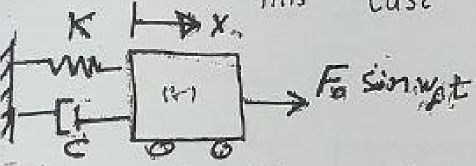
2.5.2 Damped Harmonic Excitation

Let us now consider the motion of a single spring-mass system subjected to a harmonic force, $F(t) = F_0 \cdot \sin \omega_f t$ in presence of viscous damping force. The differential equation of motion in this case will be



$$\boxed{\underbrace{m\ddot{x}}_{\text{inertia force}} + \underbrace{c\dot{x}}_{\text{damper force}} + \underbrace{kx}_{\text{spring force}} = \underbrace{F_0 \cdot \sin \omega_f t}_{\text{external force}}} \qquad (2.40)$$

The solution, will, as in the previous case, consist of a complementary solution x_c and a particular solution x_p. Hence, we may write

$$x = x_c + x_p \qquad \dots (2.41)$$

Assuming $c < c_{cr}$, the complementary solution will be the same as in case of underdamped free vibration and can be written as (See Eq. 2.27)

$$x_c = e^{-\xi \omega t}\left(A \cos \bar{\omega} t + B \sin \bar{\omega} t\right),$$

where $\bar{\omega}$ = damped natural frequency. (2.42)

The particular solution x_p may be obtained by choosing x_p in the form

$$x_p = A \sin \omega_f t + B \cos \omega_f t$$

and substituting this in Eq. (2.40). The process is a bit lengthy.

Here we will follow a more elegant approach using Euler's formula, namely

$$e^{i\omega_f t} = \cos \omega_f t + i \sin \omega_f t \qquad (2.43)$$

Using Euler's formula, we may represent the forcing function as

$$F(t) = F_0 . \, Im\left(e^{i\omega_f t}\right), \qquad \ldots \quad (2.44)$$

where $Im\left(e^{i\omega_f t}\right)$ means that only the imaginary component, i.e., $\sin \omega_f t$ will be considered.
Consequently in the resulting solution which will consist of real and imaginary parts, we will retain only the imaginary part and disregard the real part.

We will assume the particular solution as

$$x_p = C e^{i\omega_f t} \qquad \ldots \quad (2.45)$$

substituting this in Eq. (2.40), we get

$$-m\omega_f^2 C + ic\omega_f . C + k . C = F_0$$

This gives

$$C = \frac{F_0}{(k - m\omega_f^2) + ic\omega_f} \qquad (2.46)$$

Hence

$$x_p = \frac{F_0 . e^{i\omega_f t}}{(k - m\omega_f^2) + ic\omega_f} \qquad \ldots \quad (2.47)$$

Using polar form for complete x
then Eq. (2.47) can

$$x_p = \frac{F_0 e^{i\omega_f t}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}\, e^{i\theta}},$$

or

$$x_p = \frac{F_0 e^{i(\omega_f t - \theta)}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}$$

Using Euler's formula we may write

$$e^{i\omega_f t} \cdot e^{-i\theta} = \left(\cos\omega_f t + i\sin\omega_f t\right)\left(\cos\theta - i\sin\theta\right)$$

$$= \left(\cos\omega_f t\cos\theta + \sin\omega_f t\sin\theta\right) + i\left(\sin\omega_f t\cos\theta - \cos\omega_f t\cdot\sin\theta\right)$$

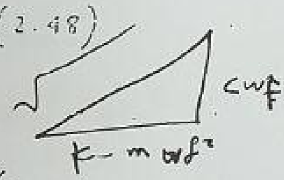
$$= \left[\cos(\omega_f t - \theta)\right] + i\left[\sin(\omega_f t - \theta)\right]$$

Substituting the above expression in Eq. (2.47), we get

$$x_p = \frac{F_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}\left[\cos(\omega_f t - \theta) + i\sin(\omega_f t - \theta)\right]$$

Retaining the imaginary part, the particular solution is finally obtained as

$$x_p = X\sin(\omega_f t - \theta) \quad \ldots\ldots \quad (2.48)$$



am pletude of vibration

where [redacted] (2.49)

and $\tan\theta = \dfrac{c\omega_f}{k - m\omega_f^2}$ (using polar diagram)

(2.50)

It may be readily seen that [redacted]

Using dimensionless ratios [redacted] and [redacted], the Eqns. (2.48) and (2.50) can be put as

$$x_p = \frac{F_o/k \ \sin(\omega_f t - \theta)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \quad \ldots\ldots \quad (2.51)$$

and [redacted] $\ldots\ldots$ (2.52)

The total response is then obtained by summing the complementary (transient response) from Eq. (2.42) and the particular solution (steady-state response) from Eq. (2.51) and may be written as

$$x = x_c + x_p$$

$$X = e^{-\xi\omega t}\left(A\cos\bar{\omega}t + B\sin\bar{\omega}t\right) + \frac{X_{st}\cdot\sin(\omega_f t - \theta)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

(2.53)

where [redacted], the static deflection of the spring due to a force F_o.

X_{st} = static displacement

In Eq. (2.53), the constants of integration A and B are to be determined from the initial conditions using the total response given by Eq. (2.53).

From an examination of Eq. (2.53), it may be seen that the transient component will die down due to presence of the exponential factor $e^{-\xi\omega t}$, leading only the steady states parts

Dynamic Magnification Factor (D):

The ratio of the steady state amplitude X (Eq. 2.49) to the static deflection x_{st} is known as the Dynamic Magnification Factor (D) and is given by

$$D = \frac{X}{x_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \quad \cdots (2.54)$$

It may be seen that the dynamic magnification factor varies with the frequency ratio $r = \frac{\omega_f}{\omega}$ and the damping ratio ξ. Parametric plots of the dynamic magnification factor are shown in Fig. 2.14.

