FLUVIAL HYDRAULICS

Flow and transport processes in channels of simple geometry

Walter H. Graf

in collaboration with M. S. Altinakar

WILEY

ABOUT THE AUTHORS

Professor at the Swiss Federal Institute of Technology in Lausanne (EPFL) since 1973, **Walter H. Graf** is in charge of its Hydraulic Research Laboratory, where M. S. Altinakar is one of his assistants. His research activities are concentrated on the hydrodynamics of free-surface flow, notably open-channel hydraulics and hydrodynamics of lakes and reservoirs. Previously W. H. Graf had teaching engagements at the University of California-Berkeley, at Cornell University and at Lehigh University in the USA. In 1995, he was named a "Guest Professor" at the prestigious Tsinghua University at Beijing, China. Over the years he served also as a visiting professor in Austria (1972, 1994), Republic of South Africa (1974), China (1984, 1994), Italy (1990, 1993), Japan (1995) and the Soviet Union (1975, 1982, 1989), where he lectured essentially on different topics of Fluvial Hydraulics. Amongst the different distinctions received could be mentioned the "Freeman Fellowship" from the American Society of Civil Engineers in 1972, the "IAHR Lecturer Award" given by the International Association of Hydraulics Research in 1988, the "guest scholarship" from the Kajima Foundation of Tokyo, Japan in 1995 and more recently, in 1997, the K. E. Hilgard Hydraulic Prize, awarded annually by the American Society of Civil Engineers.

Walter H. Graf is the author or editor of four books on *Hydraulics of Sediment Transport* (1971, McGraw Hill, USA; 1984, Water Res. Publ., USA), *Hydrodynamics of Lakes* (1979, Elsevier, NL), *Lake and Reservoir Hydraulics* (1987, Water Res. Publ., USA) and *Hydrodynamique* (1991, Ed. Eyrolles, F.; 1995, Presses polytechniques et universitaires romandes, CH). More recently the two volumes of *Fluvial Hydraulics* (1993, 1996, Presses polytechniques et universitaires romandes, CH) were produced; the last books were done with the collaboration of M. S. Altinakar.

In 1987, the XXII Congress of the International Association of Hydraulic Research (IAHR) was held in Lausanne, under the direction of W. H. Graf with the participation of M. S. Altinakar; together six volumes of proceedings were edited.

In addition to being senior assistant at the Hydraulic Research Laboratory, EPFL, Dr. **M. S. Altinakar** is principal Engineer in the hydraulics department of the Bonnard & Gardel Eng. Consulting firm headquartered in Lausanne. There, M. S. Altinakar was in charge of various large projects in hydraulic, hydropower and municipal engineering.

FLUVIAL HYDRAULICS

Flow and Transport Processes in Channels of Simple Geometry

Walter H. Graf
in collaboration with M. S. Altinakar

Laboratoire de recherches hydrauliques
Ecole polytechnique fédérale
Lausanne, Suisse

JOHN WILEY & SONS
Chichester · New York · Weinheim · Brisbane · Singapore · Toronto

Published 1998 by John Wiley & Sons Ltd,
Baffins Lane, Chichester,
West Sussex PO19 1UD, England

National 01243 779777
International (+44) 1243 779777
e-mail (for orders and customer service enquiries): cs-books@wiley.co.uk
Visit our Home Page on http://www.wiley.co.uk
or http://www.wiley.com

This book, translated by the authors, was originally published in French entitled Hydraulique Fluviale. Volume 1: Ecoulement permanent uniforme et non uniforme. Copyright © 1993 Presses polytechniques et universitaires romandes, Lausanne, Suisse. ISBN: 2-88074-261-7. Volume 2: Ecoulement non permanent et phénomènes de transport. Copyright © 1996 Presses polytechniques et universitaires romandes, Lausanne, Suisse. ISBN: 2-88074-300-1.

Other Wiley Editorial Offices

John Wiley & Sons, Inc., 605 Third Avenue,
New York, NY 10158-0012, USA

WILEY-VCH Verlag GmbH, Pappelallee 3,
D-69469 Weinheim, Germany

Jacaranda Wiley Ltd, 33 Park Road, Milton,
Queensland 4064, Australia

John Wiley & Sons (Asia) Pte Ltd, 2 Clementi Loop #02-01,
Jin Xing Distripark, Singapore 129809

John Wiley & Sons (Canada) Ltd, 22 Worcester Road,
Rexdale, Ontario M9W 1L1, Canada

British Library Cataloguing in Publication Data

A catalogue record for this book is available from the British Library

ISBN 0-471-97714-4

Printed from camera-ready-copy supplied by the authors
Printed and bound in Great Britain by Bookcraft (Bath) Ltd

This book is printed on acid-free paper responsibly manufactured from sustainable forestry, in which at least two trees are planted for each one used for paper production.

Contents

1. INTRODUCTION

In this book on *fluvial hydraulics* — here taken to be synonymous to open-channel hydraulics — we shall treat the flow and flow-related phenomena in artificial and natural channels with a free surface subjected to the atmospheric pressure.

This chapter of introduction begins with a presentation of the different types of channels as well as with the corresponding flow regimes. Subsequently, the notions of the distribution of velocity and of pressure are exposed.

A list of references as well as a list of symbols shall be presented in the final pages of this volume.

TABLE OF CONTENTS

1.1. CHANNELS

1° A channel is a transport system where water flows and where the free surface is subject to atmospheric pressure.

2° The hydraulic study of a channel often confronts the engineer with a question of the form :

for a given longitudinal bed slope, a certain discharge must be conveyed; the form and the dimensions of the channel are to be determined.

1.1.1 Kinds of Channels (see Fig. 1.1)

1° Two categories of channels are to be distinguished :

i) natural channels,
ii) artificial channels.

2° *Natural* channels are watercourses, which exist naturally on (or under) the earth, such as gullies, brooks, torrents, rivers, streams and estuaries.

The geometric and hydraulic properties of such channels are generally rather irregular. The application of the hydraulic theory gives only approximate results, since numerous assumptions have to be made.

3° *Artificial* channels are watercourses developed by men on (or under) the earth, such as open channels (navigation channels, power canals, irrigation and drainage channels) or closed channels where flow does not fill the entire section (hydraulic tunnels, aqueducts, drains, sewage canals).

The geometric and hydraulic properties of such channels are generally rather regular. The application of the hydraulic theory gives reasonably realistic results.

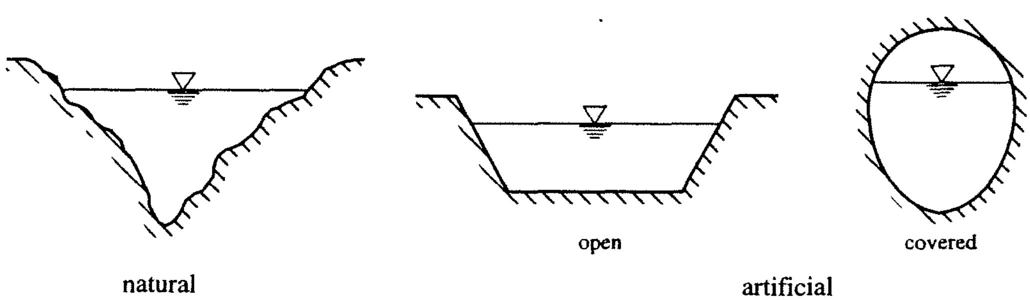


Fig. 1.1 Kinds of channels.

1.1.2 Geometry of Channels (see Fig. 1.2)

1° The (transversal) section of a channel is a section in the cross-sectional plane being normal to the direction of flow.

The section, or better the *wetted surface,* A, is the portion of the cross section occupied by the liquid.

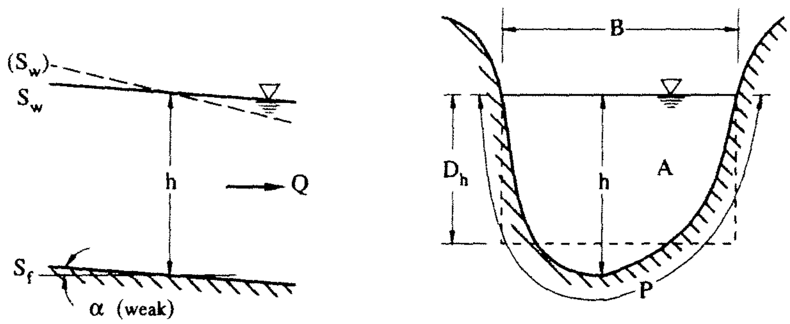


Fig. 1.2 Geometric elements of a channel section.

2° A channel, whose section does not vary and whose longitudinal slope and roughness remains constant — however the flow depth may vary — is called a *prismatic* channel; otherwise the channel is a non-prismatic one.

3° The geometric elements of a section or wetted surface, A, are the following :

i) The *wetted perimeter,* P, of the channel, being formed by the length of the line of contact between the wetted surface and the bed and the side walls, but does not include the free-water surface.

ii) The *hydraulic radius,* R_h , being the ratio of wetted surface, A, to its wetted perimeter, P , or :

$$R_h = \frac{A}{P} \tag{1.1}$$

being often used as a length of reference.

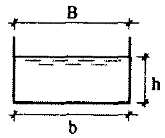
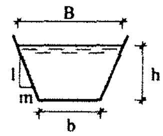
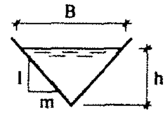
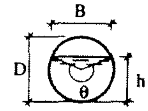
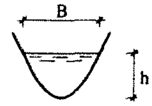
iii) The (top) *width,* B, of the channel being the width at the free surface.

iv) The *hydraulic depth,* D_h , of the channel being defined by :

$$D_h = \frac{A}{B} \tag{1.2}$$

v) The *flow depth,* h, or the water height — if not defined otherwise — is considered to be the maximum depth.

Table 1.1 Geometric elements for different sections of channels.

	 Rectangle	 Trapezoid	 Triangle	 Circle	 Parabola
Section A	$b\,h$	$(b+mh)h$	mh^2	$\frac{1}{8}(\theta - \sin\theta)\,D^2$	$\frac{2}{3}B\,h$
Wetted perimeter P	$b + 2h$	$b + 2h\sqrt{1+m^2}$	$2h\sqrt{1+m^2}$	$\frac{1}{2}\,\theta D$	$B + \frac{8}{3}\frac{h^2}{B}$ *
Hydraulic radius R_h	$\frac{b\,h}{b+2h}$	$\frac{(b+mh)\,h}{b+2h\sqrt{1+m^2}}$	$\frac{mh}{2\sqrt{1+m^2}}$	$\frac{1}{4}\left[1-\frac{\sin\theta}{\theta}\right]$ D	$\frac{2B^2h}{3B^2+8h^2}$ *
Width B	b	$b + 2mh$	$2mh$	$(\sin\theta/2)\,D$ or $2\sqrt{h\,(D-h)}$	$\frac{3}{2}\frac{A}{h}$
Hydraulic depth D_h	h	$\frac{(b+mh)\,h}{b+2mh}$	$\frac{1}{2}h$	$\left[\frac{\theta \angle \sin\theta}{\sin\theta/2}\right]\frac{D}{8}$	$\frac{2}{3}h$

* Valid for $0 < \xi \leq 1$, with $\xi = 4h/B$. If $\xi > 1$: $P = (B/2)\left[\sqrt{1+\xi^2} + 1/\xi\,\ln\left(\xi + \sqrt{1+\xi^2}\right)\right]$

4° Formulas for the geometric elements for five different types of channel sections (see *Chow*, 1959, p. 21) are given in Table 1.1. A natural watercourse might have a rather irregular geometric form, but often it can be rather well approximated by a trapezoidal or parabolic section.

5° Besides the geometric elements, the longitudinal slopes are also to be considered, namely the :

i) slope of the bed (bottom or floor), S_f,

ii) slope of the water surface (piezometric), S_w.

The value of the bottom slope depends essentially on the topography of the terrain; it is generally weak, thus may be expressed by : $S_f = \text{tg}\,\alpha \cong \sin\alpha$.

6° The wetted perimeter, P, can be composed of a fixed or immobile bed (concrete, rock) or of a mobile bed (granulates of sediments).

1.2 FLOW IN CHANNELS

1° Flow in natural or artificial channels is flow with a free surface, being the surface of separation between air and water; the pressure is there equal to the *atmospheric pressure.*

2° Flow in open channels is essentially due to the inclination (slope) of the bed, while flow in closed conduits (see *Graf & Altinakar*, 1991, chap. PP.1), is due to a difference in the charge between the sections.

1.2.1 Types of Flow

1° A classification of open-channel flow may be done according to the change of the flow depth, h or D_h, with respect to time and space :

$$D_h = f(t, x)$$

2° *Time variation* (see Fig. 1.3)

Flow is *steady* (stationary or permanent) if the average velocity of flow, U , and point velocity, u , but also the flow depth, h or D_h, remain invariable with time, in magnitude and in direction. Consequently the discharge remains constant :

$$U\,S = Q \tag{1.3}$$

between the different sections of the channel (see sect. 2.1 and eq. 2.6), supposing there is not lateral inflow or outflow.

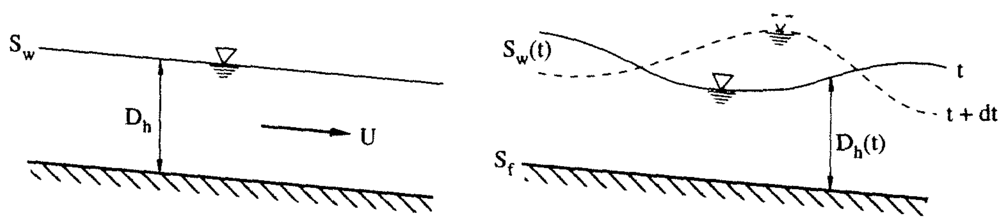


Fig. 1.3 Scheme of steady and unsteady flows.

Flow is *unsteady* if the flow depth, $D_h(t)$, as well as the other parameters vary with time. Consequently, the discharge is no more constant (see sect. 2.1 and eq. 2.1).

Strictly speaking, open-channel flow is rarely steady. However, the temporal variations are often sufficiently slow and the flow may be assumed to be steady and this at least for relatively short time intervals.

3° *Space variation* (see Fig. 1.4)

Flow is *uniform* if the flow depth, D_h , as well as the other parameters, remain unchanged at every section of the channel. The line of the bottom slope is thus parallel to the one of the free-water surface, or $S_f \equiv S_w$.

Flow is *non-uniform* or *varied* if the depth, $D_h(x)$, as well as the other parameters, vary along the length of the channel. The bottom slope is thus different from the slope of the water surface, or $S_f \neq S_w$.

Non-uniform flow can be steady or unsteady.

Varied flow can be accelerated, $dU/dx > 0$, or decelerated, $dU/dx < 0$, depending on the variation of velocity in the direction of flow.

If flow is a *gradually varied* one, the depth, $D_h(x) \cong D_h$, as well as the other parameters, vary slowly from one section to another. Over a small length of the channel, one may assume that the flow is quasi-uniform and the velocity, U, remains essentially constant.

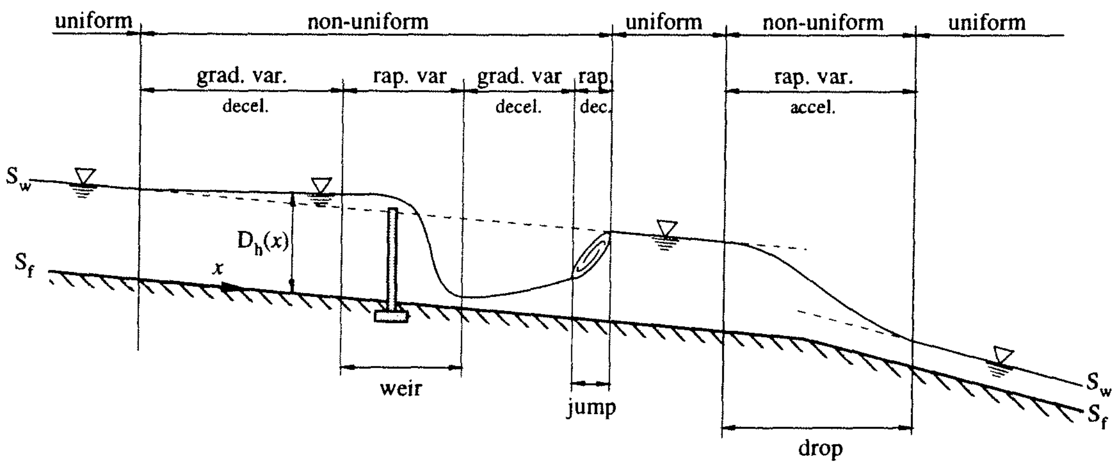


Fig. 1.4 Scheme of steady, uniform and non-uniform flows.

If flow is a *rapidly varied* one, the depth, $D_h(x)$, as well as the other parameters, change abruptly over a comparatively short distance, sometimes with a discontinuity. This happens generally in the neighbourhood of a singularity, such as at a weir or at a change of channel width, but also at an hydraulic jump or an hydraulic drop.

4° The kinds of flow one encounters in fluvial hydraulics (see Fig. 1.3 and Fig. 1.4) can be summarised as follows :

$$\text{i) Steady flow} \begin{cases} \text{uniform} \\ \text{non-uniform} \begin{cases} \text{gradually} \\ \text{rapidly} \end{cases} \end{cases}$$

$$\text{ii) Unsteady flow} \begin{cases} \text{uniform } (\textit{rare}) \\ \text{non-uniform} \begin{cases} \text{gradually} \\ \text{rapidly} \end{cases} \end{cases}$$

1.2.2 Flow Regimes

1° The physics of open-channel flow is governed basically by the interplay of the :

- inertia forces,
- gravity forces,
- friction (viscosity and roughness) forces.

2° The (reduced) equations of motion (see *Graf & Altinakar,* 1991, sect. FR.7.2) involve the following dimensionless numbers :

i) the *Froude number,* being the ratio of gravity to inertia forces, or :

$$\frac{\rho g}{\rho U_c^2 / L_c} = \frac{g L_c}{U_c^2} = Fr^{-2} \qquad \text{and} \qquad Fr = \frac{U_c}{\sqrt{g L_c}} \tag{1.4}$$

ii) the *Reynolds number,* being the ratio of friction to inertia forces, or :

$$\frac{\mu\,(U_c / L_c^2)}{\rho U_c^2 / L_c} = \frac{\nu}{U_c L_c} = Re^{-1} \qquad \text{and} \qquad Re = \frac{U_c L_c}{\nu} \tag{1.5}$$

Added to these two numbers is still :

iii) the *relative roughness,* being the ratio of the roughness height, k_s , to a characteristic length, or :

$$\frac{k_s}{L_c} \tag{1.6}$$

U_c and L_c are characteristic velocity and length; one takes often $U_c = U$ and $L_c = R_h$ or $L_c = D_h$.

In the hydraulics of open-channel flow, one generally defines these dimensionless numbers as :

$$Fr = \frac{U}{\sqrt{gD_h}} \quad ; \quad Re = \frac{4R_hU}{\nu} \quad \text{or} \quad Re' = \frac{R_hU}{\nu} \quad ; \quad \frac{k_s}{D_h} \tag{1.7}$$

3° The Reynolds number is used to classify the flow (see *Graf & Altinakar*, 1991, chap. FR.3) as follows :

- laminar flow $Re' < 500$
- turbulent flow $Re' > 2000$
- transition flow $500 < Re' < 2000$

From numerous experiments with different artificial channels (see *Chow*, 1959, p. 10) it results that flow is turbulent if the Reynolds number, Re', reaches a value of 2000.

In general, flow in open channels is a turbulent and often rough flow.

4° The Froude number is used to classify the flow (see sect. 2.3.3) as follows :

- subcritical (fluvial) flow $Fr < 1$
- supercritical (torrential) flow $Fr > 1$
- critical flow $Fr \equiv Fr_c = 1$

In general, flow in open channels can be of the three types.

5° Consequently, the combined effect of the Reynolds number, Re', and the Froude number, Fr , gives the following four regimes of flow :

- subcritical-laminar $Fr < 1$, $Re' < 500$
- subcritical-turbulent $Fr < 1$, $Re' > 2000$
- supercritical-laminar $Fr > 1$, $Re' < 500$
- supercritical-turbulent $Fr > 1$, $Re' > 2000$

A relationship depth/velocity, taken from the experiments by *Robertson et Rouse*, is given in Fig. 1.5; it is valid for very wide rectangular channels.

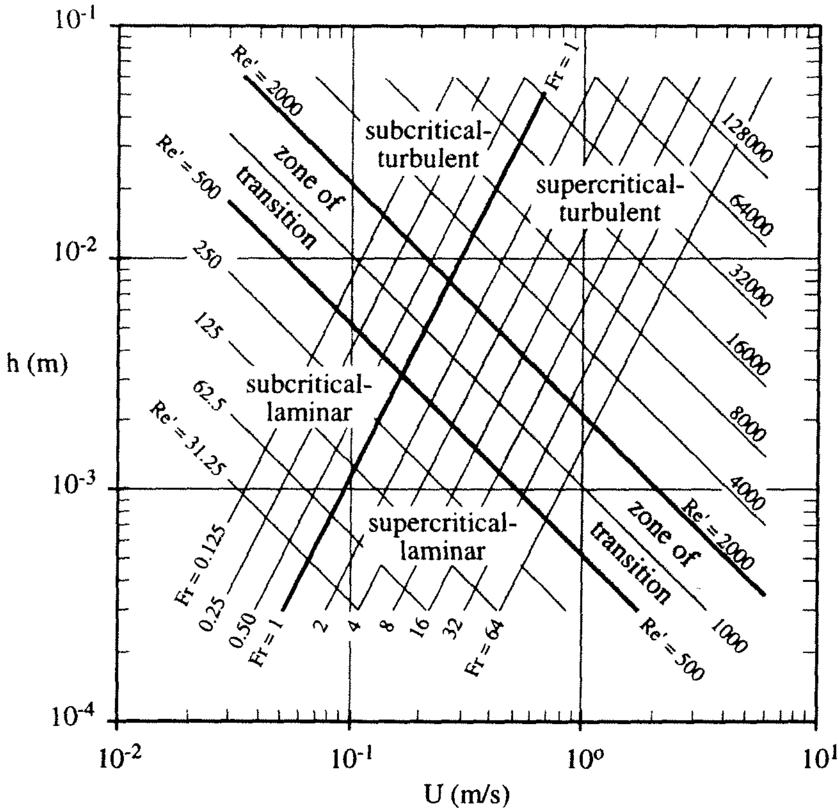


Fig. 1.5 The four regimes of open-channel flow.

1.3 DISTRIBUTION OF VELOCITY

1° In flow along a wall (the bottom of a channel), a distribution of velocity (see *Graf & Altinakar*, 1991, chap. FR. 6) is encountered. Being zero at the wall, the point velocity, u , increases rapidly towards the free surface; its maximum value is often found slightly below this free surface. The velocity profile is approximately logarithmic.

2° Steady flow depends in general on the three variables, x, y and z ; this is called *three-dimensional* flow. For a rectangular channel with a bed and vertical side walls, a schematic distribution of the point velocity, $u(x, y, z)$, is given in Fig. 1.6.

If such a channel has a large width, B — large in comparison with the depth, B > 5 h — flow is considered *two-dimensional*, with the exception of a small distance close to the vertical side walls.

Hydraulic calculations are considerably simplified, if one assumes the flow to be *one-dimensional*. The average velocity, $U(x)$, in a vertical or in a section, is expressed by :

$$U = \frac{1}{h}\int_0^h u(z)\,dz \qquad \text{or} \qquad U = \frac{1}{A}\int_0^B\int_0^h u(z)\,dz\,dy \tag{1.8}$$

3° In open channels of simple geometry, one encounters generally turbulent flow where the point velocity, $u(x, z)$, differs little from the average velocity, $U(x)$. In the steady state, such an hypothesis allows to consider the flow as one-dimensional.

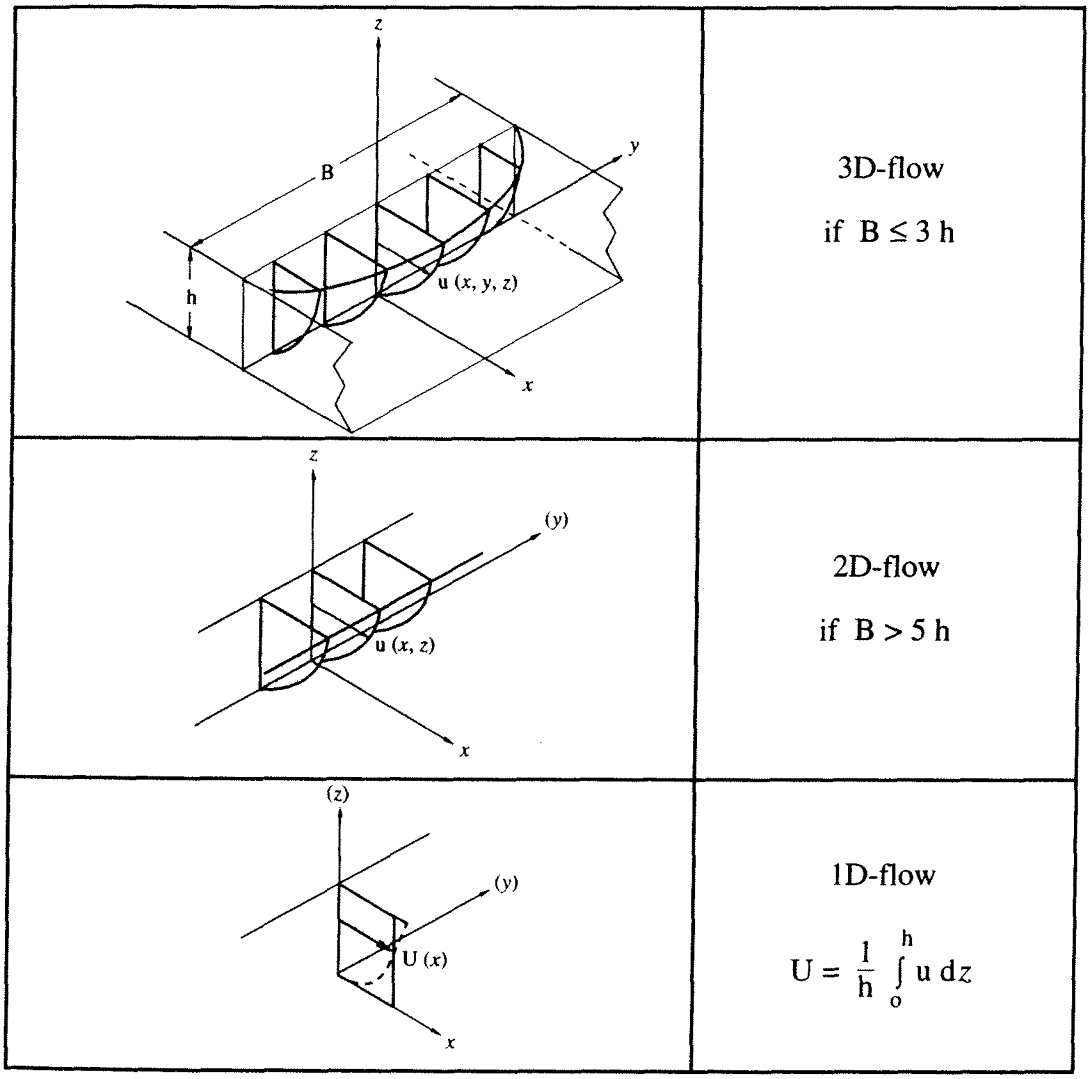


Fig. 1.6 Distribution of velocity.

4° For a determination of the average velocity, U , in a given section, the following approximate relations can be used (see Fig. 1.7) :

$$U \cong (0.8 \text{ à } 0.9)\, u_s \qquad \text{(formula of Prony)}$$
$$U \cong 0.5\,(u_{0.2} + u_{0.8}) \qquad \text{(formula of USGS)} \qquad (1.9)$$
$$U \cong u_{0.4}$$

where $u_{0.2}$, $u_{0.8}$, $u_{0.4}$ and u_s are the point velocities at given positions .

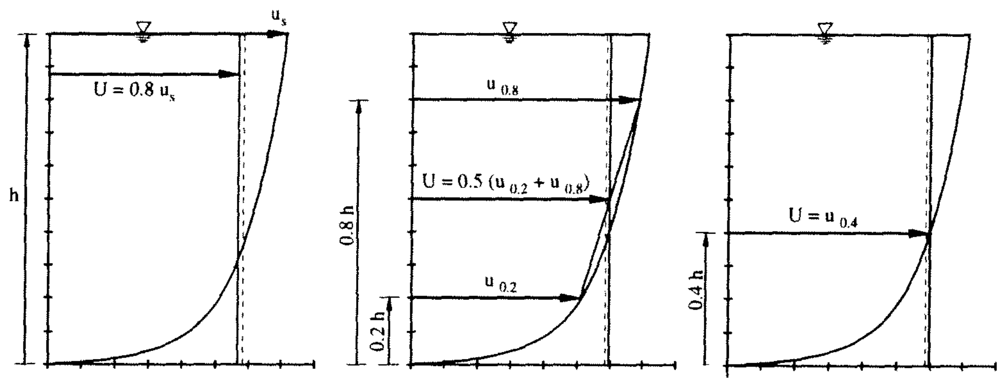


Fig. 1.7 Average velocity.

1.4 DISTRIBUTION OF PRESSURE

1° The equation of steady motion for an incompressible fluid (see *Graf & Altinakar*, 1991, p. 132), written for the normal component, n(≡ z), is :

$$U\frac{U}{r} = -\frac{1}{\rho}\frac{\partial}{\partial n}(p + \gamma z') \qquad (1.10)$$

where (U^2/r) is the centrifugal acceleration of a mass-fluid, which displaces itself on a curved line, r (see Fig. 1.8).

2° Assuming that U and r remain reasonably constant and after integration of eq. 1.10, one obtains :

$$(p + \gamma z') = -\rho \int_0^z \frac{U^2}{r}\,dn + Cte = -\rho\,\frac{U^2}{r}\,z + Cte \qquad (1.10a)$$

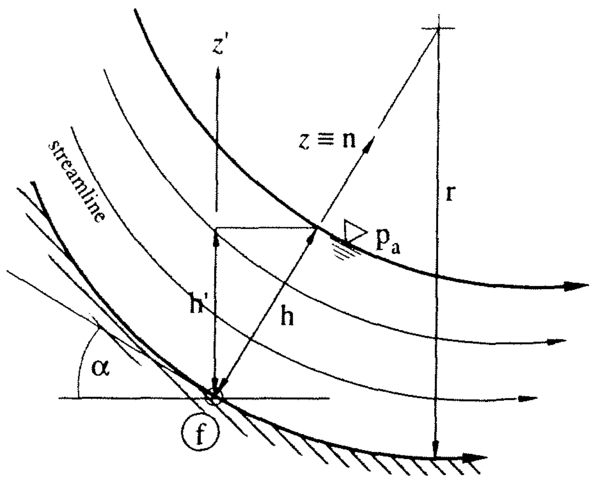


Fig. 1.8 Flow over a concave bottom.

3° Taking a point on the bottom of the channel and another one on the free surface, one respectively writes :

for $z = 0$ ($z' = 0$) : $p = p_f$ where : $p_f = Cte$

for $z = h$ ($z' = h'$) : $p = p_a$ where : $p_a + \gamma h' = -\rho \frac{U^2}{r} h + Cte$

An expression for the relative (with respect to the atmospheric pressure) pressure on the bottom of the channel, is given by :

$$p_f = \gamma h' + \rho \frac{U^2}{r} h + \underbrace{\cancel{p_a}}_{=0} \tag{1.11}$$

having an hydrostatic and an accelerating contribution.

1.4.1 Uniform Current

1° For uniform flow, when the average velocity, U, remains constant and the streamlines are reasonable rectilinear (with $r \rightarrow \infty$), the distribution of pressure is *hydrostatic* in a section, normal to the bottom (see Fig. 1.9). Thus one may write, taking $z \equiv n$ (eq. 1.10), the following :

$$0 = \frac{\partial}{\partial z} (\gamma z' + p) \tag{1.12}$$

2° An expression for the pressure, relative to the bottom, can now be given as :

$$p_f = +\gamma h' \tag{1.13}$$

which gives :

$$\left(\frac{p}{\gamma}\right)_f = h \cos\alpha \tag{1.14}$$

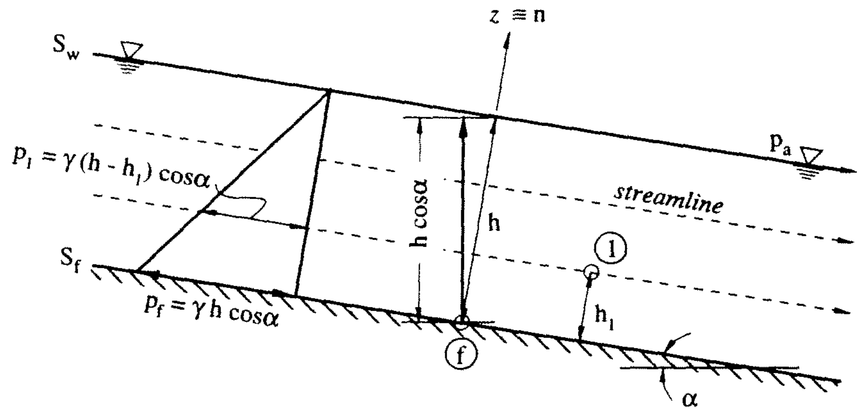


Fig. 1.9 Flow with a uniform current.

3° For the usually encountered open channels, the inclination, α , is rather weak, namely $\alpha < 6°$ or $J_f < 0.1$, implying that $\cos\alpha \simeq 1$. Consequently eq. 1.14 reduces to :

$$\left(\frac{p}{\gamma}\right)_f = h \tag{1.15}$$

where h is the flow depth in the channel.

1.4.2 Curvilinear Current

1° For flow, being (slightly) non-uniform, thus having a curvilinear current of converging or diverging type, there exists an acceleration component caused by the inertia forces. As done above, one writes :

$$\frac{U^2}{r} = -\frac{1}{\rho}\frac{\partial}{\partial n}(p + \gamma z') \tag{1.10}$$

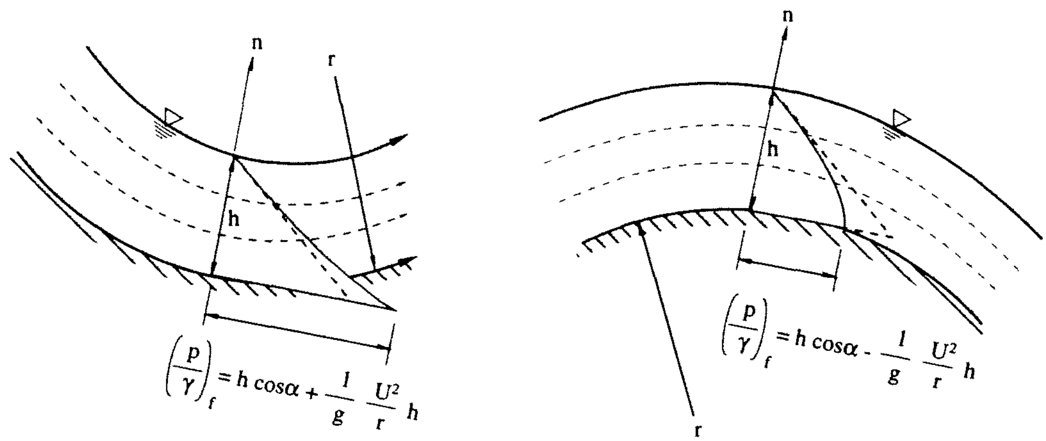


Fig. 1.10 Flow over a concave and a convex bottom.

and the expression for the pressure relative to the bottom is given by :

$$p_f = +\gamma h' \pm \rho \frac{U^2}{r} h \qquad (1.11)$$

being (+) for a concave and (–) for a convex bottom.

Subsequently one obtains :

$$\left(\frac{p}{\gamma}\right)_f = h \cos \alpha \pm \frac{1}{g} \frac{U^2}{r} h \qquad (1.16)$$

2° The distribution of pressure is no more hydrostatic (see Fig. 1.10). For an external concave current, the centrifugal force increases the pressure; while for a convex current, this force decreases the pressure. In the latter case, the pressure could get below the atmospheric pressure, thus causing separation of flow on the channel bed.

