Chapter Two

Solution Methods for Models Producing PDEs

At the end of this chapter, you should be able to:

- Classify the PDEs.
- Learn how to solve PDEs using different methods.
- Analyse engineering problems that produce PDEs.
- Apply different boundary conditions to find the final solution of PDEs.

2.1 Classification and Characteristics of Linear PDEs

The general linear equation of second order can be expressed

$$P\frac{\partial^2 z}{\partial x^2}+Q\frac{\partial^2 z}{\partial x\partial y}+R\frac{\partial^2 z}{\partial y^2}=S\text{-------}(1)$$

where the coefficients *P*, *Q*, *R* depend only on *x*, *y*, whereas *S* depends on *x*, *y*, *z*, $\partial z/\partial y$ and $\partial z/\partial y$. It is important to note here that if *P*, *Q*, *R* depend on *z*, Eq. 1 will be considered ***non-linear***. Furthermore, if *S* = 0, Eq. 1 will be called a ***homogenous*** PDE.

The terms involving second derivatives are of special importance, since they provide the basis for classification of type of PDE. By analogy, Eq. 1 can be written as:

$$ax^2+2b\,xy+c\,y^2=d\text{-------}(2)$$

Eq. 2 can be classified for constant coefficients when *P*, *Q*, *R* take values *a*, *b*, *c*, respectively. The discriminant for the case of constant coefficients *a*, *b*, *c* is defined as:

$$\Delta=b^2-4ac\text{-------}(3)$$

Accordingly, when:

$\Delta<0$: in this case the PDEs called *Elliptic* equation
$\Delta=0$: in this case the PDEs called *Parabolic* equation
$\Delta>0$: in this case the PDEs called *Hyperbolic* equation

Typical examples occurring in engineering for the above cases are:

- Fick's second law of diffusion:

$$\frac{\partial C_A}{\partial t}=D\frac{\partial^2 C_A}{\partial x^2}\ ;\qquad \text{where } C_A = \text{concentration of a species}$$

This equation is parabolic since $\Delta=0$ (*a* = 1, *b* = 0 and *c* = 0):

- Laplace's equation of heat conduction:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

This equation is elliptic since $\Delta=-4$ (a = 1, b = 0 and c = 1)

- Newton's law of wave motion:

$$\frac{\partial^2 u}{\partial t^2}=\rho\frac{\partial^2 u}{\partial y^2}$$

This equation is hyperbolic since $\Delta=4\rho$ (a = 1, b = 0 and c = $-\rho$).

2.2 Methods of solving PDEs

Three methods will be covered in this course to solve the PDEs:

1. Separation of variables
2. Combination of variables
3. Laplace Transformation

2.2.1 Separation of Variables

This is the most widely used method in applied mathematics, and its strategy is to break the dependent variable into component parts, each depending (usually) on a single independent variable; invariably, it leads to a multiple of particular solutions.

Example (1): Find the possible solutions for Fick's second law of diffusion using separation of variables method:

$$D\frac{\partial^2 C}{\partial x^2}=\frac{\partial C}{\partial t}$$

where C is a solute concentration and D is the solute diffusion coefficient through a media, and it is assumed constant.

Solution:

As a first step, we will assume that the concentration is a product of two separate functions each of which depends on one independent variable:

$$C(x,t) = X(x)\cdot T(t) ------(1)$$

In other words, Eq. 1 represents the solution of Fick's second law of diffusion. From this solution, the derivatives can be found, thus:

$$\frac{\partial C}{\partial t} = X\,\frac{\partial T}{\partial t} = X\cdot T' \quad \text{and} \quad \frac{\partial^2 C}{\partial x^2} = \frac{\partial^2 X}{\partial x^2}\,T = X''\cdot T$$

Substitute the above derivatives in the PDE gives:

$$\frac{T'}{T} = D\,\frac{X''}{X} \;------(2)$$

Since both T and X depend on a different independent variable, the ratios (T'/T and $D\,X''/X$) must equal to a constant to satisfy the equality in the equation, i.e.,

$$\frac{T'}{T} = D\,\frac{X''}{X} = constant \;-----(3)$$

There are three possible cases to solve Eq.3:

Case 1: the value of the constant is zero

In this case Eq. 3 becomes:

$$\frac{T'}{T} = D\,\frac{X''}{X} = 0$$

Now, $\frac{T'}{T} = 0 \;\Rightarrow\; T' = 0$ (by integration) $\Rightarrow T = A_1$; where A_1 is the integration constant.

and $D\,\frac{X''}{X} = 0 \;\Rightarrow\; X'' = 0$ (by integration) $\Rightarrow X' = A_2 \;\Rightarrow X = A_2 x + A_3$; where A_2 *and* A_3 are the integration constants.

According to the above, the solution (Eq.1) is:

$$\boxed{C(x,t) = A_1(A_2 x + A_3)}$$

Case 2: the value of the constant is a positive number (λ^2):

So, $\frac{T'}{T} = \lambda^2 \Rightarrow \frac{dT}{T} = \lambda^2 \cdot dt$ (by integration) $\Rightarrow 1n\,T = \lambda^2 t + A_1 \Rightarrow T = A_1 e^{\lambda^2 t}$

and $D\frac{X''}{X} = \lambda^2 \Rightarrow X'' - \frac{\lambda^2}{D}X = 0$ --------(4) (2nd order ODE).