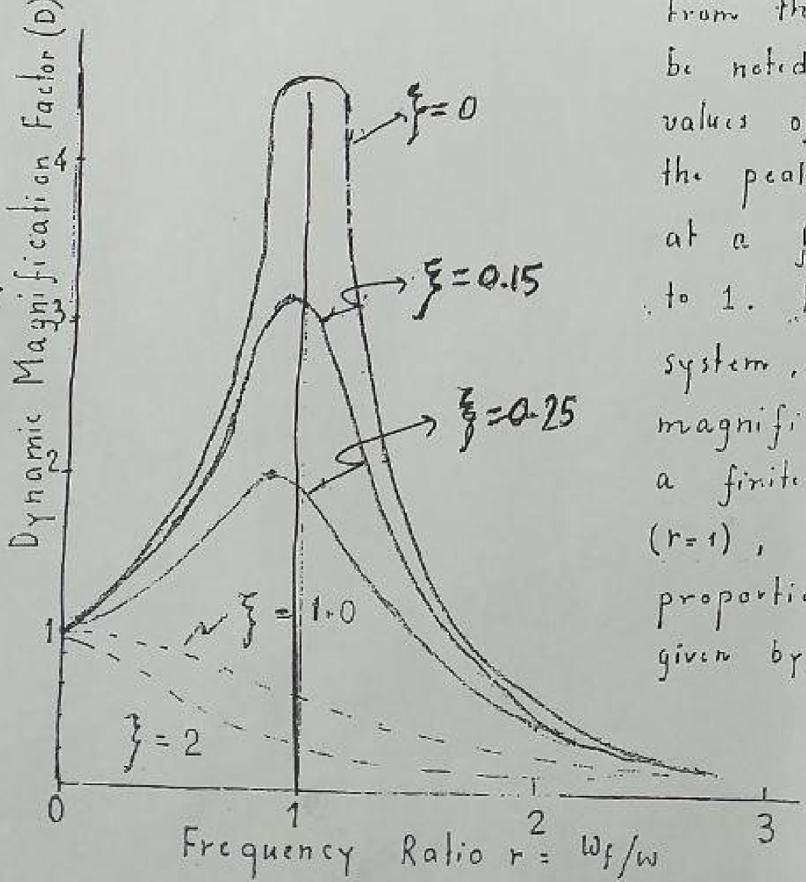


Fig. 2.14. Parametric plots of dynamic magnification factor.

From the plots it may be noted that for small values of damping ratio ξ, the peak amplitude occurs at a frequency ratio close to 1. For a damped system, the dynamic magnification factor has a finite value. At resonance (r = 1), D is inversely proportional to ξ and is given by

$$\boxed{D_{r=1} = \frac{1}{2\xi}} \quad (2.55)$$

2.6 Response to Ground Motion.

Structures subjected to ground motion by earth-quakes or other excitations such as explosions or dynamic action of machines are examples in which support motions have to be considered in the analysis of dynamic response. Let us consider a single spring-mass SDOF system (Fig. 2.15) subjected to a support motion which is harmonic and is given by

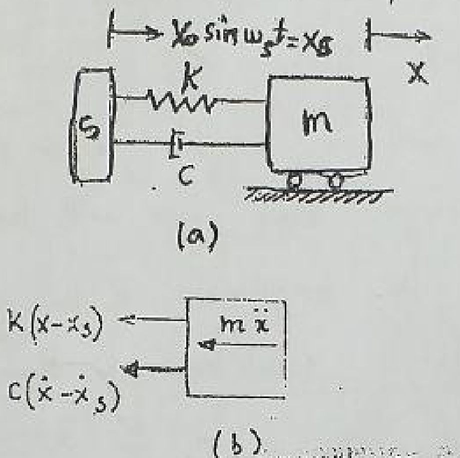


Fig. 2.15

$$x_s = x_0 \sin \omega_s t, \quad \cdots (2.56)$$

where x_0 is the maximum amplitude and ω_s is the frequency of the support motion. Considering the forces acting on the free-body in dynamic equilibrium (Fig. 2.15 b), The differential equation of motion in the horizontal direction can be written as

$$m\ddot{x} + c(\dot{x} - \dot{x}_s) + k(x - x_s) = 0 \quad \cdots (2.57)$$

Substituting expression (2.56) in Eq. (2.57) and re-arranging terms we get

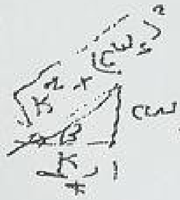
$$m\ddot{x} + c\dot{x} + kx = k x_0 \sin \omega_s t + c\,\omega_s \cdot x_0 \cos \omega_s t \quad (2.58)$$

The two harmonic terms of frequency ω_s in the right hand side of Eq. (2.58) may be combined and the equation may be re-written as

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega_s t + \beta), \quad \cdots (2.59)$$

where $F_0 = x_0 \sqrt{k^2 + (c\omega_s)^2} = x_0 k \sqrt{1 + (2r\xi)^2}$

and $\tan\beta = c\omega_s / k = 2r\xi$



$= 2r$

It may be seen that the Eq. (2.59) is of the same form as Eq. (2.40). Hence, the steady state solution of Eq. (2.59) will be similar to the solution given by Eq. (2.51) except that we will have to add the angle β in the argument of the sine function. The steady state solution can then be written as