2. HYDRODYNAMIC CONSIDERATIONS

Some fundamental notions of hydrodynamics, being the basis of open-channel hydraulics, will be exposed in this chapter.

The equations of continuity and of energy will be developed for the general case. Subsequently, the specific energy, a concept useful for the understanding of different problems, will be introduced. Elementary knowledge of gravity waves is presented.

Finally the hydrodynamic equations are developed, as well as their applications to uniform and non-uniform flow. Experimental results, being a support to the theory, are presented, such as the distribution of velocity, the characteristics of turbulence and also the friction coefficients.

TABLE OF CONTENTS

2.1 EQUATION OF CONTINUITY

1° The equation of continuity, one of the basic equations of fluid mechanics, is an expression of the conservation of mass.

The variation of a mass fluid, contained in a given volume during a certain time, must equal the sum of the mass fluid which enters, diminished by the one which leaves.

2° Studied will be a flow of an incompressible fluid, being steady, uniform and almost rectilinear, in an open channel with a free surface and having a weak bed slope (see Fig. 2.1). Considered will be two channel sections; Q will be the entering discharge.

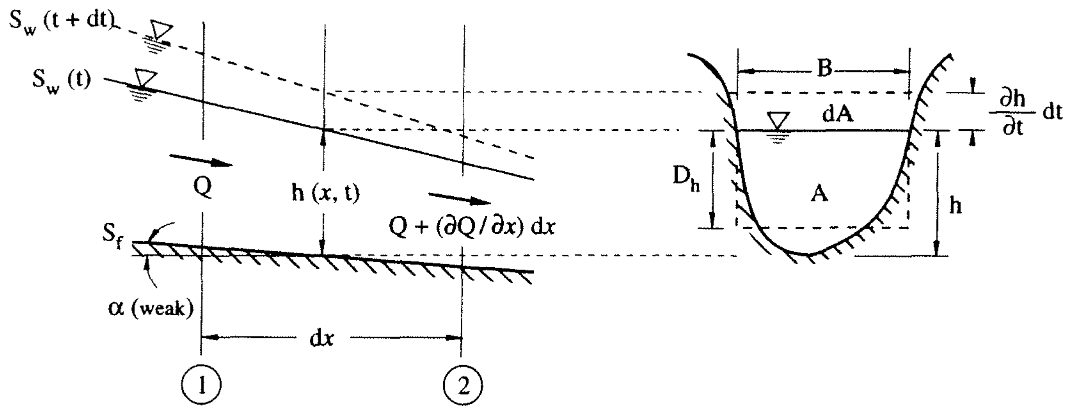


Fig. 2.1 Scheme for the equation of continuity.

The volume, entering by the first section is Qdt; the volume leaving by the second section, being at a distance, dx, from the first one, is $\left[Q + (\partial Q/\partial x)dx\right]dt$. The variation of the volume between these two sections during the time, dt, is consequently :

$$-\left(\frac{\partial Q}{\partial x}\right) dx\, dt$$

This variation of the volume is the result of a modification of the free surface, $\partial h/\partial t$, between the two sections during the time, dt; it is expressed by :

$$(B dx)\ \frac{\partial h}{\partial t}\ dt$$

where B(h) is the width of the channel at the free surface and $h(x,t)$ is the flow depth.

Assuming the fluid incompressible, the above two expressions are made equal (see *Chow*, 1959, p. 525) and one obtains:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{2.1}$$

where $dA = Bdh$.

3° For a given section, the following relation can be given :

$$Q = UA \tag{2.2}$$

where U is the average velocity in the section, A. Thus eq. 2.1 can be expressed as :

$$\frac{\partial (UA)}{\partial x} + B\frac{\partial h}{\partial t} = A\frac{\partial U}{\partial x} + U\frac{\partial A}{\partial x} + B\frac{\partial h}{\partial t} = 0 \tag{2.3}$$

Using the definition of the hydraulic depth, $D_h = A/B$, one can also write :

$$D_h\frac{\partial U}{\partial x} + U\frac{\partial D_h}{\partial x} + \frac{\partial h}{\partial t} = 0 \tag{2.4}$$

The above equations represent different forms of the equation of continuity, valid for prismatic channels (see sect. 5.1.1).

4° For a rectangular channel, eq. 2.3 is given by :

$$\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = h\frac{\partial U}{\partial x} + U\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0 \tag{2.5}$$

where $q = Q/B$ is the unit discharge.

5° For steady flow, $\partial A/\partial t = 0$, the equation of continuity, eq. 2.1, reduces to :

$$\frac{dQ}{dx} = 0 \tag{2.6}$$

6° If a supplementary discharge leaves (or enters) the channel between the two sections, eq. 2.1 can be adapted, such as :

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} \underset{(-)}{+} q_\ell = 0 \tag{2.7}$$

where q_ℓ is the supplementary discharge per unit length.

2.2 EQUATION OF ENERGY

1° The equation of energy is an expression of the first principle of thermodynamics.

2° The energy for an element of incompressible fluid, written in homogeneous quantities of length (see Fig. 2.2) — here as the height of a liquid with specific weight $\gamma = \rho g$ — in almost rectilinear flow, taken with respect to the plane of reference (PdR), is given by :

$$\frac{u^2}{2g} + \frac{p}{\gamma} + z_P = \frac{p_t}{\gamma} = \text{Cte} \tag{2.8}$$

The different terms represent :

$\dfrac{u^2}{2g}$ the velocity head

$\dfrac{p}{\gamma}$ the pressure head

z_P the elevation (position) of a point, P

$\dfrac{p_t}{\gamma} = H$ the (mechanical) energy head or the total head

$\dfrac{p}{\gamma} + z_P = \dfrac{p^*}{\gamma}$ the piezometric head

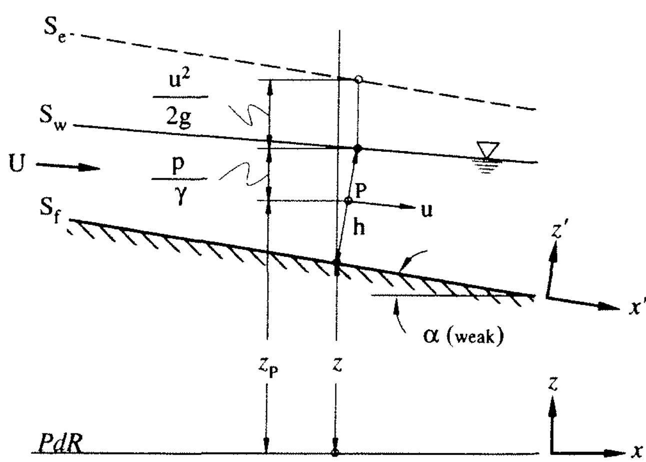


Fig. 2.2 Scheme for the equation of energy in a cross section.

3° The following assumptions shall be applied:

i) The piezometric head, p^*/γ, is supposed to be constant over a normal to the bed, implying that the distribution of pressure is hydrostatic.

ii) By considering that z gives the elevation of the bed, the slope (weak) of the channel, S_f, is given by :

$$S_f = \operatorname{tg} \alpha = -\frac{dz}{dx} \cong \sin \alpha$$

iii) If h is the flow depth, the pressure head at the bed of the channel (see eq. 1.14) is :

$$\left(\frac{p}{\gamma}\right)_f = h \cos \alpha$$

For weak slopes, $\alpha < 6°$, where $S_f < 0.1$, one may take $\cos \alpha \cong 1$. The system of the coordinates, xz, is thus almost identical with the one of the coordinates, $x'z'$, (see Fig. 2.2).

iv) In a perfect fluid, each fluid element moves with the same velocity, which is the average velocity in the section, U.

Making use of these reasonable assumptions, the total head in a section is now given by :

$$\frac{U^2}{2g} + h + z = H \tag{2.9}$$

The flow is here considered to be one-dimensional and rectilinear.

The equation of energy, eq. 2.9, is a manifestation of the principle of energy if the liquid is *perfect*. From one to another section, each of the three terms in eq. 2.9 can take a different value, but the sum, H, remains constant.

4° For flow of a *real* fluid with a free surface, being unsteady and non-uniform (gradually varied), the difference of the total head between two sections, separated by a distance, dx, (see Fig. 2.3) is given as :

$$\alpha_e \frac{U^2}{2g} + h + z = \left[\alpha_e \frac{U^2}{2g} + d\left(\alpha_e \frac{U^2}{2g}\right)\right] + [h + dh] + [z + dz] +$$
$$+ \frac{1}{g}\frac{\partial U}{\partial t} dx + \frac{1}{g}\frac{\tau_0}{\rho}\frac{dP}{dA} dx \tag{2.10}$$

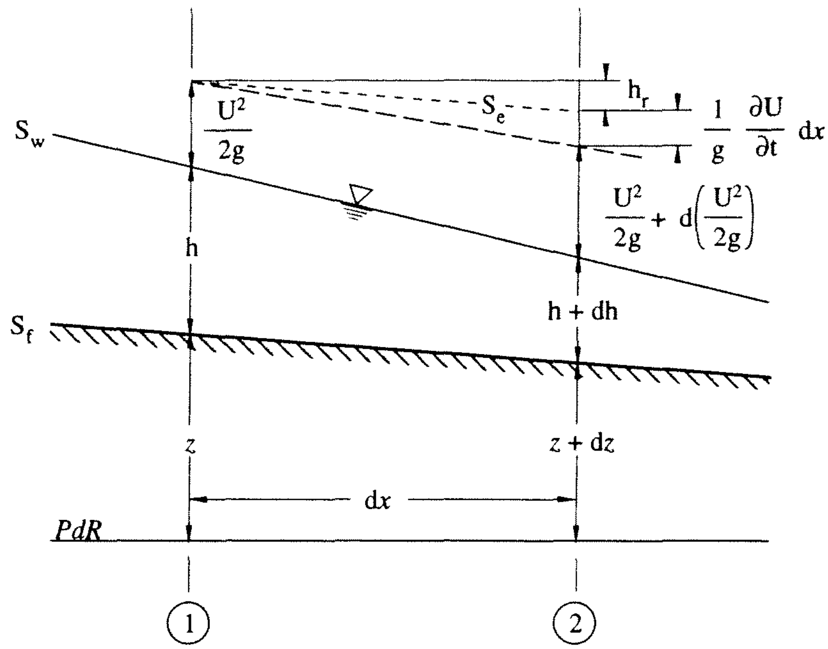


Fig. 2.3 Scheme of the equation of energy, between two sections.

i) $\frac{1}{g}\frac{\partial U}{\partial t}dx$ is the term of energy due to acceleration in the flow x-direction (see *Graf & Altinakar,* 1991, p. 137).

ii) $\frac{1}{g}\frac{\tau_o}{\rho}\frac{dP}{dA}dx = h_r$ is the term of energy or head loss due to friction (see *Graf & Altinakar,* 1991, p. 138);

The friction forces provoke a dissipation of mechanical (into thermal) energy. dP is the perimeter of an elementary surface, dA, and τ_o is the shear stress due to the frictional forces acting on the surface, $dPdx$. This term, representing the effect of friction, is usually written as h_r.

iii) The kinetic energy correction coefficient, α_e, results from the distribution of the velocity in the section (see Fig. 1.6). Its numerical values (see *Chow,* 1959, p. 28) notably for turbulent flow are very close to unity. In most common cases, the velocity head can thus be taken as :

$$\alpha_e \frac{U^2}{2g} \approx \frac{U^2}{2g}$$

where U is the average velocity in the section

The equation of energy, eq. 2.10, can thus be given as :

$$d\left(\frac{U^2}{2g} + h + z\right) = -\,h_r - \frac{1}{g}\,\frac{\partial U}{\partial t}\,dx \tag{2.11}$$

Dividing by the distance, dx, and using partial differentials, one gets :

$$\frac{1}{g}\,\frac{\partial U}{\partial t} \;+\; \frac{U}{g}\,\frac{\partial U}{\partial x} \;+\; \frac{\partial h}{\partial x} \;-\; S_f \;=\; -\,S_e \tag{2.12}$$

steady, uniform

steady, non-uniform

unsteady, non-uniform

where $h_r = S_e\,dx$ and $S_f = -\,(dz/dx)$; S_e is the energy slope.

Eq. 2.12 is the dynamic equation for unsteady and non-uniform flow.

The head loss, h_r, must be evaluated with a formula such as the one of Weisbach-Darcy, eq. 3.10, of Chézy, eq. 3.11, or of other experimenters. Such relations are only valid for steady, uniform flow; however — for lack of better information — they are also used (see *Chow*, 1959, p. 217) for unsteady and non-uniform flow.

5° The equation of continuity, eq. 2.5, and the dynamic equation of motion, eq. 2.12, form together the *equations of Barré de Saint-Venant* (see *Chow* 1959, p. 528). Despite the various simplifications made to obtain the equations of Saint-Venant, their solutions are often rather complicated. In some physical cases, which are simple but still realistic, explicit solutions are possible.

6° For flow, which is steady but non-uniform, eq. 2.12 reduces to :

$$\frac{U}{g}\,\frac{\partial U}{\partial x} + \frac{\partial h}{\partial x} - S_f = -\,S_e \tag{2.13}$$

7° For flow, which is steady and uniform, eq. 2.12 reduces to :

$$S_f = S_e \tag{2.13a}$$

The bed slope, S_f , the energy slope, S_e , and the piezometric slope of the water surface, S_w , are identical. The average velocity, U , and the flow depth, h , are constant; the equation of continuity is given with eq. 2.6.

8° The dynamic equation of motion, eq. 2.12, can also be obtained by applying the momentum equation. The resulting equation is almost the same (see *Chow*, 1959, p. 51 and *Henderson*, 1966, p. 9), i.e. : eq. 2.12.

2.3 SPECIFIC ENERGY

1° Up till now, the total head, H , in a given cross section was defined with respect to an arbitrary horizontal plane (see Fig. 2.2); for a weak bed slope one writes :

$$\frac{U^2}{2g} + h + z = H \tag{2.9}$$

If the plane of reference is now placed into the bed slope, S_f , a fraction of the total head, called the *specific energy*, H_s , is defined (see *Bakmeteff*, 1932, chap. 4); one writes now (see Fig. 2.4) :

$$\frac{U^2}{2g} + h = H_s \tag{2.14}$$

Using the equation of continuity, Q = UA, one obtains :

$$\frac{Q^2/A^2}{2g} + h = H_s \tag{2.14a}$$

2° The notion of the specific energy is often very useful; it helps to understand and to solve different problems of free-surface flow.

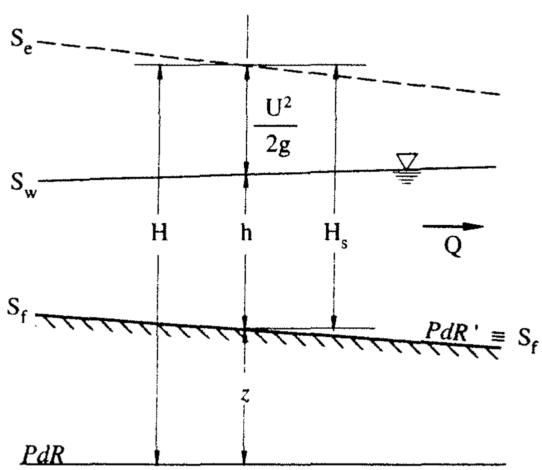


Fig. 2.4 Definition of the total head, H, and the specific energy, H_s .

3° For a given section in the channel, the area of flow, A, is a function of the flow depth, h , and eq. 2.14a gives a relation of the following form :

$$H_s = f(Q, h)$$

which allows a study of the variation of :

i)	h	with	H_s	,	for	Q = Cte
ii)	h	with	Q	,	for	H_s = Cte.

2.3.1 Specific-energy Curve

1° Eq. 2.14a gives the evolution of the specific energy, H_s, as a function of the flow depth, h, for a given discharge, $Q = UA$.

This curve (see Fig. 2.5) has two asymptotes :

i) for $h = 0$, a horizontal asymptote ,
ii) for $h = \infty$, the line $h = H_s$ is the other asymptote.

In addition, the curve has a minimum, H_{s_c} , for :

$$\frac{dH_s}{dh} = -\frac{Q^2}{gA^3}\frac{dA}{dh} + 1 = 0 \qquad (2.15)$$

Since dA/dh is equal to the width of the channel, B, at the free surface and by using the definition of the hydraulic depth, $D_h = A/B$, one obtains :

$$\frac{Q^2}{g}\frac{B}{A^3} = \frac{U^2}{gD_h} = 1 \qquad (2.15a)$$

2° For a channel with a rectangular cross section, one has $D_h = h$. The flow depth, h, which corresponds to the minimal specific energy, H_{s_c}, is called *critical depth*, h_c.

3° Following the curve, given with Fig. 2.5, one notices that for a given discharge Q = Cte , and for an arbitrary value of specific energy, H_s, — for the case when flow can take place — there are always two solutions for the flow depth, h_1 and h_2. They are called the corresponding (alternate) depths; one of which, h_1, is smaller and the other one, h_2, is larger than the critical depth, h_c. Both of these depths are indications of different regimes of flow, thus :

$h < h_c$ *supercritical* (torrential) regime

$h > h_c$ *subcritical* (fluvial) regime

$h \equiv h_c$ *critical* regime

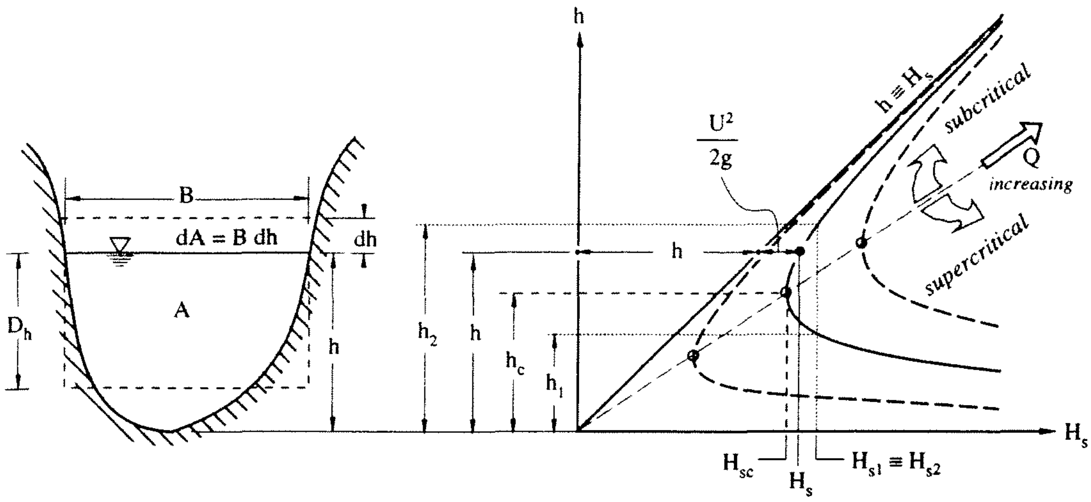


Fig. 2.5 Specific-energy curve, $H_s = f(h)$, for Q = Cte.

Each curve (see Fig. 2.5) has thus two branches. Consequently, a steady flow in a channel can exist in two different ways, both having the same specific energy, H_s :

i) in supercritical regime, where the flow depth is small and the velocity large,
ii) in subcritical regime, where the flow depth is large and the velocity small.

4° For a variation of the discharge, Q, the corresponding curves have the same form ; they follow each other for an increase in the discharge, starting at the origin, (see Fig. 2.5).

2.3.2 Discharge Curve

1° Eq. 2.14a gives also the evolution of the discharge, Q, as a function of the flow depth, h, for a given specific energy, H_s , such as :

$$Q = A \sqrt{2g\ (H_s - h)} \tag{2.16}$$

From this curve (see Fig. 2.6), one obtains :

i) for $h = 0$, $Q = 0$
ii) for $h = H_s$, $Q = 0$.

2° In addition, the curve has a maximum value, Q_{max} , for :

$$\frac{dQ}{dh} = \frac{2g\ (H_s - h)\ (dA/dh) - Ag}{\left[2g\ (H_s - h)\right]^{1/2}} = 0$$

Taking $dA/dh = B$ and $D_h = A/B$, one may write :

$$\frac{dQ}{dh} = \frac{gB\left[2\,(H_s - h) - D_h\right]}{\left[2g\,(H_s - h)\right]^{1/2}} = 0 \tag{2.17}$$

This derivative is zero, if :

$$2\,(H_s - h) - D_h = 0 \tag{2.18}$$

The values, h and D_h, which correspond to the maximum discharge, Q_{max}, represent the *critical depth*, h_c et D_{h_c}. For flows smaller than Q_{max}, one finds again the two different flow regimes (see Fig. 2.6 and also Fig. 2.5).

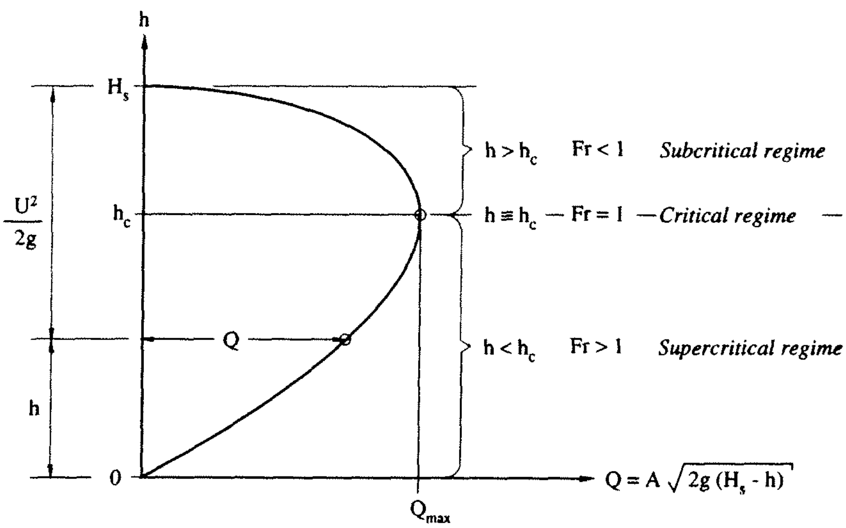


Fig. 2.6 Discharge curve, $Q = f(h)$, for H_s = Cte.

3° For a channel with a rectangular cross section, $D_h \equiv h$, eq. 2.18 becomes :

$$2\,(H_s - h) - h = 0$$

from where one obtains the critical depth ($h \equiv h_c$ and $H_s \equiv H_{s_c}$) :

$$h_c = \frac{2}{3}\,H_{s_c} \tag{2.19}$$

For a channel with a triangular or parabolic cross section, one obtains respectively :

$$h_c = \frac{4}{5}\,H_{s_c} \qquad \text{and} \qquad h_c = \frac{3}{4}\,H_{s_c} \tag{2.19a, b}$$

2.3.3 Critical Depth

1° The critical depth, h_c, in a channel is the flow depth at which :

i) the specific energy is minimal, H_{s_c}, for a given discharge (see Fig. 2.5),
ii) the discharge is maximal, Q_{max}, for a given specific energy (see Fig. 2.6).

2° It follows, that eq. 2.18 can be written as :

$$2\,(H_{s_c} - h_c) = D_{h_c}$$

and that, using eq. 2.16, the maximum discharge, Q_{max}, is given by :

$$Q_{max} = A\,\sqrt{gD_{h_c}} \tag{2.20}$$

The average velocity, which corresponds to the critical hydraulic depth, D_{h_c}, is :

$$U_c = \sqrt{gD_{h_c}} \qquad \text{or} \qquad \frac{U_c^{\,2}}{2g} = \frac{D_{h_c}}{2} \tag{2.21}$$

In critical regime, the velocity head is thus equal to half of the hydraulic depth.

3° Eq. 2.21 or eq. 2.15a could also be expressed as :

$$\frac{U_c}{\sqrt{gD_{h_c}}} = 1 \tag{2.22}$$

which is precisely the definition of the Froude number (see eq. 1.4) in critical regime ; here the Froude number, Fr, is equal to unity :

$$Fr_c = 1 \tag{2.22a}$$

Note that the Froude number, $Fr = U/\sqrt{gD_h}$, is the ratio of inertia to gravity forces per unit volume (see *Graf & Altinakar*, 1991, sect. FR. 7.3). Consequently, the Froude number classifies also the different flow regimes, such as :

$Fr > 1$	*supercritical* regime	$U > U_c$
$Fr < 1$	*subcritical* regime	$U < U_c$
$Fr = 1$	*critical* regime	$U \equiv U_c$

4° The critical velocity, U_c, is given by :

$$U_c = \sqrt{gD_{h_c}} = c \qquad (2.21a)$$

This is equal to the celerity, c, of the propagation of (superficial) infinitesimal gravity waves in a channel of hydraulic depth, D_{h_c} (see eq. 2.27 for the general definition).

5° The critical depth for a rectangular channel, $D_h \equiv h$, has been given by :

$$h_c = \frac{2}{3} H_{s_c} \qquad (2.19)$$

or equally by :

$$(H_{s_c} - h_c) = \frac{h_c}{2} = \frac{U_c^2}{2g}$$

Using the definition of the unit discharge, $q = Uh$, one obtains :

$$\frac{h_c}{2} = \frac{q^2}{2gh_c^2} \qquad \text{or} \qquad h_c = \sqrt[3]{\frac{q^2}{g}} \qquad (2.23)$$

The maximum unit discharge, q, which may exist in a channel of rectangular section is equal to :

$$q = \sqrt{gh_c^3} = \sqrt{g\left(\frac{2}{3} H_{s_c}\right)^3} \qquad (2.24)$$

6° Experience shows that flow at critical depth, h_c, is often *unstable*, presenting itself by a fluctuating water surface. This is rather evident when observing Fig. 2.5 : even small variations of energy close to the critical value, H_{s_c}, cause large variations in the flow depth, h.

7° According to eq. 2.20 and eq. 2.24, the critical hydraulic depth, D_{h_c}, or the critical flow depth, h_c, depend only on the discharge. Thus it is inviting to use this information for metering flow in open channels :

Here, two examples are given :

i) *Free Overfall* : Flow in an horizontal channel (or a broad-crested weir) discharges freely into the atmosphere; the critical section is found rather close to the brink (see sect. 4.4.2).

ii) *Venturi Canal* : An adequate reduction in the cross section of the channel is provided, where the critical regime (see sect. 4.4.2) takes place.

8° Flow goes through the critical depth, if the fluvial regime passes to the torrential one. Critical depth is also observed, if the fluvial regime is terminated by a free overfall.

2.4 GRAVITY WAVES

Flow in open channels, which is variable in time, is accompanied by gravity waves at the water surface.

2.4.1 Wave Celerity

1° Considered will be a periodic, simple wave, representing the propagation of an irrotational motion as satisfied by the equation of Laplace; the pressure at the free surface is constant and the wave amplitudes are small. A channel of rectangular cross section with uniform flow depth is filled with stagnant water; there is thus no flow.

2° The two-dimensional and progressive wave in the $^-x^+$ direction, will be given (see Fig. 2.7) by a periodic displacement of the free surface as a function of time, t, (see *Kinsman*, 1965, p. 117), such as:

$$\eta\,(x, t) = A \cos\,(2\pi\, x/L - 2\pi\, t/T) \qquad (2.25)$$

where A is the amplitude, being half of the wave height, H = 2A; L is the wave length and T is the wave period. The wave celerity is defined by :

$$c = \frac{L}{T} \qquad (2.25a)$$

3° The hydrodynamic theory for waves of *small amplitude* (see *Lamb*, 1945, pp. 254 and 366, or *Kinsman*, 1965, p. 125), i.e. : $H/L \ll 1$ and $H/h \ll 1$, gives for the apparent velocity of propagation, also called the *celerity* of a perturbation :

$$c^2 = \frac{gL}{2\pi} \tanh\left(\frac{2\pi h}{L}\right) \qquad (2.26)$$

where h is the water depth. Note that the celerity does not depend on the wave height, H.

4° This expression, eq. 2.26, reduces :

i) for *short* waves or waves of large depth, if L/h < 1, to :

$$c^2 = \frac{gL}{2\pi} \tag{2.26a}$$

ii) for *long* waves or waves of small depth, if L/h >> 1, to :

$$c^2 = gh \tag{2.27}$$

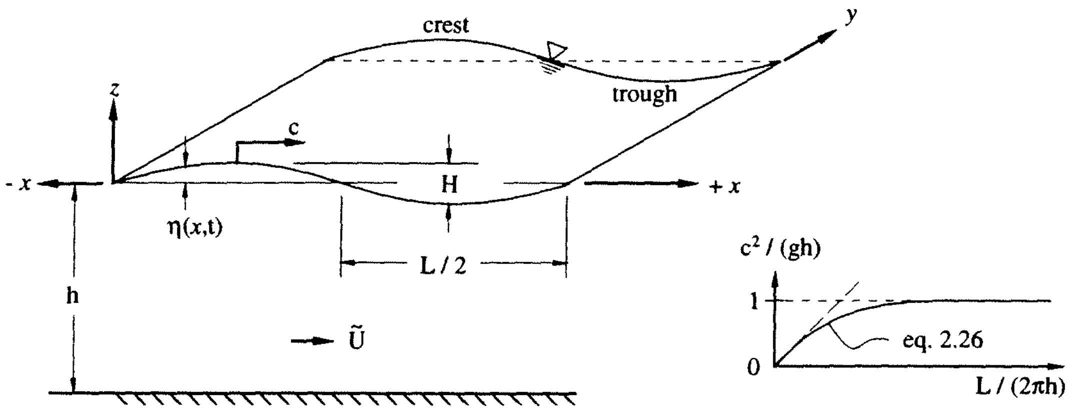


Fig. 2.7 Scheme of a surface wave.

5° If the long wave, where L/h >> 1, is not of small amplitude, thus H/h ≅ 1 , the wave celerity (see eq. 2.27) is given (see *Lamb*, 1945, p. 262) by :

$$c^2 = gh\left(1 + \frac{3}{2}\frac{A}{h}\right) \tag{2.28}$$

or also (see *Lamb*, 1945, p. 424) by :

$$c^2 = g\,(h + A) \tag{2.28a}$$

This last relation was experimentally obtained for a solitary wave.

6° The two signs, which are possible for the celerity, eq. 2.27 or eq. 2.28, show well that the wave can propagate in the direction of x^+ or x^- (see Fig. 2.7).

7° The relation, eq. 2.27, for the celerity, c , of *long* waves can also be obtained by application of the equation of continuity and of energy.

i) Consider the unsteady flow (see Fig. 2.8a) of a simple wave having an amplitude, $A \equiv \eta$. U is the liquid velocity in the section of the crest. By following the wave — one thus imagines the wave stays immobile — the flow becomes a steady flow (see Fig. 2.8b).

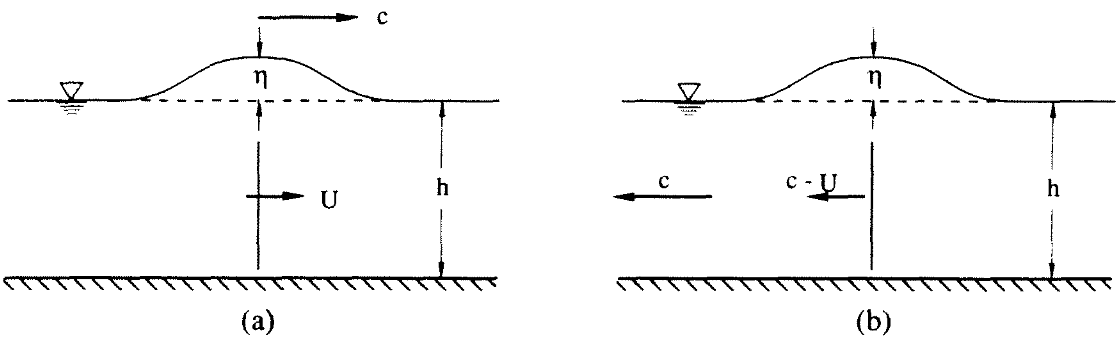


Fig. 2.8 Propagation of a wave.

ii) The equation of continuity reads now :

$$c\,h = (c - U)\,(h + \eta)$$

If $\eta \ll h$, the wave is thus infinitesimal and one may write:

$$U = c\,(\eta / h) \tag{2.29}$$

iii) The equation of energy reads :

$$h + \frac{c^2}{2g} = (h + \eta) + \frac{(c - U)^2}{2g} \tag{2.30}$$

or written otherwise :

$$\eta = \frac{(c\,U)}{g}\left(1 - \frac{U}{2c}\right) \tag{2.30a}$$

Neglecting the term (U / 2c) when compared to unity, one may write:

$$\eta = \frac{c\,U}{g} \tag{2.30b}$$

iv) Through substitution, the eqs. 2.29 and 2.30b give :

$$c^2 = gh \tag{2.27}$$

This is the celerity of wave having a small amplitude, η .

2.4.2 Wave Equation

1° In order to derive the wave equation, the equation of continuity and of motion will be applied to a situation of a wave of small amplitude, which propagates (see Fig. 2.7) in a stagnant liquid.

2° The equation of continuity, eq. 2.5, is expressed as :

$$(h+\eta)\frac{\partial \widetilde{U}}{\partial x} + \widetilde{U}\frac{\partial (h+\eta)}{\partial x} + \frac{\partial (h+\eta)}{\partial t} = 0$$

where $\widetilde{U}$ is the velocity produced by the wave and averaged over the depth. By assuming that depth variation, $\partial h/\partial x = 0$ and $\partial h/\partial t = 0$, are negligible, one may write :

$$(h+\eta)\frac{\partial \widetilde{U}}{\partial x} + \widetilde{U}\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} = 0 .$$

If the wave is of small amplitude, $\eta/h \ll 1$, and assuming that $\partial\eta/\partial x \ll 1$, one obtains :

$$h\frac{\partial \widetilde{U}}{\partial x} + \frac{\partial \eta}{\partial t} = 0 \qquad (2.31)$$

3° The dynamic equation, eq. 2.12, is expressed as :

$$\frac{1}{g}\frac{\partial \widetilde{U}}{\partial t} + \frac{\widetilde{U}}{g}\frac{\partial \widetilde{U}}{\partial x} + \frac{\partial (h+\eta)}{\partial x} - (S_f - S_e) = 0$$

While the last term shall be omitted, the second term is considered to be small compared to the first one. Since the depth variation is negligible, the above equation simplifies to :

$$\frac{1}{g}\frac{\partial \widetilde{U}}{\partial t} + \frac{\partial \eta}{\partial x} = 0 \qquad (2.32)$$

4° One sees immediately that these equations, eq. 2.32 and eq. 2.31, give the following relationship :

$$\frac{\partial^2 \widetilde{U}}{\partial t^2} = gh\frac{\partial^2 \widetilde{U}}{\partial x^2} \qquad \text{or} \qquad \frac{\partial^2 \eta}{\partial t^2} = gh\frac{\partial^2 \eta}{\partial x^2} \qquad (2.33)$$

This is the classic equation for a progressive wave (see *Lamb*, 1945, p. 255), where $c^2 = gh$ is the celerity of a long wave, previously presented with eq. 2.27. A general solution to it was given with eq. 2.25.

2.4.3 Flow with a Wave

1° It was shown that the celerity, c , with which a gravity wave, being a long one and of small amplitude, propagates in a channel of rectangular section, is given with the relation of eq. 2.27. For a channel of an arbitrary section, one writes :

$$c = \pm \sqrt{gD_h} \tag{2.27a}$$

where D_h is the hydraulic depth.