The second order differential equation can be solved using superposition theorem:

Thus, $X'' - \frac{\lambda^2}{D}X = 0 \Rightarrow m^2 - \frac{\lambda^2}{D} = 0 \Rightarrow m = \pm\frac{\lambda}{\sqrt{D}}$

Compare the roots to the Table 1 below, the solution of Eq. 4 is:

$$X(x) = A_2\ e^{\frac{\lambda}{\sqrt{D}}x} + A_3\ e^{\frac{-\lambda}{\sqrt{D}}x}$$

Table 1 Solutions of $\frac{d^2u}{dt^2} + k\frac{du}{dt} + pu = 0$

Roots of Characteristic Equation	General Solution of Differential Equation
Real, distinct: $m_1 \neq m_2$	$u(t) = c_1e^{m_1t} + c_2e^{m_2t}$
Real, double: $m_1 = m_2$	$u(t) = c_1e^{m_1t} + c_2te^{m_1t}$
Conjugate complex: $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$	$u(t) = c_1e^{\alpha t}\cos(\beta t) + c_2e^{\alpha t}\sin(\beta t)$

According to case 2, the solution (Eq.1) is:

$$\boxed{C(x,t) = A_1e^{\lambda^2 t}\ (A_2\ e^{\frac{\lambda}{\sqrt{D}}x} + A_3\ e^{\frac{-\lambda}{\sqrt{D}}x})}$$

Case 3: the value of the constant is a negative number (-β^2):

Similar to case 2: $T = A_1e^{-\beta^2 t}$

and $D\frac{X''}{X} = -\beta^2 \Rightarrow X'' + \frac{\beta^2}{D}X = 0$ --------(5)

Again, using superposition method:

$$X'' + \frac{\beta^2}{D} X = 0 \;\Rightarrow\; m^2 + \frac{\beta^2}{D} = 0 \Rightarrow m = \pm i \frac{\beta}{\sqrt{D}}$$

Compare the roots to the Table 1 below, the solution of Eq. 5 is:

$$X(x) = A_2 \cos \frac{\beta}{\sqrt{D}} x + A_3 \sin \frac{\beta}{\sqrt{D}} x$$

According to case 3, the solution (Eq.1) is:

$$\boxed{C(x,t) = A_1 e^{-\beta^2 t} \left(A_2 \cos \frac{\beta}{\sqrt{D}} x + A_3 \sin \frac{\beta}{\sqrt{D}} x\right)}$$

NOTES:

1. This example illustrates using separation of variable method, in general, to solve PDEs.

2. Always, there are three possible solutions, depending on the value of the constant appeared in Eq. 3.

3. Determining the real solution needs to apply the boundary conditions and find which one is physically possible as will be seen in the next example.

H.W 1 (deadline on 4^{th} of March 2019):

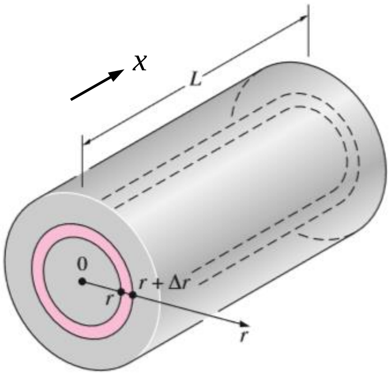
Solve the following PDEs using separation of variable method:

1) $\frac{\partial C}{\partial x} + \frac{\partial C}{\partial y} = 0$

2) $\frac{\partial C}{\partial x} + \frac{\partial C}{\partial y} - 10\, C = 0$

2.2.2 Engineering problems producing PDEs

Example (2): Consider a steel cylinder with a small length to radius ratio. The cylinder is thermally stable at T = 20 °C. In a heat treatment process, suddenly the cylinder is immersed in an oil bath such that both end are kept at T = 0 °C during the process. Find the temperature distribution at any point along the cylinder, using the appropriate assumptions?



Assumptions:

1. Heat transfer occurs only along the cylinder (i.e. no heat transfer in the radial or the angle direction but only in the x-direction).
2. The process is unsteady –state.
3. There is no heat generation.

Analysis:

Energy balance around an element with $\partial\phi, \partial r$ and ∂x gives:

$$\frac{\partial^2 T_C}{\partial x^2}+\frac{\partial^2 T_C}{\partial r^2}+\frac{1}{r}\frac{\partial T_C}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T_C}{\partial \phi^2}+\frac{\dot{q}}{K}=\frac{1}{\alpha}\frac{\partial T_C}{\partial t}$$

Take into account the proposed assumptions, the above equation reduces to:

$$\frac{\partial^2 T_C}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T_C}{\partial t} \quad ------(1)$$

and under the following conditions:

I.C: at t = 0 T_C = 20 °C or $T_C(x,0)$ = 20 °C

B.C 1: at x = 0 T_C = 0 °C or $T_C(0,t)$ = 0 °C

B.C 2: at x= L T_C = 0 °C or $T_C(L,t)$ = 0 °C

Mathematical solution of Eq.1:

Eq. 1 is a PDE and can be solved using separation of variables method (exactly as we did in the previous example for Fick's second law of diffusion). This gives the following solution:

$$T_C = X(x).T(t) \quad \text{and} \quad \frac{T'}{T}=\alpha\frac{X''}{X}=constant$$

Now, we will try the three possible solutions and test them using the I.C and B.Cs:

Case 1: the value of the constant is zero

This gives the following solution:

$$\boxed{T_C(x,t) = A_1(A_2 x + A_3)}$$

The solution can be written as:

$$T_C(x,t) = A_1^* x + A_2^* \quad ; \text{ where } A_1^* = A_1 \cdot A_2 \text{ and } A_2^* = A_1 \cdot A_3$$

Now, applying B.C 1 and B.C 2:

B.C 1: $0 = 0 + A_2^* \Rightarrow A_2^* = zero$

B.C 2: $0 = A_1^* L + 0 \Rightarrow A_1^* = zero$

It is clear from the above results that when the constant = 0, the solution is physically impossible.

Case 2: the value of the constant is a positive number (λ^2):

$$\boxed{T_C(x,t) = A_1 e^{\lambda^2 t}\, (A_2\ \ e^{\frac{\lambda}{\sqrt{\alpha}}x} + A_3\ \ e^{\frac{-\lambda}{\sqrt{\alpha}}x})}$$

The solution can be written as:

$$T_C(x,t) = e^{\lambda^2 t}\ (A_1^*\ \ e^{\frac{\lambda}{\sqrt{\alpha}}x} + A_2^*\ \ e^{\frac{-\lambda}{\sqrt{\alpha}}x})\text{; where } A_1^* = A_1 \cdot A_2 \text{ and } A_2^* = A_1 \cdot A_3$$

Now, applying B.C 1 and B.C 2:

B.C 1: $0 = A_1^* + A_2^*$

B.C 2: $0 = A_1^* e^{\frac{\lambda}{\sqrt{\alpha}}L} + A_2^* e^{\frac{-\lambda}{\sqrt{\alpha}}L}$

Applying the BCs suggests that $A_1^* = A_2^* = 0$ and, again, this situation is impossible physically.