$x = X \sin($

$$x = \frac{F_0/k \cdot \sin(\omega_s t + \beta - \theta)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \quad (2.60)$$

Substituting $F_0 = x_0\sqrt{k^2 + (c\omega_s)^2} = x_0 k \sqrt{1+(2r\xi)^2}$, in Eq. (2.60), we obtain

$$\boxed{\frac{X}{x_0} = \frac{\sqrt{1+(2r\xi)^2}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \cdot \sin(\omega_s t + \beta - \theta)} \quad (2.61)$$

Eq. (2.61) is the expression for the relative transmission of the support motion to the system. This is an important problem in vibration isolation in which a delicate equipment is desired to be protected from harmful vibrations of the supporting structure. The ratio of the amplitude of motion X of the system, to the amplitude of motion

$x = X \sin(\quad)$

x_0 of the support, is called 'transmissibility' denoted by Tr. From Eq. (2.61), Tr is obtained as

amp of mass

$$Tr = \frac{X}{x_0} = \frac{\sqrt{1+(2r\xi)^2}}{\sqrt{(1-r^2)^2+(2r\xi)^2}} \quad \dots (2.62)$$

amp of base

X = سعة الكتلة

Xo = سعة القاعدة

الأساس

A plot of transmissibility as a function of the frequency ratio r and damping ratio ξ is shown in Fig. 2.16.

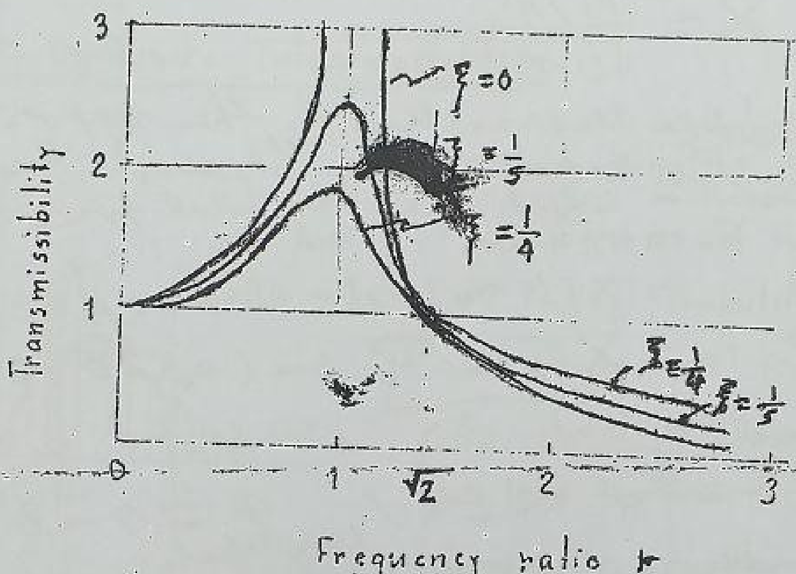


Fig. 2.16. Transmissibility vs Frequency Ratio.

It may be seen that the variation of transmissibility with frequency ratio is similar to variation of dynamic magnification factor with frequency ratio as shown in Fig. 2.14. The only striking difference is that in case transmissibility, all the curves pass through the same point at $r = \sqrt{2}$.

Eq. 2.60 provides the absolute response of the mass m to a harmonic motion of its base.

The relative motion between the mass (m) and support given by $u = x - x_s$

$m\ddot{u} + c\dot{u} + Ku = P_{eff}(t) = -m\ddot{x}_s$

CHAPTER - 3

Response of SDOF System to General Dynamic Loading

Structures are often subjected to dynamic loads which may not be harmonic or periodic. Blast انفجار loading and siesmic loading are common examples of such dynamic loads. In this chapter we shall study the response of a SDOF system to general type of dynamic force.

3.1. Duhamel's Integral Method

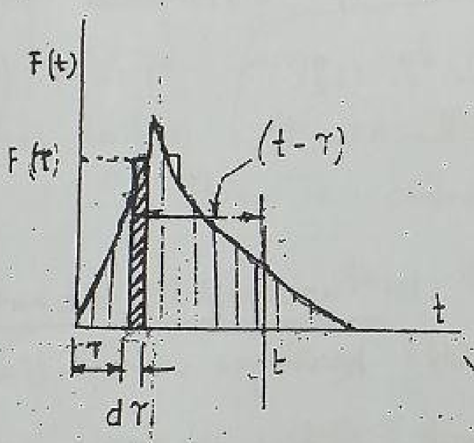


Fig. 3.1.

Duhamel تصور conceived a loading function as a summation of series impulsive loads as shown in Fig. 3.1. An impulsive loading is a load which is applied during a short time. The حاصل ضرب corresponding 'impulse' is دفع نبضي defined as the product of the force and the time of its duration. For example, the impulse of the force $F(\tau)$ at a time τ during a short interval $d\tau$ is represented by the shaded area as shown in Fig. 3.1. and is equal to $F(\tau).d\tau$. The impulse acting on the body produces a change in velocity dv and consequently a change القوة الدافعة of momentum given by $m dv$. From Newton's Law it follows :

$F = ma$

$$m \cdot dv = F(\tau) \cdot d\tau \qquad \left(\text{Since } m \frac{dv}{d\tau} = F(\tau) \right)$$

$$\text{or} \qquad dv = \frac{F(\tau)\, d\tau}{m} \qquad \ldots\ldots (3.1)$$

The impulse may be applied when the body is already in an oscillatory motion due to previous impulses. The additional motion due to the impulse $F(\tau)\, d\tau$ will then be superimposed on the previous motion. Let us, however, determine the motion due to a single impulse separately. The impulse may be اضطراب considered as a نوع / شكل sort of disturbance which imparts a velocity dv given by Eq. (3.1). This will be assumed as the initial condition due to the impulse.