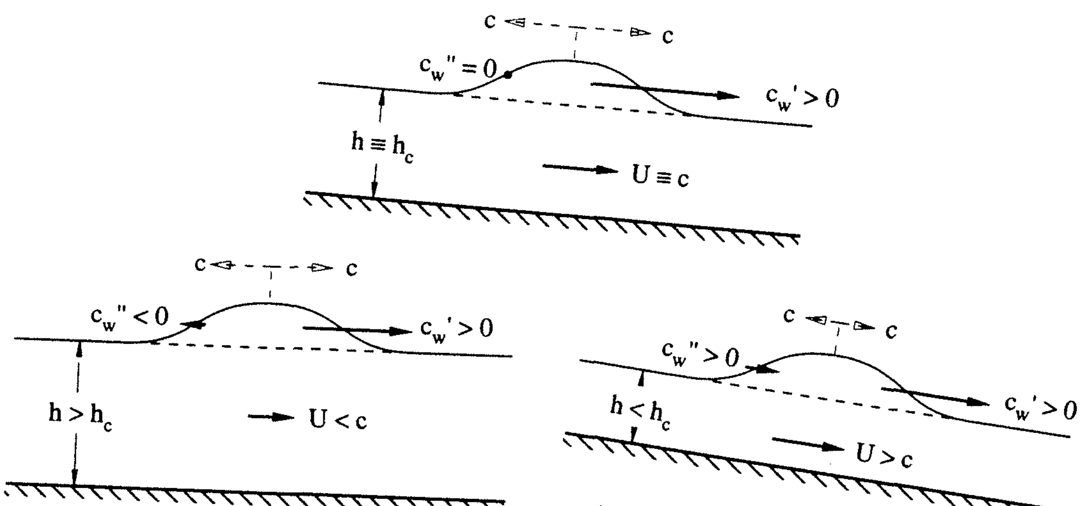


Fig. 2.9 Flow with a wave.

2° This relation, eq. 2.27a, was established for a channel where the liquid was stagnant. However the relation stays valid for the case where the liquid is in motion ; the wave superposes itself upon the flow in the channel. Consequently, the *absolute celerity*, c_w , of the wave for a channel having an average velocity, U, can be expressed as :

$$c_w = U \pm \sqrt{gD_h} \tag{2.34a}$$

and for a channel of rectangular section :

$$c_w = U \pm \sqrt{gh} = U \pm c \tag{2.34}$$

3° The absolute celerity, c_w, being the velocity with respect to the bed, has evidently two values :

$$c_w' = U + c \quad , \quad c_w'' = U - c \tag{2.34b}$$

Thus one may distinguish two plus one cases (see Fig. 2.9) :

i) $U < c$, where the wave with celerity, c_w', propagates downstream and where the wave with celerity, c_w'', propagates upstream; the flow regime is fluvial.

ii) $U > c$, where the wave with celerity, c_w', propagates downstream and where the wave with celerity, c_w'', propagates downstream as well; the flow regime is torrential.

iii) At a flow depth, at which the current velocity, U , and the wave celerity, c, are the same, thus :

$$U \equiv c = \sqrt{g h_c}$$

the flow is in critical regime (see sect. 2.3.3) ; h_c being the critical depth.

4° Flow with a gravity wave, which is long but not of small amplitude, will be treated later (see sect. 5.6).

2.5 HYDRODYNAMIC EQUATIONS

2.5.1 Equations of Motion

1° For flow (see Fig. 2.10), which is two-dimensional, plane, $\vec{V}(\bar{u}, 0, \bar{w})$, and turbulent, the equations of motion and of continuity (see *Graf & Altinakar*, 1991, p. 275 or *Rotta*, 1972, p. 129) can be written as :

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} &= -\frac{1}{\rho}\frac{\partial \bar{p}^*}{\partial x} + \\ &+ \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) - \left[\frac{\partial}{\partial x}(\overline{u'^2}) + \frac{\partial}{\partial z}(\overline{u'w'})\right] \\ \frac{\partial \bar{w}}{\partial t} + \bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{w}\frac{\partial \bar{w}}{\partial z} &= -\frac{1}{\rho}\frac{\partial \bar{p}^*}{\partial z} + \\ &+ \nu\left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial z^2}\right) - \left[\frac{\partial}{\partial x}(\overline{u'w'}) + \frac{\partial}{\partial z}(\overline{w'^2})\right] \end{aligned} \tag{2.35}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{2.35a}$$

- $\bar{u}$ and $\bar{w}$ are the average point velocities in the x and z -direction ;
- u' and w' are the velocities due to fluctuations ;
- $\rho\overline{u'^2}$, $\rho\,\overline{u'w'}$, etc. are the supplementary (or Reynolds) stresses due to the turbulence;
- $\overline{p^*}$ is the average (point) driving pressure.

These equations, eqs. 2.35, are known as the *Reynolds equations*. In the absence of turbulence they reduce to the *Navier-Stokes* equations, valid notably for laminar flow.

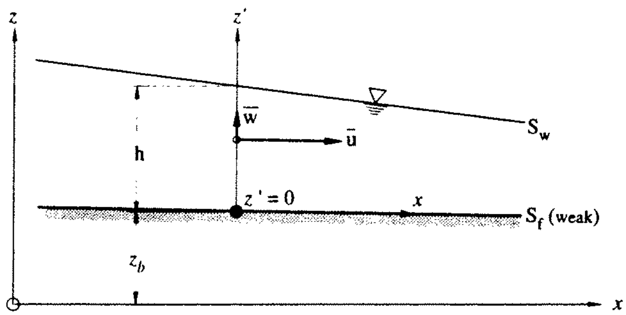


Fig. 2.10 Scheme for the equations of motion.

2° For free-surface flow on weak slopes, $S_f \ll 1$, the terms of the driving pressure, $\overline{p^*}(x,z) = \bar{p}(x,z) + g\rho z$, are expressed as :

$$\frac{\partial \overline{p^*}}{\partial x} = \frac{\partial \bar{p}}{\partial x} - \rho g\, S_f$$

$$\frac{\partial \overline{p^*}}{\partial z} = \frac{\partial \bar{p}}{\partial z} + \rho g \tag{2.36}$$

where the bed slope is defined as : $S_f = -\dfrac{\partial z_b}{\partial x}$.

The bed of the channel is defined by z_b, but in the following it will be used without an index, thus written as $S_f = -\dfrac{\partial z}{\partial x} = -\dfrac{dz}{dx}$.

3° Steady free-surface flow may be considered as being flow at high Reynolds numbers. It is thus possible to use the approximations which are developed for boundary-layer flow (see *Graf & Altinakar*, 1991, sect. CL.1).

By considering the order of magnitude of each term in eqs. 2.35 and 2.35a, and by keeping only the terms of the highest order (see *Graf & Altinakar*, 1991, p. 351, or *Rotta*, 1972, p. 130), one has :

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p^*}}{\partial x} + \nu\frac{\partial^2 \bar{u}}{\partial z^2} - \frac{\partial}{\partial x}(\overline{u'^2}) - \frac{\partial}{\partial z}(\overline{u'w'})$$

$$0 = -\frac{1}{\rho}\frac{\partial \overline{p^*}}{\partial z} - \frac{\partial}{\partial z}(\overline{w'^2}) \tag{2.37}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{2.35a}$$

4° In eqs. 2.37, the second one can be integrated and written as :

$$0 = -\frac{1}{\rho}\,\overline{p^*}(x, z') + \frac{1}{\rho}\,\overline{p^*}_f(x, z'=0) - \overline{w'^2}(z') + \overline{w'^2}_f(z'=0)$$

$\overline{p^*}_f$ being the driving pressure at the bed, $z' = 0$; due to the no-slip condition one takes $\overline{w'^2}_f = 0$. Thus one may write :

$$\overline{p^*}(x, z') = \overline{p^*}_f(x, 0) - \rho\overline{w'^2}(z') \tag{2.38}$$

or also :

$$\bar{p} + \rho g z' = \bar{p}_f + \rho g 0 - \rho\overline{w'^2}$$

With $\bar{p}_f = \rho g h$, an expression for the pressure is obtained :

$$\bar{p} = \rho g (h - z') - \rho\overline{w'^2} \tag{2.38a}$$

The driving pressure is consequently not constant over the flow depth, but will be slightly modified by the Reynolds stress.

The derivation of eq. 2.38 with respect to x, gives :

$$\frac{\partial \overline{p^*}}{\partial x} = \frac{\partial \overline{p^*}_f}{\partial x} - \rho\frac{\partial}{\partial x}(\overline{w'^2}) \tag{2.39}$$

where the last term is often neglected; using eq. 2.36, it can be written as :

$$\frac{\partial \overline{p}^*}{\partial x} \cong \frac{\partial \overline{p}^*_f}{\partial x} = \frac{\partial \overline{p}_f}{\partial x} - g\rho S_f = g\rho\left(\frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x}\right) \tag{2.40}$$

5° Upon substitution of eq. 2.39 and eq. 2.40 into the first of eqs. 2.37 one gets :

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial \overline{u}}{\partial z} - \overline{u'w'}\right) + \frac{\partial}{\partial x}\left(\overline{w'^2} - \overline{u'^2}\right) \tag{2.41}$$

The last term, which is due to the normal Reynolds stress, is also often neglected (see *Rotta,* 1972, p. 130). If one defines the total tangential stress by :

$$\tau_{zx} = \rho\left(\nu\frac{\partial \overline{u}}{\partial z} - \overline{u'w'}\right) \tag{2.42}$$

one can write eq. 2.41 as :

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x}\right) + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \tag{2.41a}$$

This equation is also valid for laminar flow, where :

$$\tau_{zx} = \mu\frac{\partial \overline{u}}{\partial z} \tag{2.42a}$$

6° Free-surface flow, being unsteady, can thus be represented (see *Grishanin,* 1969, p. 59) by a system of equations such as :

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x}\right) + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \tag{2.41b}$$

$$\overline{p} = \rho g(h - z') - \rho\overline{w'^2} \tag{2.38a}$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{w}}{\partial z} = 0 \tag{2.35a}$$

Note, that the pressure is not quite hydrostatic.

2.5.2 Uniform Flow

1° It will be assumed that flow is steady, unidirectional, $\vec{V}(\bar{u}, 0, 0)$, and uniform on the average.

The equations of motion, eq. 2.41a and eq. 2.38a, and of continuity, eq. 2.35a, reduce to :

$$0 = + g\, S_f + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \tag{2.43}$$

$$\bar{p} = \rho g\, (h - z') - \rho \overline{w'^2} \tag{2.38a}$$

$$\frac{\partial \bar{u}}{\partial x} = 0 \tag{2.44}$$

where the total tangential stress is expressed by :

$$\tau_{zx} = \mu \frac{\partial \bar{u}}{\partial z} - \rho \overline{u'w'} \tag{2.42}$$

2° After integration over the flow depth, h , one obtains :

i) the equation of motion, eq. 2.43, as being :

$$0 = g\, S_f \int_o^h dz + \frac{1}{\rho} \int_o^h \frac{\partial}{\partial z} \tau_{zx}\, dz$$

$$0 = + g\, S_f\, (h - 0) + \frac{1}{\rho}(0 - \tau_o)$$

and consequently one has :

$$\tau_o = \rho g\, S_f\, h \tag{2.45}$$

with τ_o as the stress due to friction, called the wall or bed-shear stress; the ratio, $\tau_o/\rho g h$, gives the energy slope, being now expressed by :

$$S_f = S_e \tag{2.45a}$$

In eq. 2.43 one notes that the longitudinal pressure gradient, namely the longitudinal component of the gravity (see eq. 2.36), provides the driving force of a uniform flow; the tangential stress (see eq. 2.42) is the dissipating force.

ii) the equation of continuity, eq. 2.44, as being :

$$\int_0^h \frac{\partial \bar{u}}{\partial x} dz = \frac{\partial}{\partial x} \int_0^h \bar{u}\, dz - \bar{u}_h \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial}{\partial x} (Uh) = 0 \tag{2.46}$$

where $Uh = q$ is the unit discharge and U the average velocity; $\bar{u}_h$ is the velocity at the water surface.

For steady flow, one writes :

$$\frac{\partial q}{\partial x} = U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x} = 0 \tag{2.46a}$$

The equation of motion, eq. 2.45a (see eq. 2.13a), and of continuity, eq. 2.46a (see eq. 2.6), in their integral form, form together the simplified equations of Saint-Venant for a steady and uniform flow (see sect. 2.2).

3° To obtain the distribution (see Fig. 2.11) of the total shear stress, $\tau_{zx}(z)$, the following equation must be integrated over the flow depth :

$$\frac{\partial \tau_{zx}}{\partial z} = \frac{\partial \overline{p^*}}{\partial x} = -\rho g S_f \tag{2.43a}$$

As boundary conditions serve :

$z' \cong 0$ ($z'/h = 0.05$) $\Rightarrow$ $\tau_{zx} = \tau_o$

$z' = h$ $\Rightarrow$ $\tau_{zx} = 0$

thus the following is obtained :

$$\tau_{zx}(z') = \rho g S_f (h - z') \tag{2.47}$$

or written in dimensionless form :

$$\frac{\tau_{zx}(z')}{\tau_o} = \left(\frac{h - z'}{h}\right) \tag{2.47a}$$

This gives (see *Monin* et *Yaglom*, 1971, p.268) a linear (triangular) distribution, being valid for turbulent flow with eq. 2.42, and for laminar flow with eq. 2.42a.

Nevertheless, for very small distances from the bed, $z'/h \lesssim 0.05$, the shear-stress distribution may be considered to be constant (see *Graf* & *Altinakar*, 1991, sect. 6.1) :

$$\frac{\tau_{zx}(z')}{\tau_o} = 1$$

The zone very close to the bed, $z'/h \lesssim 0.20$, where the shear stress is constant, is called the *inner region* (see *Hinze*, 1975, p.503 and *Monin* et *Yaglom*, 1971, p.311) where the total shear-stress variation becomes negligible.

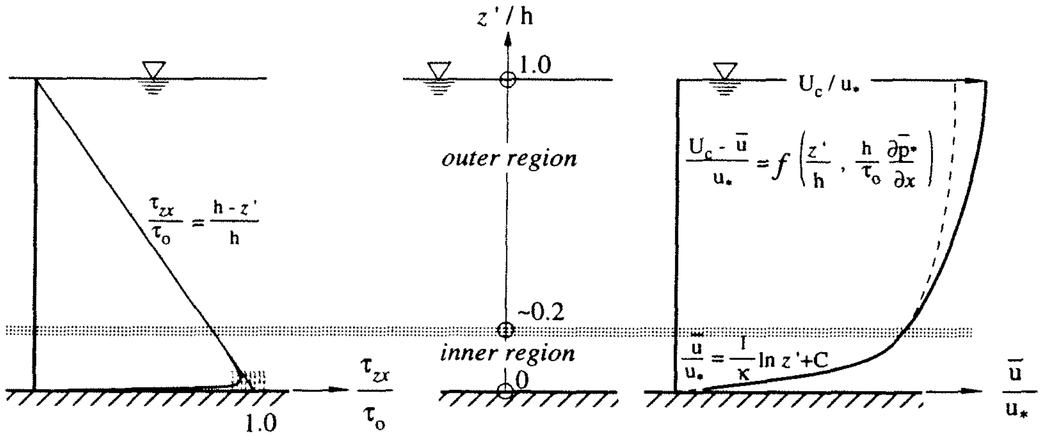


Fig. 2.11 Scheme of the distribution of shear stress, $\tau_{zx}(z)$, and of velocity, $\bar{u}(z)$; for uniform flow.

4° The system of equations, eq. 2.43, eq. 2.38a and eq. 2.44, cannot be used to obtain the distribution of the velocity, $\bar{u}(z)$, (see Fig. 2.11) since the Reynolds stresses are not known. Semi-empirical methods have to be exploited.

i) In the *inner region*, the pressure gradient, $(\partial \bar{p}^* /\partial x)$, — being very weak in uniform flow — in eq. 2.43a may be neglected (see *Monin* et *Yaglom*, 1971, p.268); one may write :

$$\frac{\partial \tau_{zx}}{\partial z} = 0 \tag{2.43b}$$

and consequently :

$$\tau_{zx} = \mu \frac{\partial \bar{u}}{\partial z} - \rho \overline{u'w'} = Cte \tag{2.42}$$

To find an expression for this equation, eq. 2.42, one may use the semi-empirical method of the mixing length, $l = \kappa z'$, proposed by *Prandtl* (see *Graf & Altinakar*, 1991, p. 280), such as :

$$\tau_{zx} = \mu \frac{d\bar{u}}{dz} + \rho \kappa^2 z'^2 \left(\frac{d\bar{u}}{dz}\right)^2 \tag{2.48}$$

where κ is Karman's universal constant.

Outside a very thin region — the viscous region — situated very close to the bed, the shear stress due to viscosity can be neglected, thus one has :

$$\tau_{zx} = \rho \kappa^2 z'^2 \left(\frac{d\bar{u}}{dz}\right)^2 \tag{2.49}$$

Since the shear stress, τ_{zx}, remains constant in the vertical — more or less correct in the inner region — and equal to the wall-shear stress, τ_o, one may write :

$$\tau_{zx} \equiv \tau_o = \rho \kappa^2 z'^2 \left(\frac{d\bar{u}}{dz}\right)^2$$

After separation of variables, the following differential equation is obtained :

$$d\bar{u} = \frac{\sqrt{\tau_o/\rho}}{\kappa} \frac{dz}{z'}$$

which, upon integration, renders :

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln z' + C \tag{2.50}$$

where $u_* = \sqrt{\tau_o/\rho}$ is the friction velocity. The value of the integration constant, C, must be determined experimentally; in this way the type of the surface (bed), being smooth or rough, will enter.

This *logarithmic law* (*of the wall*), eq. 2.50, is only valid in the inner region, $z'/h \lesssim 0.2$, where the shear stress remains constant and the influence of a possible pressure gradient can be neglected (see *Rotta*, 1972, p. 153). The logarithmic law is universal (see *Monin* et *Yaglom*, 1971, p. 311), being the same for boundary-layer flow, as well as for flow in pipes and in open channels.

In the *inner* region, $z'/h \lesssim 0.2$, the velocity, $\bar{u}(z)$, whose variation is considerable, depends also on the wall-shear stress, on the fluid properties, on the type of the wall (bed) and on the distance from the wall; thus :

$$\bar{u} = f(\tau_0, \rho , \mu , k_s , z').$$

ii) In the *outer region*, $0.2 \lesssim z'/h \lesssim 1.0$, the velocity, $\bar{u}(z)$, whose variation is weak, depends also on the maximum velocity, on the flow depth and the driving pressure (see eq. 2.40), but does not depend on the viscosity or on the type of the wall (bed); thus :

$$\bar{u} = f(U_c , h , \frac{\partial \bar{p}^*}{\partial x} \; ; \; \tau_0 , \rho , z').$$

In the outer region, a good agreement with experimental data is not possible, since $\tau_{zx} \neq \tau_0$. The logarithmic law, eq. 2.50, must be modified with a function which depends on the flow depth, h, and notably on an dimensionless pressure gradient, $(h / \tau_0)(\partial \bar{p}^* / \partial x)$. The velocity distribution is given here by the *law of velocity defect* (see *White*, 1974, p. 477), such as :

$$\frac{U_c - \bar{u}}{u_*} = f(\frac{z'}{h} , \frac{h}{\tau_0} \frac{\partial \bar{p}^*}{\partial x})$$

Amongst the different relations available (see *Hinze*, 1975, p. 630 et p. 697), the one of Coles shall here be used :

$$\frac{U_c - \bar{u}}{u_*} = \frac{1}{\kappa} \ln (\frac{\delta}{z'}) + \frac{\Pi}{\kappa} (2 - \tilde{\omega}) \tag{2.51}$$

where the function, known as wake function, is defined by :

$$\tilde{\omega} = 2 \sin^2 (\frac{\pi}{2} \frac{z'}{\delta})$$

The wake parameter of Coles, Π, depends notably on the gradient of the longitudinal pressure :

$$\Pi = f(\beta)$$

where

$$\beta = \frac{h}{\tau_o} \frac{\partial \overline{p^*}}{\partial x} \tag{2.52}$$

Since this parameter, Π, must remain constant for flow in equilibrium, the β-value is an equilibrium parameter (see *White*, 1974, p. 477). The value of $(2\Pi/\kappa)$ represents the deviation from the logarithmic part of eq. 2.51 for $(z'/\delta) = 1$ (see *Graf & Altinakar*, 1991, p. 288).

The height, $z' = \delta$ ($\leq$ h) , is the position in the flow section (see Fig. 2.13) where the maximum velocity, U_c , is measured; if U_c is on the water surface, the flow depth, $\delta \equiv h$, is to be taken.

This empirical relation, eq. 2.51, whose validity is made evident by experiments, is valid in the outer region as well as in the inner one, but not in the viscous region. However this relation is valid for both smooth and rough surfaces.

Nevertheless, as a first (and often good) approximation, the logarithmic law can often be applied over the entire flow depth, h (see *Monin* et *Yaglom*, 1971, p. 298); this is especially so if the flow is uniform having a weak pressure gradient.

iii) The distribution of velocity — called universal since it is independent of the Reynolds number — as given with eq. 2.50 and eq. 2.51, is complete but also complex. For practical purpose, one may also use a simple empirical relation (see *Graf & Altinakar*, 1991, p. 289) of the type :

$$\frac{\bar{u}}{\bar{u}_R} = \left(\frac{z'}{z_R}\right)^q$$

where $\bar{u}_R(z_R)$ is a reference velocity, which was for example previously measured. The variation of $1/10 < q < 1/6$ depends on the Reynolds number; often $q = 1/7$ is taken.

2.5.3 Non-uniform Flow

1° It will be assumed that flow is steady, two-dimensional, $\vec{V}(\bar{u}, 0, \bar{w})$, and gradually varied (non-uniform).

The equations of motion, eq. 2.41a and eq. 2.38a, and of continuity, eq. 2.35a, reduce to :

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -g\left(\frac{\partial h}{\partial x} - S_f\right) + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \qquad (2.53)$$

$$\bar{p} = \rho g(h - z') - \rho\overline{w'^2} \qquad (2.38a)$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad (2.35a)$$

Flow, which is (gradually) non-uniform, may be considered unidirectional if the variation of flow depth, $\partial h/\partial x$, is weak (see *Grishanin*, 1969, p. 59 - 62).

2° After integration over the flow depth, h , (see *Grishanin*, 1969) one obtains :

i) the equation of motion, eq. 2.53, as being :

$$\int_0^h \bar{u}\frac{\partial \bar{u}}{\partial x}dz + \int_0^h \bar{w}\frac{\partial \bar{u}}{\partial z}dz = -g\int_0^h \frac{\partial h}{\partial x}dz + gS_f\int_0^h dz + \frac{1}{\rho}\int_0^h \frac{\partial \tau_{zx}}{\partial z}dz$$

$$\beta_u U h\frac{\partial U}{\partial x} + U^2 h\frac{\partial \beta_u}{\partial x} = -gh\frac{\partial h}{\partial x} + g S_f h + \frac{1}{\rho}(-\tau_o)$$

where $\beta_u = (1/U^2h)\int_0^h \bar{u}^2 dz$ is the correction coefficient (of Boussinesq) of the velocity distribution, which is usually taken as $\beta_u = Cte \cong 1$ for turbulent flow. Assuming that the energy slope is given by $S_e = \tau_o/\rho g h$, one may now write :

$$\frac{1}{g}U\frac{\partial U}{\partial x} + \frac{\partial h}{\partial x} - S_f = -S_e \qquad (2.54)$$

ii) the equation of continuity, eq. 2.35a, as being :

$$\int_0^h \frac{\partial \bar{u}}{\partial x} dz + \int_0^h \frac{\partial \bar{w}}{\partial z} dz = \left(\frac{\partial}{\partial x} \int_0^h \bar{u}\, dz - \bar{u}_h \frac{\partial h}{\partial x} \right) + \bar{u}_h \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial}{\partial x} (Uh) = 0 \tag{2.46}$$

For steady, non-uniform (but also uniform) flow, one writes :

$$\frac{dq}{dx} = U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x} = 0 \tag{2.46a}$$

Note, that the term, $\bar{u}_h (\partial h/\partial x) = \bar{w}_h$, gives the equation of the streamline at the water surface.

The equation of motion, eq. 2.54 (see eq. 2.13) and of continuity, eq. 2.46a (see eq. 2.6), form together the simplified equations of Saint-Venant for a steady and non-uniform flow (see sect. 2.2).

3 ° Furthermore, one may postulate :

i) $$\frac{\partial h}{\partial x} - S_f = \frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x} = \frac{1}{\rho g} \frac{\partial \bar{p}^*}{\partial x} \tag{2.40}$$

ii) using the equation of continuity, eq. 2.46a :

$$\frac{\partial U}{\partial x} = - \frac{U}{h} \frac{\partial h}{\partial x}$$

iii) using the definition of head loss (see point 2.2, 4°) :

$$S_e = \frac{\tau_o}{\rho g\, h}$$

In this way, the equation of motion, eq. 2.54, can be expressed as:

$$\rho\, U^2 \frac{\partial h}{\partial x} = \tau_o \left(1 + \frac{h}{\tau_o} \frac{\partial \bar{p}^*}{\partial x} \right) \tag{2.54a}$$

This equation may be compared with the Karman equation (see *Graf & Altinakar*, 1991, sect. CL.4 : eq. CL.25a) for boundary-layer flow.

The dimensionless longitudinal gradient of the driving pressure :

$$\beta = \frac{h}{\tau_o} \frac{\partial \overline{p^*}}{\partial x} \tag{2.52}$$

which stands for the ratio of the forces due to the driving pressure and due to friction, defines the equilibrium parameter. This parameter can be used to classify non-uniform flow, being flow with a pressure gradient. Note, however, that also uniform flow (see eq. 2.43a) is flow with a (weak) pressure gradient, where $\beta = -1$.

4° To obtain the distribution (see Fig. 2.12) of velocity, $\bar{u}(z')$, one should distinguish two regions, as was also done with uniform flow.

i) In the *inner region*, one finds that the logarithmic law of the wall :

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln z' + C \tag{2.50}$$

remains valid (see *White*, 1974, p. 473), and this as long as the driving-pressure gradient is weak (see eq. 2.40), being either positive or negative, $\pm (\partial \overline{p^*}/\partial x)$. One can explain (see *Tennekes* et *Lumley*, 1972, p. 185) this, by assuming the inertia term in eq. 2.53 is negligible and that the term of the driving pressure is weak compared to the term of the Reynolds stress; a zone of quasi-constant stress is thus delimited.

Nevertheless the integration constant, C, may depend upon the pressure gradient (see *Tennekes* et *Lumley*, 1972, p. 186). The thickness of the inner region, $z'/h \leq 0.2$, will now depend on the pressure gradient (see *White*, 1974, p. 473). The inner region can disappear in decelerating flow with strong positive pressure gradients, when flow separation occurs.

ii) In the *outer region*, one finds (experimentally) that the law of velocity defect, the one of Coles :

$$\frac{U_c - \bar{u}}{u_*} = \frac{1}{\kappa} \ln \left(\frac{\delta}{z'}\right) + \frac{\Pi}{\kappa} (2 - \tilde{\omega}) \tag{2.51}$$

remains valid. Depending on the gradient of the driving pressure (see eq. 2.40), one has to make the following distinction (see Fig. 2.12) :

- a positive (unfavourable) pressure gradient, $\partial \overline{p^*}/\partial x > 0$, being always accompanied by a decrease of the average velocity in the direction of the flow (*deceleration*); the velocity profiles get less uniformly distributed;

- a negative (favourable) pressure gradient, $\partial\overline{p^*}/\partial x < 0$, being usually accompanied by an increase of the average velocity in the direction of flow (*acceleration*); the velocity profiles get more uniformly distributed.

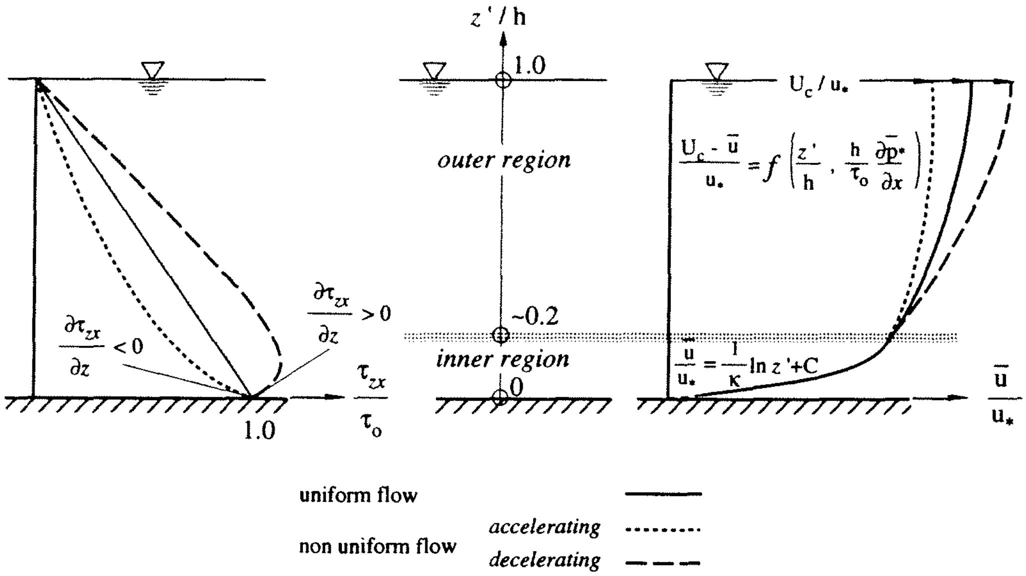


Fig. 2.12 Scheme of the distribution of the shear stress, $\tau_{zx}(z')$, and of the velocity, $\bar{u}(z')$, in non-uniform flow.

5° To obtain the distribution (see Fig. 2.12) of the total shear stress, $\tau_{zx}(z)$, the equation of motion, eq. 2.53, must be integrated (see *Rotta*, 1972, p. 240). The boundary conditions are the following :

$$z' = 0 \quad \Rightarrow \quad \tau_{zx} = \tau_o$$

$$z' = h \quad \Rightarrow \quad \tau_{zx} = 0$$

Consequently, one obtains :

i) Close to the wall, $z' << h$, where the non-slip conditions, $\bar{u} = 0$ and $\bar{w} = 0$, are valid, eq. 2.53 is written as :

$$\frac{\partial \tau_{zx}}{\partial z} = \rho g \left(\frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x}\right) = \frac{\partial \overline{p^*}}{\partial x} \qquad (2.53a)$$

which subsequently gives (see *White*, 1974, p. 474) :

$$\tau_{zx} = \tau_o + (\frac{\partial \overline{p^*}}{\partial x}) z$$

Depending on the pressure gradient, one has :

- for a positive pressure gradient, $\partial \overline{p^*} / \partial x > 0$, where the flow is *decelerating* :

$$\frac{\partial \tau_{zx}}{\partial z} > 0 \qquad \text{and} \qquad \tau_{zx} > \tau_o$$

the total stress has its maximum value, $\tau_{zx} \equiv \tau_{max}$, at a certain distance from the wall;

- for a negative pressure gradient, $\partial \overline{p^*} / \partial x < 0$, where the flow is generally *accelerating* :

$$\frac{\partial \tau_{zx}}{\partial z} < 0 \qquad \text{and} \qquad \tau_{zx} < \tau_o$$

the total stress has its maximum value, $\tau_{zx} \equiv \tau_{max}$, at the wall.

ii) Far from the wall, beyond the point where $\tau_{zx} \equiv \tau_{max}$, the distribution of total shear stress is monotone (see *Rotta*, 1972, p.240) and this up to the water surface, where $\tau_{zx} = 0$.

2.6 DISTRIBUTION OF VELOCITY

1° The experimental results, to support the theory developed in chap. 2.5, will now be presented.

It is taken that the flow of a real and incompressible fluid is completely developed along (the bed of) the channel. Assumed will be that the flow is two-dimensional, but unidirectional in the x-direction, being steady and uniform or non-uniform.

2° A direct consequence of a real-fluid flow is the manifestation of the (point) velocity profile, $u(z')$, where the z' is the distance measured from the bed of the channel.

By integration of the velocity profile the average velocity, U, across the flow section is obtained.

3° Between the dimensionless average velocity, U/u_*, and friction coefficient, f, there exists (see *Graf & Altinakar*, 1991, p. 433) the following relationship (see eq. 3.8) :

$$\frac{U}{u_*} = \sqrt{\frac{8}{f}} \tag{2.55}$$

where $u_* = \sqrt{\tau_o/\rho}$ is the friction velocity.

4° A summary of the velocity distribution, $u(z')$, of the average velocity, U, and of the friction coefficient, f, for uniform flow, both laminar and turbulent, is given in Table 2.1.

2.6.1 Laminar Flow

1° Uniform, steady and laminar flow in a channel of a large width, $R_h \equiv h$, has been studied in great detail (see *Graf & Altinakar*, 1991, p. 257); it is a special case of the *Couette* flow.

2° The distribution of the velocity, $u(z')$, for two-dimensional flow (see Fig. 1.6) is given by a parabolic relation :

$$\frac{u(z')}{u_*} = \frac{1}{2\mu\, u_*} \left(-\gamma \frac{dh}{dx}\right) (2hz' - z'^2) \tag{2.56}$$

where $h = h + z_b$ and (dh/dx) is the slope of the water surface, given herewith as $S_w \equiv S_f = -(dz_b/dx)$. Using the friction velocity, given as $u_*^2 = gh\, S_f$ (see eq. 3.7), this equation, eq. 2.56, becomes :

$$\frac{u(z')}{u_*} = \left(\frac{u_* z'}{\nu}\right) \left(1 - \frac{z'}{2h}\right) \tag{2.56a}$$

3° The average velocity, U , in the flow section, A, is given (see *Graf & Altinakar*, 1991, p. 257) by :

$$\frac{U}{u_*} = \frac{1}{u_*} \frac{g}{3\nu} S_f h^2 = \frac{1}{3} \left(\frac{u_* h}{\nu}\right) \tag{2.57}$$

a relationship which expresses a proportionality between the average velocity, U, and the bed slope, S_f.

4° Flow is considered to be laminar, if the Reynolds number is :

$$Re' = \frac{Uh}{\nu} \leq 500 \qquad \text{or} \qquad Re = \frac{4Uh}{\nu} \leq 2000$$

As long as flow stays laminar, the roughness of the bed of the channel is of no consequence.

5° The friction coefficient, f, is obtained by combining eq. 2.55 and eq. 2.57; that is :

$$\sqrt{\frac{8}{f}} = \frac{U}{u_*} = \frac{1}{3}\left(\frac{u_* h}{\nu}\right)\frac{U}{U} = \frac{1}{3}\,Re'\sqrt{\frac{f}{8}}$$

or written also as :

$$f = \frac{24}{Re'} \qquad \text{or} \qquad f = \frac{6}{Re} \tag{2.58}$$

The coefficient, $B_l = 24$, is valid notably in two-dimensional flow, when the width of the channel is large and having an aspect ratio of $B/h > 5$. For channels which are less large, $B/h < 5$, or for channels being not rectangular, this coefficient may be smaller, $14 < B_l < 24$ (see *Chow*, 1959, p. 11).

2.6.2 Turbulent, smooth Flow

1° The universal velocity distribution for turbulent, smooth flow was developed using the concept of the mixing length (see point 2.5.2, 4°, or *Graf & Altinakar*, 1991, pp. 280 - 289).

2° The distribution of the velocity, $u(z')$, — one shall take now $u \equiv \bar{u}$ for the point-average velocity, or the bar will no more be used — is logarithmic (see eq. 2.50); it is given by :

$$\frac{u(z')}{u_*} = \frac{1}{\kappa}\ln\left(\frac{z' u_*}{\nu}\right) + B_s \tag{2.59}$$

The numerical constants, obtained from numerous experiments with *uniform* flow (see *Reynolds*, 1974, p. 187) are :

$$\kappa = 0.4 \qquad ; \qquad B_s = 5\,(\pm 25\%)$$

3° For *non-uniform* flows, the numerical constants are only slightly different (see *Reynolds*, 1974, p. 187 and *Cardoso* et *al.*, 1989).

4° This relation, eq. 2.59, is only valid close to the surface (bed), delimited by :

$$35 \leq \frac{z' u_*}{\nu} \leq 200 \qquad \text{or} \qquad \frac{z'}{h} \leq 0.2$$

but experiments have shown good agreement over the entire flow depth, h. The region delimited by $(z'/h) \leq 0.2$, is the inner region (see Fig. 2.11) where the shear stress remains essentially constant.

5° Upon integration of eq. 2.59, one obtains an expression for the average velocity :

$$\frac{U}{u_*} = \frac{1}{\kappa} \ln\left(\frac{R_h u_*}{\nu}\right) + \overline{B}_s \tag{2.60}$$

The constant of integration obtained from numerous experiments (see *Keulegan*, 1938) is given as :

$$\overline{B}_s = 3.5$$

but it depends slightly on the geometry of the cross section and on the Froude number (see *Chow*, 1959, p. 205).

