e^{mt} + w^2 e^{mt} = 0 \qquad \div e^{mt}$$

$$m^2 + w^2 = 0 \implies m^2 = -w^2$$

$$m = \sqrt{-w^2} \implies m = iw$$

$$\therefore X = e^{(0)t}\left[A\cos wt + B\sin wt\right]$$

$$X = A\cos wt + B\sin wt$$

$$\dot{X} = -AW\sin wt + BW\cos wt$$

at $t = 0$ $\quad x = x_0 \quad + \quad \dot{x} = V_0$

$x_0 = A\cos(0) + B\sin(0) \rightarrow A = x_0$

$\dot{x} = V_0 = -Aw\sin(0) + Bw\cos(0)$

$B = \frac{V_0}{w}$

$\therefore x = x_0\cos wt + \frac{V_0}{w}\sin wt$

$\dot{x} = -x_0 w\sin wt + V_0\cos wt$

w = is natural frequency rad/sec

$w = \sqrt{\frac{k}{m}}$

$T = \frac{2\pi}{w}$ sec/cycle Period time

$f = \frac{1}{T} = \frac{w}{2\pi}$ cycle/sec or hertz

Let $\tan\alpha = \frac{x_0}{\frac{V_0}{w}}$ then

$x = \frac{V_0}{w}\tan\alpha\cos wt + \frac{V_0}{w}\sin wt$

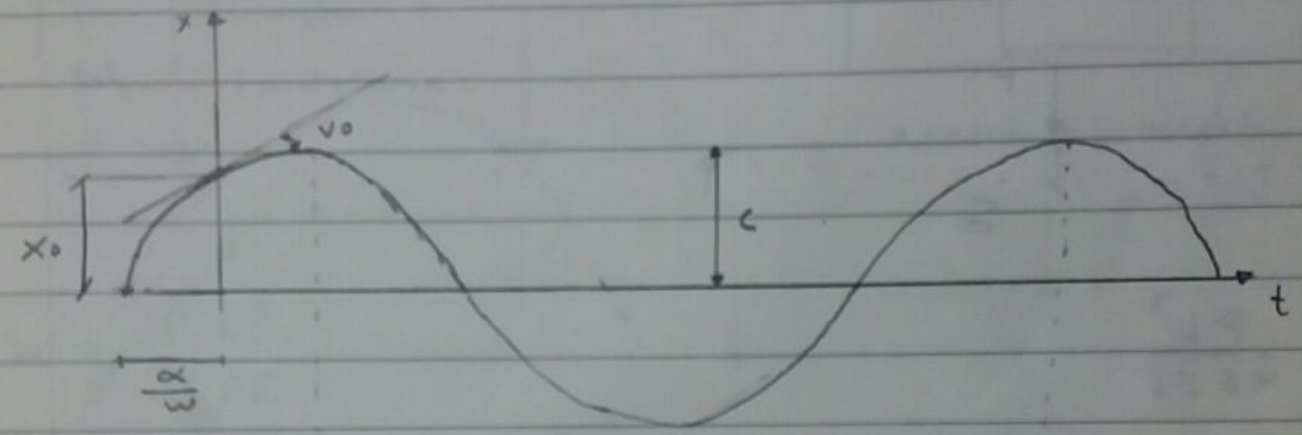
$x = \frac{V_0}{w}\frac{\sin\alpha}{\cos\alpha}\cos wt + \frac{V_0}{w}\sin wt$

$x = \frac{V_0}{w\cos\alpha}\left[\sin\alpha\cos wt + \sin wt\cos\alpha\right]$

$$x = \frac{v_0}{w\cos\alpha}\sin(wt+\alpha)$$

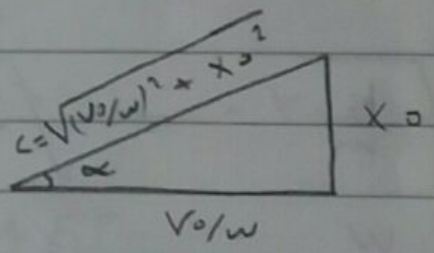
assume $C = \frac{v_0}{w\cos\alpha}$ — 3