We know that the undamped vibration of a S.D.O.F. system having an initial displacement x_0 and initial velocity v_0 is given by Eq. (2.7a) as

$$\boxed{x = \cancel{x_0 \cos \omega t} + \frac{v_0}{\omega} \sin \omega t.}$$

In case of an impulse we will assume that due to this disturbance, there is an initial velocity dv only. $x_0 = 0$ Then the motion due to a single impulse at a time 't' after τ when the impulse is applied, will be

$$dx(t) = \frac{dv}{\omega} \sin \omega (t - \tau)$$

Using relation (3.1) in above, we get

$$\boxed{dx(t) = \frac{F(\tau)}{m\omega} \cdot \sin\omega\,(t-\tau)\,d\tau} \quad (3.2)$$

This differential motion is the contribution (مساهمة أو الأثر) of a single impulse applied at any time τ. Therefore, the total displacement due to a series of continuous short impulses acting on the body from time $t=0$ to time t, will be given by the summation or integral of the differential motions $dx(t)$ during the interval 0 to t, that is,

$$\boxed{x(t) = \frac{1}{m\omega} \int_0^t F(\tau) \cdot \sin\omega\,(t-\tau)\,d\tau} \quad (3.3)$$

This is called the 'Duhamel Integral', which respresents the total displacement produced by the exciting $F(\tau)$ acting on the system; it includes both the steady-state and the transient components of the motion.

If the function $F(\tau)$ can not be expressed analytically, the integral of Eq. (3.3) can always be evaluated approximately by suitable numerical methods.

If the mass had already a motion due to some initial displacement x_0 and velocity v_0, the total motion will be obtained by superposition. Thus velocity v_0 at $t=0$, the total motion will

$$\boxed{x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \cdot \sin \omega t + \frac{1}{m\omega} \int_0^t F(\tau) \cdot \sin \omega (t-\tau)\, d\tau.} \quad (3.4)$$

<u>Example. 1.</u> : Constant Force

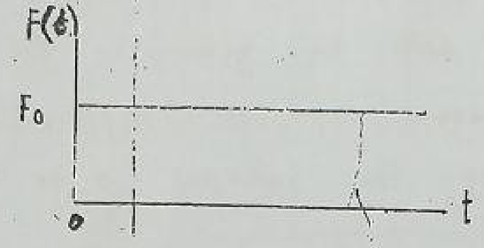


Fig. 3.2

Consider the case of a constant force of magnitude F_0 applied suddenly at $t=0$ and the force continues to act. (Fig. 3.2)

$F(\tau) = F_0$

The displacement at any time 't' is given by Eq. (3.3) as

$\tau = 0 \longrightarrow t$

$$x(t) = \frac{1}{m\omega} \int_0^t F_0 \cdot \sin \omega (t-\tau)\, d\tau .$$

Integrating we get

$\omega^2 = \frac{K}{m}$

$$x(t) = \frac{F_0}{m\omega^2} \Big[\cos \omega (t-\tau) \Big]_0^t$$

$\cos w(0) = 1$

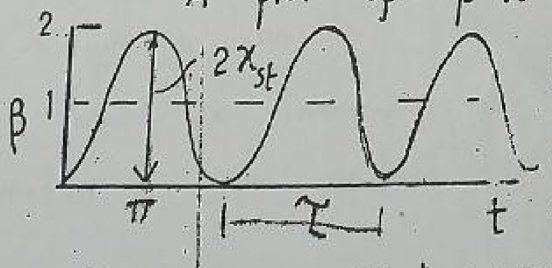
$\boxed{X_{st} = \frac{F_0}{K}}$

$$X_t = \frac{F_0}{K} (1 - \cos \omega t) = x_{st} (1 - \cos \omega t). \quad (3.5)$$

dynamic ~~load factor~~

Then <u>DLF</u> $\beta = \dfrac{x(t)}{x_{st}} = 1 - \cos \omega t .$

A plot of β vs t, is shown below:



It may be seen that the max. displacement $= 2x_{st}$.

This means that the displacement due to a suddenly applied force F_0 is twice the displacement due to a force F_0, applied statically (slowly)

τ = natural period

Example. 2: Rectangular Pulse

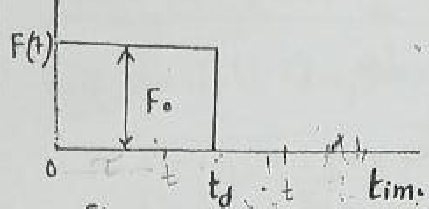


Fig. 3.3.

Let us now consider a constant force suddenly applied and acting during a limited time duration t_d as shown in the figure (3.3)

We have to consider two cases.

Case (a) : $0 < t \leq t_d$

Upto the time t_d, the displacement is given by Eq. (3.5). Thus

$$= x(t) = x_{st}\,(1 - \cos \omega t).$$

Case (b) : $t > t_d$.