6° The friction coefficient, f, can be now obtained in combining eq. 2.55 with eq. 2.60; this gives :

$$\sqrt{\frac{8}{f}} = \frac{U}{u_*} = \frac{1}{\kappa} \ln\left(\frac{R_h u_*}{\nu}\right) + \overline{B}_s$$

and for $\overline{B}_s = 3.5$, one has :

$$\sqrt{\frac{1}{f}} = 2.03 \log\left(\mathrm{Re}' \sqrt{f}\right) + 0.32 \tag{2.61}$$

or putting $\mathrm{Re}' = \mathrm{Re}/4$:

$$\sqrt{\frac{1}{f}} = 2.03 \log\left(\mathrm{Re} \sqrt{f}\right) - 0.88 \cong 2 \log\left(\frac{\mathrm{Re}}{3} \sqrt{f}\right) \tag{2.61a}$$

The above relations, eq. 2.61, are valid for turbulent flow, $\mathrm{Re}' > 500$, in a channel having smooth walls, $(u_* k_s/\nu) < 5$.

2.6.3 Turbulent, rough Flow

1° The universal velocity distribution for turbulent, rough flow was developed, using the concept of the mixing length (see point 2.5.2, 4 °, or *Graf & Altinakar*, 1991, pp. 280-289).

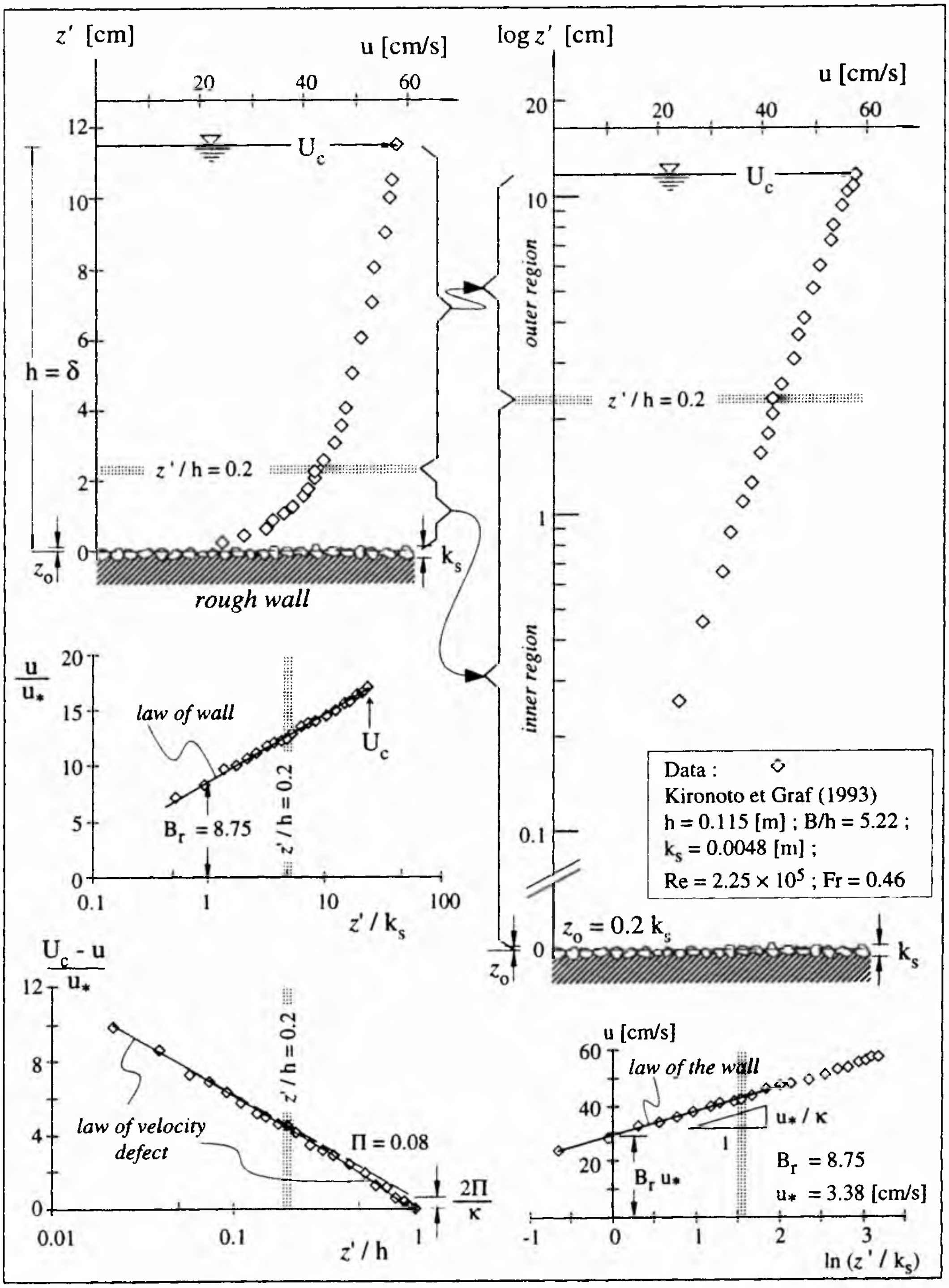


Fig. 2.13 Velocity profile, u(z'); *uniform*, rough flow.

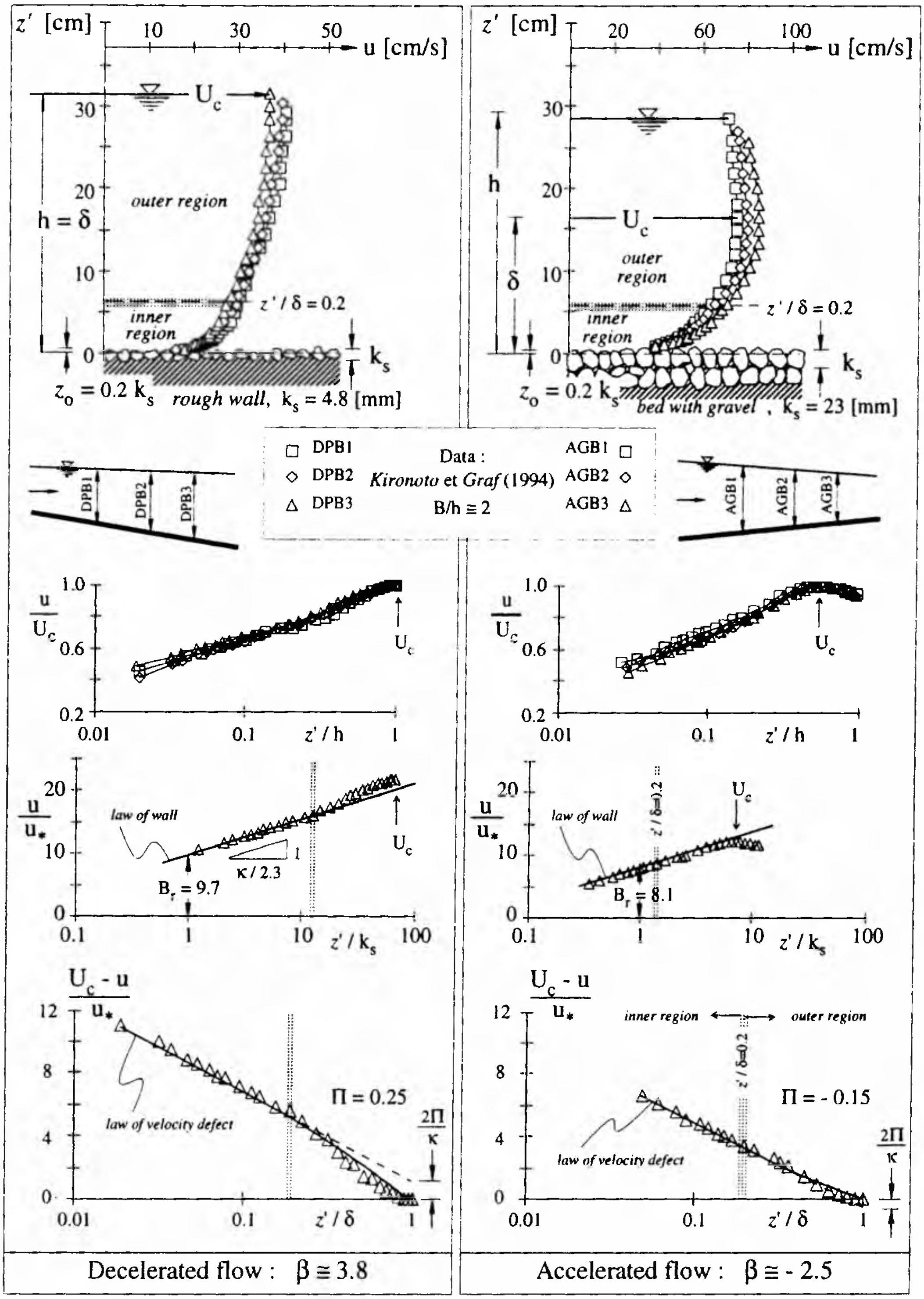


Fig. 2.14 Velocity profile, u(z'); *non-uniform*, rough flow.

2° As an example, the distribution of the velocity — measured in a laboratory flume by *Kironoto et Graf* (1993 et 1994) — is given with Fig. 2.13 for uniform and with Fig. 2.14 for non-uniform flow. Different coordinates are here used :

- u *vs* z' : is the original (measured) profile;
- u *vs* $\log z'$: shows the logarithmic form of the original profile;
- $\frac{u}{U_c}$ *vs* $\frac{z'}{h}$: shows the auto-similarity, necessary for non-uniform flow in equilibrium;
- u *vs* $\ln\left(\frac{z'}{k_s}\right)$: used to determine the values of u_* and B_r, if $\kappa = 0.4$ is imposed;
- $\frac{u}{u_*}$ *vs* $\frac{z'}{k_s}$: gives a dimensionless representation of the law of wall;
- $\frac{U_c - u}{u_*}$ *vs* $\frac{z'}{\delta}$: gives the dimensionless representation of the law of velocity deficit.

3° The distribution of the velocity, $u(z')$ (see Fig. 2.13), is logarithmic (see eq. 2.50); it is given by :

$$\frac{u(z')}{u_*} = \frac{1}{\kappa} \ln\left(\frac{z'}{k_s}\right) + B_r \qquad (2.62)$$

k_s being the equivalent or standard uniform roughness (see *Graf & Altinakar*, 1991, p. 287 et sect. 3.2.1). The numerical constants, which are obtained from numerous experiments (see *Reynolds*, 1974, p. 187 and *Kironoto* et *Graf*, 1993) for *uniform* flow, are given as :

$$\kappa = 0.4 \qquad ; \qquad B_r = 8.5\ (\pm 15\%)$$

The vertical distance, z', is measured from a level which passes slightly below the peaks of the roughness (see Fig. 2.13); in general one takes $z_0 \cong -0.2\ k_s$ (see *Graf*, 1991 or *Hinze*, 1975, p. 637). The relation of eq. 2.62, just as the one of eq. 2.59, is actually only valid within the inner region, $z'/h \leq 0.2$ (see Fig. 2.11 and Fig. 2.13), but an extension into the outer region is often possible.

4° For *non-uniform* flow, the constant, B_r, is slightly different (see *Kironoto* et *Graf*, 1994): it is larger in decelerating flow and smaller in accelerating flow (see Fig. 2.14). The same tendency is observed for *unsteady* flow (see *Tu* et *Graf*, 1992).

5° After integration of eq. 2.62, one obtains the following expression for the average velocity :

$$\frac{U}{u_*} = \frac{1}{\kappa} \ln \left(\frac{R_h}{k_s}\right) + \overline{B_r} \tag{2.63}$$

The constant of integration, obtained from different experiments (see *Keulegan*, 1938, p. 722) is :

$$\overline{B_r} = 6.25$$

being almost independent of the geometrical form of the channel. For flow with large Froude numbers, $Fr > 1$, the value of $\overline{B_r}$ diminishes (see *Chow*, 1959, p. 205).

6° The friction coefficient, f, obtained by combining eq. 2.55 with eq. 2.63, is written as :

$$\sqrt{\frac{8}{f}} = \frac{U}{u_*} = \frac{1}{\kappa} \ln \left(\frac{R_h}{k_s}\right) + \overline{B_r}$$

substitution of $\overline{B_r} = 6.25$, gives :

$$\sqrt{\frac{1}{f}} = 2.03 \log \left(\frac{R_h}{k_s}\right) + 2.2 \tag{2.64}$$

This relation is valid for turbulent flow, $Re' > 2 \cdot 10^4$, in channels with completely rough surfaces, $(k_s u_*/\nu) > 70$.

7° Between smooth surface flow, delimited by $(k_s u_*/\nu) < 5$, and rough surface flow, delimited by $(k_s u_*/\nu) > 70$, there exists the transition region, where the experiments of Nikuradse (see *Graf & Altinakar*, 1991, p. 427) are used to make the connection.

8° The friction coefficient, f, for flow over smooth, transition and rough surfaces, is given by the relation of Colebrook et White (see *Graf & Altinakar*, 1991, p. 436) which was adapted for channels by *Silberman* et *al.* (1963, p. 104), such as :

$$\sqrt{\frac{1}{f}} = -2.0 \log \left(\frac{k_s/R_h}{a_f} + \frac{b_f}{Re\sqrt{f}} \right) \tag{2.65}$$

where $12 < a_f < 15$ and $0 < b_f < 6$, being established for sections of different geometrical shapes and $Re = 4\, R_h U/\nu$. For very wide channels it is recommended to take : $a_f = 12$ and $b_f = 3.4$. If $k_s = 0$, the relation reduces to eq. 2.61a, valid for smooth surfaces; if $Re \to \infty$, it reduces to eq. 2.64, valid for completely rough surfaces.

9° Outside the inner region (see Fig. 2.11) and up to the entire flow depth, h, is the outer region, delimited by $0.2 < (z'/h) < 1.0$. In this region, the flow is conditioned by the maximum velocity, $u = U_c$, as well as by a possibly existing longitudinal pressure gradient. The distribution of the velocity deviates slightly from the logarithmic law. It is approximately given (see *Graf*, 1991 et *Hinze*, 1975) by a law of velocity deficit :

$$\frac{U_c - u}{u_*} = 9.6 \left(1 - \frac{z'}{h} \right)^2 \tag{2.66}$$

being valid for turbulent flow, both for smooth and rough surfaces.

Nevertheless, the logarithmic law given by eq. 2.59 and eq. 2.62 can be used over the entire flow depth, if one does not desire a too high precision.

10° The distribution of the velocity over the entire flow depth — with exception of the viscous region — , thus in the zone of $0.01 < (z'/h) < 1.00$ (see Fig. 2.11 and Fig. 2.13), is given by the following law of velocity deficit (see point 2.5.2, 4 °) :

$$\frac{U_c - u}{u_*} = \frac{1}{\kappa} \ln \left(\frac{\delta}{z'} \right) + \frac{\Pi}{\kappa} \left(2 - \tilde{\omega} \right) \tag{2.51}$$

where Π is the wake parameter of Coles, which depends notably on the longitudinal pressure gradient, β, (see eq. 2.52). For *uniform* flow over smooth and rough surfaces (see *Kironoto* et *Graf*, 1993) in a channel having a weak pressure gradient, namely the bottom slope, one takes :

$$\Pi \cong 0.2$$

having a variation of $-0.1 \lesssim \Pi \lesssim 0.3$.

For flow (boundary-layer) without pressure gradient, $(\partial \overline{p^*} / \partial x) = 0$, one takes $\Pi \cong 0.55$ (see *Hinze*, 1975, p. 697).

Table 2.1

Summary of velocity profile, u, of average velocity, U, and of friction coefficient, f, for steady, *uniform* flow in channels.

	$$\frac{u}{u_*} = \frac{1}{2\mu u_*} (\gamma S_f)(2hz' - z'^2)$$ $$\frac{U}{u_*} = \frac{1}{3}\left(\frac{h u_*}{\nu}\right)$$ $$f = 24/Re'$$
↑ LAMINAR / ↓ TURBULENT Flow	$$Re' = \frac{Uh}{\nu} \cong 500$$
smooth $\frac{u_* k_s}{\nu} < 5$	$$\frac{u}{u_*} = \frac{1}{\kappa} \ln\left(\frac{z' u_*}{\nu}\right) + 5.5$$ $$\frac{U}{u_*} = \frac{1}{\kappa} \ln\left(\frac{R_h u_*}{\nu}\right) + 3.5$$ $$\frac{1}{\sqrt{f}} = 2 \log\left(Re'\sqrt{f}\right) + 0.32$$
smooth *transition* *rough*	$$\frac{U_c - u}{u_*} = \frac{1}{\kappa} \ln\left(\frac{\delta}{z'}\right) + \frac{\Pi}{\kappa}(2 - \tilde{\omega})$$ $$\frac{1}{\sqrt{f}} = -2 \log\left(\frac{k_s/R_h}{a_f} + \frac{b_f}{4Re'\sqrt{f}}\right)$$
rough $\frac{u_* k_s}{\nu} > 70$	$$\frac{u}{u_*} = \frac{1}{\kappa} \ln\left(\frac{z'}{k_s}\right) + 8.5$$ $$\frac{U}{u_*} = \frac{1}{\kappa} \ln\left(\frac{R_h}{k_s}\right) + 6.25$$ $$\frac{1}{\sqrt{f}} = 2 \log\left(\frac{R_h}{k_s}\right) + 2.2$$

11 ° For *non-uniform* flow (see Fig. 2.12 and Fig. 2.14), eq. 2.51 remains still valid, but the wake parameter, Π, has no more a constant value. Such flows must remain in equilibrium, namely the velocity profile must stay auto-similar. An empirical relationship of the type :

$$\Pi = f(\beta)$$

is proposed, where β is the equilibrium parameter :

$$\beta = \frac{h}{\tau_o} \frac{\partial \overline{p^*}}{\partial x} \tag{2.52}$$

which characterizes the longitudinal pressure gradient. Depending on this parameter, β, one finds (see *Kironoto* et *Graf*, 1994) that :

$\beta < -1$: flow is *accelerating*, and the wake parameter is $-1.0 < \Pi < 0.2$;

$\beta > -1$: flow is *decelerating*, and the wake parameter is $\Pi > 0.2$;

$\beta = -1$: flow is *uniform* having a weak pressure gradient, and the wake parameter is $\Pi \cong 0.2$.

The same tendency was observed (see *Tu* et *Graf*, 1992) for *unsteady* flow.

12 ° For two-dimensional flow, the maximum velocity, U_c. occurs at the water surface, $\delta \equiv h$. For three-dimensional flow, the maximum velocity, U_c, may occur below the water surface, $\delta < h$ (see Fig. 2.15); secondary flow is evident. To parametrise this, one may use a ratio of $(h - \delta)/h$ (see Fig. 2.15). The aspect ratio of $B/h \cong 5$ is the limiting value.

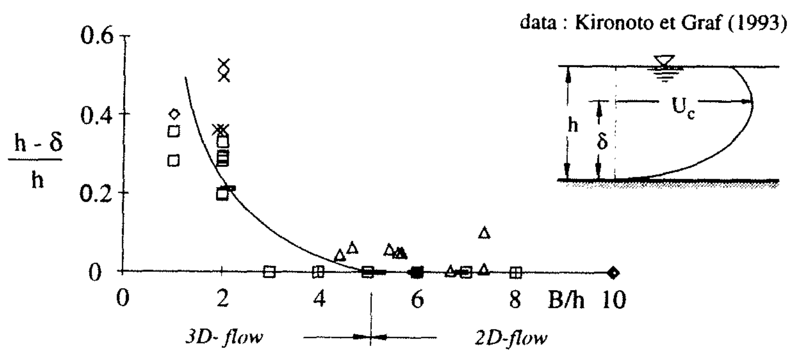


Fig. 2.15 The position of the maximum velocity, U_c ;
in two- and three- dimensional flow.

2.6.4 Turbulence Characteristics

1° Flow at large Reynolds numbers ceases to be laminar; it becomes turbulent. In each point of the flow, the instantaneous velocity, u_i and w_i, is subject to variation in direction and in intensity. The velocity varies around a mean value defined by :

$$u_i = u + u' \quad , \qquad w_i = w + w'$$

The fluctuation components, u' and w', are by definition weak as compared to the respective mean values, u and w.

2° The equations of motion for laminar flow — the Navier-Stokes equation (see *Graf & Altinakar*, 1991, sect. FR.1) — are modified by the supplementary stresses due to the turbulence; these are the Reynolds equations, eq. 2.35, (see *Graf & Altinakar*, 1991, sect. FR.5). The supplementary stresses have the form of :

$$\rho\,\overline{u'^2} \quad , \quad \rho\,\overline{w'^2} \quad , \quad \rho\,\overline{u'w'} \quad , \quad \text{etc.}$$

Semi-empirical methods are used to express these stresses.

The characterization of the structure of turbulence is based here on experiments done in channels; some of these will be presented.

3° *Intensity of turbulence*

The temporal mean value of the velocity fluctuations are by definition zero :

$$\overline{u'} = 0 \; , \quad \overline{w'} = 0$$

This is however not the case for the mean quadratic values, $\overline{u'^2}$. The RMS-value (*Root-Mean-Square*), $\sqrt{\overline{u'^2}}$ and $\sqrt{\overline{w'^2}}$, is commonly used.

The ratio of the RMS-value and the friction velocity (see *Graf & Altinakar*, 1992, p. 267) is used to define the intensity of turbulence :

$$\frac{\sqrt{\overline{u'^2}}}{u_*} = f(z) \quad , \qquad \frac{\sqrt{\overline{w'^2}}}{u_*} = f(z) \tag{2.67}$$

which varies in space.

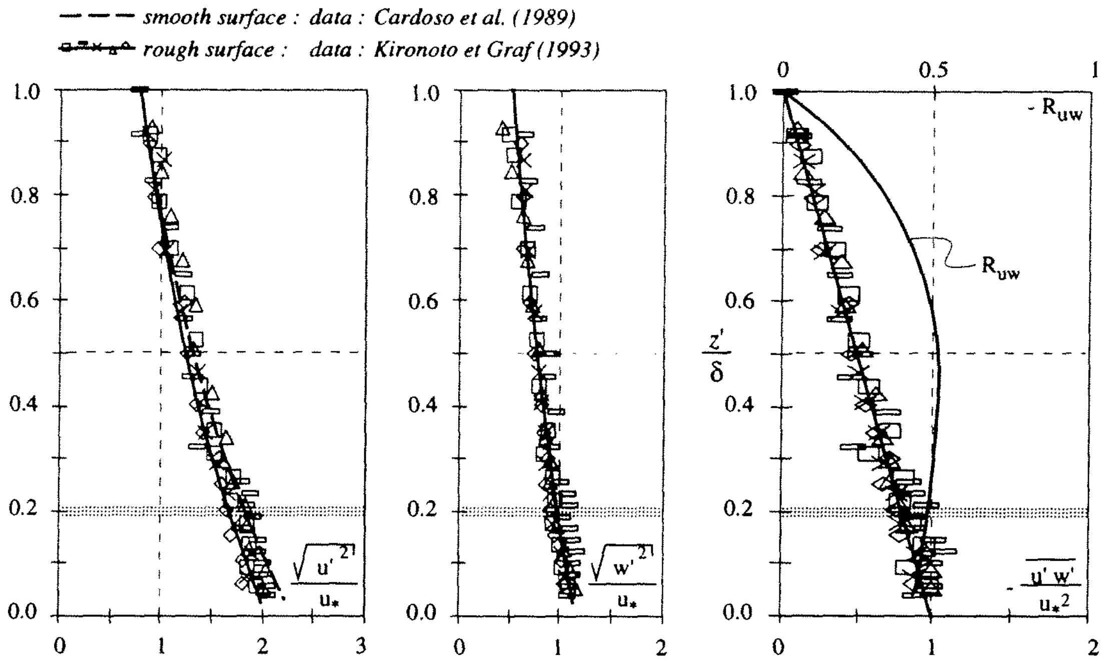


Fig. 2.16 Distribution of the normal and tangential stress of Reynolds; for *uniform* flow.

The vertical distribution of the normal stress, expressed as turbulence intensity, is given in Fig. 2.16 for *uniform* flow; the measurements were performed in the center of the channel having an aspect ratio of 2.1 < B/h < 6.9. One notices that :

i) for channels with smooth (see *Cardoso* et *al.*, 1989) and rough surface (see *Kironoto* et *Graf*, 1993), the distribution are reasonably the same;

ii) close to the bed (surface), in the inner region, one has :

$$\sqrt{\overline{u'^2}} \cong 1.8\, u_* \qquad ; \qquad \sqrt{\overline{w'^2}} \cong 1.0\, u_*$$

iii) up to the flow depth, the distribution stays monotone; at the surface one has :

$$\sqrt{\overline{u'^2}} \cong \sqrt{\overline{w'^2}} \cong 0.6\, u_*$$

and the turbulence becomes isotropic.

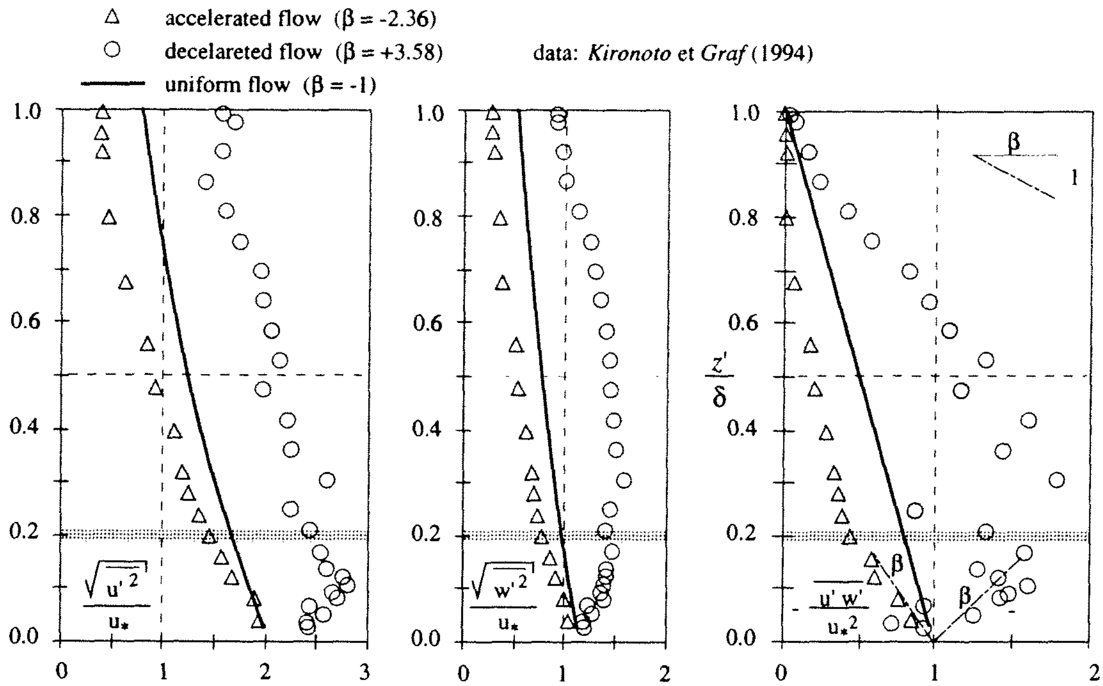


Fig. 2.17 Distribution of the normal and tangential stress of Reynolds; for *non-uniform* flow.

For *non-uniform* flow in equilibrium, the distribution of the normal stress is given in Fig. 2.17; the measurements (see *Kironoto* et *Graf*, 1994) are performed in the center of the channel having an aspect ratio of $B/h \cong 2$. One notices that :

i) for *accelerating* flow, $\beta < -1$, the turbulence intensity is smaller than for uniform flow, $\beta = -1$. The maximum value is at the bed and diminishes towards the water surface. Consequently, in accelerating flow the turbulence is suppressed (see *Hinze*, 1959, p. 66);

ii) for *decelerating* flow, $\beta > -1$, the turbulence intensity is larger than for uniform flow, $\beta = -1$. The maximum value is above the bed and diminishes towards the water surface. Consequently, in decelerating flow the turbulence is enhanced.

Similar experimental observations have also been communicated (see *Bradshaw*, 1978, p. 68) for boundary-layer flow with pressure gradients.

4° *Reynolds stress*

The total tangential stress, eq. 2.42, are well represented by the supplementary stress, notably for flow at large Reynolds numbers; one writes :

$$\tau_{zx} = -\rho\, \overline{u'w'} \tag{2.42b}$$

The vertical distribution of the supplementary stress, in short the Reynolds stress, is given in Fig. 2.16 for experiments with *uniform* flow. One finds — as one has already seen in point 2.5.2, 3° — that :

i) the distribution is linear, or :

$$\frac{\tau_{zx}}{\tau_o} = \left(\frac{\delta - z'}{\delta}\right) \tag{2.47b}$$

ii) with the boundary conditions being :

$$z' \cong 0 \qquad \tau_{zx} \cong \tau_o$$
$$z' \cong \delta \qquad \tau_{zx} = 0$$

For *non-uniform* flow in equilibrium, the distribution of the Reynolds stress is given with Fig. 2.17. One finds — as one has already seen in point 2.5.3, 5° — that :

i) in *accelerating* flow, $\beta < -1$, the Reynolds stress, whose distribution is concave, diminishes;

ii) in *decelerating* flow, $\beta > -1$, the Reynolds stress, whose distribution is convex, increases;

iii) consequently, the energy dissipation, namely the head loss, is larger in decelerating than in accelerating flow;

iv) close to the bed, the gradient to the curve of distribution is given by :

$$\frac{\partial \tau_{zx}}{\partial z} = \frac{\partial \overline{p^*}}{\partial x} \approx \beta\left(\frac{\tau_o}{h}\right) \tag{2.53b}$$

which is in agreement with arguments advanced in point 2.5.3, 5° (see Fig. 2.12).

Similar observations have also been done for unsteady flow with a free surface (see *Tu* et *Graf*, 1992) as well as for boundary-layer flow having pressure gradients (see *Bradshaw*, 1978, p. 68).

A dependence between the velocity fluctuations, u' et w', is often given with a correlation coefficient :

$$-R_{uw} = \frac{\overline{u'w'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{w'^2}}} \tag{2.68}$$

The distribution of this coefficient for *uniform* flow — for *non-uniform* flow it is similar — is given with Fig. 2.16. One sees that :

i) over a large fraction of the flow depth, $0.1 < z'/\delta < 0.6$, the fluctuation are reasonably well correlated : $R_{uw} \cong -0.5$;

ii) this correlation diminishes close to the bed and close to the water surface.

5° *Mixing length*

The Reynolds stress, eq. 2.42b, can also be expressed by :

$$\tau_{zx} = \rho\, l^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2 = \rho\, \nu_t \left(\frac{\partial \bar{u}}{\partial z}\right) \tag{2.49}$$

where l, known as Prandtl's mixing length, is the distance over which the fluid mass displaces itself. ν_t is the mixing coefficient of Boussinesq which has dimensions of the kinematic viscosity, but its value is very large, such as $\nu_t >> \nu$.

The vertical distribution of the mixing length, l, in *uniform* flow — it is similar in *non-uniform* flow — is given with Fig. 2.18; one finds that :

i) close to the surface (bed), in the inner region, $z'/\delta \lesssim 0.2$, it is given as :

$l = \kappa z'$ where $\kappa = 0.4$

ii) in a large part of the outer region, $0.5 \lesssim z'/\delta \lesssim 1.0$, it is given as :

$l/\delta \cong 0.12$

but the data show a large spread.

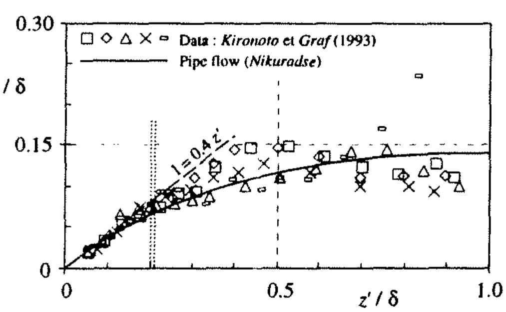


Fig. 2.18 Distribution of the mixing length.

6° *Energy spectrum*

The spectrum of kinetic energy (see *Graf & Altinakar*, 1991, p. 273) provides important information about the turbulence of the flow, namely about the energy distribution of the eddies having different sizes and frequencies.

The spectral function, E(n), gives the turbulent kinetic energy, $\overline{u'^2}(n)$ or $\overline{w'^2}(n)$, for a range of frequencies, dn ; or :

$$\overline{u'^2} = \int_0^\infty E(n)\, dn \tag{2.69}$$

or written in normalized form, $F(n) = E(n)/\overline{u'^2}$, as :

$$\int_0^\infty F(n)\, dn = 1 \tag{2.70}$$

F(n) has units of time, [t] , and n has units of frequency, $[t^{-1}]$.

The function, F(n) , — known as the *turbulence spectrum* — is a representation of the way the energy is distributed with the frequency, n. By a transformation — use is made of Taylor's hypothesis of frozen turbulence — of the frequency, n , into a wave number, $(2\pi/u)\, n = k$, the wave-number spectrum is obtained :

$$\int_0^\infty F(k)\, dk = 1 \tag{2.70a}$$

F(k) has units of length, [m], and k has units of length, $[m^{-1}]$.

With Fig. 2.19 is shown a one-dimensional frequency spectrum for longitudinal, u', and vertical, w', velocity fluctuations for uniform flow in an open channel with a free surface, at different levels, *z'*. These spectra are rather typical for *uniform* flow over smooth and rough surfaces, having $10^4 < Re < 10^6$ (see *Kironoto* et *Graf*, 1993) and also for *non-uniform* flow (see *Kironoto* et *Graf*, 1994).

The energy spectrum is in general rather wide and is usually delimited in three zones (see Fig. 2.19b):

i) The turbulence structure of the largest eddies has no universality; it is anisotropic and depends largely on the flow conditions, thus on the flow Reynolds number. The *macro scale* (see *Reynolds,* 1974, p. 79) of the turbulence, $k_1 = \Lambda^{-1}$, is the upper limit.

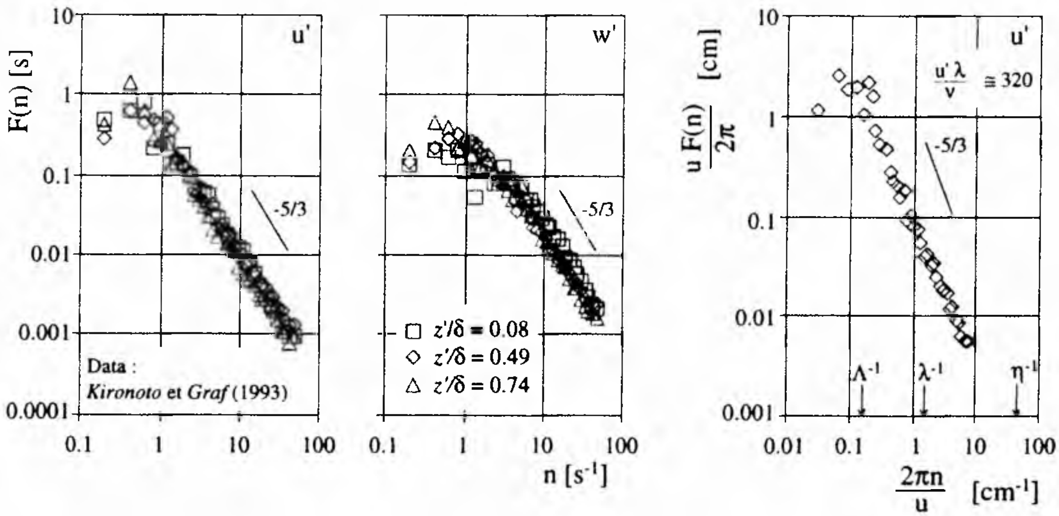


Fig. 2.19 Turbulence spectrum for different levels, z'/δ;
in uniform flow.

ii) The turbulence structure of the very smallest eddies, where viscous dissipation dominates, has universality; the lower limit (see *Reynolds,* 1974, p. 99) is given by the Kolmogoroff scale, $k_3 = \eta^{-1} = (\nu^3/\varepsilon)^{1/4}$, where ν is the kinematic viscosity and ε is the dissipated turbulent energy, expressed by :

$$\varepsilon = 15\ \mu\ (\overline{u'^2}/\lambda^2).$$

iii) In the *inertial zone*, if the flow Reynolds number is high, the large eddies disintegrate into smaller eddies and so on (in cascades), and all this by action of the inertia forces. The resulting energy spectrum has universality, thus being independent of the Reynolds number. This part of the spectrum is described by *Kolmogoroff*'s hypothesis, which shows by way of a dimensional analysis, that the spectral function, F(k) , can be given by a law of the type (see *Reynolds,* 1974, p. 99) :

$$F(k) \propto \varepsilon^{2/3}\, k^{-5/3}$$

In this zone, $k_1 < k << k_3$, the turbulence is quasi isotropic and the spectrum is in equilibrium: the *micro-scale* (see *Reynolds,* 1974, p. 79) of the turbulence, $k_2 = \lambda^{-1}$, falls into this zone.