$$\therefore \quad x = C\sin(wt+\alpha)$$



$$T = \frac{2\pi}{w} \text{ sec/cycle}$$

$$C = \frac{v_0}{w\cos\alpha}$$

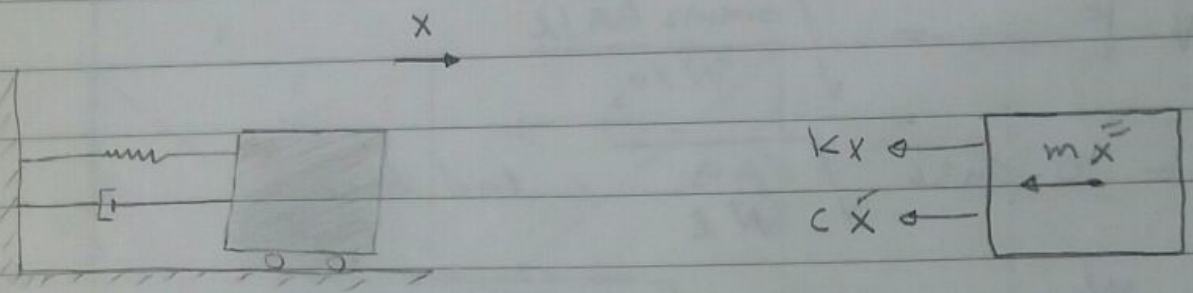


$$C = \frac{v_0}{w} \cdot \frac{v_0/w}{\sqrt{(\frac{v_0}{w})^2 + (x_0)^2}}$$

$$C = \frac{v_0}{w} \times \frac{\sqrt{(\frac{v_0}{w})^2 + (x_0)^2}}{\frac{v_0}{w}}$$

$$\therefore \quad C = \sqrt{(\frac{v_0}{w})^2 + (x_0)^2}$$

Ex.8 Derive the displacement formula and natural frequency of Damped free Vibration for single degree of freedom.



$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \text{---------- (1)}$$

assume $x = Q e^{Pt}$ sub in (1)

$$m \cdot Q \cdot P^2 e^{Pt} + c \cdot Q \cdot P \cdot e^{Pt} + k \cdot Q \cdot e^{Pt} = 0 \quad \div Q e^{Pt}$$

$$mP^2 + cP + k = 0$$

$$P = \frac{-c \mp \sqrt{c^2 - 4mk}}{2m}$$

$$P = \frac{-c}{2m} \mp \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

if $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0$ (critical damping) $c = c_{cr}$

then $P = \frac{-c}{2m}$ $\therefore x = C_1 e^{\frac{-c}{2m}t} + C_2 t e^{\frac{-ct}{2m}}$

at $t = 0$ $x = x_0$ $\therefore x_0 = C_1 + 0$ $\therefore \boxed{C_1 = x_0}$

$$\dot{x} = C_1 \cdot \left(\frac{-c}{2m}\right) e^{\frac{-c}{2m}t} + C_2 \left(t \cdot \left(\frac{-c}{2m}\right) e^{\frac{-c}{2m}t} + e^{\frac{-ct}{2m}}\right)$$

$$\dot{x} = x_0 \left(\frac{-c}{2m}\right) e^{\frac{-c}{2m}t} + C_2 e^{\frac{-c}{2m}t} \left(\frac{-ct}{2m} + 1\right)$$

at $t = 0$ $\dot{x} = v_0$

$$V_0 = x_0\left(\frac{-c}{2m}\right) + c_2 \Longrightarrow \boxed{c_2 = V_0 + x_0\frac{c}{2m}}$$

$$\therefore x = x_0 e^{\frac{-c}{2m}t} + \left(V_0 + x_0\frac{c}{2m}\right)t e^{\frac{-c}{2m}t}$$

$$\boxed{x = \left(V_0 t + x_0\left(1 + \frac{c}{2m}t\right)\right) e^{\frac{-c}{2m}t}}$$