At a time t after t_d, the displacement is given by

$$x(t) = \frac{1}{m\omega} \int_0^{t_d} F_0 . \sin \omega (t - \tau)\, d\tau$$

$$= \frac{F_0}{m\omega^2} \Big[\cos \omega (t - \tau) \Big]_0^{t_d}$$

$$x_{st}$$

$$= \frac{F_0}{k} \Big[\cos \omega (t - t_d) - \cos \omega t \Big] \quad \ldots (3.6)$$

Dynamic Load Factor (β) :

The Dynamic Load Factor $\boxed{\beta = \frac{x(t)}{x_{st}}}$

In Example. 2, β is given by

$t = t_d$ عوض For $t < t_d$: $\beta = 1 - \cos \omega t$ (3.7)

$x_{st} = \frac{F_0}{K}$

and for $t > t_d$: $\beta = \cos \omega (t - t_d) - \cos \omega t$.

t_1 – duration of load

It is often convenient to express β in non-dimensional form as below.

Putting $\omega = \frac{2\pi}{T}$ in Eqs. (3.7), we find

for $t < t_d$: $\beta = 1 - \cos 2\pi \left(\frac{t}{T}\right)$;

for $t > t_d$: $\beta = \cos 2\pi \left(\frac{t}{T} - \frac{t_d}{T}\right) - \cos 2\pi \frac{t}{T}$. (3.8).

From Eqns (3.8) it is evident that β depends on $\frac{t_d}{T}$, that is, the ratio of duration of the load to the natural period.

For different values of $\frac{t_d}{T}$, we can find value of maximum β from Eq. (3.8). A plot of β_{max} vs $\frac{t_d}{T}$, is called the response spectrum as shown in Fig. 3.4.

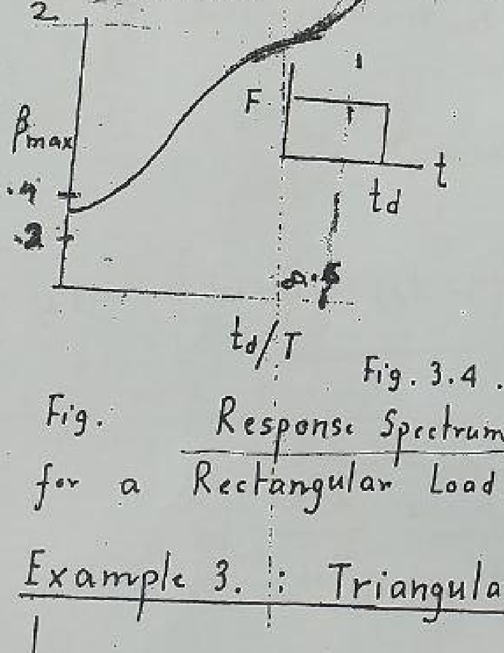


Fig. 3.4.
Fig. Response Spectrum for a Rectangular Load.

Similar curves may be obtained for different types of loading. These are called response spectral charts which are useful for design purposes.

max β for $\frac{t_d}{T} \geq 0.5$ is the same as if the load duration had been infinite.

Example 3. : Triangular Load.

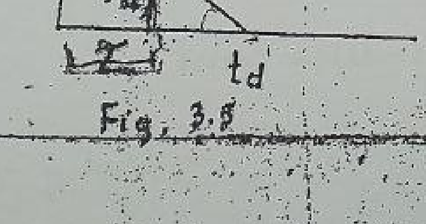


Fig. 3.5

As shown in Fig. 3.5, the load function is given by

$F(\tau) = F_0\left(1 - \frac{\tau}{t_d}\right)$

As before, there will be two cases:

Case (a) : $t < t_d$

The displacement will be given by

$$\boxed{x(t) = \frac{1}{m\omega}\int_0^t F_0\left(1-\frac{\tau}{t_d}\right)\sin\omega(t-\tau)\,d\tau}$$

$$= \frac{F_0}{m\omega}\left[+\left(1-\frac{\tau}{t_d}\right)\cdot\frac{1}{\omega}\cdot\cos\omega(t-\tau)\Big|_0^t + \int_0^t \frac{1}{\omega}\cdot\cos\omega(t-\tau)\cdot\left(-\frac{1}{t_d}\right)d\tau\right]$$

$$= \frac{F_0}{m\omega^2}\left[\left(1-\frac{\tau}{t_d}\right)\cos\omega(t-\tau)\Big|_0^t - \frac{1}{t_d}\cdot\frac{1}{\omega}\cdot\sin\omega(t-\tau)\Big|_0^t\right]$$

$$= \frac{F_0}{K}\left[\left(1-\frac{t}{t_d}\right) - \left\{\cos\omega t - \frac{\sin\omega t}{t_d\cdot\omega}\right\}\right]$$

$$x(t) = \frac{F_0}{K}\left(1-\cos\omega t\right) + \frac{F_0}{K\,t_d}\left(\frac{\sin\omega t}{\omega} - t\right) \qquad (3.9)$$

This gives

$$\beta = \frac{x(t)}{x_{st}} = 1 - \frac{t}{t_d} - \cos\omega t + \frac{1}{\omega\cdot t_d}\cdot\sin\omega t \qquad (3.10)$$

Case (b) : $t \geq t_d$ $\longrightarrow \tau = t_d$

$$x(t) = \frac{1}{m\omega}\int_0^{t_d} F_0\left(1-\frac{\tau}{t_d}\right)\sin\omega(t-\tau)\,d\tau$$

$$= \frac{F_0}{K}\left[\left(1-\frac{\tau}{t_d}\right)\cos\omega(t-\tau) - \frac{1}{t_d\cdot\omega}\cdot\sin\omega(t-\tau)\right]_0^{t_d}$$

$$= \frac{F_0}{K\omega t_d}\left\{\sin\omega t - \sin\omega(t-t_d)\right\} - \frac{F_0}{K}\cdot\cos\omega t$$

This gives

$$\beta = \frac{x(t)}{x_{st}} = \frac{1}{\omega t_d}\left\{\sin\omega t - \sin\omega(t-t_d)\right\} - \cos\omega t \qquad (3.11)$$

See Fig 4.5

Numerical Evaluation of Duhamel's Integral

In many practical problems, as for example, in case of blast or earthquake loading, the load function which is usually is known from experimental data, it is difficult to express the load by analytical function. In such cases, the response is determined by numerical evaluation of Duhamel's integral. We will discuss the numerical method for undamped and damped SDOF systems separately.