3. UNIFORM FLOW

Flow in a channel is considered as uniform and steady, if the flow depth remains invariable in the flow direction as well as in time. In fluvial hydraulics, uniform flow is taken as the base (reference) for all other considerations, and this despite the fact that truly uniform flow is rarely encountered in reality.

In this chapter, the equations of continuity and of motion will be developed. Subsequently are presented the different relationships for the determination of the coefficients of friction for fixed and mobile channel beds. Knowledge about this coefficient is paramount in all kinds of problems of fluvial hydraulics. The calculation of the discharge for flow over a fixed as well as a mobile bed is elaborated. Elementary knowledge about flow in curves as well as instabilities at the free-water surface will be exposed.

TABLE OF CONTENTS

3.1 HYDRODYNAMIC EQUATIONS

3.1.1 Notion of Uniformity

1° Flow is considered as uniform and steady (see sect. 1.2.1) if the flow depth, h or D_h, as well as other hydraulic parameters such as the average velocity, the discharge, the roughness and the channel slope, remain invariable in different cross sections of the channel along the axis of flow. The streamlines are rectilinear and parallel and the vertical pressure distribution is hydrostatic. The slope of the bed, S_f, of the water surface, S_w, and of the energy-grade line, S_e, are the same.

2° Truly uniform flow is rather rare in natural, but also in artificial channels. Uniform flow is only possible in very long prismatic channels and this far from the upstream and downstream boundary conditions (see Fig. 3.1).

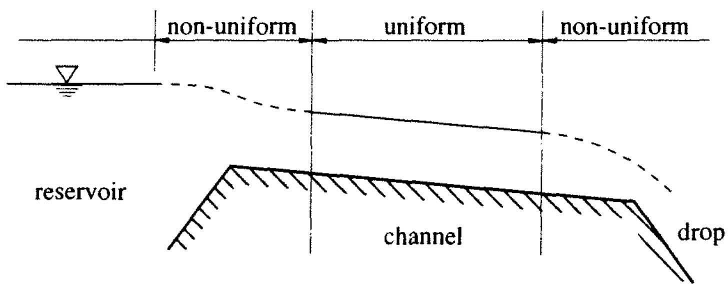


Fig. 3.1 Uniform flow between boundary conditions.

3° Despite the fact that uniform flow occurs rarely, this type of flow is usually taken as the standard (reference) flow for any theoretical and experimental study of other types of flow, but notably for the understanding of the flow resistance.

3.1.2 Equation of Continuity

1° As long as flow is uniform and steady, the cross section of the flow, A, remains the same in direction, x, and in time, t. The equation of continuity (see sect. 2.1) was given as :

$$\frac{\partial(UA)}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{2.1}$$

but becomes now :

$$\frac{\partial(UA)}{\partial x} = 0 \tag{3.1}$$

where Q = UA is the discharge and U is the average velocity.

2° Consequently, the discharge remains constant :

$$Q = Cte \tag{3.2}$$

Between two cross sections (see Fig. 3.2), one has :

$$A_1U_1 = Q = A_2U_2 \tag{3.2a}$$

and with $U_1 = U_2$ and $A_1 = A_2$, one writes : $Q = UA$.

3.1.3 Equation of Motion

1° Consider a prismatic channel (see Fig. 3.2). The liquid in motion provokes a friction force at the wetted perimeter :

$$F_F = \tau_o P\, dx$$

by an action of the longitudinal component of the gravity force :

$$F_G = \gamma A\, dx\; \sin\alpha = W \sin\alpha$$

In uniform flow, there exists an equilibrium between these forces :

$$\tau_o P\, dx = \gamma A\, dx\; \sin\alpha \tag{3.3}$$

Consequently, one obtains an expression for :

$$\tau_o = \gamma \frac{A}{P} \sin\alpha \tag{3.4}$$

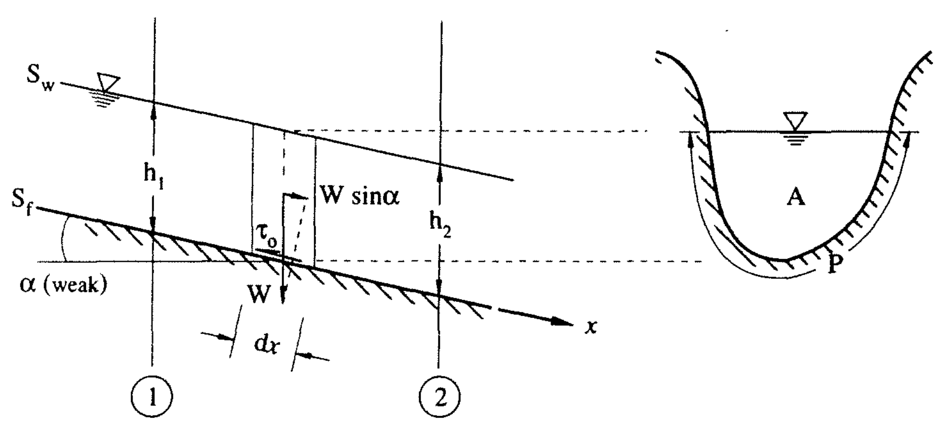


Fig. 3.2 Scheme of uniform flow.

The quotient of the wetted cross section , A , and its wetted perimeter, P , defines the hydraulic radius, R_h. The angle, α , is usually very small; thus one may write $\sin\alpha \cong \mathrm{tg}\ \alpha = S_f$. Above relation, eq. 3.4, now reads :

$$\tau_o = \gamma\ R_h\ S_f \tag{3.5}$$

where τ_o is the tension due to the friction forces, called the shear stress, which acts on the wetted surface (wall and bed). Note that eq. 3.5 can be obtained directly from eq. 2.10 or eq. 2.12, by considering the uniformity of the flow.

2° In hydrodynamics, one defines :

$$\tau_o/\rho = u_*^2 \tag{3.6}$$

where u_* is the friction velocity. Thus one can also write :

$$u_* = \sqrt{g\ R_h\ S_f} \tag{3.7}$$

Instead of the shear stress, $\tau_o = \rho\ u_*^2$, one may also use the definition of the *friction coefficient* (see *Graf & Altinakar*, 1991, p. 433) which is given by :

$$f = \frac{\tau_o}{\rho U^2/8} = 8\left(\frac{u_*}{U}\right)^2 \tag{3.8}$$

3° Upon substitution of eq. 3.8 into eq. 3.5, one obtains:

$$(f/8)\ \rho U^2 = \tau_o = \rho g\ R_h\ S_f$$

or, written otherwise :

$$S_f = f\frac{1}{4R_h}\ \frac{U^2}{2g} \tag{3.9}$$

This relation is known as the equation of *Weisbach-Darcy* (see *Graf & Altinakar*, 1991, sect. FR. 2.1 and sect. PP. 2); it reveals itself as very useful for flow in pipes. The coefficient, f, of friction (head loss) depends on the Reynolds number and the relative roughness, but also on the form of the cross section.

The equation of Weisbach-Darcy can also be written as :

$$U = \sqrt{8g/f}\ \sqrt{R_h\ S_f} \tag{3.10}$$

an expression which is frequently given in the form of :

$$U = C \sqrt{R_h S_f} \tag{3.11}$$

This is called the relationship of *Chézy*, where C is the resistance coefficient of Chézy.

4° In uniform regime (see eq. 3.10 and eq. 3.11), the flow depth, h, which corresponds to the hydraulic radius, R_h, is defined as being the *normal flow depth*, $h \equiv h_n$.

5° Different formulae have been elaborated over the years to render expressions for the friction (resistance) coefficients. Herewith some more common formulae will be presented, namely :

i) coefficient of Weisbach-Darcy (see sect. 3.2.1),
ii) coefficient of Chézy (see sect. 3.2.2),
iii) coefficient of Manning-Strickler (see sect. 3.2.3),
iv) coefficient of friction for mobile bed (see sect. 3.2.6).

3.2 COEFFICIENT OF FRICTION

1° It will certainly be useful, to express the friction coefficient, f, for laminar and turbulent flow with the equation of Weisbach-Darcy, eq. 3.10. However, the channel data presently available contain major limitations since channels should be of circular cross section and the roughness should be the standard one.

2° The relation of Chézy, eq. 3.11, is also rather useful as long as the flow in the channel is truly turbulent; this is often the case.

3° These two approaches, eq. 3.10 and eq. 3.11, give satisfactory results, notably for practical problems, if applied correctly and respecting their possible limitations. The ASCE (see *Silberman* et *al.*, 1963) recommended however the use of the equation of Weisbach-Darcy.

The precision which is obtained with these formulae, eq. 3.10 or eq. 3.11, is nevertheless strongly dependent upon the choice of the friction coefficient, f or C.

4° Artificial and particularly natural channels have all types of form of the cross section. No parameter exists which would well take care of the variability in form; the use of the hydraulic radius is often not sufficient.

5° An estimation of the friction coefficient for a fixed or immobile bed is already difficult; but still more difficult will be an estimation for a mobile bed.

3.2.1 Coefficient of Weisbach-Darcy

1° In above equations, eqs. 3.9 and 3.10, the definition of friction coefficient, f, is analogous to the one given for circular pipes. For pipes having an industrial roughness, a universal formulation is given (see *Graf & Altinakar*, 1991, sect. PP. 2) by :

i) the diagram of Moody-Stanton or

ii) the relation of Colebrook-White, for turbulent flow.

2° For cross sections, which are geometrically close to circular sections, one may readily use the experiments performed on pipes. However, some modifications are necessary; the hydraulic radius (see *Graf & Altinakar*, 1991, p. 439) should be written as follows :

$$4R_h = 4\frac{A}{P} \tag{3.12}$$

Thus $4R_h$ becomes the characteristic length, which is to be used in the definition of the Reynolds number, the relative roughness and the equation of Weisbach-Darcy, respectively :

$$\mathrm{Re} = \frac{4R_h U}{\nu} \quad ; \quad \frac{k_s}{4R_h} \quad ; \quad S_f = f\frac{1}{4R_h}\frac{U^2}{2g}$$

For the roughness, k_s , in artificial channels, the equivalent roughness, established for industrial pipes, may be taken.

3° The use of the diagram of Moody-Stanton (see *Graf & Altinakar*, 1991, Fig. PP.9) with $\mathrm{Re} = 4R_hU/\nu$ and $k_s/4R_h$ gives values for f for laminar and turbulent flow. Subsequently, one obtains the average velocity, U, using eq. 3.10, or the bed slope, $S_f = S_w$, of the channel, using eq. 3.9.

4° Instead of using the diagram of Moody-Stanton, one may also take the semi-empirical *relation of Colebrook-White* (see *Graf & Altinakar*, 1991, p. 436), valid only for turbulent flow, which is written for channels as follows (see eq. 2.65) :

$$\sqrt{\frac{1}{f}} = -2\log\left(\frac{k_s/R_h}{a_f} + \frac{b_f}{\mathrm{Re}\sqrt{f}}\right) \tag{3.13}$$

with $12 < a_f < 15$ and $0 < b_f < 6$, established for different kinds of cross sections, as well as for different types of roughnesses (see *Silberman* et *al.*, 1963, p. 104).

The equivalent roughnesses, k_s, established for industrial pipes, but considered valid also for artificial channels, are given in Table 3.1. A more complete tabulation is given by *Wallisch* (1990, pp. 235-250).

For channels or watercourses whose bed is made up of a granulate, one generally takes $k_s \equiv d_{50}$; d_{50} being the diameter equal to 50% of grains in the granulometric curve.

The importance of the form of the cross section can be somehow taken care of by a factor, which multiplies the hydraulic radius R_h ; thus (ϕR_h) and (ϕRe) replace R_h and Re in eq. 3.13. One takes (see *Ghetti* , 1981) :

- for a rectangular (B = 2 h) section $\phi = 0.95$
- for a large trapezoidal section $\phi = 0.80$
- for a triangular (equilateral) section $\phi = 1.25$

5° However, it must be pointed out that the results obtained with the diagram of Moody-Stanton or the *relation of Colebrook-White* will only be good approximations.

For channels, being very large and rectangular or very different from circular sections, above methods are less applicable.

Table 3.1 Equivalent roughness for industrial pipes.

Types of wall		*Uniform equivalent roughness* k_s [mm]
glass, copper, brass		< 0.001
lead		0.025
pipes, steel	new old	0.03 à 0.1 0.4
wrought iron	new old coated	0.25 1.0 à 1.5 0.1
concrete	smooth rough	0.3 à 0.8 < 3.0
wood		1.0 à 2.5
riveted steel		0.9 à 9
stone, worked rough		8 à 15
rock		90 à 600

6° Natural or artificial channels are usually of large dimensions. Consequently, the Reynolds number, $Re = 4R_hU/\nu$, and the roughness, k_s, have also large values. This implies that the turbulent flow is often also a rough one; the value of the friction coefficient, f, remains constant and is no more dependent on the Reynolds number.

This is a justification for using the relation of Chézy, eq. 3.10 or eq. 3.11, where the coefficient of Chézy depends only on the relative roughness, $C = f(k_s/R_h)$, (see eq. 3.13); thus one may write :

$$C = \sqrt{8g}\left(\frac{1}{\sqrt{f}}\right) = \sqrt{8g}\left[2\cdot\log\left(\frac{a_f}{k_s/R_h}\right)\right] \tag{3.13a}$$

Subsequently, taking $a_f = 12.7$, one obtains (see also eq. 2.63) :

$$\sqrt{\frac{8}{f}} = 5.6\log\left(\frac{R_h}{k_s}\right) + 6.25 \tag{3.13b}$$

For a roughness due to large grains, $R_h/d_{50} \leq 10$, one should take (see *Graf* et *al.*, 1987) :

$$\sqrt{\frac{8}{f}} = 5.75\log\left(\frac{R_h}{d_{50}}\right) + 3.25 \tag{3.13c}$$

where $k_s \equiv d_{50}$, with d_{50} as the median grain diameter.

7° In rough channels of large width, $R_h \approx h$, the friction coefficient, f, can be obtained making in-situ measurements of point velocities (see *Graf*, 1966) and assuming a logarithmic distribution (see sect. 2.63) :

$$\frac{u}{u_*} = 5.75\log\frac{30\,z}{k_s}$$

The point velocities, $u_{0.2}$ and $u_{0.8}$, at two elevations, $z' = 0.2\,h$ and $z' = 0.8\,h$, situated on the same vertical (see Fig. 1.7), are given by :

$$u_{0.8} = 5.75\,u_*\log(24h/k_s) \qquad u_{0.2} = 5.75\,u_*\log(6h/k_s)$$

By elimination of u_* in these two relations and putting $(u_{0.8}/u_{0.2}) = \zeta$, one gets :

$$\log\frac{h}{k_s} = \frac{0.78\zeta - 1.38}{1 - \zeta} \tag{3.19}$$

The average velocity, U , for a turbulent rough flow (see eq. 2.63) was given by :

$$\frac{U}{u_*} = 5.75 \log \frac{h}{k_s} + 6.25$$

A substitution of eq. 3.19 into this equation – making use of the definition of eq. 3.8 – renders :

$$\frac{U}{u_*} = \frac{1.78\,(\zeta + 0.95)}{(\zeta - 1)} \tag{3.20}$$

Subsequently one obtains for the coefficient of friction :

$$\sqrt{\frac{1}{f}} = \frac{1.78}{\sqrt{8}} \; \frac{(\zeta + 0.95)}{(\zeta - 1)} \tag{3.21}$$

and also (see eq. 3.13a) :

$$C = \sqrt{8g}\sqrt{\frac{1}{f}} = 1.78\sqrt{g}\;\frac{(\zeta + 0.95)}{(\zeta - 1)} \tag{3.21a}$$

The coefficients of friction, f or C , are thus obtained in an experimental way for very wide channels, where $h \cong D_h = R_h$, using the hypothesis of a logarithmic velocity distribution.

3.2.2 Coefficient of Chézy

1° For turbulent, rough flow the *formula of Chézy* :

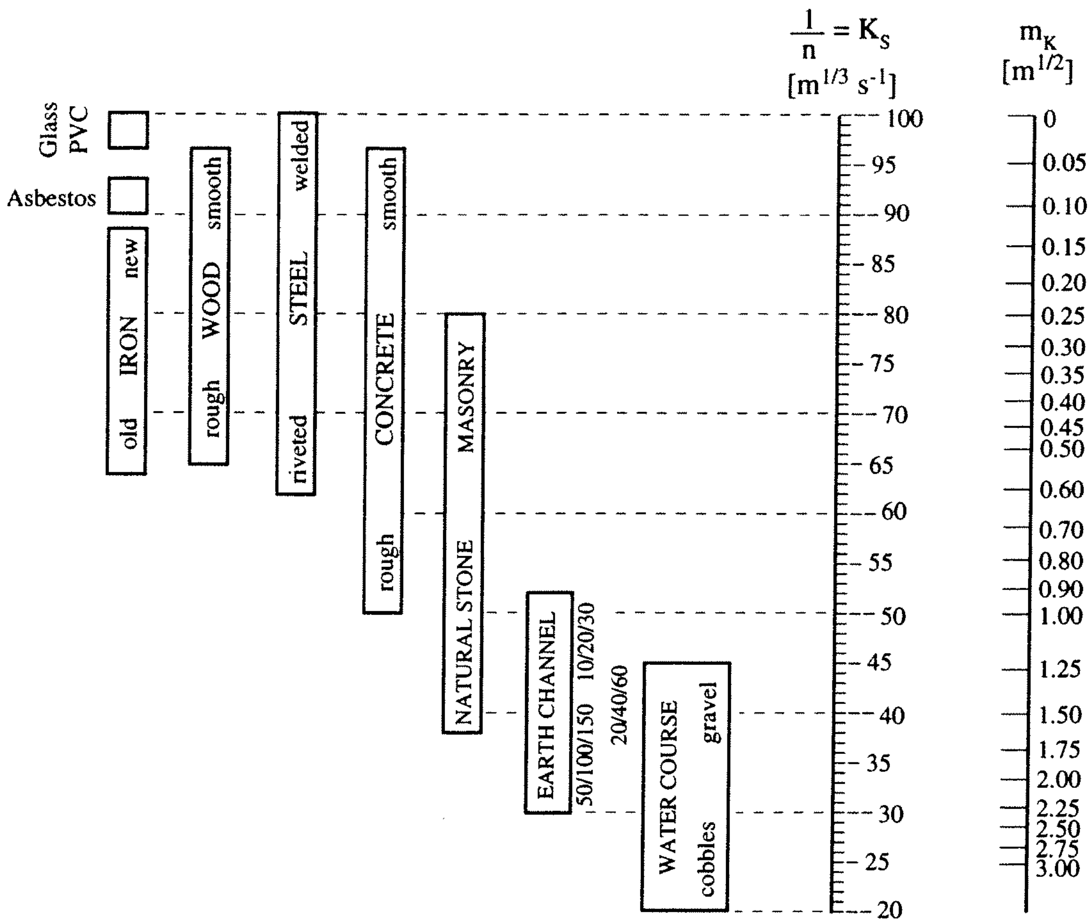
$$U = C\sqrt{R_h S_f} \tag{3.11}$$

can be used. However, it cannot be used for laminar or turbulent smooth flow.

The coefficient of Chézy, C $[m^{1/2}/s]$, is a dimensional expression; the numerical values use as unity the meter [m] and the second [s].

Different formulae, being all of empirical nature, have been advanced for the determination of the coefficient of Chézy, C ; all of which make use of the hydraulic radius, R_h .

Table 3.2 Coefficients of roughness of Manning, of Strickler and of Kutter.



2° The *formula of Bazin* considers C as being a function of the hydraulic radius, R_h [m], and of a coefficient, m_B [$m^{1/2}$], which characterises the roughness of the walls and the bed. Established with data from small artificial channels, this relation reads :

$$C = \frac{87}{1 + \left(m_B / \sqrt{R_h} \right)} \tag{3.14}$$

The coefficient of Bazin varies from $m_B = 0.06$, for a smooth bed, to $m_B = 1.75$, for a bed made up of stones or covered with vegetation.

3° The *(simplified) formula of Kutter,* established with data from artificial channels as well as from larger rivers, has a similar form, being :

$$C = \frac{100}{1 + \left(m_K / \sqrt{R_h} \right)} \tag{3.15}$$

where m_K [$m^{1/2}$] is the coefficient of Kutter. Some values for m_K are given in Table 3.2.

4° In the practice, one prefers presently the exponential relations and one uses commonly the *formula of Manning-Strickler* in the form of :

$$U = K_s R_h^{2/3} S_f^{1/2} \tag{3.16}$$

with

$$C = K_s R_h^{1/6} = \frac{1}{n} R_h^{1/6} \tag{3.17}$$

Here K_s [$m^{1/3}s^{-1}$] is the coefficient of *Strickler* and n [$m^{-1/3}s^{1}$] is the coefficient of *Manning*. Above relation, eq. 3.16, was elaborated using numerous measurements, performed in both natural and artificial channels. The values of n and K_s are given in Table 3.2. More detailed tables are available (see *Wallisch* 1990, pp. 252-267).

5° There exist other exponential relations, such as :

i) *formula of Forchheimer* : $C = \frac{1}{n} R_h^{1/5}$

ii) *formula of Pavloski* (see *Grishanin*, 1990, p. 45): $C = \frac{1}{n} R_h^{q}$

for $R_h \le 1$ [m] : $q = 1.5 \sqrt{n}$

$R_h > 1$ [m] : $q = 1.3 \sqrt{n}$

3.2.3 Coefficient of Manning

1° The most popular formula is presently the one of *Manning-Strickler*, often called shortly the *formula of Manning* :

$$U = \frac{1}{n} R_h^{2/3} S_f^{1/2} \tag{3.16}$$

This is a rather simple relationship, but it must be used only for turbulent, rough flow, thus for flow at large Reynolds numbers. In such a case, the coefficient of Manning, n , stays constant for a given roughness, while the coefficient of Chézy, C , depends (see eq. 3.17) on the relative roughness, ($R_h^{1/6}$/n) .

2° Complete tabulations of the coefficient of Manning, n , have been presented by *Crause* (1951, p. 38), *Chow* (1959, pp. 110-113) and *Graf* (1984, pp. 306-309). Furthermore, *Chow* (1959, pp. 115-123) and *Barnes* (1967) provide photos of different natural and artificial channels as a visual support, to facilitate the choice of the coefficient of Manning in the range of 0.012 < n < 0.15.

Indicative values are summarised in Table 3.2.

It must be pointed out that the values of the coefficient of Manning are the same both in the metric and in the English system. In the latter case, one has to use the following relation :

$$C = \frac{1.48}{n} R_h^{1/6} \tag{3.17a}$$

3° For watercourses, where the bed and walls are made up of a non-cohesive granulate, the *formula of Strickler* (see *Strickler*, 1923, pp. 11-15) may be used :

$$K_s = \frac{21.1}{d_{50}^{1/6}} \qquad \text{or} \qquad K_s = \frac{26}{d_{90}^{1/6}} \tag{3.18}$$

where d_{50} or d_{90} [m] are the diameters, being equal to 50% or 90% of the grains in the granulometric curve.

4° The influence of vegetation on the coefficient of friction is extensively treated by *Chow* (1959, pp. 179-184) and *Wallisch* (1990, p. 229).

3.2.4 Composite Roughness

1° The coefficients of friction, f, n and C , are valid as long as the entire wetted perimeter has the same roughness; thus the wetted section is homogeneous.

2° In sections where the wetted perimeter is *not homogeneous*, the bed and the side walls have different roughnesses (see Fig. 3.3); thus it becomes necessary to compute an equivalent coefficient of friction.

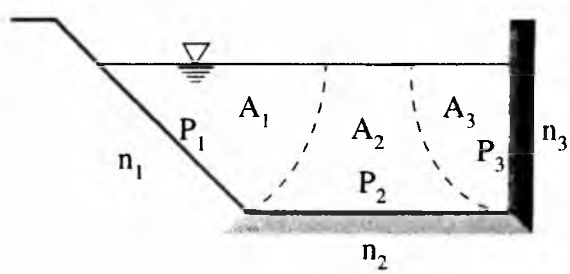


Fig. 3.3 Section of composite roughness.

3° According to *Einstein* (see *Chow*, 1959, p. 136), one divides – in a reasonable way – the wetted surface, A , in N parts, each one having its wetted perimeter, P_1 , P_2 ......P_N , and its coefficient of friction, n_1 , n_2 .....n_N. Furthermore, one assumes that the average velocity of each particular section, A_1 , A_2 .....A_N, is the same and thus also the same as the average velocity of the entire section, $U_1 = U_2 = \dots\dots\dots = U_N \equiv U$.

4° Using, for example, the formula of Manning, eq. 3.16, on writes :

$$U = \frac{1}{n}\left(\frac{A}{P}\right)^{2/3} S_f^{1/2} = \frac{1}{n_1}\left(\frac{A_1}{P_1}\right)^{2/3} S_f^{1/2} = \dots\dots\dots\dots = \frac{1}{n_N}\left(\frac{A_N}{P_N}\right)^{2/3} S_f^{1/2}$$

If one assumes that $A^{2/3} = \sum_1^N A_N^{2/3}$, the equivalent coefficient of friction for a composite roughness can be computed, being :

$$n = \left[\frac{\sum_1^N (P_N\, n_N^{3/2})}{P}\right]^{2/3} \tag{3.22}$$

3.2.5 Bed Forms

1° Natural, but also artificial channels may have a *mobile bed*, defined as being a channel bed composed of solid particles (non-cohesive granulate, alluviums), which displace themselves by the action of the flow. The bed may become covered with *bed forms*, commonly called *dunes* (see Fig. 3.4). These solid particles are characterised by the density, ρ_s , the median diameter, $d \equiv d_{50}$, and their granulometric distribution.

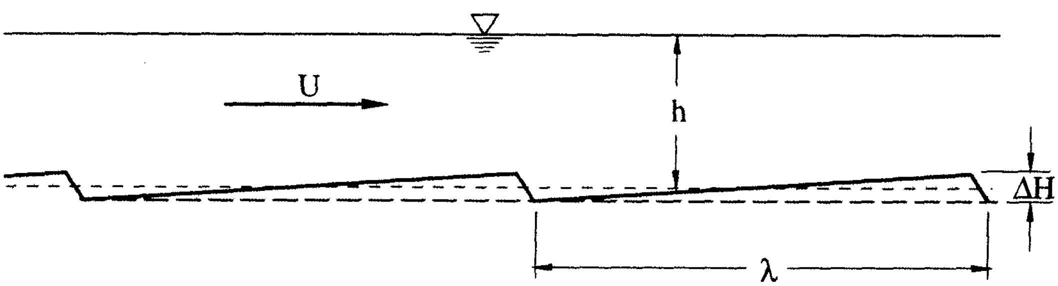


Fig. 3.4 Scheme of a channel bed with a series of dunes.

2° A mobile bed presents successively various aspects, which correspond to the different types of bed deformations. These are usually classified into three regimes, by using the Froude number, Fr (see Fig. 3.5) :

i) Fr < 1 : The bed remains rather flat, and this till the velocity becomes critical (see sect. 3.4.2) and the sediment (solid) transport begins. Consequently, *mini-dunes* or ripples appear, followed by the *dunes* of growing dune length, λ .

Regime	Transport of sediments	Bed form
$Fr < 1$	no	flat
	yes	mini-dune dune
$Fr \cong 1$	yes	flat
$Fr > 1$	yes	anti-dune

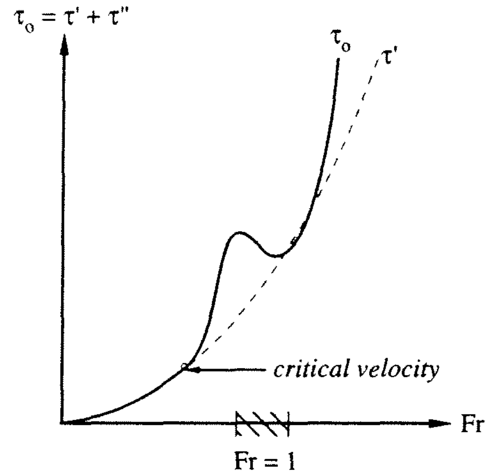


Fig. 3.5 Regime of flow over a mobile bed.

ii) $Fr \approx 1$: As the flow velocity increases, these dunes, already rather long, are washed out and tend to disappear. In this state of *transition*, the bed is once more a flat one.

iii) $Fr > 1$: With a further increase of the flow velocity, another kind of dunes appear, commonly called *anti-dunes*, which, contrary to the dunes, travel usually into the upstream direction. The water surface becomes wavy and the sediment transport is very strong.

3° The geometry of a dune (idealised, since sometimes they are not well apparent) is approximated by a triangular form of length, λ, and of height, ΔH (see Fig. 3.4).

Indicatives relations (see *Graf*, 1984, p. 283), made dimensionless by the flow depth, are given as :

$$\frac{\Delta H}{h} < \frac{1}{6} \quad ; \quad \frac{\lambda}{h} \approx 5 \tag{3.23}$$

4° The presence of bed forms will cause an increase in the flow resistance. For the calculation of the total shear stress on the bed, τ_o , one assumes (see *Graf*, 1984, p. 303) that the contribution of the roughness due to the particles, τ' , and the one due to the bed forms , τ'' , is additive, namely :

$$\tau_o = \tau' + \tau'' \tag{3.24}$$

or (see eq. 3.5) :

$$\gamma R_h S_f = \gamma (R_h' + R_h'') S_f$$

where R_h' and R_h'' are the hydraulic radius due to the particle roughness and to the bed forms, respectively.

Using the definition of the friction velocity and of the coefficient of friction, eq. 3.6 and eq. 3.8, one writes :

$$u_*^2 = (u_*')^2 + (u_*'')^2$$

$$f = f' + f'' \tag{3.25}$$

but also :

$$n = n' + n'' \quad ; \quad C = C' + C'' \tag{3.26}$$

5° The total shear stress, τ_0, (see eq. 3.24) varies as a function of the Froude number, Fr . This variation is schematically shown in Fig. 3.5.

3.2.6 Coefficient of Friction, mobile Bed

1° A quantification of friction coefficient for flow over a mobile bed has, up to now, not been very successful over a large range of flow parameters.

2° There exist methods where one determines directly the *entire* coefficient of friction, f or n .

There exist other methods, where one calculates the coefficient of friction due to the grain roughness, f' or n', using the formulae presented above (see sects. 3.2.1 to 3.2.3). Subsequently one determines the coefficient of friction due to the bed forms, f'' or n'', using other types of formulae.

3° A selection from the different existing formulae for a direct calculation is given in the following :

i) The determination of the entire coefficient of friction can be done using an exponential relation of the Chézy type (see eq. 3.11) :

$$U = K_T R_h^x S_f^y \tag{3.27}$$

Sugio (1972) studied extensively watercourses, having $0.1 < d_{50}$ [mm] < 130 , and artificial channels, having $0.2 < d_{50}$ [mm] < 7.0 ; proposed was :

$$U = K_T R_h^{0.54} S_f^{0.27} \tag{3.27a}$$

It should be noted that the exponent of $y = 0.27$ is very different from the one used in the relation of Chézy or of Manning, where $y = 0.50$ for channels with fixed beds.

The values for K_T are : $K_1 = 54$ for mini-dunes
$K_2 = 80$ for dunes
$K_3 = 110$ for the upper regime
$K_4 = 43$ for rivers with meanders.

This relation, eq. 3.27a, is simple to use and compares itself favourably with other formulae advanced for channels in *regime*, namely in equilibrium (see *Graf*, 1984, chap. 10), which are also presented by *Sugio* (1972, p. 24).

ii) The following relationship, presented by *Grishanin* (1990, p. 59), expresses the coefficient of Chézy, C , used in eq. 3.11, as being :

$$C = 5.25 \left(\frac{Ug}{\sqrt[3]{gv}}\right)^{1/2} \left(\frac{D_h}{B}\right)^{1/6} \tag{3.28}$$

It was established for Russian rivers, having $0.1 < d_{50}\,[\mathrm{mm}] < 0.44$ and $3 \times 10^{-6} < S_f < 2.2 \times 10^{-4}$.

iii) Yet another relationship, presented by *Grishanin* (1990, p. 69) and obtained from different (35) Russian rivers, was given as :

$$U = \frac{1}{M_G^2} \left(\frac{g}{B}\right)^{1/2} D_h \tag{3.29}$$

where M_G is a local non-dimensional invariant, $M_G = 0.91 \pm 0.12$, for channel beds of sand.

iv) Using a large series of artificial (laboratory) channels as well as natural watercourses, having grain diameters of $0.11 < d_{50}\,[\mathrm{mm}] < 1.35$ and bed slopes of $3 \times 10^{-6} < S_f < 3.7 \times 10^{-2}$, the following relation (see eq. 3.24) was proposed by *Brownlie* (1983, p. 975) :

$$\tau_* = \frac{\tau_o}{d_{50}\,(\gamma_s - \gamma)} = w\ (q_* S_f)^x\ S_f^y\ \sigma^z \left(\frac{\rho}{\rho_s - \rho}\right) \tag{3.30}$$

where $q_* = q/\sqrt{g d_{50}^3}$; q is the unit discharge, σ the standard deviation of the grains in the granulometric distribution and $\gamma_s = \rho_s g$ is the specific weight of the granulate. The coefficients were obtained by a statistical analysis, being :

- for channels with mobile beds, having mini-dunes and dunes :

$$w = 0.37 \quad , \quad x = 0.65 \quad , \quad y = 0.09 \quad , \quad z = 0.11$$

- for channels with mobile beds, being flat (see Fig. 3.5) or having anti-dunes :

$$w = 0.28 \quad , \quad x = 0.62 \quad , \quad y = 0.09 \quad , \quad z = 0.08$$

4° Different empirical relationships have been elaborated for the calculation of the friction velocity and of the coefficient of friction, u_*'' and f'', being due to bed forms. Here are given two relations :

i) The relation proposed by *Einstein-Barbarossa* is given usually in graphical form (see Fig. 3.6), where a large spread is evident. Used were many observations from American rivers, having $0.19 < d_{35}$ [mm] < 4.3 and $1.49 \times 10^{-4} < S_w < 1.72 \times 10^{-3}$. This relationship is expressed (see *Graf*, 1984, p. 310) by :

$$\frac{U}{u_*''} = f\left(\frac{\rho_s-\rho}{\rho}\,\frac{d_{35}}{R_h'\,S_f}\right) = f(\psi') \tag{3.31}$$

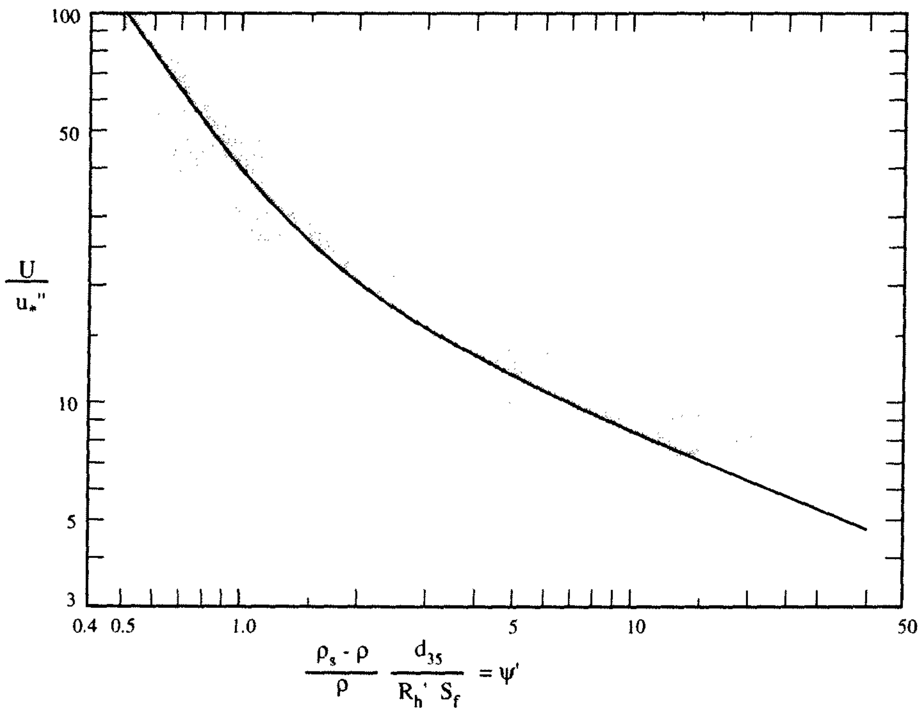


Fig. 3.6 Friction velocity, u_*'', due to bed forms for mobile bed ; after Einstein-Barbarossa.

ii) The relationship proposed by *Alam-Kennedy* is given as :

$$f'' = f\left(\frac{R_h}{d_{50}}, \frac{U}{\sqrt{g d_{50}}}\right) \tag{3.32}$$

Also this one is usually (see *Yalin*, 1972, p. 280) given in a graphical form (see Fig. 3.7), where a large spread (not shown) is evident. A great many data from artificial (laboratory) channels, having $0.04 < d_{50}[\mathrm{mm}] < 0.54$, and American rivers, having $0.08 < d_{50}[\mathrm{mm}] < 0.45$, have been used.