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0 \longrightarrow \frac{c}{2m} = \sqrt{\frac{k}{m}}$$

$$\boxed{c_{cr} = 2\sqrt{km}}$$

If $\left[\left(\frac{c}{2m}\right)^2 - \frac{k}{m}\right]$ is negative or $c < c_r$

$$p_1, p_2 = \frac{-c}{2m} \mp i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

if $\alpha = \frac{-c}{2m}$ and $\bar{w} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

$$\therefore p_1 = \alpha + i\bar{w}, \quad p_2 = \alpha - i\bar{w}$$

$$x = c_1 e^{(\alpha + i\bar{w})t} + c_2 e^{(\alpha - i\bar{w})t}$$

$$x_1 = e^{(\alpha + i\bar{w})t} = e^{\alpha t}(\cos\bar{w}t + i\sin\bar{w}t)$$

$$x_2 = e^{(\alpha - i\bar{w})t} = e^{\alpha t}(\cos\bar{w}t - i\sin\bar{w}t)$$

$$\frac{1}{2}(x_1 + x_2) = e^{\alpha t} \cos \bar{w} t$$

$$\frac{1}{2i}(x_1 - x_2) = e^{\alpha t} \sin \bar{w} t$$

$$\therefore x = e^{\alpha t} \left[A \cos \bar{w} t + B \sin \bar{w} t \right]$$

at $t = 0$ $x = x_0$

$$x_0 = \left\{ A(1) + B(0) \right\} \quad \therefore A = x_0$$

$$\dot{x} = e^{\alpha t} \left[-A\bar{w} \sin \bar{w} t + B\bar{w} \cos \bar{w} t \right] + \alpha e^{\alpha t} \left[A \cos \bar{w} t + B \sin \bar{w} t \right]$$

$$\dot{x} = e^{\alpha t} \left[A(\alpha \cos \bar{w} t - \bar{w} \sin \bar{w} t) + B(\alpha \sin \bar{w} t + \bar{w} \cos \bar{w} t) \right]$$

at $t = 0$ $\dot{x} = v_0$

$$v_0 = \left[A(\alpha) + B(\bar{w}) \right] \Rightarrow v_0 = x_0 \alpha + B\bar{w}$$

$$\therefore B = \frac{v_0}{\bar{w}} - \frac{x_0 \alpha}{\bar{w}}$$

$$\boxed{x = e^{\alpha t} \left[x_0 \cos \bar{w} t + \left(\frac{v_0 - x_0 \alpha}{\bar{w}} \right) \sin \bar{w} t \right]}$$

$$\alpha = \frac{-c}{2m} \times \frac{w}{w} = \frac{-c}{2m} \frac{\sqrt{k/m}}{\sqrt{k/m}} = \frac{-c\sqrt{k/m}}{2\sqrt{km}}$$

$$\alpha = \frac{-cw}{c_{cr}}$$

if $\xi = \frac{c}{c_{cr}}$ $\therefore \boxed{\alpha = -\xi w}$

$$x = e^{-\xi\omega t}\left[x_0 \cos\bar{\omega}t + \left(\frac{v_0 + x_0\xi\omega}{\bar{\omega}}\right)\sin\bar{\omega}t\right]$$

Also $\bar{\omega} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

$$\bar{\omega} = \sqrt{\frac{k}{m}}\sqrt{1 - \frac{c^2}{4km}}$$

$$\bar{\omega} = \sqrt{\frac{k}{m}}\sqrt{1 - \left(\frac{c}{2\sqrt{km}}\right)^2}$$

$$\bar{\omega} = \omega\sqrt{1 - \left(\frac{c}{c_c}\right)^2}$$

$$\therefore \boxed{\bar{\omega} = \omega\sqrt{1 - \xi^2}}$$

if $\tan\beta = \frac{(v_0 + x_0\xi\omega)}{\bar{\omega}\, x_0}$ then

$$x = e^{-\xi\omega t}\left[x_0 \cos\bar{\omega}t + x_0 \tan\beta \sin\bar{\omega}t\right]$$

$$x = e^{-\xi\omega t} x_0\left[\cos\bar{\omega}t + \frac{\sin\beta}{\cos\beta}\sin\bar{\omega}t\right]$$