I. Undamped System:

As derived earlier, for zero initial conditions, the dynamic displacement is obtained for a general load function $F(\tau)$, using Duhamel's integral as

$$\boxed{x(t) = \frac{1}{m\omega} \int_0^t F(\tau) \cdot \sin\omega(t-\tau)\, d\tau} \qquad (3.11)$$

Using $\sin\omega(t-\tau) = \sin\omega t \cdot \cos\omega\tau - \cos\omega t \cdot \sin\omega\tau$

in Eq. (3.11), we write

$$x(t) = \frac{1}{m\omega}\left[\sin\omega t \int_0^t F(\tau) \cdot \cos\omega\tau\, d\tau - \cos\omega t \int_0^t F(\tau) \cdot \sin\omega\tau\, d\tau\right]$$

or
$$\boxed{x(t) = \frac{1}{m\omega}\left[A(t) \cdot \sin\omega t - B(t) \cdot \cos\omega t\right]} \qquad (3.12)$$

where
$$A(t) = \int_0^t F(\tau) \cos\omega\tau\, d\tau \qquad (a)$$

$$(3.13)$$

and
$$B(t) = \int_0^t F(\tau) \cdot \sin\omega\tau\, d\tau \qquad (b)$$

It is necessary to evaluate the integrals $A(t)$ and $B(t)$ by some appropriate numerical method. For this purpose, we may use Trapezoidal Rule or Simpson's Rule, dividing the time interval into n equal time increments. An alternative approach which gives better results is discussed below:

approximate

An alternative method : (exact)

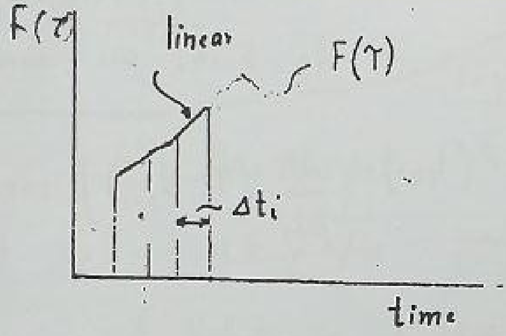


Fig . 3.5

In this method, the forcing function is assumed to be given by summation of piece-wise linear functions. (Fig. 3.5) In that case, there need not be any approximation in the integration process.

For convenience, the two integrals $A(t)$ and $B(t)$ will be expressed in incremental form, viz.,

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau) \cdot \cos \omega\tau \, d\tau \rightarrow \Delta A \tag{3.14}$$

and
$$B(t_i) = B(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau) . \sin \omega\tau \, d\tau \quad \Delta B \tag{3.15}$$

where $A(t_i)$ and $B(t_i)$ represent the values of the integrals at time t_i.

Assuming that the forcing function $F(\tau)$ is approximated by piece-wise linear functions as shown in Fig, 3.6 , we may write

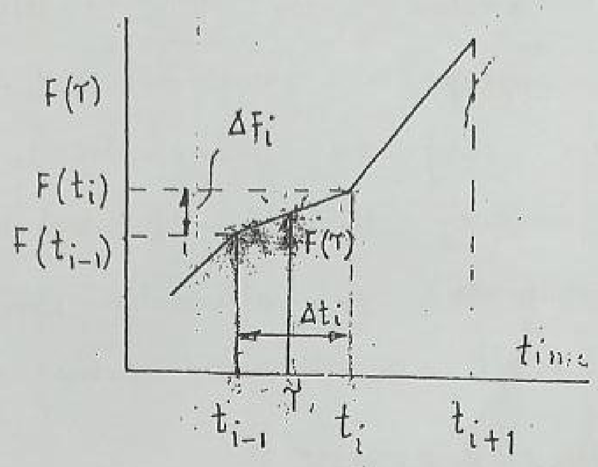


Fig. 3.6

$$F(\tau) = F(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i}(\tau - t_{i-1})$$

$$t_{i-1} \leq \tau \leq t_i \quad (3.16)$$

where

$$\Delta F_i = F(t_i) - F(t_{i-1})$$

and $\Delta t_i = t_i - t_{i-1}$.

Substituting relation (3.16) in Eq. 3.14, we write

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left[F(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i}(\tau - t_{i-1}) \right] \cos\omega\tau \, d\tau$$

$$= A(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1} \cdot \frac{\Delta F_i}{\Delta t_i} \right)\left(\sin\omega t_i - \sin\omega t_{i-1}\right)/\omega$$

$$+ \frac{\Delta F_i}{\omega^2 \cdot \Delta t_i}\left[\left(\cos\omega t_i - \cos\omega t_{i-1}\right) + \omega\left(t_i \cdot \sin\omega t_i - t_{i-1}\sin\omega t_{i-1}\right) \right] \quad (3.17)$$

Following similar procedure we get

$$B(t_i) = B(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1} \cdot \frac{\Delta F_i}{\Delta t_i} \right)\left(\cos\omega t_{i-1} - \cos\omega t_i\right)/\omega$$

$$+ \frac{\Delta F_i}{\omega^2 \Delta t_i}\left[\left(\sin\omega t_i - \sin\omega t_{i-1}\right) - \omega\left(t_i \cos\omega t_i - t_{i-1} \cdot \cos\omega t_{i-1}\right) \right] \quad (3.18)$$

Using the expressions (3.17) and (3.18), we can evaluate $\Delta A(t_i) = A(t_i) - A(t_{i-1})$ and also $\Delta B(t_i) = B(t_i) - B(t_{i-1})$ for different time steps Δt_i and force increments ΔF_i. Then from Eqn. (3.12), we can find the displacements $x(t)$ at different times.