Case 3: the value of the constant is a negative number (-β^2):

$$\boxed{T_C(x,t) = A_1 e^{-\beta^2 t}\ (A_2 \cos\frac{\beta}{\sqrt{\alpha}}x + A_3 \sin\frac{\beta}{\sqrt{\alpha}}x)}$$

The solution can be written as:

$$T_C(x,t) = e^{-\beta^2 t}\ (A_1^* \cos\frac{\beta}{\sqrt{\alpha}}x + A_2^* \sin\frac{\beta}{\sqrt{\alpha}}x)\,;\ \text{where}\ A_1^* = A_1 \cdot A_2\ \text{and}\ A_2^* = A_1 \cdot A_3$$

Now, applying B.C 1 and B.C 2:

$$\text{B.C 1:}\ \ 0 = A_1^*$$

$$\text{B.C 2:}\ \ 0 = A_2^* \sin\frac{\beta}{\sqrt{\alpha}}L$$

Now, B.C 2 suggests either $A_2^* = 0$ (which is impossible) or $\sin\frac{\beta}{\sqrt{\alpha}}L = 0$ which is possible when $\frac{\beta}{\sqrt{\alpha}}L = n\pi$ where n = 0 ,1, ,2 ,3 4 , …….

Thus, from B.C 2: $\frac{\beta}{\sqrt{\alpha}}L = n\pi \Rightarrow \frac{\beta}{\sqrt{\alpha}} = \frac{n\pi}{L}$ and $\beta = \sqrt{\alpha}(\frac{n\pi}{L})$. Substitution of these terms in the general solution gives:

$$T_C(x,t) = e^{-\alpha\left[\frac{n\pi}{L}\right]^2 t}\left[A_2^* \sin\left(\frac{n\pi}{L}\right)x\right]\ \ \textbf{Or}\ \ T_C(x,t) = A_2^*\, e^{-\alpha\left[\frac{n\pi}{L}\right]^2 t} \sin\left(\frac{n\pi}{L}\right)x$$

The above equation can be expressed as a series:

$$T_C(x,t) = \sum_{n=1}^{\infty} A_n^*\, e^{-\alpha\left[\frac{n\pi}{L}\right]^2 t} \sin\left(\frac{n\pi}{L}\right)x$$

Now, it is necessary to find the value of A_n^* . This constant can be found by applying I.C:

$$\text{I.C:}\ \ 20 = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi}{L}\right)x\,;\ where\ \ A_n^* = \frac{2}{L}\int_0^L 20\sin\left(\frac{n\pi}{L}\right)x.dx\ \ (\text{Fourier sine series})$$

Let’s take L = 10 cm, this means:

$$A_n^* = 0.2\int_0^{10} 20\sin\left(\frac{n\pi}{10}\right)x.dx \ \Rightarrow\ A_n^* = \frac{40}{n\pi}\left[-\cos\left(\frac{n\pi}{10}\right)x\right]_0^{10}$$

$$\Rightarrow\ A_n^* = \frac{40}{n\pi}\left[-(-1)^n + 1\right]$$

Example (3): Repeat Example 2, using the following conditions:

$$\frac{\partial^2 T_C}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_C}{\partial t}$$

I.C: at $t = 0$ $T_C = 10\,x^2$ or $T_C\,(x,0) = 10\,x^2$

B.C 1: at $x = 0$ $T_C = 0$ °C or $T_C\,(0,t) = 0$ °C

B.C 2: at $x = L$ $T_C = 0$ °C or $T_C\,(L,t) = 0$ °C

SOLUTION:

The solution is exactly as in Example 2, but the analysis of case 3 will be different since the initial condition is different:

Case 3: the value of the constant is a negative number (-β^2):

$$\boxed{T_C(x,t) = A_1 e^{-\beta^2 t}\,(A_2 \cos\frac{\beta}{\sqrt{\alpha}}x + A_3 \sin\frac{\beta}{\sqrt{\alpha}}x)}$$

Again, the solution can be written as:

$$T_C(x,t) = e^{-\beta^2 t}\,(A_1^* \cos\frac{\beta}{\sqrt{\alpha}}x + A_2^* \sin\frac{\beta}{\sqrt{\alpha}}x)\,;\ \text{where}\ A_1^* = A_1 \cdot A_2\ \text{and}\ A_2^* = A_1 \cdot A_3$$

Now, applying B.C 1 and B.C 2:

B.C 1: $0 = A_1^*$

B.C 2: $0 = A_2^* \sin\frac{\beta}{\sqrt{\alpha}}L$

Now, B.C 2 suggests either $A_2^* = 0$ (which is impossible) or $\sin\frac{\beta}{\sqrt{\alpha}}L = 0$ which is possible when $\frac{\beta}{\sqrt{\alpha}}L = n\pi$ where n = 1, 2 ,3, 4 , …….