$$x = \frac{e^{-\xi\omega t} x_0}{\cos\beta}\left[\cos\bar{\omega}t\cos\beta + \sin\bar{\omega}t\sin\beta\right]$$

$$x = \frac{e^{-\xi\omega t}\cdot x_0}{\cos\beta}\cos(\bar{\omega}t - \beta)$$

assume $c = \frac{x_0}{\cos\beta}$

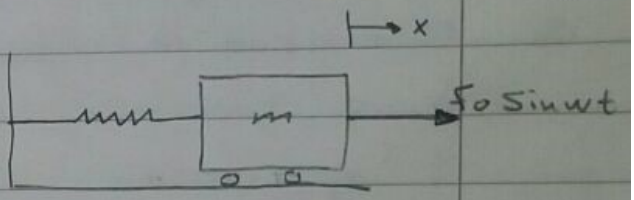
$$\therefore \boxed{x = c\, e^{-\xi\omega t}\cos(\bar{\omega}t - \beta)}$$

$$C = \frac{X_0}{\cos\beta} \Rightarrow C = \frac{X_0}{\sqrt{X_0^2 + \left(\frac{V_0 + X_0 \xi w}{w}\right)^2}}$$

$$\boxed{C = \sqrt{X_0^2 + \frac{(V_0 + X_0 \xi w)^2}{\bar{w}^2}}}$$

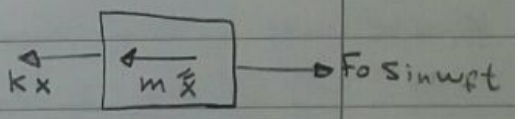
$$T_0 = \frac{2\pi}{\bar{w}}, \quad \boxed{T_0 = \frac{2\pi}{w\sqrt{1-\xi^2}}}$$

Ex.10 Derive the displacement formula for forced vibration (harmonic Loading).



Sol

$$m\ddot{x} + kx = F_o \sin \omega_f t$$



$$x = x_c + x_p$$

$$x_c = A\cos \omega t + B \sin \omega t$$

$$x_p = C \sin \omega_f t$$

$$\dot{x_p} = C \omega_f \cos \omega_f t$$

$$\ddot{x_p} = -C \omega_f^2 \sin \omega_f t$$

$$-mC\omega_f^2 \sin \omega_f t + kC \sin \omega_f t = F_o \sin \omega_f t \quad \div \sin$$

$$C(k - m\omega_f^2) = F_o$$

$$C\left(1 - \frac{m}{k}\omega_f^2\right) = F_o/k$$

$$C\left(1 - \left(\frac{\omega_f}{\omega}\right)^2\right) = F_o/k$$

$$\therefore C = \frac{F_o/k}{\left(1 - \left(\frac{\omega_f}{\omega}\right)^2\right)},$$

if $r = \frac{\omega_f}{\omega}$ $\qquad \boxed{C = \frac{F_o/k}{(1 - r^2)}}$

$$\therefore x_p = \frac{F_o/k}{(1 - r^2)} \sin \omega_f t$$

W.Sayghal

$$X = A\cos\omega t + B\sin\omega t + \frac{F_0/k}{(1-r^2)}\sin\omega_f t$$

at $t = 0 \quad X = 0$

$$0 = A + 0 + 0 \quad \therefore A = 0$$

$$x = B\sin\omega t + \frac{F_0/k}{(1-r^2)}\sin\omega_f t$$

$$\dot{x} = B\omega\cos\omega t + \frac{F_0/k}{(1-r^2)}\omega_f\cos\omega_f t$$

at $t = 0 \quad \dot{x} = 0$

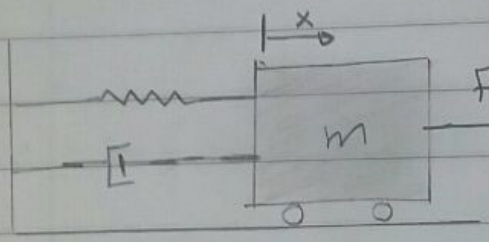
$$0 = B\omega + \frac{F_0/k}{(1-r^2)}\omega_f$$