In this figure, Fig. 3.7, the relation of Einstein-Barbarossa corresponds to the region where the lines of the values of $U/\sqrt{g\, d_{50}}$ stay reasonably horizontal, namely for $R_h/d_{50} > 3 \times 10^3$.

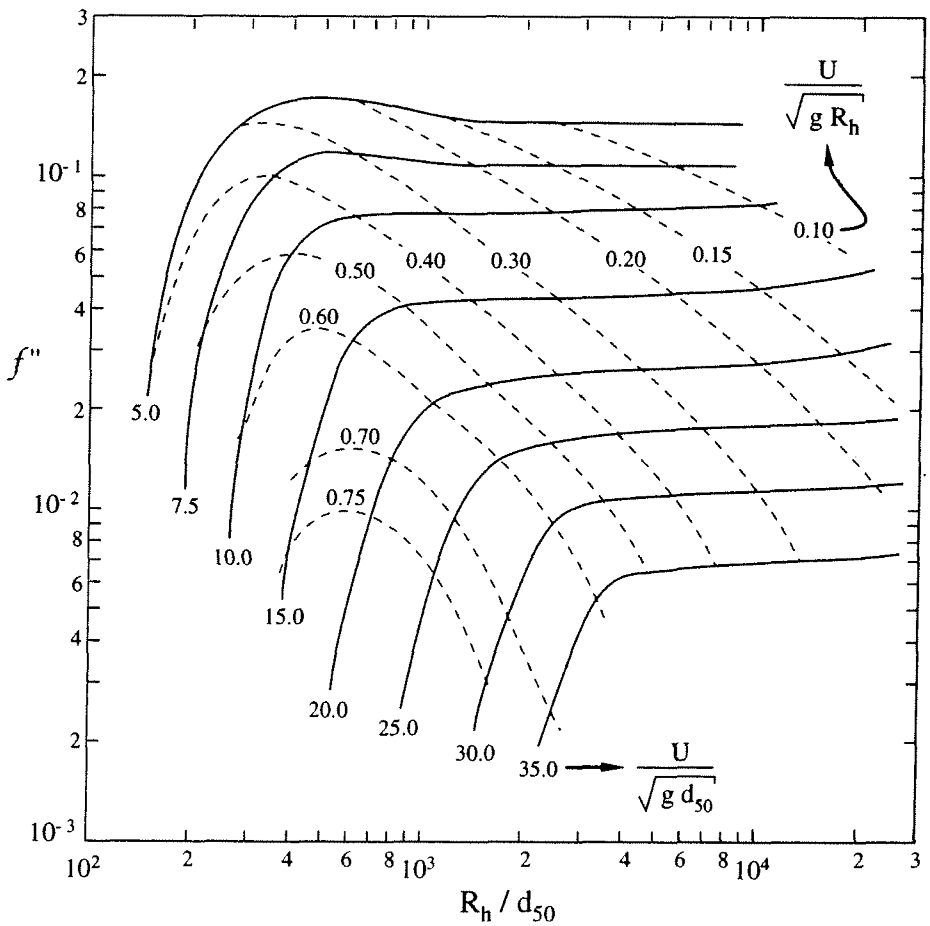


Fig. 3.7 Coefficient of friction, f'', due to bed forms for mobile bed; after Alam-Kennedy.

iii) Some more relations have been presented in *Graf* (1984, pp. 303-320) and

3.3 DISCHARGE CALCULATION, FIXED BED

1° The study of uniform flow in a watercourse is a common task for the hydraulic engineer.

2° Determination of the discharge, Q, in a channel with a fixed bed requires the knowledge of the channel geometry, of the roughness coefficient and of the bed slope.

3° Assumed will be that the walls (bed and side walls) of the channel are fixed or immobile, thus not subject to erosion.

3.3.1 Conveyance

1° The discharge, Q, at uniform flow, given by eq. 3.2a, using the corresponding velocity, given by eq. 3.16, can be expressed as :

$$Q = UA = \frac{1}{n} R_h^{2/3} S_f^{1/2} A \tag{3.33}$$

2° The values of the wetted section, A, and of the hydraulic radius, R_h, are determined and given by the flow depth, h. Furthermore, the nature of the wall roughness, n, is taken to be known. The following expression can be formed:

$$K(h) = \frac{1}{n} R_h^{2/3} A \tag{3.34}$$

known as the *conveyance* of the channel (see *Bakhmeteff*, 1932, p. 13), being only a function of the flow depth, $h \equiv h_n$. This depth is known as the *normal depth*, h_n, for the given discharge, Q. Thus, above expression, eq. 3.33, yields :

$$Q = K(h)\sqrt{S_f} \tag{3.35}$$

or

$$Q/\sqrt{S_f} = f(h) \tag{3.35a}$$

For a given form (shape) of the section, this relation can be obtained and plotted point by point (see Fig. 3.8). One can readily calculate the conveyance for geometrically simple sections; for complex ones, a graphical solution is necessary.

The normal depth, h_n, increases with the discharge, Q. For identical channels, but having different slopes, S_f, the normal depth increases if the bed slope decreases.

3° The conveyance, K, characterises the channel; it represents a measure of the capacity of water transport through the cross section.

The curve of the normal depths (see Fig. 3.8) will be found to be rather useful in solving different kinds of problems : if two of the three parameters, h_n , Q and S_f, (see eq. 3.35a) are known, the third one can be found; à priori, the roughness of the walls is taken to be known.

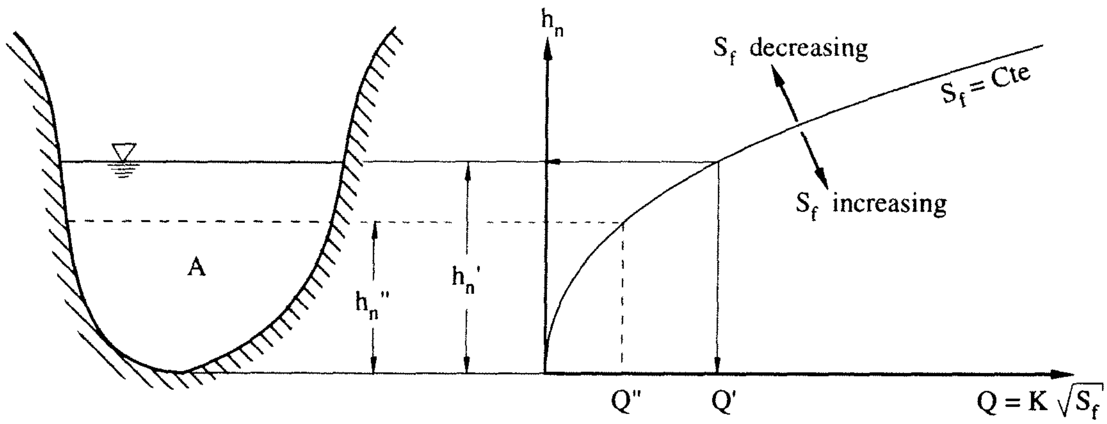


Fig. 3.8 Curve of conveyance or of normal depth.

3.3.2 Normal Depth

1° The normal depth, h_n , (see eq. 3.36) is the flow depth at uniform flow of discharge, Q , at a given bed slope, S_f . (All geometric elements of the cross section, which correspond to the normal depth, h_n , are known as normal elements, such as : R_{h_n} , A_n or P_n .)

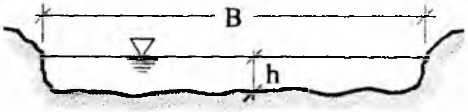
2° The normal depth of a channel of a given geometry is calculated using the relation for the discharge :

$$Q = UA = \frac{1}{n} R_h^{2/3} S_f^{1/2} A \tag{3.33}$$

This relation shows that uniform flow is only possible in a channel whose bed slope is descending, $S_f > 0$. In a horizontal channel, $S_f = 0$, the normal depth would be infinite.

3° For a natural watercourse and for rectangular channels whose width, B, is very large (see Fig. 3.9), one takes $R_h \approx h$ as the hydraulic radius. The relation of the discharge, eq. 3.33, can now be written as :

$$Q = UA = (C h^{1/2} S_f^{1/2}) (h B) \tag{3.33a}$$



$$h = D_h$$
$$B >> h$$
$$R_h = \frac{B\,h}{B + 2h} \approx \frac{B\,h}{B} \approx h$$

Fig. 3.9 Section of a channel, having a large width.

For the normal or uniform depth, $h \equiv h_n$, one obtains :

$$h_n = \left(\frac{q^2}{C^2 S_f}\right)^{1/3} \qquad \text{where } q = Q/B \tag{3.36}$$

3.3.3 Composite Section

1° A cross section of a channel can be composed of different subsections (see Fig. 3.10), of which each one can have a different roughness and a different bed slope.

This is frequently the case during floods, when the flow leaves the channel and enters into the overflow section of the channel.

2° Such a case can be approximately treated by applying the formula of discharge for each subsection :

$$Q = Q_c + Q_o = \frac{1}{n_c} A_c R_{h_c}^{2/3} \left(\frac{\Delta h}{L_c}\right)^{1/2} + \frac{1}{n_o} A_o R_{h_o}^{2/3} \left(\frac{\Delta h}{L_o}\right)^{1/2} \tag{3.37}$$

Note that the wetted perimeters, P_c and P_o , should be calculated for the lines of contact between water and bed.

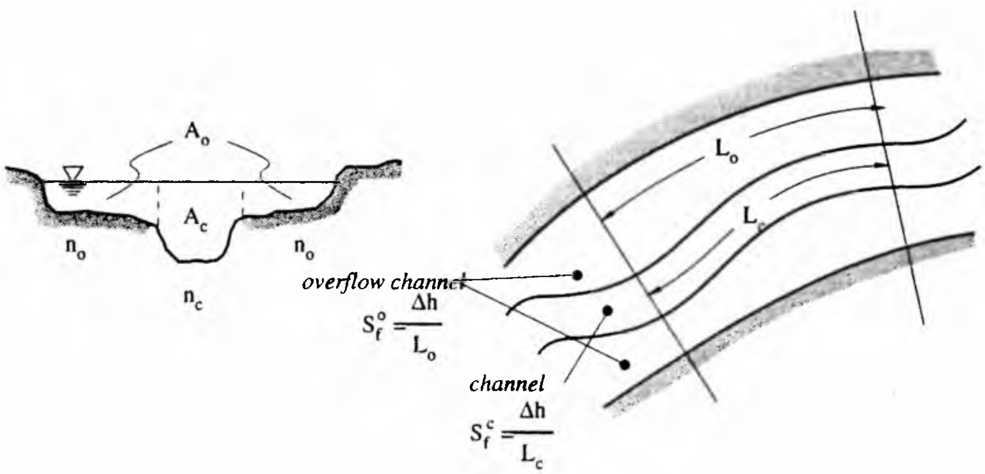


Fig. 3.10 Composite section.

3.3.4 Section of maximum Discharge

1° The construction of a channel with a given slope, S_f, and a given roughness, n, which should convey a certain discharge, Q, will be less expensive if the cross section, A, is the smallest possible.

2° Take the formula of discharge :

$$Q = UA = \frac{1}{n} R_h^{2/3} S_f^{1/2} A \tag{3.33}$$

where for $(S_f^{1/2}/n) = Cte$ one writes :

$$Q = Cte \, (A^{5/3} P^{-2/3})$$

For a wetted cross section, A , being constant, the above expressions show that the discharge will be maximal, $Q \Rightarrow Q_{max}$, if the hydraulic radius is maximal, $R_h \Rightarrow R_{hmax}$; thus if the wetted perimeter is minimal, $P \Rightarrow P_{min}$.

3° Amongst all geometrical forms possible, the cross section of a *semi-circular* form will give a P_{min} for a given constant cross section, A . This is given (see Fig. 3.11) by :

$$A = \frac{\pi r^2}{2} \quad , \qquad P = \pi r \quad , \qquad R_h = \frac{r}{2} = \frac{h}{2}$$

The semi-circular form can however be only realised (constructed) with artificial channels, made of metal, concrete or wood.

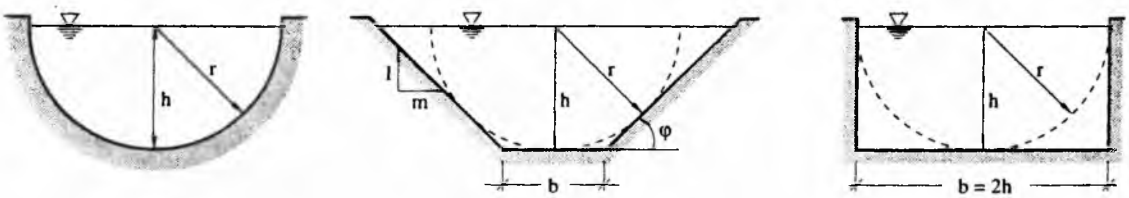


Fig. 3.11 Sections of maximum discharge.

4° For channels in an alluvium, one should also take into account the angle of repose, φ , as well as various constraints due to construction. Consequently, a *trapezoidal* form (see *Crausse*, 1951, p. 51) may be the most reasonable one (see Fig. 3.11) ; where one defines the wetted section, A, and the perimeter, P, such as (see Table 1.1) :

$$A = h\,(b + mh) \quad , \qquad P = b + 2h\sqrt{1 + m^2} \qquad \text{where } m = \mathrm{ctg}\,\varphi \,.$$

Subsequently, one takes dA as being zero, since the section, A , remains constant :

$$dA = h\,db + (b + 2\,mh)\,dh = 0$$

If one puts the wetted perimeter, P , as being minimal, this yields :

$$dP = db + 2\sqrt{1 + m^2}\,dh = 0$$

By elimination of db and dh in above equations, one gets :

$$b = 2h\,(\sqrt{1 + m^2} - m)$$

This value, b, can be put in above relations for A and for P, and one obtains an expression for the hydraulic radius, or :

$$R_h = h/2$$

which remains independent from the angle of repose, φ .

5° It should be remarked that for m = 0 the trapeze becomes a rectangle (see Fig. 3.11) such as :

$$b = 2h$$
$$R_h = h/2$$

For a *rectangular* channel, where $b \equiv B$, the ratio width/depth must be $(B/h) = 2$.

3.4 DISCHARGE CALCULATION, MOBILE BED

1° Artificial and natural channels, whose flow moves in an alluvium, composed of a (non-cohesive) granulate, are channels of *mobile bed*. The discharge will be calculated using the coefficient of roughness for a mobile bed (see sect. 3.2.6).

2° In such channels, the velocity (in the vicinity of the bed) should :

i) not be superior to a certain critical value, otherwise there is a risk of erosion of the solid particles on the bed : this is the permissible maximum velocity or *velocity of erosion*, usually also called the *critical velocity;*

ii) not be inferior to a certain critical value, otherwise there is a risk of deposition or sedimentation of the solid particles which are possibly suspended in the flow : this is the permissible minimum velocity or *velocity of sedimentation.*

3° The flow velocity, U, to be selected for a 'good functioning' of the channel, must lie between the velocity of erosion, $U_E \equiv U_{cr}$, and the one of sedimentation, U_D :

$U_D < U < U_E$.

4° It is evident in Fig. 3.12, that the two velocities, U_D and U_E, will have distinctly different values .

3.4.1 Sedimentation Velocity

1° The allowable minimum velocity or velocity of sedimentation, U_D, is the minimum velocity which is necessary to transport the flow containing solid particles in suspension.

2° Recommended (see *Chow*, 1959, p. 158 and *Crausse*, 1951, p. 16) is to take the following approximate values :

$0.25 < U_D$ [m/s] < 0.9

depending on fine or very coarse material.

3° The diagram (see Fig. 3.12) which was established by *Hjulstrom* (see *Graf*, 1984, p. 88) delimits the zone of sedimentation as a function of the diameter of the (monodispersed) granulate.

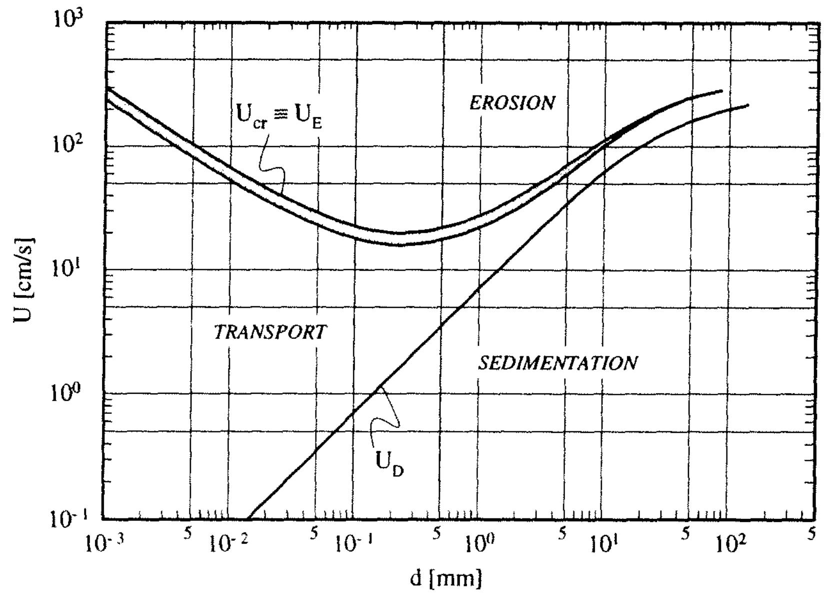


Fig. 3.12 Velocity of sedimentation and of erosion, U_D et U_{cr}, for a uniform granulate,

4° In an experimental study with granulates of $0.49 < d_{50}\,[\mathrm{mm}] < 3.02$, *Graf* et *Pazis* (1977) expressed the critical values by the shear stress, τ_o. It was found that

$$\tau_{o_D} < \tau_{o_E}$$

The difference was however shown to be negligible for this range of granulates; one may thus readily take $\tau_{o_D} \cong \tau_{o_E}$.

3.4.2 Critical Velocity

1° There will be erosion of the bed (and the walls) when one exceeds a certain critical value, expressed with :

i) the average critical velocity, U_{cr}, or the critical velocity, $u_{b_{cr}}$, at or close to the bed,

ii) the critical shear stress, $\tau_{o_{cr}}$.

2° From an hydraulic view point, it is more reasonable to use the shear stress, τ_o, as a criterion of erosion.

The shear stress was earlier defined as :

$$\tau_o = \gamma\, R_h\, S_f \tag{3.5}$$

and the average velocity as :

$$U = C\sqrt{R_h\, S_f} \tag{3.11}$$

This gives the following ratio, showing the relation between the shear stress, τ_o, and the velocity, U, or :

$$\frac{U}{\sqrt{\tau_o/\rho}} = \frac{C}{\sqrt{g}}$$

In fluvial hydraulics, one uses rather often (see *Graf*, 1984, p.91) a dimensionless form of the shear stress, τ_*, or :

$$\frac{\tau_o}{(\gamma_s-\gamma)d} \equiv \tau_* = \frac{\gamma\, R_h\, S_f}{(\gamma_s-\gamma)d} \tag{3.38}$$

where d is the diameter of the granulate (to be specified); γ_s and γ are the specific weight of the granulate and of water respectively. With this relation, one compares the flow parameters , R_h and S_f, with the granulometric parameters, d and (s_s-1).

3° Amongst the different formulae, which one finds in the literature (see *Graf*, 1984, chap. 6), only three will be presented herewith, namely the ones proposed by *Hjulstrom*, by *Neill* and by *Shields*.

4° In an analysis of available data from (monodispersed) uniform granulates, *Hjulstrom* used the average flow velocity, U , instead of the velocity close to the bed, u_b , by assuming that $u_b = 0.4U$. On Fig. 3.12, one can see the limiting zone, where erosion is encountered. This diagram of :

$$U_{cr} = f(d)$$

shows that fine sand ($d \cong 0.1$ mm) is rather easily eroded; the strong resistance to erosion for silt ($d \cong 0.01$ mm) is attributed to the cohesion between the particles.

5° For erosion of a bed composed of a uniform granulate of large diameter, *Neill* proposes the following relation :

$$\frac{\rho\, U_{cr}^2}{gd\,(\rho_s - \rho)} = 2.5\left(\frac{d}{D_h}\right)^{-0.2} \tag{3.39}$$

being valid for $0.01 < (d / D_h) < 1.0$.

6° Relying on some concepts of the hydrodynamics, *Shields* developed a relation between the dimensionless shear stress, τ_* (see eq.3.38), and the friction/particle Reynolds number, $Re_* = u_* d/\nu$, such as :

$$\tau_* \equiv \frac{\tau_o}{(\gamma_s - \gamma)\, d} = f\left(\frac{u_* d}{\nu}\right) \tag{3.40}$$

where $u_* = \sqrt{\tau_o/\rho}$. *Shields* has determined the form of this relation, using experimental data. An average curve (see *Graf*, 1984, p. 96), reasonably well defined (despite an important scattering), characterises the begin of erosion, expressed by τ_{*cr} . For the particle diameter, one takes usually $d \equiv d_{50}$. It is to be seen (see Fig. 3.13) that these critical values fall roughly in the range of :

$$0.03 < \tau_{*cr} < 0.06$$

The determination of τ_{*cr} is done using the above relation, eq. 3.40, by successive approximations. It must be underlined that the criterion of *Shields* is of great importance for the hydraulic engineer.

7° Since a direct use of the relation of *Shields*, eq. 3.40, is not a simple one, *Yalin* (1972, p. 82) has proposed an interesting combination of terms, such as :

$$\frac{Re_*^2}{\tau_*} = \frac{d^3 g}{\nu^2} \frac{(\rho_s - \rho)}{\rho}$$

Rather than using the Reynolds number, Re_* , it is now proposed to use a dimensionless diameter of the granulate, given by :

$$d_* = d \left(\frac{\rho_s - \rho}{\rho} \frac{g}{\nu^2} \right)^{1/3}$$

Consequently, above relation, eq. 3.40, can be expressed as :

$$\tau_* = f(d_*) \tag{3.40a}$$

which is given with Fig. 3.13; one usually takes $d = d_{50}$.

If the properties of the fluid, ρ and ν , and of the granulate, ρ_s and d , are known, one can readily determine the corresponding value of τ_{*cr} and subsequently of $\tau_{o_{cr}}$.

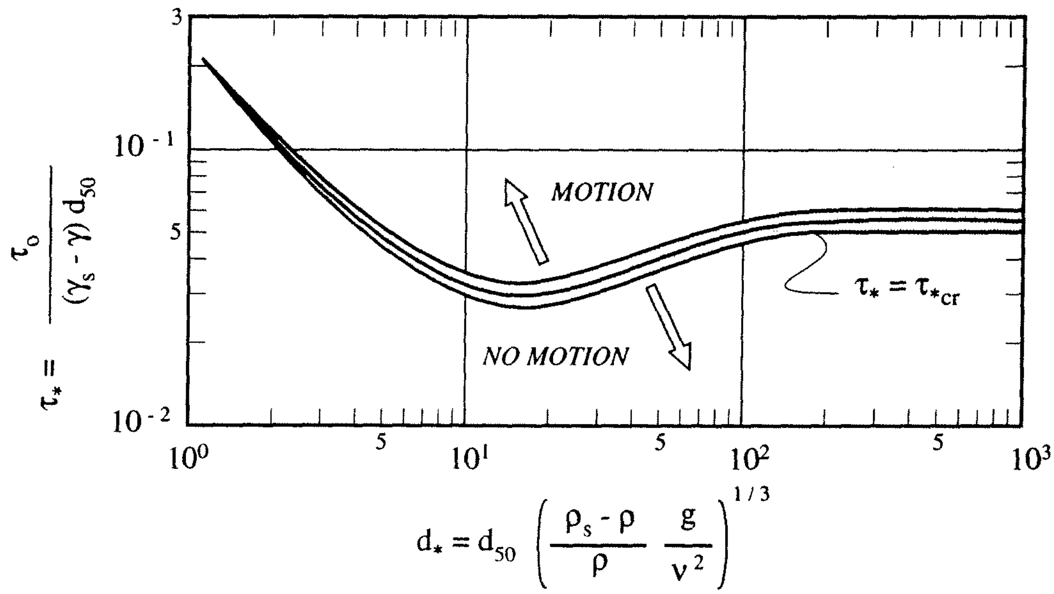


Fig. 3.13 Dimensionless shear stress, τ_* , as a function of the dimensionless diameter, d_* , after Shields-Yalin.

8° For cohesive material, the determination of the critical values, U_{cr} or τ_{*cr}, represents a difficult task; the specialised literature (see *Graf*, 1984, chap. 12, and *Raudkivi*, 1976, chap. 9) should be consulted.

3.4.3 Distribution of Shear Stress

1° The shear stress, τ_o, is given by :

$$\tau_o = \gamma S \frac{1}{P} S_f = \gamma R_h S_f \tag{3.5}$$

For a channel of large width (see Fig. 3.9), when $R_h \cong h$, one writes :

$$\tau_o = \gamma h S_f \tag{3.5a}$$

2° However, it must be remarked that the shear stress, τ_o, is distributed over the wetted perimeter, P. A typical distribution for a trapezoidal channel (see *Chow*, 1959, p. 169) is given with Fig. 3.14.

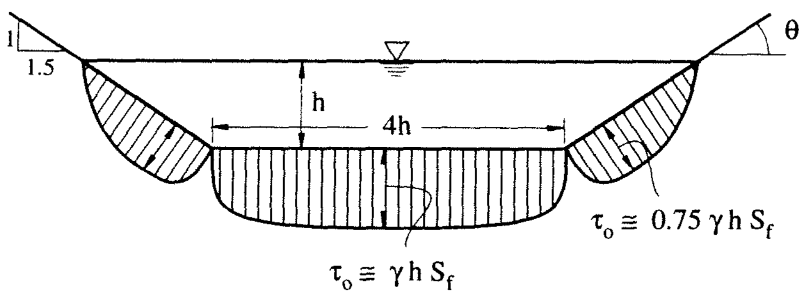


Fig. 3.14 Distribution of the shear stress in a trapezoidal channel.

3° An expression for the shear stress on the channel side walls, $(\tau_{o_{cr}})^w$, was proposed by *Forchheimer* and subsequently by *Lane* (see *Graf*, 1984, p. 116), being of the following form :

$$(\tau_{o_{cr}})^w = \tau_{o_{cr}} \left[\cos\theta \left(1 - \mathrm{tg}^2\theta / \mathrm{tg}^2\varphi\right)^{1/2}\right] \tag{3.41}$$

$\tau_{o_{cr}}$ is the critical shear stress on the bed – given for example with Fig. 3.13 – , θ is the inclination of the side wall(s), and φ is the angle of repose. The latter depends on the granulometry and on the cohesion (see *Graf*, 1984, p. 115); it varies such as $20° < \varphi < 40°$. Evidently : $(\tau_{o_{cr}})^w < \tau_{o_{cr}}$, and for stable side walls : $\theta < \varphi$.

3.4.4 Stable Section

1° A *stable* cross section of a channel with a mobile bed, thus erodible, is a section where there is no erosion over the entire wetted perimeter, P.

2° An *ideal stable* cross section – with a maximal discharge and a minimal wetted perimeter – can be calculated with a method advanced by *Glover* et *Lane* (see *Graf*, 1984, p. 119). The form of such a section (see Fig. 3.15) can be determined as follows :

i) Assumed will be that the angle of the side wall at the water surface is identical to the angle of repose, $\theta_s \equiv \varphi$.

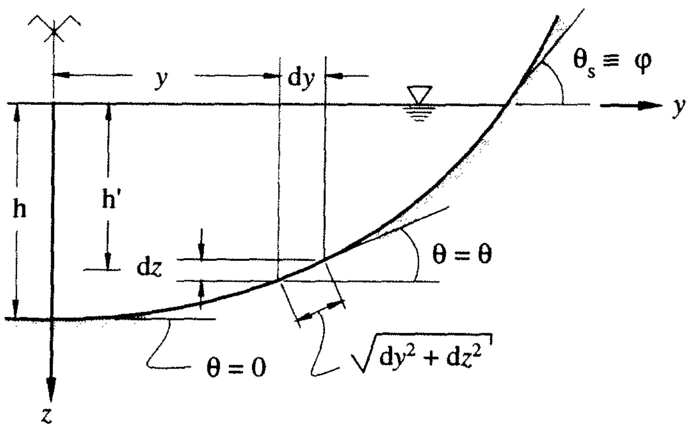


Fig. 3.15 Ideal stable section.

ii) The shear stress on an element on the bed situated at the side wall is :

$$(\tau_o^{w}) = \gamma\, h'\, S_f\, (dy / \sqrt{dy^2 + dz^2}) = \gamma\, h'\, S_f \cos\theta$$

iii) Subsequently, one assumes that this shear stress, (τ_o^{w}) , be critical, $(\tau_{o_{cr}})^{w}$, over the entire wetted perimeter ; using now the expression of eq. 3.41, one writes :

$$\gamma\, h'\, S_f \cos\theta \Leftarrow (\tau_o^{w}) \equiv (\tau_{o_{cr}})^{w} \Rightarrow \gamma\, h\, S_f \left[\cos\theta (1 - tg^2\theta / tg^2\varphi)^{1/2}\right]$$

where h is the maximum water depth situated at $y = 0$.

iv) After mathematical manipulations and taking $(dz/dy) = tg\theta$, one obtains :

$$\left(\frac{dz}{dy}\right)^2 + \left(\frac{h'}{h}\right)^2 tg^2\varphi - tg^2\varphi = 0.$$

v) The solution of this differential equation is :

$$h' = h \cos\left(\frac{tg\varphi}{h}\, y\right) \tag{3.42}$$

which gives the geometry of the ideal stable cross section, being *sinusoidal.*

vi) The other hydraulic parameters of such a section are deduced as being :

$$\begin{aligned} A &= 2\,h^2 / tg\varphi \\ B &= \pi\, h / tg\varphi \\ U &= \frac{1}{n}\, S_f^{\,1/2}\, (h \cos\varphi / E)^{2/3} \end{aligned} \tag{3.42a}$$

with $h = \tau_o / \gamma\, S_f$ and $E(\sin\varphi)$ being an elliptic integral, approximated by $E \approx (\pi/2)\,(1 - \frac{1}{4}\sin^2\varphi)$.

vii) The discharge, Q_i , which *can* be conveyed through this *ideal stable* section, is evidently given by $Q_i = UA$.

3° If the discharge, Q , which *must* be conveyed through such a section is different from the ideal discharge , Q_i , thus $Q \neq Q_i$, a corrective calculation must be done :

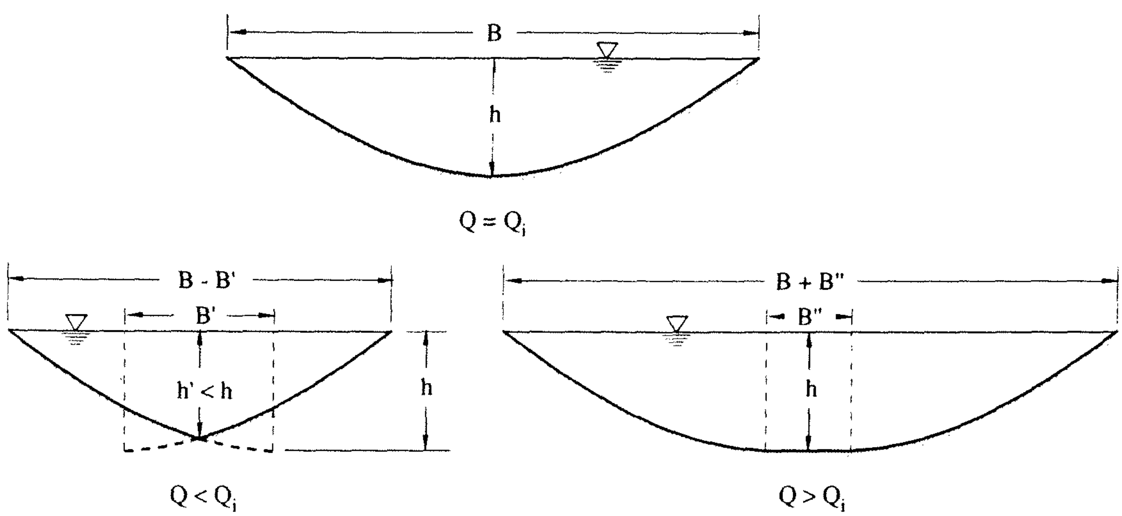


Fig. 3.16 Ideal stable section for different widths.

i) For $Q < Q_i$, the width, B , must be reduced by B', to be computed with :

$$B' = B\,(1 - \sqrt{Q/Q_i})$$

ii) For $Q > Q_i$, the width, B , must be increased by B", to be computed with :

$$B'' = n\,(Q - Q_i) / (h^{5/3} S_f^{\,1/2})$$

The effect of this change in width on the geometry of the channel section is shown in Fig. 3.16 (see *Chow*, 1958, p. 177).

3.5 FLOW IN CURVES

1° A curve or bend, positioned in a rectangular channel, causes a change in the flow direction.

2° If the discharge, Q , remains constant along the curve, the flow velocity , U , as well as the wetted section, A , remain also constant. The sectional distribution of the flow depth, $h(y)$, will be responsible for a transversal water slope and a super-elevation, Δz , at the outside of the curve.

3° The distribution of velocity in the curve can be approximated by the one of a free vortex (see *Graf & Altinakar*, 1991, p. 196). The velocity has a maximum at the inside of the curve (see Fig. 3.17).

3.5.1 Super-elevation

1° In a curve the streamlines will no longer stay parallel and the flow becomes three-dimensional. This is a complex physical phenomenon, for which an adequate analysis seems difficult.