Thus, from B.C 2: $\frac{\beta}{\sqrt{\alpha}}L = n\pi \Rightarrow \frac{\beta}{\sqrt{\alpha}} = \frac{n\pi}{L}$ and $\beta = \sqrt{\alpha}(\frac{n\pi}{L})$. Substitution of these terms in the general solution gives:

$$T_C(x,t) = e^{-\alpha\left[\frac{n\pi}{L}\right]^2 t}\left[A_2^* \sin\left(\frac{n\pi}{L}\right)x\right] \quad \textbf{Or} \quad T_C(x,t) = A_2^*\, e^{-\alpha\left[\frac{n\pi}{L}\right]^2 t} \sin\left(\frac{n\pi}{L}\right)x$$

The above equation can be expressed as a series:

$$T_C(x,t) = \sum_{n=1}^{\infty} A_n^*\, e^{-\alpha\left[\frac{n\pi}{L}\right]^2 t} \sin\left(\frac{n\pi}{L}\right)x ------ (1)$$

Now, it is necessary to find the value of A_n^* . This constant can be found by applying I.C:

$$\text{I.C: } 10\,x^2 = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi}{L}\right)x \,;\; \textit{where}\;\; A_n^* = \frac{2}{L}\int_0^L 10\,x^2 \sin\left(\frac{n\pi}{L}\right)x.dx \;(\textit{Fourier sine series})$$

Let's take L = 10 cm, this means:

$$A_n^* = 0.2\int_0^{10} (10\,x^2) \sin\left(\frac{n\pi}{10}\right)x.dx \quad ;\text{integration by part as shown in the below table}$$

$$\Rightarrow A_n^* = 0.2\left[-10\,x^2\left(\frac{10}{n\pi}\right)\cos\left(\frac{n\pi}{10}\right)x + 20x\left(\frac{10}{n\pi}\right)^2 \sin\left(\frac{n\pi}{10}\right)x + 20\left(\frac{10}{n\pi}\right)^3 \cos\left(\frac{n\pi}{10}\right)x\right]_0^{10}$$

$$\Rightarrow A_n^* = 0.2\left[-1000\left(\frac{10}{n\pi}\right)\cos n\pi + 20\left(\frac{10}{n\pi}\right)^3 \cos n\pi - 20\left(\frac{10}{n\pi}\right)^3\right]$$

$$\Rightarrow A_n^* = -\left(\frac{2000}{n\pi}\right)\cos n\pi + 4\left(\frac{10}{n\pi}\right)^3 (\cos n\pi - 1)$$

Integration by part

$10\,x^2$	$\sin\left(\frac{n\pi}{10}\right)x$
$20\,x$	$-\left(\frac{10}{n\pi}\right)\cos\left(\frac{n\pi}{10}\right)x$
20	$-\left(\frac{10}{n\pi}\right)^2 \sin\left(\frac{n\pi}{10}\right)x$
zero	$\left(\frac{10}{n\pi}\right)^3 \cos\left(\frac{n\pi}{10}\right)x$

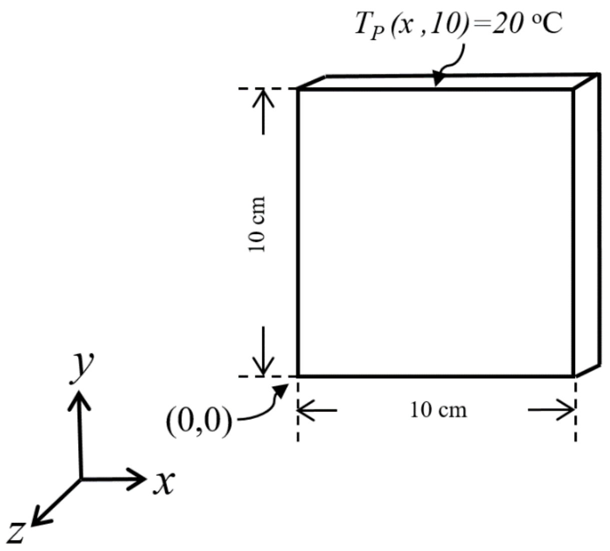
Substitute A_n^* in Eq. 1, the final solution is:

$$T_C(x,t) = \sum_{n=1}^{\infty}\left[-\left(\frac{2000}{n\pi}\right)\cos n\pi + 4\left(\frac{10}{n\pi}\right)^3 (\cos n\pi - 1)\right] e^{-\alpha\left[\frac{n\pi}{L}\right]^2 t} \sin\left(\frac{n\pi}{L}\right)x$$

Example (4): The faces of a thin square copper plate ($10 \times 10\,\text{cm}^2$) are perfectly insulated. The upper side is kept at 20 °C and the other sides are kept at 0 °C. Find the steady-state temperature $T_P(x, y)$ in the plate (solve the Laplace equation given in page 3).

SOLUTION:

Assumptions:
1. Two-dimensional heat transfer (only in *x* and *y*-directions).
2. No heat generation.
3. steady-state heat transfer.



Analysis:
The general heat transfer equations in a solid body:

$$\frac{\partial^2 T_P}{\partial x^2} + \frac{\partial^2 T_P}{\partial y^2} + \frac{\partial^2 T_P}{\partial z^2} + \frac{\dot{q}}{K} = \frac{1}{\alpha}\frac{\partial T_P}{\partial t} -----(1)$$

Since the problem is a 2-D heat transfer in a solid at steady-state condition, Eq. 1 reduces to:

$$\frac{\partial^2 T_P}{\partial x^2} + \frac{\partial^2 T_P}{\partial y^2} = 0 -----(2)$$

and under the following conditions:

B.C 1: at $x = 0$, $T_P = 0$ °C or $T_P\,(0, y) = 0$ °C

B.C 2: at $x = 10$, $T_P = 0$ °C or $T_P\,(10, y) = 0$ °C

B.C 3: at $y = 0$, $T_P = 0$ °C or $T_P\,(x, 0) = 0$ °C

B.C 4: at $y = 10$, $T_P = 20$ °C or $T_P\,(x,10) = 20$ °C

Mathematical solution of Eq.2:

Using the separation of variables method gives:

$$T_P\ (x,y) = X(x).Y(y) -----(3)$$

The derivatives can be found, thus:

$$\frac{\partial^2 T_P}{\partial x^2} = X''\cdot Y \quad \text{and} \quad \frac{\partial^2 T_P}{\partial y^2} = X\ \cdot Y''$$

Substitute the above derivatives in the PDE (Eq.2) gives:

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Since both *X* and *Y* depend on a different independent variable, the ratios (X''/X and Y''/Y) must equal to a constant to satisfy the equality in the equation, i.e.,

$$\frac{X''}{X} = -\frac{Y''}{Y} = constant \ -----(4)$$

As in the previous examples, there are three possible cases:

Case 1: the value of the constant is zero

In this case Eq. 4 becomes:

$$\frac{X''}{X} = \frac{Y''}{Y} = 0$$

Now, $\frac{X''}{X} = 0 \ \Rightarrow \ X'' = 0$ (by integration) $\Rightarrow X' = A_1 \Rightarrow X = A_1 x + A_2$; where A_1 *and* A_2 are the integration constants.

and $-\frac{Y''}{Y} = 0 \ \Rightarrow \ Y'' = 0$ (by integration) $\Rightarrow Y' = A_3 \ \Rightarrow Y = A_3 y + A_4$; where A_3 *and* A_4 are the integration constants.