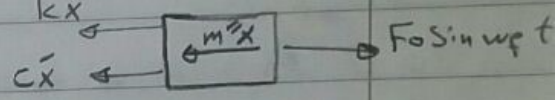
$$B = \frac{-F_0/k}{(1-r^2)}\frac{\omega_f}{\omega} = \frac{-F_0/k}{(1-r^2)}\cdot r$$

$$X = \frac{-F_0/k}{(1-r^2)}\cdot r\sin\omega t + \frac{F_0/k}{(1-r^2)}\sin\omega_f t$$

$$\boxed{X = \frac{F_0/k}{(1-r^2)}\left[\sin\omega_f t - r\sin\omega t\right]}$$

Ex.11 Derive the displacement formula for Damped Harmonic Excitation.





$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_f t$$

This equation is nonhomogenous

$$X = X_c + X_p \qquad \text{assum. } c < c_r$$

From Ex.8 $\quad X_c = e^{-\xi \omega t}(A \cos \bar{\omega} t + B \sin \bar{\omega} t)$

$$X_p = C_1 e^{i\omega_f t}$$

$$m \times C_1 (i)^2 (\omega_f)^2 e^{i\omega_f t} + C \cdot C_1 (i)(\omega_f) e^{i\omega_f t} + k \times C_1 e^{i\omega_f t} = F(t)$$

$$C_1 e^{i\omega_f t}\left[-m\omega_f^2 + ic\omega_f + k\right] = F(t)$$

$$f(t) = F_0 e^{i\omega_f t}$$

$$C_1 e^{i\omega_f t}\left[-m\omega_f^2 + ic\omega_f + k\right] = F_0 e^{i\omega_f t}$$

$$C_1 = \frac{F_0}{(k - m\omega_f^2) + ic\omega_f}$$

$$Xp = \frac{Fo\ e^{iwft}}{(k - mwf^2) + icwf}$$

if $z = x + iy = (k - mwf^2) + i(cwf)$ complex number in cartesian form

in polar form $z = re^{i\theta}$

$$r = \sqrt{x^2 + y^2} = \sqrt{(k - mwf^2)^2 + (cwf)^2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{cwf}{(k - mwf^2)}$$

$$\therefore Xp = \frac{Fo\ e^{iwft}}{\sqrt{(k - mwf^2)^2 + (cwf)^2}\ e^{i\theta}}$$

$$Xp = \frac{Fo\ e^{i(wft - \theta)}}{\sqrt{(k - mwf^2)^2 + (cwf)^2}}$$

$$Xp = \frac{Fo}{\sqrt{(k - mwf^2)^2 + (cwf)^2}} \left(\cos(wf - \theta) + i\sin(wf - \theta)\right)$$

if we retaining the imaginary part, then;

$$Xp = \frac{Fo}{\sqrt{(k - mwf^2)^2 + (cwf)^2}} \sin(wft - \theta) \times \frac{k}{k}$$

$$Xp = \frac{Fo/k\ \sin(wft - \theta)}{\sqrt{\left(1 - \frac{m}{k}wf^2\right)^2 + \left(\frac{c}{k} \times \frac{2\sqrt{m}}{2\sqrt{m}}wf\right)^2}}$$

$$XP = \frac{Fo/k \quad \sin(\omega_f t - \phi)}{\sqrt{\left(1 - \frac{\omega_f^2}{\omega^2}\right)^2 + \left(\frac{2c}{c_{cr}} \frac{\omega_f}{\omega}\right)^2}}$$

if $\xi = \frac{c}{c_{cr}}$ and $r = \frac{\omega_f}{\omega}$

$$\therefore XP = \frac{Fo/k \cdot \sin(\omega_f t - \phi)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\tan\phi = \frac{2\xi r}{1-r^2}$$

$$X = XP + X_c = \frac{Fo/k \sin(\omega_f t - \phi)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} + e^{-\xi\omega t}(A\cos\bar{\omega}t + B\sin\bar{\omega}t)$$