Example 1. Determine the dynamic response of a tower subjected to a blast loading as shown in Fig. 3.7. Neglect damping.

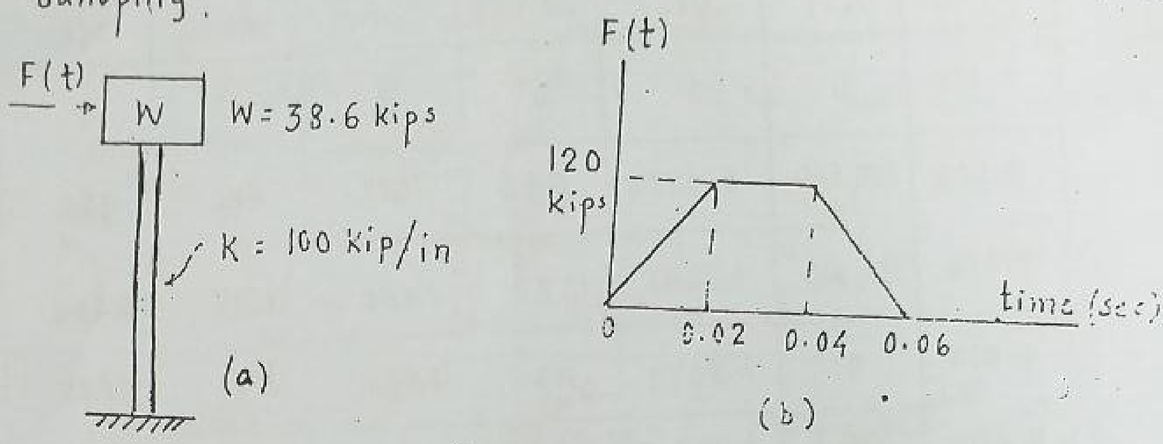


Fig. 3.7

For this system, the natural frequency 'ω' is

$$\omega = \sqrt{K/m} = \sqrt{100{,}000/100} = 31.62 \text{ rad/sec.}$$

$$\left(\text{Note : } m = \frac{38.6 \times 10^3}{386} = 100 \text{ lbs} \right)$$

The load can be assumed as piece-wise linear during time step $\Delta t = 0.02$ sec. The scheme of computation will be as below:

1. Using available values at time t_{i-1}, compute $\Delta A(t_i)$ and $\Delta B(t_i)$ from expressions (3.17) and (3.18).

$$\Delta A = A(t_i) - A(t_{i-1}) = \int_{t_{i-1}}^{t_i} F(\tau) \cos \omega\tau \, d\tau$$

$$\Delta B = B(t_i) - B(t_{i-1}) = \int \quad " \quad \sin \quad " \quad "$$

2. Compute $A(t_i) = A(t_{i-1}) + \Delta A(t_i)$ and $B(t_i) = B(t_{i-1}) + \Delta B(t_i)$.

3. Compute $x(t_i) = \left[A(t_i) \sin \omega t - B(t_i) \cos \omega t \right] \times \frac{1}{m\omega}$.

The calculations are presented in a tabular form in Table. 3.1.

$\omega = 31.62$

Table. 3.1

t (sec)	F(T)	ωt	$\Delta A(t)$	A(t)	$\Delta B(t)$	B(t)	x(t) (ins)
0.000	0	0	0	0	0	0	0
0.020	120,000 lb	0.6324	1082	1082	486	486	0.078
0.040	120,000	1.2645	1376	2458	1918	2404	0.512
0.060	0	1.8974	113	2571	1181	3585	1.134
0.080	0	2.5298	0	2571	0	3585	$\boxed{1.395}$
0.100	0	3.1623	0	2571	0	3585	1.117

From the calculations as shown in the above, we may note the following:

$F(t) = 0$

(i) Since the blast terminates at $t = 0.060$ sec, the values of A(t) and B(t) remain constant after this time.

(ii) The maximum displacement occurs at $t = 0.080$ sec and the value is 1.395 inch.

Assignment:

1. Write a computer program in BASIC to determine dynamic displacement $x(t)$ for the above problem.
2. Verify the results as shown in Table. 3.1.
3. Obtain values of $x(t)$ by using Trapezoidal Rule and compare the results.
4. Plot the response vs time.

II. Damped System

In this case, we will assume that the system is underdamped, that is $c < c_{cr}$ and there will be oscillatory motion.

For initial conditions, at $t = 0$, $x = x_0$ and $v = v_0$, the displacement is (Eq. 2.27)

$$x(t) = e^{-\xi \omega t}\left[\cancel{x_0}\cos\bar{\omega} t + \frac{v_0 + \cancel{x_0 \xi \omega}}{\bar{\omega}}\sin\bar{\omega} t\right]$$

where ω = natural frequency without damping

and $\bar{\omega}$ = damped frequency.