According to the above, the solution (Eq.2) is:

$$\boxed{T_P(x,y) = (A_1 x + A_2)(A_3 y + A_4)}$$

Applying the boundary conditions:

$$\text{B.C 1:}\quad 0 = A_2(A_3 y + A_4) \Rightarrow A_2 = 0$$
$$\text{B.C 2:}\quad 0 = (10A_1)(A_3 y + A_4) \Rightarrow A_1 = 0$$

Again, the solution is not physically possible.

Case 2: the value of the constant is a positive number (λ^2):

In this case Eq. 4 becomes:

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2$$

Now, $\frac{X''}{X} = \lambda^2 \;\Rightarrow\; X'' - \lambda^2 X = 0$ (2nd ODE)

The second order differential equation can be solved:

Thus, $X'' - \lambda^2 X = 0 \;\Rightarrow\; m^2 - \lambda^2 = 0 \Rightarrow m = \pm\lambda$

Compare the roots to the Table 1 given in page 5, the solution is:

$$X(x) = A_1\, e^{\lambda x} + A_2\, e^{-\lambda x}$$

For Y variable:

$$Y'' + \lambda^2 Y = 0 \;\Rightarrow\; m^2 + \lambda^2 = 0 \Rightarrow m = \pm\, i\lambda \;\;\Rightarrow\; Y(y) = A_3 \sin \lambda y + A_4 \cos \lambda y$$

According to the above, the solution (Eq.2) is:

$$T_P\,(x, y) = (A_1\, e^{\lambda x} + A_2\, e^{-\lambda x})(A_3 \sin\, \lambda y + A_4 \cos\, \lambda y)$$

Applying the boundary conditions:

$$\text{B.C 1:}\quad 0 = (A_1 + A_2)(A_3 \sin \lambda y + A_4 \cos \lambda y) \Rightarrow A_1 + A_2 = 0$$
$$\text{B.C 2:}\quad 0 = (A_1\, e^{\lambda 10} + A_2\, e^{-\lambda 10})(A_3 \sin \lambda y + A_4 \cos \lambda y) \Rightarrow A_1\, e^{\lambda 10} + A_2\, e^{-\lambda 10} = 0$$

To satisfy B.C 1 and 2, A_1 and A_2 must be zero and that gives impossible physical solution.

Case 3: the value of the constant is a negative number ($-\beta^2$)

In this case Eq. 4 becomes:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\beta^2$$

Now, $\frac{X''}{X} = \lambda^2 \;\Rightarrow\; X'' + \beta^2 X = 0$ (2nd ODE)

The second order differential equation can be solved:

Thus, $X'' + \beta^2 X = 0 \;\Rightarrow\; m^2 + \beta^2 = 0 \Rightarrow m = \pm\, i\beta$

Compare the roots to the Table 1 given in page 5, the solution is:

$$X(x) = A_1 \sin \beta x + A_2 \cos \beta x$$

For Y variable:

$Y'' - \beta^2 Y = 0 \;\Rightarrow\; m^2 - \beta^2 = 0 \Rightarrow m = \pm\,\beta \;\;\Rightarrow\; Y(y) = A_3 e^{\beta y} + A_4 e^{-\beta y}$

According to the above, the solution (Eq.2) is:

$$T_P(x, y) = (A_1 \sin \beta x + A_2 \cos \beta x)(A_3 e^{\beta y} + A_4 e^{-\beta y}) \;------\; (5)$$

Applying the boundary conditions yields,

B.C 1: $0 = (0 + A_2)(A_3 e^{\beta y} + A_4 e^{-\beta y}) \Rightarrow A_2 = 0$
B.C 2: $0 = (A_1 \sin \beta 10)(A_3 e^{\beta y} + A_4 e^{-\beta y}) \Rightarrow \beta = n\pi/10$

Thus, Eq.5 becomes:

$$T_P(x, y) = (A_1 \sin \frac{n\pi}{10} x)(A_3 e^{\frac{n\pi}{10} y} + A_4 e^{-\frac{n\pi}{10} y})$$

Or,

$$T_P(x, y) = \sin \frac{n\pi}{10} x \left(A_1^* e^{\frac{n\pi}{10} y} + A_2^* e^{-\frac{n\pi}{10} y} \right); \text{ where } A_1^* = A_1 \cdot A_3 \text{ and } A_2^* = A_1 \cdot A_4$$

From B.C 3:

$$0 = \left(A_1^* + A_2^*\right) \Rightarrow A_1^* = -A_2^*$$

Now, $T_P(x,y)$ equation can be written as:

$$T_P(x,y) = A_1^* \sin\frac{n\pi}{10}x\left(e^{\frac{n\pi}{10}y} - e^{-\frac{n\pi}{10}y}\right) ----- (6)$$

Since $\sinh\frac{n\pi}{10}y = \dfrac{e^{\frac{n\pi}{10}y} - e^{-\frac{n\pi}{10}y}}{2}$, Eq. 6 can be expressed as:

$$T_P(x,y) = 2A_1^* \sin\frac{n\pi}{10}x \sinh\frac{n\pi}{10}y ----- (7)$$

Eq. 7 can be expressed as a series of *n*:

$$T_P(x,y) = \sum_1^n 2A_n^* \sin\frac{n\pi}{10}x \cdot \sinh\frac{n\pi}{10}y ------ (8)$$