2° The method proposed by *Kozeny* (1953, p. 223) puts forward the following arguments and this for turbulent flow (see Fig. 3.17) :

i) Assumed is that the head loss, h_r^c , – following a streamline, s , – can be expressed as :

$$S_e = \frac{h_r^c}{L_c} = \lambda\, u_s^2$$

where λ is a factor of proportionality and $L_c = \alpha r$ is the length of the curve; α being the angle and r the radius of the curve. Thus one may write :

$$u_s = \sqrt{\frac{h_r^c}{\lambda\ \alpha}} \cdot \sqrt{\frac{1}{r}} = \frac{\kappa}{\sqrt{r}}$$

ii) For $r = r_o$, one takes $u_s = u_a \equiv U$; this implies that the axial velocity, u_a, is identical to the average velocity, U , of the cross section. One obtains now $u_a \sqrt{r_o} = \kappa$ and an expression for the distribution of the velocity, such as :

$$\frac{u_s}{u_a} = \sqrt{\frac{r_o}{r}}$$

iii) If r_2 and r_1 are the outside and inside radius, the corresponding velocities are given by :

$$u_2 = u_a \sqrt{r_o/r_2} \qquad \text{and} \qquad u_1 = u_a \sqrt{r_o/r_1}$$

iv) The super-elevation (see Fig. 3.17) can now be calculated as being :

$$\Delta z = \frac{u_1^2}{2g} - \frac{u_2^2}{2g} = \frac{u_a^2}{2g}\left(\frac{r_o}{r_1} - \frac{r_o}{r_2}\right)$$

Since $B = (r_2 - r_1)$ is the width of the curve, one can also write :

$$\Delta z = \frac{B\ r_o}{r_1\ r_2}\ \frac{U^2}{2g} \tag{3.43}$$

If the channel width, B , is small compared to the radius of the curve, r_o , one gets the simplified expression of :

$$\Delta z = \frac{B}{r_o}\ \frac{U^2}{2g} \tag{3.43a}$$

v) The transversal water profile is convex; one can write :

$$\Delta z_2 = \frac{U^2}{2g}\left(1 - \frac{r_o}{r_2}\right) \qquad \text{and} \qquad \Delta z_1 = \frac{U^2}{2g}\left(\frac{r_o}{r_1} - 1\right)$$

The super-elevation, $\Delta z = \Delta z_2 + \Delta z_1$, given with eq. 3.43 has its maximal value, $\Delta z = \Delta z_{max}$, usually observed for fluvial flow, $Fr < 1$, at the entrance of the curve and for supercritical flow, $Fr > 1$, at the exit of the curve.

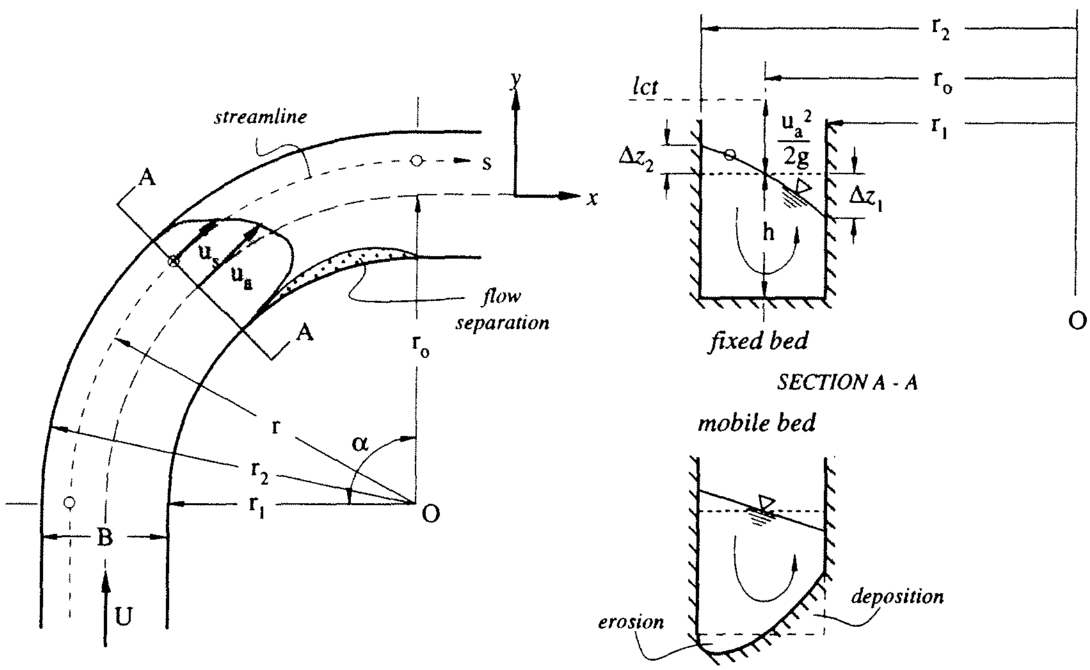


Fig. 3.17 Flow in a curve.

3° One can also define a coefficient of super-elevation by :

$$K^c = \Delta z \;/\; (U^2 / 2g) \tag{3.44}$$

which, taking eq. 3.43, is : $K^c = B\, r_o / (r_1\, r_2)$.

Apmann (1973, p. 73) proposed the following empirical expression, after analysing flow in curves in artificial and natural channels, or :

$$K^c = \frac{5}{4}\, \mathrm{tgh}\left(\frac{r_o\, \alpha}{B}\right) \ln\left(\frac{r_2}{r_1}\right) \tag{3.44a}$$

4° The super-elevation, Δz , can readily be used for the determination of the discharge (see *Apmann*, 1973, p. 70) :

$$Q = A \sqrt{2g\; \Delta z \;/\; K^c}$$

This relation is of great use in the determination of flood discharges, which usually leave traces (marks) at their largest occurring flow depth, thus Δz .

5° The super-elevation at the outside of the curve (see Fig. 3.17) causes a vertical downward current, which comes back to the water surface at the inside of the curve. Such a secondary current will superpose itself on the primary flow and result in helicoidal flow over the entire reach of the curve.

6° If the outside side wall is of mobile material, erosion will take place; on the inside there will be deposition (see Fig. 3.17).

3.5.2 Supercritical Flow

1° Gravity waves (see sect. 2.4.3) will establish themselves in a curve (see Fig. 3.18), notably if the flow is supercritical, $Fr > 1$ (see *Ippen*, 1950, p.563). For a rectangular channel, the celerity of a gravity wave is given by :

$$c^2 = gh \tag{2.27a}$$

2° At the entrance of the curve at the points A and A', one observes under an angle, β:

i) positive perturbations (waves) along the line ABD ,

ii) negative perturbations (waves) along the line A'BC ,

ii) but no perturbations appear in the zone ABA' .

This angle, β , is approximately defined as being :

$$\sin\beta = \frac{c}{U} = \frac{\sqrt{gh}}{U} = \frac{1}{Fr} \tag{3.45}$$

where Fr is Froude number of the flow upstream of the curve.

3° Consequently, the flow depth varies :

i) increasingly along the line AC, having a maximum at C ,

ii) decreasingly along the line A'D, having a minimum at D .

The maximum (+) or minimum (-) flow depth can be calculated (see *Ippen* 1950, pp.551 and 564) as being :

$$h_{\substack{max \\ min}} = h\, Fr^2 \sin^2(\beta \pm \theta / 2) \tag{3.46}$$

The central angle, θ , is determined using geometrical considerations, or :

$$\text{tg}\,\theta = \frac{B}{(r_o + B/2)\,\text{tg}\,\beta} \tag{3.46a}$$

4° These maxima/minima are then reflected from one to the other side in the curve, such as is indicated in Fig. 3.18. The next maxima/minima appear after an interval of 2θ. This swinging, referred to as *cross waves*, can continue well beyond the end of the curve.

5° The maximum super-elevation, $(\Delta z' + \Delta z)$, due to the gravity waves, can be twice the super-elevation, Δz , obtained with eq. 3.43. It is calculated by :

$$\Delta z' = \frac{B}{r_o}\frac{U^2}{2g} \tag{3.47}$$

The water surface and the resulting transversal slopes are shown schematically with Fig. 3.18.

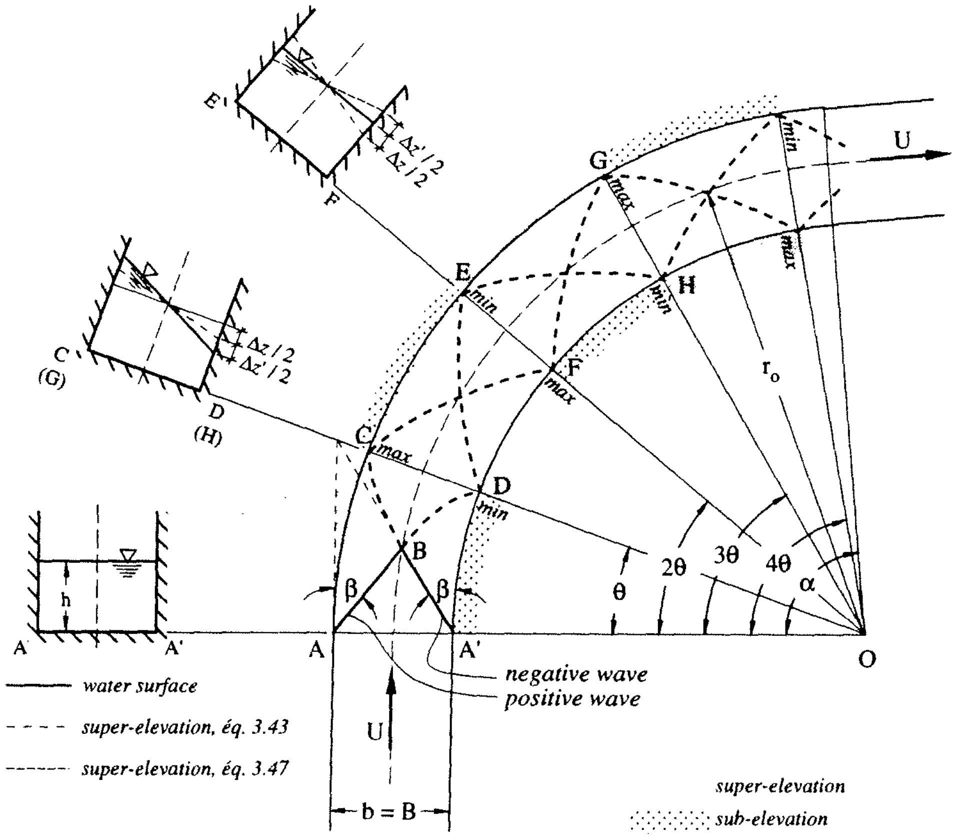


Fig. 3.18 Supercritical flow in a curve.

6° If the super-elevations get very large, methods are available to suppress it (see *Naudascher*, 1987, p.206), like the construction of transition curves or the installation of steps on the channel bed.

3.5.3 Head Loss

1° In flow over a curve, one encounters not only a head loss due to friction, h_r, but also one due to the curvilinear flow, h_r^c (see sect. 3.5.1).

2° This additional head loss is usually expressed by :

$$h_r^c = \zeta_c \frac{U^2}{2g} \tag{3.48}$$

where ζ_c is a coefficient which depends on :

$$\zeta_c = f(Fr, \; Re, r_o / B, h / B, \alpha)$$

α being the angle of the curve, Fr and Re are the number of Froude and of Reynolds, respectively. According to numerous experiments, one takes (see *Chow*, 1958, p. 443) :

$$0.1 \leq \zeta_c \leq 1.1$$

where the larger values of ζ_c are for curves of $r_o / B = 0.5$.

3.6 INSTABILITY AT SURFACE

1° If the channel slope is very high and/or if the flow is supercritical, the water surface can become unstable. The normal flow depth, h_n, must now be considered as an average value.

2° Such an instability is characterised by :

i) a series of gravity waves of small flow depth, eq. 2.27, called *roll waves*, progressing downstream, and

ii) a breaking of these waves, causing an *air entrainment.*

3.6.1 Roll Waves

1° An instability at the water surface is evidenced by the formation of roll waves. The uniform steady flow becomes locally an unsteady one.

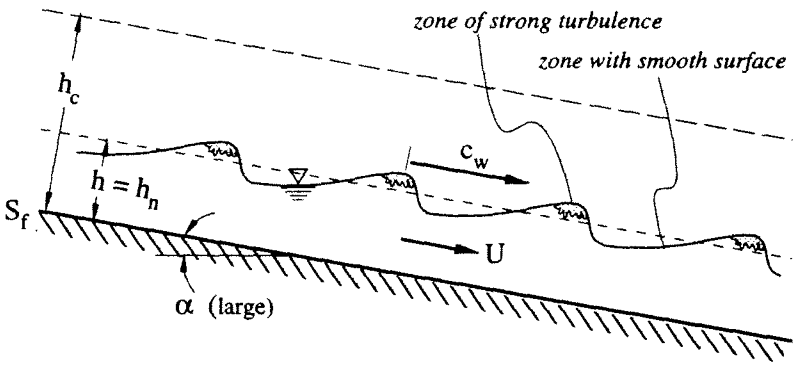


Fig. 3.19 Roll waves.

2° The roll waves are superposed on the uniform flow (see Fig. 3.19). They displace themselves towards the downstream – increasing in height and then collapsing – with an absolute celerity, c_w , being larger than the flow velocity, U, or :

$$c_w = U + \sqrt{gh} \quad > \quad U$$

3° There is no simple criteria available to determine the geometrical dimensions of this type of waves. Their height can, however, attain dimensions of the order of magnitude of the prevailing flow depth (see *French*, 1986, p. 625).

The crests of the roll waves are zones of strong turbulence, while the rest of the waves remains remarkably smooth.

4° Some theoretical considerations for a determination of the (in)stability of uniform flow are presented in *Liggett* (1975, chap. 6).

5° However, there exist useful practical criteria to determine the instability, which is responsible for the creation of roll waves. The geometry of the channel and the type of flow are taken in consideration.

i) Taken is the Froude number, $Fr = U / \sqrt{gh}$; the flow of a large channel is unstable (see *Albertson* et *al.*, 1960, p.355), if :

$Fr > 2$	for turbulent, rough flow,
$Fr \geq 1.5$	for turbulent, smooth flow,
$Fr \geq 0.5$	for laminar flow.

ii) Taken is the number of Vedernikov (see *Chow*, 1959, p. 210), defined as :

$$Ve = x\, f_g\, Fr \tag{3.49}$$

where x is an exponent of the hydraulic radius, R_h , in eq. 3.11 (x = 2 for laminar flow; x = 2/3 or x = 1/2 for turbulent flow, taking the relation of Manning or of Chézy, respectively). f_g is a shape factor given by :

$$f_g = 1 - R_h \frac{dP}{dS}$$

being $f_g = 1$ for very wide channels and $f_g = 0$ for very narrow channels. If :

$$Ve < 1$$

flow can remain stable and any wave at the surface will be depressed. If :

$$Ve \geq 1$$

stable flow is impossible; unsteady flow will prevail and existing waves will amplify and form the roll waves.

6° Roll waves form themselves not only in uniform flow, but are also encountered in non-uniform flow.

3.6.2 Air Entrainment

1° For large channel slopes, S_f , – such as exist also on the downstream face of a weir – the flow is usually supercritical and gravity waves appear at the water surface. These waves will break and entrain air into the water. The turbulence will diffuse (mix) the air bubbles across the entire flow depth; and water droplets will escape into the air.

2° In flow of such an air-water mixture, it becomes a bit difficult to define the flow depth; the water surface is often covered by *white water.*

3° The schematic distribution of the concentration of air :

$$C(z) = \frac{\text{volume (air)}}{\text{volume (air + water)}}$$

is given with Fig. 3.20. Two regions are to be distinguished : bubbles in the water and droplets in the air.

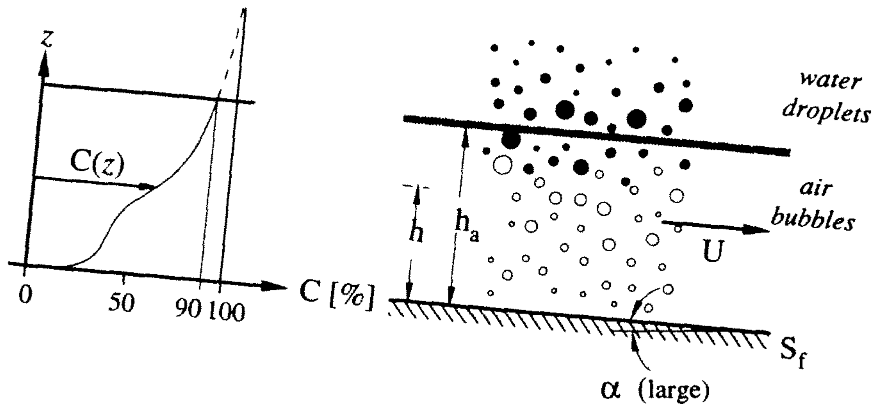


Fig. 3.20 Flow with air entrainment.

4° The equivalent flow depth of water (without the air) is defined by :

$$h = \int_0^\infty (1-C)\,dz \tag{3.50}$$

and the average velocity of water by :

$$U = q/h$$

where q is the unit discharge of the water.

The depth of the mixture, which is the depth where the concentration is equal to C = 90%, is given (see *Wood*, 1985, p.21) by :

$$h_a = \frac{h}{(1 - \overline{C})} \tag{3.51}$$

where $\overline{C}$ is the average concentration in the cross section, given (see *Henderson*, 1966, p. 185) by the relation :

$$\overline{C} = 0.7 \log (S_f / q^{1/5}) + 0.9 \tag{3.52}$$

which was obtained from experimental studies performed by *Straub* et *Anderson* for $0.14 < q[m^2/s] < 0.93$. Usually (see *Wood*, 1985, p.21) one takes :

$\overline{C} = 0.14$ for slopes of $S_f = 7.5°$
$\overline{C} = 0.71$ for slopes of $S_f = 75°$

5° The air entrainment, or the aspiration of air on the downstream face of a weir, begins at a point where the boundary layer is completely developed, $\delta \equiv h$ (see *Wood*, 1985, p. 18).

3.7 EXERCISES

3.7.1 Problems, solved

Ex. 3.A

A channel is to be built of medium-quality concrete to convey a discharge of $Q = 80$ [m^3/s]. The channel should have a trapezoidal cross section with a bottom width of $b = 5$ [m] and side slopes of $m = 3$. The channel slope will follow the geomorphology of the terrain which is determined to be $S_f = 0.1$ %. It is admitted that the flow is uniform and the water has a temperature of $T = 10$ [°C].

i) Calculate the flow depth using both the coefficient of Manning and the coefficient of Weisbach-Darcy.

ii) Verify whether the flow is laminar or turbulent and subcritical or supercritical.

SOLUTION :

The geometrical characteristics of a trapezoidal cross section are (see Table 1.1) :

wetted surface : $A = (b + mh)\,h = (5 + 3h)\,h$

hydraulic radius : $R_h = \dfrac{(b + mh)\,h}{b + 2h\sqrt{1+m^2}} = \dfrac{(5 + 3h)\,h}{5 + 2h\sqrt{1+3^2}}$

i) a) *Calculation of normal flow depth using the coefficient of Manning:*

If one uses the Manning-Strickler formula, eq. 3.16, to express the average flow velocity, the discharge in the channel is given by :

$$Q = U\,A = \frac{1}{n}\,R_h^{2/3}\,S_f^{1/2}\,A$$

The coefficient of Manning is obtained from the Table 3.2 :

For a medium-quality concrete, one can take : $n = \dfrac{1}{K_s} \cong \dfrac{1}{70} = 0.0143$ [$m^{-1/3}s$]

By introducing the expressions for A and R_h , as well as the values of n and S_f into the equation for the discharge, Q , one obtains :

$$Q = 80 = 70\left(\frac{(5 + 3h)\,h}{5 + 2h\sqrt{10}}\right)^{2/3}\sqrt{0.001}\;(5 + 3h)\,h$$

This equation can be solved by trial-and-error :

h [m]	Q [m^3/s]
2.20	69
2.50	91
2.36	*80*

The *normal depth* is therefore : $h = h_n = 2.36$ [m]

b) *Calculation of normal flow depth using the coefficient of Weisbach-Darcy* :

The average flow velocity in a channel is given by :

$$U = \sqrt{8g/f}\ \sqrt{R_h\, S_f} \tag{3.10}$$

The discharge is therefore : $Q = U\,A = \sqrt{\frac{8g}{f}}\ \sqrt{R_h\, S_f}\ A$

The coefficient of Weisbach-Darcy, f, can be obtained (either by using the Moody-Stanton diagram or) by using the Colebrook-White formula :

$$\sqrt{\frac{1}{f}} = -2\log\left(\frac{k_s/R_h}{a_f} + \frac{b_f}{Re\sqrt{f}}\right) \tag{3.13}$$

For a trapezoidal channel one generally uses : $a_f = 12$ and $b_f = 2.5$. However this equation can only be solved by trail-and-error.

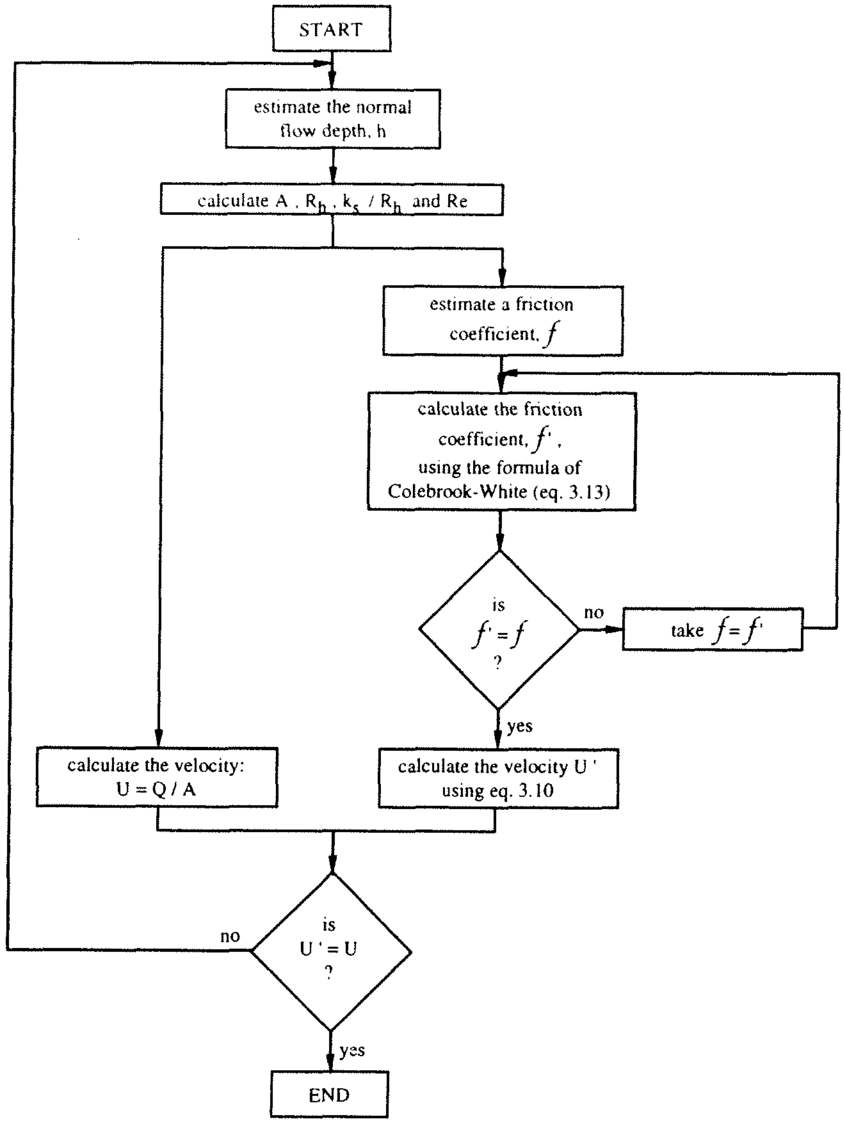
The uniform equivalent roughness, k_s, is obtained from the Table 3.1 :
For a medium-quality concrete, one can take : $k_s = 0.001$ [m] .

It is evident that the velocity, U', calculated with eq. 3.10 for an *estimated* normal depth, h_n , should also satisfy the relationship $U = Q/A$, where A is the wetted surface corresponding to h_n. Using this fact, a trial-and-error type calculation can be devised for calculating the normal depth. The flowchart of this trial-and-error calculation is given hereafter. As it can be seen the algorithm is based on two nested calculation loops. The internal loop calculates the friction coefficient, f, whereas the external loop calculates the normal depth, h_n .

This procedure can be programmed on a microcomputer using a spreadsheet program. The calculation sheet presented below simulates the sequence of calculations depicted in the flowchart. The order of execution of the nested loops is shown in the leftmost column. The detailed explanations of the other columns, numbered from 1 to 10, are given in the table.

The iteration starts with an estimation of $h_n = 2.20$ [m] in the external loop. The internal loop is here executed three times for this same value of h_n. Although the estimated and the calculated values of the Weisbach-Darcy friction coefficient are equal: $f = f' = 0.01419$, at the end of the third iteration, the velocities, U and U', are still different : $\Delta U \neq 0$. Consequently, the calculations must continue by next taking $h_n = 2.50$ [m] and then $h_n = 2.35$ [m]. After the first execution of the internal loop with $h_n = 2.35$ [m] and $f = 0.02$, one finds already that $f' = 0.01402$ and $\Delta U = 0.00$. The iterations can be stopped here without waiting for the convergence of the value of f. The following two calculation lines only help to refine the value of f.

The *normal depth* is therefore : $h = h_n = 2.35$ [m]



ii) For $h_n = 2.36$ [m] , $R_h = 1.43$ [m] and $U = 2.81$ [m/s] , one has :

Reynolds number : $$Re' = \frac{R_h\, U}{\nu} = \frac{(1.43)\,(2.81)}{1.31 \times 10^{-6}} = 3.1 \times 10^{6} > 2000$$

The flow is therefore *turbulent.*

Froude number : $$Fr = \frac{U}{\sqrt{g h_n}} = \frac{2.81}{\sqrt{(9.81)\,(2.36)}} = 0.58 < 1$$

The flow is therefore *fluvial.*

Calculation sheet for determining the normal flow depth
by using the coefficient of Weisbach-Darcy,
calculated using the formula of Colebrook-White (eq. 3.13)

$Q = 80$ [m³/s] | $S_f = 0.001$ [-] | | $a_f = 12$ [-]
$b = 5.0$ [m] | $k_s = 0.001$ [m] | $m = 3.0$ [-] | $b_f = 2.5$ [-]

For $T = 10$ [°C] $\nu = 1.31 \times 10^{-6}$ [m²/s] (*Graf & Altinakar*; 1991, Table I.3)

iteration number for h_n and f		1 h_n [m]	2 A [m²]	3 R_h [m]	4 U [m/s]	5 k_s / R_h [-]	6 Re [-]	7 f [-]	8 f' [-]	9 U' [m/s]	10 ΔU [m/s]
1	1	2.20	25.52	1.35	3.13	7.41×10^{-4}	1.29×10^{7}	0.02000	0.01417		
	2							0.01417	0.01419		
	3							0.01419	0.01419	2.73	0.40
2	1	2.50	31.25	1.50	2.56	6.66×10^{-4}	1.18×10^{7}	0.02000	0.01388		
	2							0.01388	0.01389		
	3							0.01389	0.01389	2.91	-0.35
3	1	2.35	28.32	1.43	2.83	7.01×10^{-4}	1.23×10^{7}	0.02000	0.01402	2.82	0.00
	2							0.01402	0.01403	2.82	0.00
	3							0.01403	0.01403	2.82	0.00

col.	symbol	explanations	expression
1	h_n	*estimated* normal depth	h
2	A	wetted surface	$(b + mh)\,h$
3	R_h	hydraulic radius	$\dfrac{(b + mh)\,h}{b + 2h\sqrt{1+m^2}}$
4	U	average velocity :	Q / A
5	k_s / R_h	relative roughness	k_s / R_h
6	Re	Reynolds number	$4\,U\,R_h / \nu$
7	f	*estimated* friction coefficient	
8	f	friction coefficient calculated using eq. 3.13 :	
9	U'	average velocity obtained using eq. 3.10 :	$\sqrt{8g / f'}\ \sqrt{R_h S_f}$
10	ΔU	difference between the velocities	U - U'

Expression for col. 8:

$$\left[-2\log\left(\frac{k_s/R_h}{a_f} + \frac{b_f}{Re\sqrt{f}}\right)\right]^{-2}$$

Ex. 3.B

The river *Happy* has a variable discharge in the range of $10 < Q\ [m^3/s] < 1000$. In the city of Ste-Justice, the width of the bed is $b = 90$ [m] and the non erodible banks have a slope of 1:1. A bridge crosses the river (without causing any obstruction to the flow) and it is planned to install a measuring gauge at the mid-span of the bridge. A grain-size analysis of the bed material yielded : $s_s = 2.65$ [-] and $d_{50} = 0.32$ [mm], $d_{35} = 0.29$ [mm] and $d_{90} = 0.48$ [mm]. The water temperature is $T = 14$ [°C]. A survey of the river bed showed that the bed slope is $S_f = 0.0005$ [-].

i) Determine the stage-discharge curve, $Q = f(h_n)$, by assuming that the flow is turbulent rough.

ii) At what depth will erosion and deposition begin to occur ?

SOLUTION :

For water at $T = 14$ [°C] (interpolating the values in Table I.3, in *Graf & Altinakar*, 1991), one has : $\rho = 999.1\ [kg/m^3]$ and $\nu = 1.186 \times 10^{-6}\ [m^2/s]$.

Using the definition of the specific density (see *Graf & Altinakar*, 1991, p.9), one writes:

$$s_s = \rho_s / \rho_{eau} \quad \Rightarrow \quad \rho_s = \rho_{eau}\, s_s = 1000 \times 2.65 = 2650\ [kg/m^3]$$

where $\rho_{eau} = 1000\ [kg/m^3]$ is the density of water at $p_a = 1$ [atm] and $T = 3.98$[°C].

i) The bed (and banks) of a river are generally mobile, composed of erodible granular material. If it is desired to obtain the hydraulic radius, R_h , or the bed shear stress, τ_0 , it becomes necessary to make a distinction (see eq. 3.24) between the contribution of the grain roughness, R_h' or τ', and the one of the bed forms, R_h'' or τ'' .

The calculations can readily be programmed on a microcomputer using a spreadsheet program. The tabular computation sheet prepared in this way is presented below. Each line of this table represents the computation of the discharge, Q, and some other useful parameters for a given water depth, h. The detailed explanations concerning all the columns are given on the bottom of the table. A brief description of the computation sequence is presented hereafter.

The calculations on a line start by *assuming* a value for the hydraulic radius due to the grain roughness, R_h' . The values of R_h' should be chosen in a way to cover the whole range of the possible discharges, $10 < Q\ [m^3/s] < 1000$. The calculations in the columns 2 and 3 are straightforward. It is interesting to note that, according to the method of Einstein-Barbarossa, the average velocity, U, is calculated using only R_h'. The hydraulic radius due to bed forms, R_h'' , calculated in the columns 4 to 7, influences only the flow depth. It is to be noted that the value of U/u_*'' is obtained from Fig. 3.6 with the value of ψ'.

Computation sheet for determining the stage-discharge curve . . .

$b = 90$ [m] $m = 1$ $S_f = 0.0005$ [-]	$T = 14$ [°C] $\rho = 999.1$ [kg / m^3] $\nu = 1.186 \times 10^{-6}$ [m^2/s]	$\rho_s = 2650$ [kg/m^3]

1 R_h' [m]	2 u_*' [m/s]	3 U [m/s]	4 ψ' [-]	5 $\frac{U}{u_*''}$ [-]	6 u_*'' [m/s]	7 R_h'' [m]	8 R_h [m]	9 u_* [m/s]	10 h [m]
0.02	0.01	0.16	47.92	4.5	0.04	0.26	0.28	0.04	0.29
0.05	0.02	0.29	19.17	6.6	0.04	0.39	0.44	0.05	0.44
0.10	0.02	0.45	9.58	8.7	0.05	0.55	0.65	0.06	0.65
0.15	0.03	0.58	6.39	10.4	0.06	0.63	0.78	0.06	0.79
0.20	0.03	0.69	4.79	11.9	0.06	0.68	0.88	0.07	0.89
0.40	0.04	1.05	2.40	18.3	0.06	0.67	1.07	0.07	1.09
0.60	0.05	1.33	1.60	23.4	0.06	0.66	1.26	0.08	1.29
0.80	0.06	1.58	1.20	32.4	0.05	0.49	1.29	0.08	1.32
1.00	0.07	1.81	0.96	42.6	0.04	0.37	1.37	0.08	1.41
1.25	0.08	2.06	0.77	56.2	0.04	0.27	1.52	0.09	1.57
1.50	0.09	2.30	0.64	73.1	0.03	0.20	1.70	0.09	1.76
2.00	0.10	2.72	0.48	107.2	0.03	0.13	2.13	0.10	2.23
2.50	0.11	3.11	0.38	163.0	0.02	0.07	2.57	0.11	2.71
3.00	0.12	3.46	0.32	6844.0	0.00	0.00	3.00	0.12	3.19

col.	symbol	explanations	expression
1	R_h'	hydraulic radius due to grain roughness (*assumed* value)	
2	u_*'	friction velocity due to grain roughness, eq. 3.24,	$\sqrt{g\ R_h'\ S_f}$
3	U	average velocity in the cross section	$u_*'\sqrt{8/f'}$
		with (see eq. 3.13b) : $\sqrt{8/f'} = 5.6 \log (R_h'/k_s) + 6.25$	
4	ψ'	parameter of Einstein-Barbarossa, eq. 3.31,	$\frac{\rho_s-\rho}{\rho}\ \frac{d_{35}}{R_h'\ S_f}$
5	$\frac{U}{u_*''}$	ratio of velocities corresponding to ψ' (see eq. 3.31 and Fig. 3.6)	
6	u_*''	friction velocity due to bed forms	$U / (U/u_*'')$
7	R_h''	hydraulic radius due to bed forms	$(u_*'')^2 / (g\,S_f)$
8	R_h	total hydraulic radius, eq. 3.24,	$R_h' + R_h''$
9	u_*	friction velocity , eq. 3.7,	$\sqrt{g\ R_h\ S_f}$

. . . using the method of *Einstein-Barbarossa*

$d_{35} = 0.00029$ [m]

$U_{cr} = 0.2$ [m/s] $\Leftarrow$ $k_s = d_{50} = 0.00032$ [m] $\Rightarrow$ $d_* = 7.21$ [-] $\Rightarrow$ $\tau_{*cr} = 0.04$ [-]

(see Fig. 3.12) $d_{90} = 0.00048$ [m] (see Fig. 3.13)

11 A [m²]	12 P [m]	13 Q [m³/s]	14 $\frac{U}{\sqrt{g\ h}}$ [-]	15 $\frac{R_h}{d_{50}}$ [-]	16 $\frac{U}{\sqrt{g\ R_h}}$ [-]	17 $\frac{U}{\sqrt{g\ d_{50}}}$ [-]	18 τ_o [N/m²]	19 τ_* [-]	20 Notes
25.83	90.81	4.2	0.10	889	0.10	2.88	1.39	0.27	¥
40.20	91.26	11.7	0.14	1377	0.14	5.18	2.16	0.42	† ¥
59.25	91.85	26.5	0.18	2016	0.18	7.99	3.16	0.61	† ¥
71.92	92.24	41.4	0.21	2437	0.21	10.27	3.82	0.74	† ¥
81.34	92.53	55.8	0.23	2747	0.23	12.25	4.31	0.83	† ¥
99.28	93.08	103.7	0.32	3333	0.32	18.65	5.23	1.01	† ¥
118.03	93.66	157.4	0.37	3938	0.38	23.80	6.18	1.19	† ¥
120.78	93.74	191.3	0.44	4026	0.45	28.26	6.31	1.22	† ¥
128.53	93.98	232.4	0.49	4274	0.49	32.28	6.70	1.29	† ¥
144.00	94.45	297.3	0.53	4765	0.53	36.84	7.47	1.44	† ¥
161.62	94.98	371.6	0.55	5317	0.56	41.04	8.34	1.61	† ¥
205.28	96.30	559.3	0.58	6662	0.60	48.63	10.45	2.02	† ¥
251.41	97.67	780.9	0.60	8044	0.62	55.44	12.61	2.43	† ¥
297.05	99.02	1026.7	0.62	9375	0.64	61.69	14.70	2.84	† ¥

col.	symbol	explanations	expression
11	A	wetted surface (see Table 1.1)	$(b + mh)\ h$
12	P	wetted perimeter (see Table 1.1)	$b + 2h\sqrt{1 + m^2}$
13	Q	discharge (see eq. 3.2a)	U A
14	$U / \sqrt{g\,h}$	Froude number using the flow depth	Fr
15	R_h / d_{50}	relative depth	
16	$U / \sqrt{g\,R_h}$	Froude number using the hydraulic radius	
17	$U / \sqrt{g\,d_{50}}$	parameter proposed by Alam-Kennedy (see eq. 3.32 and Fig. 3.7)	
18	τ_o	total bed shear stress, eq. 3.6,	$\rho\ u_*^2$
19	τ_*	dimensionless bed shear stress, eq. 3.38,	$\frac{\tau_o}{(\gamma_s - \gamma)\ d_{50}}$
20	†	$U > U_{cr} \Rightarrow$ erosion according to Hjulstrom's criteria (see Fig. 3.12)	
	¥	$\tau_* > \tau_{*cr} \Rightarrow$ motion according to Shields' criteria (see Fig. 3.13)	

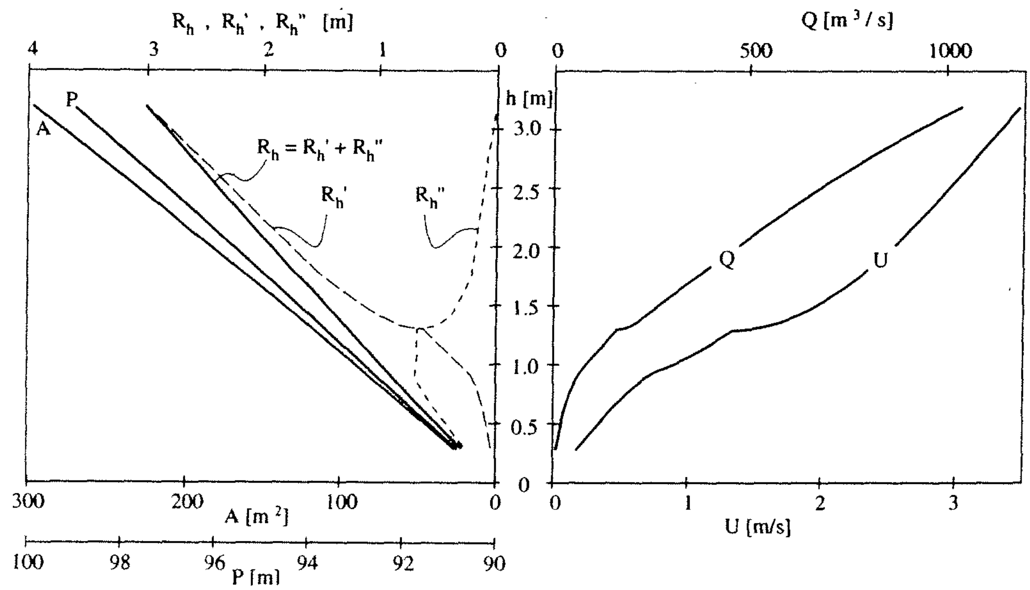
The friction coefficient due to bed forms, f'', can be evaluated using either the method of Einstein-Barbarossa, or the method of Alam-Kennedy (see sect. 3.2.6). The second method is more general but, it necessitates an iterative solution; the calculations are also more elaborate. The method of Einstein-Barbarossa, on the other hand, relies on a straightforward and simple calculation and it will be used for solving the present problem. However, as was already mentioned (see sect. 3.2.6), according to Alam-Kennedy (see Fig. 3.7), the relationship of Einstein-Barbarossa is valid in the region where the friction coefficient due to the bed forms, f'', does not depend on the relative depth, R_h/d_{50}. At the end of the calculation, a verification must be made to check that the calculated f''-value lies in that region.