For an impulsive load $F(\tau)\,d\tau$, we will put $x_0 = 0$ and $v_0 = dv = \frac{F(\tau)\,d\tau}{m}$. Then the differential response due to $F(\tau)\,d\tau$ at time 't' will be

$$dx(t) = e^{-\xi\omega(t-\tau)} \cdot \frac{F(\tau)\,d\tau}{m\bar{\omega}} \cdot \sin\bar{\omega}(t-\tau).$$

Summing the responses for a load function continuously acting from 0 to t, the total (X) given by Duhamel Integral will be

$$x(t) = \frac{1}{m\bar{\omega}}\int_0^t F(\tau)\cdot e^{-\xi\omega(t-\tau)}\sin\bar{\omega}(t-\tau)\,d\tau$$

$$\boxed{x(t) = \frac{1}{m\bar{\omega}}\int_0^t e^{-\xi\omega t}\cdot F(\tau)\cdot e^{\xi\omega\tau}\cdot\sin\bar{\omega}(t-\tau)\,d\tau.} \quad (3.19)$$

For numerical evaluation of the above integral we proceed as in case of undamped system.

As before we write

$$\sin\bar{\omega}(t-\tau) = \sin\bar{\omega}t\cos\bar{\omega}\tau - \cos\bar{\omega}t . \sin\bar{\omega}\tau .$$

Using this in Eq. 3.19, we obtain

$$x(t) = \left[A_D(t)\sin\bar{\omega}t - B_D(t)\cos\bar{\omega}t \right] \frac{e^{-\xi\omega t}}{m\bar{\omega}} \qquad (3.20)$$

where $$A_D(t) = \int_0^t F(\tau) . e^{\xi\omega\tau} . \cos\bar{\omega}\tau \, d\tau \qquad (3.21)$$

and $$B_D(t) = \int_0^t F(\tau) . e^{\xi\omega\tau} . \sin\bar{\omega}\tau . d\tau . \qquad (3.22)$$

Again for a piece-wise linear function, the integrals (3.21) and (3.22) may be expressed in incremental form as

$$A_D(t_i) = A_D(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau) . e^{\xi\omega\tau} . \cos\bar{\omega}\tau . d\tau \qquad (3.23)$$

and $$B_D(t_i) = B_D(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau) . e^{\xi\omega\tau} . \sin\bar{\omega}\tau . d\tau \qquad (3.24)$$

Assuming that the forcing function $F(\tau)$ is approximated by a piece-wise linear function as shown in Fig (3.6), we may write

$$F(\tau) = F(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i}(\tau - t_{i-1}) . \qquad (3.25)$$

Using relation (3.25) in Eqn. (3.23) and (3.24)

We obtain

$$A_D(t_i) = A_D(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1} \cdot \frac{\Delta F_i}{\Delta t_i}\right) I_1 + \frac{\Delta F_i}{\Delta t_i} \cdot I_4$$

(3.26)

$$B_D(t_i) = B_D(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1} \cdot \frac{\Delta F_i}{\Delta t_i}\right) I_2 + \frac{\Delta F_i}{\Delta t_i} \cdot I_3$$

where

(3.27)

$$I_1 = \int_{t_{i-1}}^{t_i} e^{\xi\omega\tau} \cdot \cos\bar{\omega}\tau \, d\tau = \frac{e^{\xi\omega\tau}}{(\xi\omega)^2 + \bar{\omega}^2} \left[\xi\omega \cdot \cos\bar{\omega}\tau + \bar{\omega} \cdot \sin\bar{\omega}\tau\right] \Bigg|_{t_{i-1}}^{t_i}$$

$$I_2 = \int_{t_{i-1}}^{t_i} e^{\xi\omega\tau} \cdot \sin\bar{\omega}\tau \, d\tau = \frac{e^{\xi\omega\tau}}{(\xi\omega)^2 + \bar{\omega}^2} \left(\xi\omega \cdot \sin\bar{\omega}\tau - \bar{\omega} \cdot \cos\bar{\omega}\tau\right) \Bigg|_{t_{i-1}}^{t_i}$$

$$I_3 = \int_{t_{i-1}}^{t_i} \tau \cdot e^{\xi\omega\tau} \cdot \sin\bar{\omega}\tau \, d\tau = \left(\tau - \frac{\xi\omega}{(\xi\omega)^2 + \bar{\omega}^2}\right) I_2' + \frac{\bar{\omega}}{(\xi\omega)^2 + \bar{\omega}^2} \cdot I_1' \Bigg|_{t_{i-1}}^{t_i}$$

$$I_4 = \int_{t_{i-1}}^{t_i} \tau \cdot e^{\xi\omega\tau} \cdot \cos\bar{\omega}\tau \, d\tau = \left(\tau - \frac{\xi\omega}{(\xi\omega)^2 + \bar{\omega}^2}\right) I_1' - \frac{\bar{\omega}}{(\xi\omega)^2 + \bar{\omega}^2} I_2' \Bigg|_{t_{i-1}}^{t_i}$$

where again I_1' and I_2' are the integrals I_1 and I_2 before their evaluation at the limits.

Finally, the substitution of Eqns. (3.26) and (3.27) into Eq. (3.20) gives the displacement at time t_i as

$$\boxed{x(t_i) = \frac{e^{-\xi\omega t_i}}{m\bar{\omega}} \left\{ A_D(t_i) \sin\bar{\omega} t_i - B_D(t_i) \cdot \cos\bar{\omega} \cdot t_i \right\}}$$

(3.28)