Re-arrange Eq. 8:

$$T_P(x,y) = \sum_1^n (2A_n^* \sinh\frac{n\pi}{10}y)\sin\frac{n\pi}{10}x ------ (9)$$

Now, apply B.C 4:

$$20 = \sum_1^n (2A_n^* \sinh\frac{n\pi}{10}10)\sin\frac{n\pi}{10}x$$

Comparing the above equation to Fourier sine series, A_n^* can be found:

$$2A_n^* \sinh\frac{n\pi}{10}10 = \frac{2}{10}\int_0^{10} 20\ \sin\frac{n\pi}{10}x\,dx$$

$$\Rightarrow 2A_n^* \sinh \frac{n\pi}{10} 10 = \frac{40}{n\pi}\left(-\cos\frac{n\pi}{10}x\right)_0^{10}$$

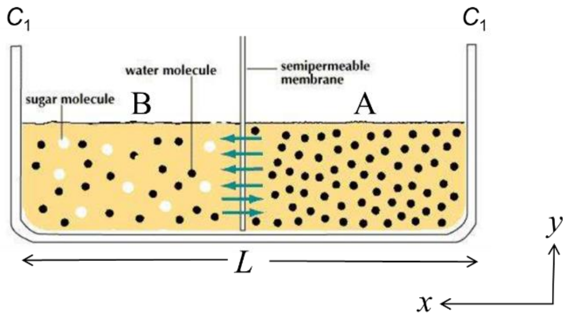
$$\Rightarrow 2A_n^* \sinh \frac{n\pi}{10} 10 = \frac{40}{n\pi}\left(-\cos n\pi + 1\right) \Rightarrow A_n^* = \frac{\frac{20}{n\pi}\left(-\cos n\pi + 1\right)}{\sinh n\pi}$$

Now substitute A_n^* in Eq. 9:

$$\boxed{T_P(x,y) = \sum_1^n \left(\frac{\frac{20}{n\pi}\left(-\cos n\pi + 1\right)}{\sinh n\pi}\right) \sinh \frac{n\pi}{10} y \,.\, \sin \frac{n\pi}{10} x}$$

Example (5): (diffusion process) two chambers (A and B) are separated by a thin semipermeable membrane. The water molecules transfer from chamber A to B due to the difference in concentration. This process obeys Fick's second law of diffusion which is:

$$D\frac{\partial^2 C_W}{\partial x^2} = \frac{\partial C_W}{\partial t}$$



where:

C_W = water concentration (mole/m^3).
D = water diffusion coefficient (m^2/s).

Using Fick's second law of diffusion, find water concentration distribution throughout chambers A and B under the following conditions:

I.C: at $t = 0$, $C_W = C_0$
B.C 1: at $x = 0$, $C_W = C_1$
B.C 2: at $x = L$, $C_W = C_1$

Note: C_0 and C_1 are constant values.

Solution:

It is clear by inspection that conditions must be homogeneous in the x domain, if a direct analytical solution is required. To ensure getting homogenous conditions at x = 0 and x = L, let's define a new variable (η, for example) as:

$$\eta\,(x,t) = C_W\,(x,t) - C_1 \;----(1)$$

From Eq. (1), Fick's second law of diffusion can be written as:

$$D\frac{\partial^2 \eta}{\partial x^2} = \frac{\partial \eta}{\partial t}$$

And the boundary conditions can be modified as:

I.C: at $t = 0$, $\eta = C_0 - C_1$

B.C 1: at $x = 0$, $\eta = 0$

B.C 2: at $x = L$, $\eta = 0$

Now, the equation and the boundary conditions are simplified such that they can be solved directly using the same technique discussed in the previous examples.

Thus, using the separation of variables method gives:

$$\eta\,(x,t) = X(x).T(t) - - - - - (3)$$

The derivatives can be found, thus:

$$\frac{\partial^2 \eta}{\partial x^2} = X'' \cdot T \quad \text{and} \quad \frac{\partial^2 \eta}{\partial t} = X \cdot T'$$

Substitute the above derivatives in the PDE (Eq.2) gives:

$$D\frac{X''}{X} = \frac{T'}{T}$$

As in the previous examples, case 1 and 2 give illogical solution. Thus, we will start with case 3 where the constant is a negative value (-β^2).

$$D\frac{X''}{X} = \frac{T'}{T} = -\beta^2 - - - - - (2)$$

The solution of Eq. 2 is:

$$\boxed{\eta(x,t) = e^{-\beta^2 t}\ (A_1^* \cos\frac{\beta}{\sqrt{D}}x + A_2^* \sin\frac{\beta}{\sqrt{D}}x) - - - - (3)}$$

Now, apply the modified boundary conditions:

B.C 1: $A_1^* = 0$

B.C 2: $\sin\frac{\beta}{\sqrt{D}}L = \sin n\pi \Rightarrow \beta = \sqrt{D}\,\frac{n\pi}{L}$

Thus, from B.C 1 and 2, Eq. 3 reduces to:

$$\eta(x,t) = e^{-D\left[\frac{n\pi}{L}\right]^2 t} A_2^* \sin(\frac{n\pi}{L})x, \quad \text{where n = 0, 1, 2, 3 …..}$$

Or

$$\eta(x,t) = \sum_1^{\infty} A_n^* \, e^{-D\left[\frac{n\pi}{L}\right]^2 t} \cdot \sin(\frac{n\pi}{L})x ----- (4)$$

To find A_n^* , I.C can be applied. Thus,

$$C_0 - C_1 = \sum_1^{\infty} A_n^* \sin(\frac{n\pi}{L})x \,; \; where \; A_n^* = \frac{2}{L}\int_0^L \left(C_0 - C_1\right) \sin\,(\frac{n\pi}{L})x\,.dx$$