$e^{-\xi\omega t}$ will decrease with time then we can neglect the term $e^{-\xi\omega t}(A\cos\bar{\omega}t + B\sin\bar{\omega}t)$

$$\therefore \boxed{X = \frac{Fo/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega_f t - \phi)}$$

amplitude $= X = \frac{Fo/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$

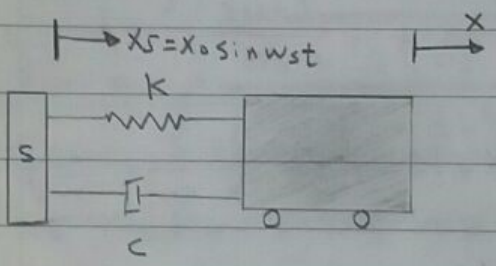
$X_{st} = \frac{Fo}{k}$ static

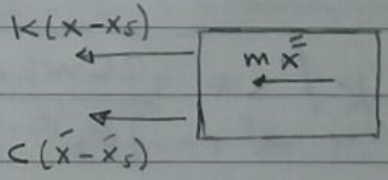
$$\therefore X = X_{st} \times \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$X = X_{st} \times D$

where D = Dynamic magnification factor

Ex.12 Derive the displacement formula for response to support motion.





$$m\ddot{x} + c(\dot{x} - \dot{x}_s) + k(x - x_s) = 0$$

$$m\ddot{x} + c\dot{x} + kx = kx_s + c\dot{x}_s$$

$$m\ddot{x} + c\dot{x} + kx = kx_0 \sin \omega_s t + c\omega_s x_0 \cos \omega_s t$$

$$R.H.S = kx_0 \sin \omega_s t + c\omega_s x_0 \cos \omega_s t$$
$$= x_0 (k \sin \omega_s t + c\omega_s \cos \omega_s t)$$

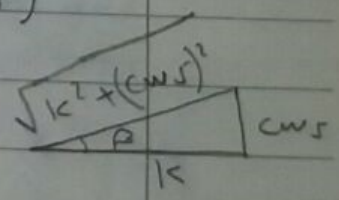
let $\beta = \tan^{-1} \frac{c\omega_s}{k} \rightarrow c\omega_s = k \tan \beta$

$$R.H.S = x_0 (k \sin \omega_s t + k \tan \beta \cos \omega_s t)$$
$$= x_0 k \left(\sin \omega_s t + \frac{\sin \beta}{\cos \beta} \cos \omega_s t \right)$$

$$= \frac{x_0 k}{\cos \beta} (\sin \omega_s t \cos \beta + \cos \omega_s t \sin \beta)$$

$$= \frac{x_0 k}{\cos \beta} \sin (\omega_s t + \beta)$$

$$= x_0 \sqrt{k^2 + (c\omega_s)^2} \sin (\omega_s t + \beta)$$



$$\cos \beta = \frac{k}{\sqrt{k^2 + (c\omega_s)^2}}$$

if $f_0 = x_0 \sqrt{k^2 + (c\omega_s)^2}$, then

R.H.S. $= F_0 \sin(\omega_s t + \beta)$

$$F_0 = X_0 \sqrt{k^2 + (c\omega_s)^2} = X_0 k \sqrt{1 + \left(\frac{c\omega_s}{k}\right)^2}$$

$$= X_0 k \sqrt{1 + \left(\frac{2c\omega_s \sqrt{km}}{2k\sqrt{m}}\right)^2} = X_0 k \sqrt{1 + \left(\frac{2}{c_{cr}} \cdot \frac{c}{c_{cr}} \cdot \frac{\omega_s}{\omega}\right)^2}$$

if $\xi = \frac{c}{c_{cr}}$ and $r = \frac{\omega_s}{\omega}$

$$\therefore \quad F_0 = X_0 k \sqrt{1 + (2\xi r)^2}$$

The diff. equation becomes:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega_s t + \beta)$$

this eq. is similar to eq. of Damped harmonic system

The solution is:

$$X = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega_s t + \beta - \phi)$$

$$\tan\phi = \frac{2\xi r}{1 - r^2}$$

or $$\boxed{X = \frac{X_0 \sqrt{1 + (2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega_s t + \beta - \phi)}$$

$$\frac{\hat{X}}{X_0} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_s t + \beta - \phi)$$

$$\boxed{\frac{X_0}{X_0} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}} = T_r = \text{transmissibility}$$