From the integration, A_n^* can be defined as:

$$\Rightarrow \; A_n^* = \frac{2L(C_0 - C_1)}{n\pi} \left[-(-1)^n + 1\right]$$

Substitute A_n^* in Eq. 4, gives:

$$\eta(x,t) = \sum_1^{\infty} \frac{2L(C_0 - C_1)}{n\pi} \left[-(-1)^n + 1\right] \cdot e^{-D\left[\frac{n\pi}{L}\right]^2 t} \cdot \sin(\frac{n\pi}{L})x$$

Since $\eta\,(x,t) = C_W\,(x,t) - C_1$ (from Eq. 1), the above equation can be written as:

$$\boxed{\frac{C_W\,(x,t) - C_1}{C_0 - C_1} = \sum_1^{\infty} \frac{2L}{n\pi} \left[-(-1)^n + 1\right] \cdot e^{-D\left[\frac{n\pi}{L}\right]^2 t} \cdot \sin(\frac{n\pi}{L})x}$$

2.2.3 Solution of INHOMOGENEOUS PDEs using Separation of Variables Method

In example 5 (page 18), we have used elementary change of variables as a method to convert certain inhomogeneous boundary conditions to homogeneous form. In certain cases, the boundary inhomogeneity cannot be removed by elementary substitution. In other cases, the defining equation itself is not homogeneous (could give an example ?). Both sets of circumstances lead to inhomogeneous equations.

A fairly general way of coping with inhomogeneous PDE is to apply *the concept of deviation variables*. This technique is best illustrated by way of examples below.

Example (5): solve the following PDE:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$$

I.C: at $t = 0$, $y = 9x$

B.C 1: at $x = 0$, $y = 10$

B.C 2: at $x = 5$, $y = 15$

Solution:

To solve the above PDEs, we will use the concept of deviation variables. The concept is based on assuming that,

$$y(x,t) = \eta(x,t) + f(x) \quad ---- (1)$$

It is easy to find the derivatives:

$$\frac{\partial y}{\partial t} = \frac{\partial \eta}{\partial t} + 0 \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}$$

Substitute the derivatives in the PDE,

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{\partial \eta}{\partial t} ------(2)$$

To force the above equation to be as the original PDE (i.e, $\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$), the term $\frac{\partial^2 f}{\partial x^2}$ must equal zero, i.e.:

$$\frac{\partial^2 f}{\partial x^2} = 0 ----- (3)$$

Eq.2 can be solved:

$$\frac{\partial^2 f}{\partial x^2} = 0 \Rightarrow \quad m^2 = 0 \quad \Rightarrow \; m1 = m2 = 0 \quad \Rightarrow \quad f = C_1 x + C_2 ---- (4)$$

Now, one can use Eq. 4 and the BCs to find *C1* and *C2* and to convert the B.Cs to a homogenous set of conditions, therefore:

From B.C 1 and Eq.4, Eq.1 can be written as:

$$y(x,t) = \eta(x,t) + \underbrace{C_1 x + C_2}_{f(x)} \Rightarrow \; 10 = \eta(x,t) + C_1 \,(0) + C_2$$

Note for homogenous BCs, $\eta(x,t) = 0$. This means $C_2 = 10$.

From B.C 2 and Eq.4, Eq.1 can be written as:

$$15 = \eta(x,t) + C_1 \,(5) + \underbrace{10}_{C_2}$$

Again, $\eta(x,t)$ must equal zero and, thus $C_1 = 1$ and Eq.4 becomes:

$$f(x) = x + 10$$

Now, applying the I.C:

$$9x = \eta(x,0) + \underbrace{x+10}_{f(x)} \;\;\Rightarrow\;\; \eta(x,0) = 8x - 10$$

From the above simplifications, PDE (Eq. 3) and BCs can be re-written as:

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{\partial \eta}{\partial t} - - - (5)$$

I.C: at $t = 0$, $\eta = 8x - 10$

B.C 1: at $x = 0$, $\eta = 0$

B.C 2: at $x = 5$, $\eta = 0$

Now, Eq.5 can be solved directly as we did in the previous examples:

$$\eta(x,t) = X(x) \cdot T(t)$$

Then,

$$\frac{X''}{X} = \frac{T'}{T} = constant$$

It must be know by now that the zero and positive value of constant (λ^2)give physically impossible solution. So, we will assume that the constant is a negative number ($-\beta^2$) and proceed the solution:

$$\frac{X''}{X} = \frac{T'}{T} = -\beta^2$$

Thus, the solution is:

$$\boxed{\eta(x,t) = A_1 e^{-\beta^2 t} \,(A_2 \cos \beta x + A_3 \sin \beta x)}$$

Again, the solution can be written as:

$$\eta(x,t) = e^{-\beta^2 t}\,(A_1^* \cos\beta x + A_2^* \sin\beta x);\ \text{ where } A_1^* = A_1 \cdot A_2 \text{ and } A_2^* = A_1 \cdot A_3$$

Now, applying B.C 1 and B.C 2:

$$\text{B.C 1:}\quad 0 = A_1^*$$
$$\text{B.C 2:}\quad 0 = A_2^* \sin\beta 5$$

Now, B.C 2 suggests either $A_2^* = 0$ (which is impossible) or $\sin\beta 5 = 0$ which is possible when $\beta\,5 = n\pi$ where n = 1, 2 ,3, 4 , …….

Thus, from B.C 2: $\beta\,5 = n\pi \Rightarrow \beta = \dfrac{n\pi}{5}$. Substitution of these terms in the general solution gives:

$$\eta(x,t) = e^{-\left[\frac{n\pi}{5}\right]^2 t}\left[A_2^* \sin\left(\frac{n\pi}{5}\right)x\right] \quad \textbf{Or} \quad \eta(x,t) = A_2^*\, e^{-\left[\frac{n\pi}{5}\right]^2 t} \sin\left(\frac{n\pi}{5}\right)x$$

The above equation can be expressed as a series:

$$\eta(x,t) = \sum_{n=1}^{\infty} A_n^*\, e^{-\left[\frac{n\pi}{5}\right]^2 t} \sin\left(\frac{n\pi}{5}\right)x \text{------} (6)$$

Now, it is necessary to find the value of A_n^* . This constant can be found by applying I.C:

$$\text{I.C}:\ 8x - 10 = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi}{5}\right)x\,;\ where\ \ A_n^* = \frac{2}{5}\int_0^5 (8x-10)\cdot \sin\left(\frac{n\pi}{5}\right)x.dx\ \left(\mathit{Fourier\ sine\ series}\right)$$

$$A_n^* = \frac{2}{5}\int_0^5 (8x-10)\sin\left(\frac{n\pi}{5}\right)x.dx \quad ;\text{integration by part as shown in the below table}$$

Integration by part

$8x$-10	$\sin\left(\frac{n\pi}{5}\right)x$
8	$-\left(\frac{5}{n\pi}\right)\cos\left(\frac{n\pi}{5}\right)x$
zero	$-\left(\frac{5}{n\pi}\right)^2 \sin\left(\frac{n\pi}{5}\right)x$

$$\Rightarrow A_n^* = 0.4\left[-(8x-10)\left(\frac{5}{n\pi}\right)\cos\left(\frac{n\pi}{5}\right)x + 8\left(\frac{5}{n\pi}\right)^2 \sin\left(\frac{n\pi}{5}\right)x\right]_0^5$$

$$\Rightarrow A_n^* = 0.4\left[-30\left(\frac{5}{n\pi}\right)\cos n\pi - 10\left(\frac{5}{n\pi}\right)\right]$$

$$\Rightarrow A_n^* = \left(\frac{20}{n\pi}\right)\left[-3\cos n\pi - 1\right]$$

Substitute A_n^* in Eq. 6, the final solution is:

$$\eta(x,t)=\sum_{n=1}^{\infty}\left[\left(\frac{20}{n\pi}\right)\left[-3\cos n\pi-1\right]\right]e^{-\left[\frac{n\pi}{5}\right]^2 t}\sin\left(\frac{n\pi}{5}\right)x$$

Now, the equation can be written in terms of $y(x,t)$ using Eq.1:

$$y(x,t)=\underbrace{\sum_{n=1}^{\infty}\left[\left(\frac{20}{n\pi}\right)\left[-3\cos n\pi-1\right]\right]e^{-\left[\frac{n\pi}{5}\right]^2 t}\sin\left(\frac{n\pi}{5}\right)x}_{\eta(x,t)}+\underbrace{x+10}_{f(x)}$$

Where $\cos n\pi=(-1)^n$

- Sheet No. 1 -

Q1. Solve the following equation (wave equation):

$$K\frac{\partial^2 V}{\partial x^2}=\frac{\partial^2 V}{\partial t^2}$$, where V is a string displacement.

I.C: $t = 0$, $V = x$
B.C 2: $x = 0$, $V = 0$
B.C 3: $x = L$, $V = 0$

NOTE: The above equation represents the vibration of a string, like a violin or guitar string and is known as Wave Equation.

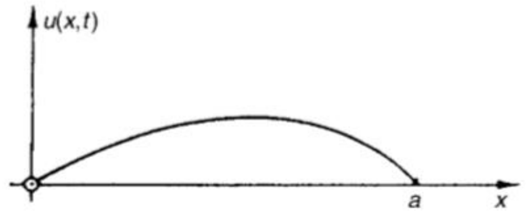


Figure 1 String fixed at the ends.

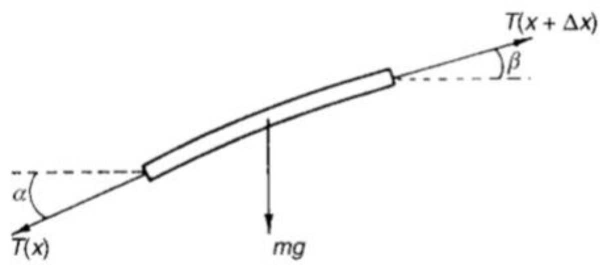
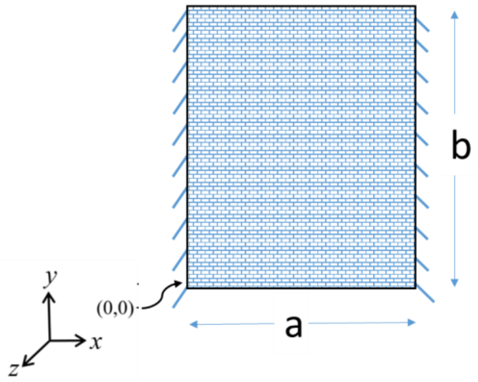


Figure 2 Section of string showing forces exerted on it. The angles are $\alpha = \phi(x,t)$ and $\beta = \phi(x+\Delta x,t)$.

Q2. Left and right sides of an iron plate are perfectly insulated. The upper side is kept at 10 °C and the opposite side is kept at 0 °C. Find the steady-state temperature $T_P(x,y)$ in the plate.



Q3. Unsteady-state heat transfer in the radial direction of a sphere can be represented by the following PDE:

$$\frac{\partial^2 T_S}{\partial r^2} + \frac{2}{r}\frac{\partial T_S}{\partial r} = \frac{1}{\alpha}\frac{\partial T_S}{\partial t}$$

Find $T_S\ (r,t)$ under the following conditions:

I.C: $t = 0$, $T_S = 10$ °C

B.C 2: $r = 0$, $\dfrac{\partial T_S}{\partial r} = 0$

B.C 3: $r = 1$, $T_S = 0$

Q4. (**Inhomogeneous PDE**) solve the following PDE:

$$\frac{\partial^2 y}{\partial x^2} + 2x = \frac{\partial y}{\partial t}$$

Boundary conditions:

I.C: at $t = 0$, $y = 5$

B.C 1: at $x = 0$, $y = 0$

B.C 2: at $x = L$, $y = 0$