The response of motion can be put in the following forms:

Let $u = x - x_s$ (Relative motion)

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{x}_s \qquad \text{——— (1)}$$

$$x_s = X_0 \sin \omega_s t \implies \ddot{x}_s = -X_0 \omega_s^2 \sin \omega_s t$$

eq (1) becomes $m\ddot{u} + c\dot{u} + ku = m X_0 \omega_s^2 \sin \omega_s t$

this equation is similar to equation of damped harmonic system, then the solution is:

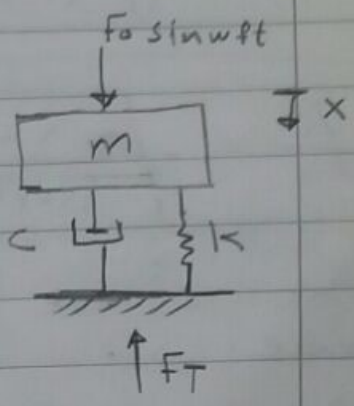
$$u = \frac{m X_0 \omega_s^2 / k \sin(\omega_s t - \phi)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}, \quad \tan\phi = \frac{2\zeta r}{1-r^2}$$

$$\frac{m\omega_s^2}{k} = \frac{\omega_s^2}{\omega^2} = r^2$$

$$\therefore \boxed{u = \frac{X_0 r^2 \sin(\omega_s t - \phi)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}}$$

$$U = \frac{X_0 r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \text{or} \quad \boxed{\frac{U}{X_0} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}}$$

EX.13 Derive the formula for force transmitted to the foundation.



$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega_f t$$

$$X = \frac{F_o / k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \phi)$$

where $\tan\phi = \frac{2\zeta r}{1-r^2}$

or $x = X \sin(\omega_f t - \phi)$

$$F_T = kx + c\dot{x} = kX \sin(\omega_f t - \phi) + c X \omega_f \cos(\omega_f t - \phi)$$

$$F_T = X\left[k \sin(\omega_f t - \phi) + c\omega_f \cos(\omega_f t - \phi)\right]$$

$$F_T = X\sqrt{k^2 + (c\omega_f)^2} \sin(\omega_f t - \phi + \beta)$$

$$F_T = X\sqrt{k^2 + (c\omega_f)^2} \sin(\omega_f t - \phi)$$

where $\phi = \phi - \beta$

where $\tan\phi = \tan(\phi - \beta) = \frac{\tan\phi - \tan\beta}{1 + \tan\phi \tan\beta}$

$$\tan\beta = \frac{c\omega_f}{k} * \frac{2\sqrt{m}}{2\sqrt{m}} = \frac{2c}{2\sqrt{km}} \frac{\omega_f \sqrt{m}}{\sqrt{k}}$$

$$\tan\beta = \frac{2c}{c_{cr}} \frac{\omega_f}{\omega} = 2\zeta r$$

$$\tan\phi = \frac{\frac{2\zeta r}{1-r^2} - 2\zeta r}{1 + \frac{2\zeta r}{1-r^2} * 2\zeta r} = \frac{\frac{2\zeta r - 2\zeta r + 2\zeta r^3}{1-r^2}}{\frac{1-r^2 + 4\zeta^2 r^2}{1-r^2}}$$

$$\tan\phi = \frac{2\xi r^3}{1 - r^2 + 4\xi^2 r^2}$$

$$F_T = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \cdot k\sqrt{1 + (2\xi r)^2} \times \sin(\omega_f t - \phi)$$

$$F_T = \frac{F_0\sqrt{1 + (2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega_f t - \phi)$$

the maximum force (A_T) can be written as:

$$A_T = F_0 \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\frac{A_T}{F_0} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = T_r$$