After calculating the total hydraulic radius, $R_h = R_h' + R_h''$, the flow depth, h, can be obtained using the geometrical relationships. For a trapezoidal cross section (see Table 1.1), the problem is reduced to finding the positive square root of the following quadratic equation :

$$m\,h^2 + (b - 2\,R_h\sqrt{1 + m^2}\,)\,h - b\,R_h = 0$$

The calculations for the remaining columns are explained in the computation sheet. The Froude numbers in column 14 show that the flow is subcritical for all flow depths. Using the values in columns 15 to 17 it can now be checked on Fig 3.7 that all the points fall into the region where the value of f'' is independent from R_h/d_{50}.

The stage-discharge curve, $Q = f(h_n)$, as well as the variation of other useful parameters, U, P, A, R_h', R_h'', R_h, are plotted on the following figure. On this figure, it is interesting to observe the evolution of the curves corresponding to R_h' and R_h''.



ii) a) According to the *criteria of Hjulstrom*, for a given grain size, the critical velocities for the erosion and the sedimentation can be obtained from Fig. 3.12 :

for $d_{50} = 0.00032$ [m] $\Rightarrow$ $U_E = U_{cr} \cong 0.2$ [m/s] and $U_D \cong 0.03$ [m/s]

As it can be seen in the column 20 of the computation sheet, for the average velocities calculated in column 3. one has :

always $U > U_D$: sediment transport takes place,
always(except for h = 0.29 [m]) $U > U_E$: erosion of the bed should be expected.

It is to be noted that the erosion of the bed *does not mean* the formation of a scour hole in the river bed. Since the sediment transported from the upstream compensates globally the erosion, one should rather talk about a *transport of sediments*.

b) According to the *criteria of Shields*, the initiation of erosion should be verified using the total bed-shear stress, τ_* (see eq. 3.40), whose critical value, τ_{*cr} , for a given dimensionless grain diameter, d_*, is obtained from Fig. 3.13; namely:

$$d_{50} = 0.00032 \text{ [m]} \Rightarrow d_* = d_{50}\left(\frac{\rho_s-\rho}{\rho}\frac{g}{\nu^2}\right)^{1/3} = 7.21 \text{ [-]} \quad \Rightarrow \quad \tau_{*cr} = 0.04 \text{ [-]}$$

As it can be seen in the column 20 of the computation sheet, the calculated dimensionless total bed-shear stress values (column 19) are :

always $\tau_* > \tau_{*cr}$: erosion of the bed should be expected.

In order to compute the flow conditions corresponding to $\tau_* = \tau_{*cr}$, one has to consider the values of $R_h < 0.28$ [m]. The flow depth at which the erosion starts can be calculated as being $h_{cr} = 0.05$ [m], which is too shallow for a river of this importance.

Ex. 3.C

A channel excavated in earth should convey a water discharge of $Q = 57$ [m^3/s] at an average temperature of $T = 14$ [°C]. The bed slope, $S_f = 0.001$ [-] , is given; it will be assumed that the banks will have side slopes of 1.5 horizontal for 1 vertical. A grain-size analysis yielded : $d_{50} = 37$ [mm], $\varphi = 37°$, $s_s = 2.65$ [-] and $n = 0.02$ [m$^{-1/3}$s]. What should be the dimensions of this channel, if no erosion is allowed either at the bottom or on the banks ?

SOLUTION :

For water at $T = 14$ [°C] (interpolating the values in Table I.3, in *Graf & Altinakar*, 1991), one has : $\rho = 999.1$ [kg/m^3] and $\nu = 1.186 \times 10^{-6}$ [m^2/s] .

Using the definition of the specific density (see *Graf & Altinakar*, 1991, p.9), one writes:

$s_s = \rho_s / \rho_{eau} \quad \Rightarrow \quad \rho_s = \rho_{eau}\, s_s = 1000 \times 2.65 = 2650$ [kg/m^3]

where $\rho_{eau} = 1000$ [kg/m^3] is the density of water at $p_a = 1$ [atm] and $T = 3.98$[°C].

The stability of the banks requires that the side slopes, θ, should be *smaller* than the angle of repose, $\varphi = 37°$ (see sect. 3.4.3) :

$\mathrm{tg}\theta = 1 / 1.5 = 0.667 \quad \Rightarrow \quad \theta = 33.7° < 37° \quad \Rightarrow$ the banks are thus stable.

The critical bed-shear stress on the banks can be calculated using eq. 3.41 :

$$(\tau_{o_{cr}})^w = \tau_{o_{cr}}\left[\cos\theta\left(1 - \frac{\mathrm{tg}^2\theta}{\mathrm{tg}^2\varphi}\right)^{1/2}\right] = \tau_{o_{cr}}\left[\cos(33.7)\left(1 - \frac{\mathrm{tg}^2(33.7)}{\mathrm{tg}^2(37)}\right)^{1/2}\right] = 0.39\,\tau_{o_{cr}}$$

To determine the critical shear stress, the Shields criteria will now be used (see sect. 3.4.2).

The dimensionless grain diameter corresponding to $d_{50} = 0.037$ [m] is :

$$d_* = d_{50}\left(\frac{\rho_s - \rho}{\rho}\,\frac{g}{\nu^2}\right)^{1/3} = 0.037\left(\frac{2650 - 999.1}{999.1}\,\frac{9.81}{(1.186 \times 10^{-6})^2}\right)^{1/3} = 835 \text{ [-]}$$

According to Shields' criteria, the critical value of the shear stress at the bed, τ_{*cr}, is obtained from Fig. 3.13. In the present case, this value falls in the region where $d_* > 2 \times 10^2$ [-] , having a constant value of $\tau_{*cr} \cong 0.055$ [-] $\cong$ Cte.

The critical shear stress for the *bed* is (see eq. 3.38) :

$$\tau_{o_{cr}} = \tau_{*cr}\, g\,(\rho_s - \rho)\, d_{50} = 0.055 \times 9.81 \times (2650 - 999.1) \times 0.037 \cong 33 \text{ [N/m}^2\text{]}$$

On the *banks*, however, the critical value is already reached for :

$$(\tau_{o_{cr}})^w = 0.39\,\tau_{o_{cr}} = 0.39 \times 33 = 12.9 \text{ [N/m}^2\text{]}$$

The flow depth should be chosen not to exceed these critical values. By taking the critical value of shear stress at the bed as the design criteria and by assuming a wide channel the flow depth can readily be calculated using eq. 3.5a :

$$h = \frac{\tau_{o_{cr}}}{\rho g\, S_f} = \frac{33}{999.1 \times 9.81 \times 0.001} = 3.37 \text{ [m]}$$

According to Fig. 3.14, for a wide channel with a trapezoidal cross section having side slopes of m = 1.5, the maximum value of the shear stress on the banks is : $\tau_o = 0.75\ \rho g\ h\ S_f$. By taking this value as the critical value, one concludes that the flow depth should not exceed :

$$h = \frac{(\tau_{o_{cr}})^w}{0.75\ \rho g\ S_f} = \frac{12.9}{0.75 \times 999.1 \times 9.81 \times 0.001} = 1.75\ [m]$$

One should therefore choose h = 1.75 [m] as the *maximum flow depth.*

Now the channel width, b , should be determined such that for a discharge of Q = 57 [m^3/s] the uniform flow can be established at a flow depth of h = 1.75 [m]. The discharge in a channel is calculated using eq. 3.33 :

$$Q = U\ A = \frac{1}{n}\ R_h^{2/3}\ S_f^{1/2}\ A$$

where U represents the average flow velocity. Given that both the hydraulic radius, R_h , and the wetted surface, A , depend on the channel width, b, which is what we are trying to calculate, the above relationship can not be solved directly for b. Therefore, a trial-and-error calculation should be carried out by varying b until the desired discharge is obtained. The computation sheet for the trial-and-error calculation is presented below.

Computation sheet for determining the channel width, b , by trial-and-error.

m = 1.5 [-] $S_f = 0.001$ h = 1.75 [m] n = 0.02 [$m^{-1/3}s$]

b [m]	A = h (b + mh) [m^2]	$P = b + 2h\sqrt{1+m^2}$ [m]	$R_h = A / P$ [m]	U (eq. 3.16) [m/s]	Q = UA [m^3/s]
5.00	13.34	11.31	1.18	1.77	23.56
10.00	22.09	16.31	1.35	1.94	42.77
14.00	29.09	20.31	1.43	2.01	58.46
13.63	28.45	19.94	1.43	2.00	57.00

The channel width should therefore be : b = 13.63 [m]. In calculating the shear stress at the bed the channel was assumed to be a wide one. Given that b > h, this hypothesis is now justified.

The uniform flow velocity is U = 2.0 [m/s] (see the above computation sheet). This velocity can be compared with the critical velocity for the erosion according to Hjulstrom. As it can be read on Fig. 3.12, the erosion velocity for $d_{50} = 0.037$ [m] is $U_{cr} \cong 2$ [m/s] approximately. Given that the uniform flow depth was selected considering the critical shear stress on the banks, the flow is at the limit of the beginning of erosion in accordance with the previous calculations. To have a better security against the erosion it is necessary to choose a flow depth smaller than the one corresponding to $(\tau_{o_{cr}})^w$.

Ex. 3.D

An artificial channel is going to be constructed in a mountainous region. The slope of the channel imposed by the terrain is $S_f = 0.01$ [-]. This channel should convey a discharge of $Q = 30$ [m^3/s] at a temperature of $T = 14$ [°C], *without causing any erosion.* The grain-size analysis has shown that the granular material is non cohesive with $d_{50} = 50$ [mm] and $s_s = 2.65$ [-]. The angle of repose of this material is $\varphi = 37°$. The Manning coefficient is estimated to be $n = 0.025$ [$m^{-1/3}s^1$].

i) What should be the dimensions of this channel which should have a rectangular section with the sides made of wooden boards ? The use of two different approaches is suggested : the critical velocity, U_{cr} , and the critical shear stress, $\tau_{o_{cr}}$.

ii) What will be the dimensions of the channel, if it were to be constructed entirely in its bed material having an ideal stable cross section ?

iii) Compare the dimensions of the channel, obtained using the different methods.

SOLUTION :

For water at $T = 14$ [°C] (interpolating the values in Table I.3, in *Graf & Altinakar*, 1991), one has : $\rho = 999.1$ [kg/m^3] and $\nu = 1.186 \times 10^{-6}$ [m^2/s] .

Using the definition of the specific density (see *Graf & Altinakar*, 1991, p.9), one writes:
$s_s = \rho_s / \rho_{eau} \quad \Rightarrow \quad \rho_s = \rho_{eau}\, s_s = 1000 \times 2.65 = 2650$ [kg/m^3]
where $\rho_{eau} = 1000$ [kg/m^3] is the density of water at $p_a = 1$ [atm] and $T = 3.98$[°C].

i) a) *Design of the channel using the critical velocity criteria :*

The critical velocity, U_{cr} , according to Hjulstrom (see Fig. 3.12) is :

$$d_{50} = 0.05 \text{ [m]} \quad \Rightarrow \quad U_{cr} = 2.5 \text{ [m/s]}$$

The hydraulic radius corresponding to this velocity can be calculated using the formula of Manning-Strickler :

$$U \equiv U_{cr} = \frac{1}{n} R_h^{2/3} S_f^{1/2} \tag{3.16}$$

With $n = 0.025$ [$m^{-1/3}s^1$] and $S_f = 0.01$ [-], one finds :

$$R_h = \left(\frac{U\ n}{S_f^{1/2}} \right)^{3/2} = \left(\frac{2.5 \times 0.025}{0.01^{1/2}} \right)^{3/2} = 0.494 \text{ [m]}$$

The wetted surface is (see eq. 1.3) : $A = Q / U = 30 / 2.5 = 12\ [m^2]$

The wetted perimeter is (see eq. 1.1) : $P = A / R_h = 12 / 0.494 = 24.3\ [m]$

Given that (see Table 1.1) : $A = b\,h$ and $P = b + 2h$,

the *dimensions* of the rectangular channel are : $b = \mathit{23.1}\ [m]$ and $h = \mathit{0.52}\ [m]$

b) *Design of the channel using the critical shear-stress criteria:*

The dimensionless grain diameter corresponding to $d_{50} = 0.05\ [m]$ is (see sect. 3.4.2) :

$$d_* = d_{50}\left(\frac{\rho_s - \rho}{\rho}\ \frac{g}{v^2}\right)^{1/3} = 0.05\left(\frac{2650 - 999.1}{999.1}\ \frac{9.81}{\left(1.186 \times 10^{-6}\right)^2}\right)^{1/3} = 1129\ [-]$$

Since d_* is known, the critical shear stress at the bed, τ_{*cr} , according to the Shields criteria, can be obtained from Fig. 3.13. In the present case, the calculated value of d_* falls outside of the limits of this figure. Nevertheless, it can be assumed that for $d_* > 2 \times 10^2$ [-] the critical shear stress has a constant value, being $\tau_{*cr} \cong 0.055$ [-] $\cong$ Cte.

Given the definition of τ_* (see eq. 3.40) : $\tau_{o_{cr}} = \tau_{*cr}\,(\gamma_s - \gamma)\,d = \tau_{*cr}\,g\,(\rho_s - \rho)\,d_{50}$

The bed-shear stress is given by eq. 3.5 : $\tau_{o_{cr}} = \gamma\,R_h\,S_f = g\,\rho\,R_h\,S_f$

Combining these two expressions, the hydraulic radius can be calculated :

$$R_h = \frac{\tau_{*cr}\,d_{50}}{S_f}\ \frac{(\rho_s - \rho)}{\rho} = \frac{0.055 \times 0.05}{0.01}\ \frac{(2650 - 999.1)}{999.1} = 0.45\ [m]$$

The velocity corresponding to this hydraulic radius can now be calculated using the formula of Manning-Strickler, eq. 3.16. With $n = 0.025\ [m^{-1/3}s^{1}]$ and $S_f = 0.01$ [-], one has :

$$U \equiv U_{cr} = \frac{1}{n}\ R_h^{2/3}\,S_f^{1/2} = \frac{1}{0.025}\ (0.45)^{2/3}\,(0.01)^{1/2} = 2.35\ [m/s]$$

The remaining part of the calculations are the same as in the first case (see above) :

$A = Q / U = 30 / 2.35 = 12.8\ [m^2]$ and $P = A / R_h = 12.8 / 0.45 = 28.4\ [m]$

The *dimensions* of the rectangular channel are : $b = \mathit{27.5}\ [m]$ and $h = \mathit{0.47}\ [m]$

ii) The critical dimensionless shear stress at the bed, τ_{*cr}, according to the Shields criteria has already been obtained above :

$$d_{50} = 0.05 \text{ [m]} \quad \Rightarrow \quad d_* = 1129 \text{ [-]} \quad \Rightarrow \quad \tau_{*cr} \cong 0.055 \text{ [-]}$$

By using the definition of τ_* (see eq. 3.40), one finds :

$$\tau_{o_{cr}} = \tau_{*cr}\, g\, (\rho_s - \rho)\, d_{50} = 0.055 \times 9.81 \times (2650 - 999.1) \times 0.05 = 44.5 \text{ [N/m}^2\text{]}$$

The maximum depth in the middle of ideal section will be (see sect. 3.4.4) :

$$h = \frac{\tau_{o_{cr}}}{\gamma\, S_f} = \frac{\tau_{o_{cr}}}{g\, \rho\, S_f} = \frac{44.5}{9.81 \times 999.1 \times 0.01} = \mathit{0.45} \text{ [m]}$$

The expression for the ideal section is given by eq. 3.42 :

$$h' = h \cos\left(\frac{tg\varphi}{h}\, y\right) = 0.45 \cos\left(\frac{tg(37)}{0.45}\, y\right) \quad \Rightarrow \quad h' = 0.45 \cos(1.675\, y)$$

Other characteristics of the section are calculated using eqs. 3.42a :

$$A = 2\, h^2 / tg\varphi = 2 \times (0.45)^2 / tg(37) = 0.54 \text{ [m}^2\text{]}$$

$$B = \pi\, h / tg\varphi = \pi \times 0.45 / tg(37) = 1.88 \text{ [m]}$$

Approximating the elliptic integral by :

$$E \approx (\pi/2)\,(1 - \tfrac{1}{4}\sin^2\varphi) = (\pi/2)\left(1 - \frac{\sin^2(37)}{4}\right) = 1.429 \text{ [-]}$$

one obtains :

$$U = \frac{1}{n}\, S_f^{1/2} \left(\frac{h \cos\varphi}{E}\right)^{2/3} = \frac{1}{0.025}\,(0.01)^{1/2} \left(\frac{0.45 \times \cos(37)}{1.429}\right)^{2/3} = 1.59 \text{ [m/s]}$$

The discharge, Q_i, which *can* be conveyed through this ideally stable section is :

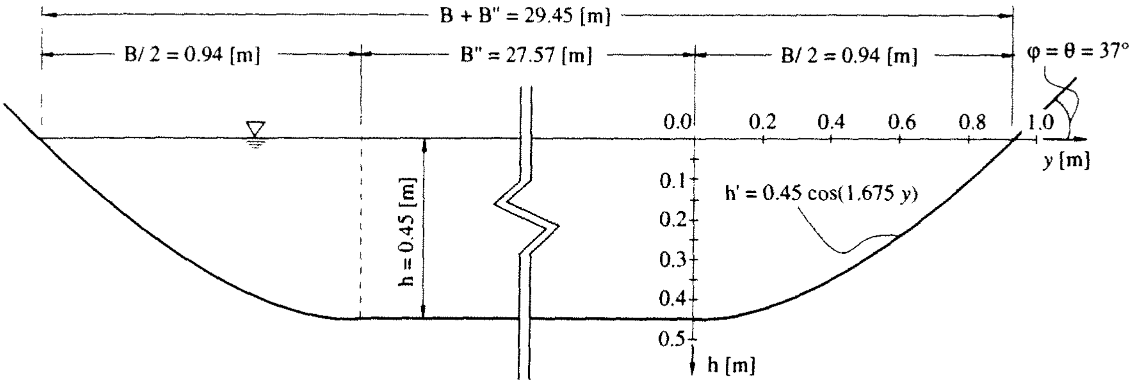
$$Q_i = UA = 1.59 \times 0.54 = 0.86 \text{ [m}^3\text{/s]}$$

This discharge is considerably less than the *required* discharge of $Q = 30$ [m^3/s] . Therefore the wetted surface should be increased by adding a rectangular central part which has a flow depth of $h = 0.45$ [m] and a width of :

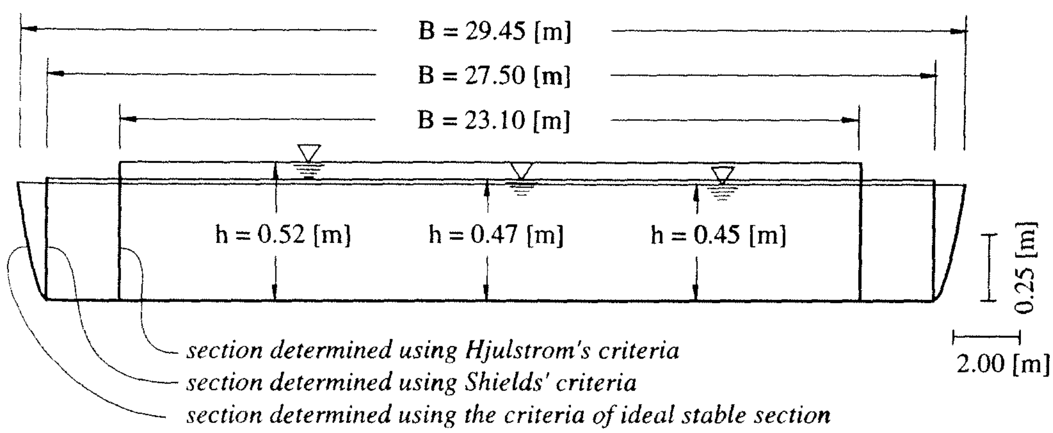
$$B'' = n \frac{(Q - Q_i)}{h^{5/3} J_f^{1/2}} = 0.025 \frac{(30 - 0.86)}{(0.45)^{5/3} \; (0.01)^{1/2}} = 27.57 \text{ [m]}$$

The total width of the channel will thus be : $B + B'' = 1.88 + 27.57 = \mathit{29.45}$ [m]

This section is drawn below :



iii) The figure below shows the superposition of the sections, which were determined using the three different methods. It is interesting to note the differences in the dimensions of channels designed according to the erosion criteria and the one designed according to the criteria of an ideal stable section. Each of the three methods rely on different assumptions and a perfect agreement of the results should not be expected.



Ex. 3.E

In a riveted steel channel, the uniform flow is established at a depth equal to 70% of the critical depth. This channel has a rectangular cross section with a width of b = 9 [m]. An average velocity of U = 12 [m/s] has been assumed for its design.

i) What bed slope should the channel have for this flow ?

ii) What flow regime should one expect ?

iii) What is the shear stress at the bottom of the canal ?

iv) Verify if there is air entrainment into the flow and if so determine its influence on the flow depth.

v) After a long straight reach, the channel makes a $\alpha = 60°$ curve with a radius of curvature of $r_o = 100$ [m] . How much super-elevation should one expect ? Will cross waves develop in the curved reach ?

SOLUTION :

i) Assuming a turbulent rough, uniform flow regime the average velocity in the channel can be calculated using the formula of Manning-Strickler :

$$U = K_s R_h^{2/3} S_f^{1/2} \tag{3.16}$$

from which one can obtain the bed slope : $S_f = \left[U / (K_s R_h^{2/3})\right]^2$

The average velocity, U, of the flow is specified. From Table 3.2 the friction coefficient is estimated as being $K_s \cong 65$ [$m^{1/3}s^{-1}$] .

For a rectangular channel (see also Table 1.1), one has : $R_h = \dfrac{b\, h_n}{b + 2h_n}$

The width of the channel, b, is given, but the normal depth, $h = h_n$, has to be calculated using the following relationship: $h_n = 0.7\, h_c$. The critical flow depth, h_c, for a rectangular channel is given by :

$$\frac{h_c}{2} = \frac{q^2}{2gh_c^2} \qquad \text{or} \qquad h_c = \sqrt[3]{\frac{q^2}{g}} \tag{2.23}$$

where the unit discharge, q, for the uniform flow is :

$q = Q / b = (U\, A) / b = (U\, bh_n) / b = Uh_n = U\,(0.7\, h_c)$

One can therefore write :

$$h_c = \sqrt[3]{\frac{q^2}{g}} = \sqrt[3]{\frac{(0.7\, Uh_c)^2}{9.81}} = \sqrt[3]{\frac{[(0.7)\,(12)\,h_c]^2}{9.81}} = 1.93\, h_c^{2/3}$$

which yields : $h_c = 7.10$ [m] and $h_n = 0.7\,(7.10) = 5.03$ [m]

The discharge for uniform flow, $h = h_n$, is :

$Q = (h\, b)\, U = (5.03 \times 9.0)\, 12 = 543$ [m³/s]

One can now compute : $R_h = \dfrac{b\, h_n}{b + 2h_n} = \dfrac{9\,(5.03)}{9 + 2\,(5.03)} = 2.38$ [m]

and the bed slope : $S_f = \left[\dfrac{U}{K_s\, R_h^{2/3}}\right]^2 = \left[\dfrac{12}{(65)\,(2.38)^{2/3}}\right]^2 = 0.0107$ [-].

For such a uniform flow, the bed slope should be : $S_f = \mathit{0.0107}$ [-].

ii) Given that : $h_c > h_n \quad \Rightarrow \quad$ the flow is supercritical, namely : $Fr > 1$.

For a water temperature of $T = 20$ [°C], the viscosity is $\nu = 1.004 \times 10^{-6}$ [m²/s] (see *Graf & Altinakar*, 1991, Table I.3). The Reynolds number is then (see eq. 1.7) :

$$Re' = \frac{R_h\, U}{\nu} = \frac{(2.38)\,(12)}{1.004 \times 10^{-6}} = 2.8 \times 10^7 > 2000 \quad \Rightarrow \qquad \text{the flow is turbulent.}$$

Moreover, on the Moody-Stanton diagram (see *Graf & Altinakar*, 1991, p.438), it can be verified that the flow is in the region of turbulent rough flow, where the friction coefficient is independent of the Reynolds number.

The flow regime in this channel is therefore *supercritical and turbulent rough.*

iii) The bed-shear stress can be calculated using eq. 3.5. By taking $\rho = 998.2$ [kg/m³] for water at 20 [°C] (see *Graf & Altinakar*, 1991, Table I.3) one has :

$$\tau_o = \rho\, u_*^{\,2} = \gamma\, R_h\, S_f = \rho g\, R_h\, S_f = 998.2 \times 9.81 \times 2.38 \times 0.0107 = \mathit{249.4}\ [\text{N/m}^2]$$

iv) The Froude number for the flow is :

$$Fr = \frac{U}{\sqrt{g\, h_n}} = \frac{12}{\sqrt{9.81 \times 5.03}} = 1.71\ [\text{-}]$$

For a turbulent rough flow the surface instabilities are expected only for $Fr > 2$. Given that in the present case $Fr < 2$, there will be no roll-wave forming on the free-water surface.

Since the channel slope is weak, it can also be surmised that there will not be any air entrainment. This can be verified using the following equation to compute the mean air concentration in the cross section :

$$\overline{C} = 0.7 \log (\sin\alpha / q^{1/5}) + 0.9 \tag{3.52}$$

The unit discharge is : $q = Q / b = U\, h_n = 12 \times 5.03 \cong 60.4$ [m²/s]

The calculated value of the mean air concentration is negative, which is an impossibility. It can therefore be concluded that no air entrainment takes place across the water surface.

v) The Froude number, Fr = 1.71, calculated above indicates that the flow is supercritical. The cross waves will form in the curved reach. The angle formed between the upstream tangent and the positive and negative waves is given by eq. 3.45 :

$$\sin\beta = \frac{1}{Fr} = \frac{1}{1.71} \qquad \Rightarrow \qquad \beta = 35.8°$$

By using now data for the curve, $r_0 = 100$ [m] and $B = b = 9$ [m], the central angle between a successive maximum and minimum can be calculated using eq. 3.46a :

$$tg\,\theta = \frac{B}{(r_0 + B/2)\,tg\,\beta} = \frac{9}{(100 + 9/2)\,tg(35.8)} = 0.119 \qquad \Rightarrow \qquad \theta = 6.8°$$

Knowing these values, β and θ, one can then calculate the maximum and minimum water depths on the concave and convex side walls of the channel, respectively by using eq. 3.46 :

$$h_{max} = h_n\, Fr^2 \sin^2(\beta + \theta/2) = 5.03 \times (1.71)^2 \times \sin^2(35.8 + 6.8/2) = 5.88 \text{ [m]}$$

$$h_{min} = h_n\, Fr^2 \sin^2(\beta - \theta/2) = 5.03 \times (1.71)^2 \times \sin^2(35.8 - 6.8/2) = 4.22 \text{ [m]}$$

The super-elevation is: $h_{max} - h_{min} = 5.88 - 4.22 = 1.66$ [m]

It is interesting to compare this value with the one obtained using eqs. 3.43a and 3.47 (see Fig. 3.18) :

$$\Delta z + \Delta z' = 2\,\frac{B}{r_0}\,\frac{U^2}{2g} = 2\,\frac{9}{100}\,\frac{12^2}{2 \times 9.81} = 1.32 \text{ [m]} < 1.66 \text{ [m]}$$

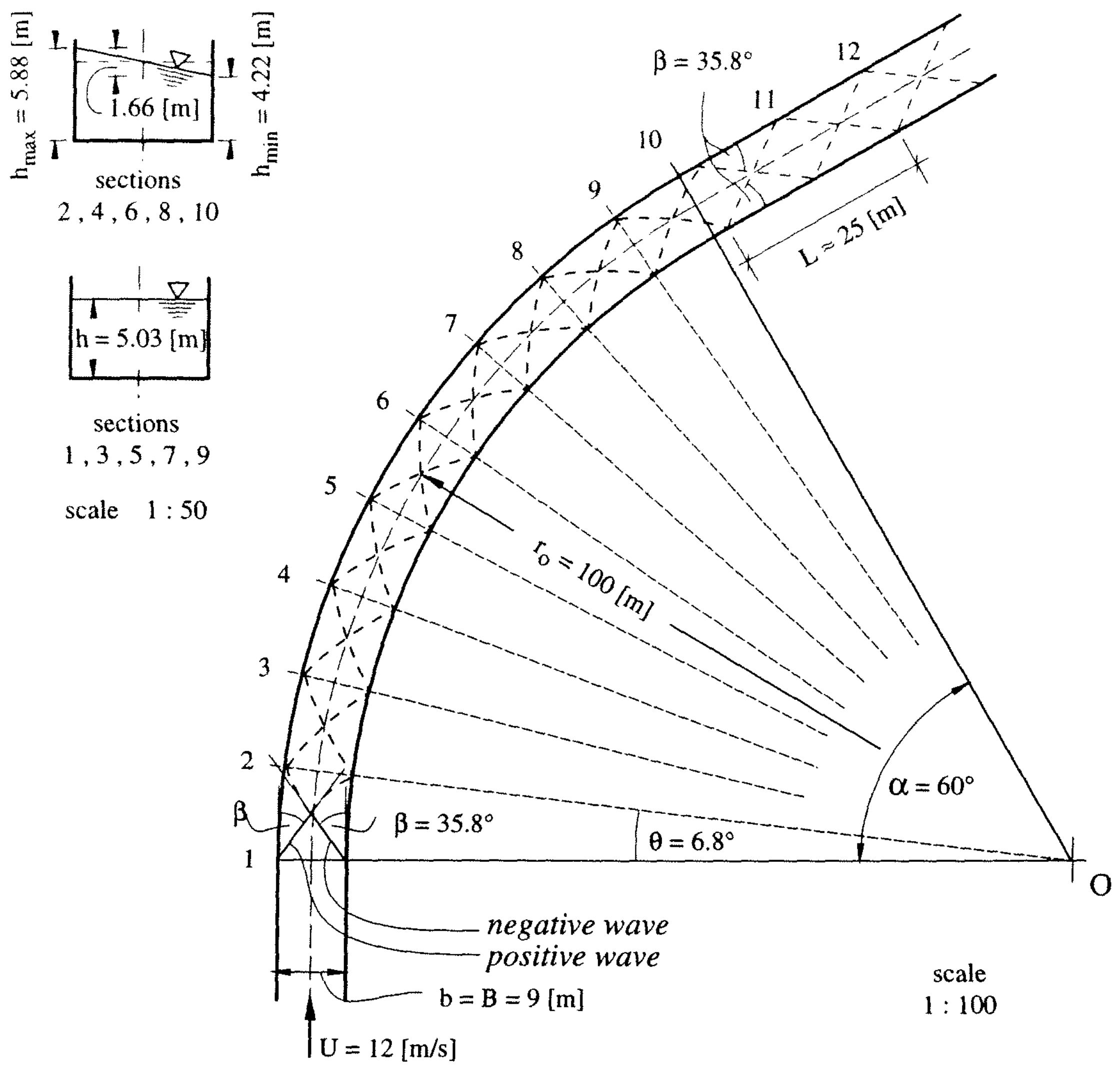
One can see that the first relationship yields a value larger than the second one. For a better safety a super-elevation of 1.66 [m] can be assumed.

The super-elevation with respect to the normal depth, $h_n = 5.03$ [m] , is :

$h_{max} - h_n = 5.88 - 5.03 = 0.85$ [m]

This value will be used in dimensioning the channel cross section in the curved reach.

The formation of cross waves in the curved reach and the downstream straight reach as well as the channel cross sections at different central angles are represented in the following figure. It is important to emphasize that this figure represents a rather simplified image of the reality. A more realistic representation of the water surface can be calculated using the method of characteristics which will be studied in sect. 5.2.2.

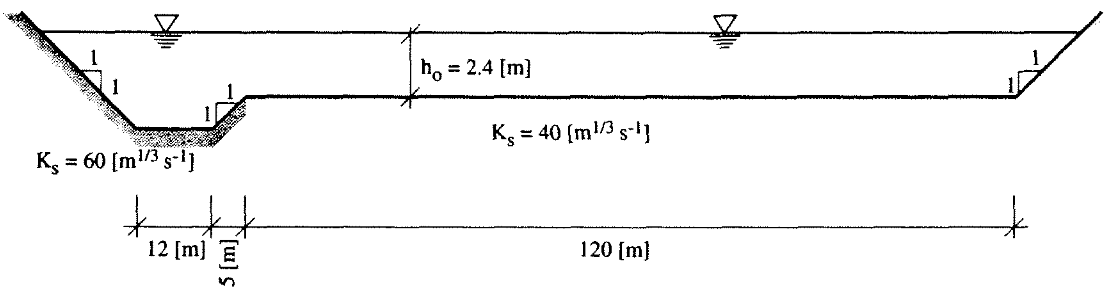


The situation downstream of the curved reach is difficult to predict by a simple method. At the end of the curved reach, at the point of passage from a curved channel to a straight channel, new positive and negative waves are created. According to the ratio between the total angle of the curve, α, and the central angle between a successive maxima and minima, θ, these new waves can be *in* phase or *out* of phase with the existing waves in the curved reach. In some cases a cross-wave pattern of considerable amplitude continues to exist in the downstream channel foı some distance before being gradually attenuated by the friction. In the present case, given that the water surface is inclined at the end of the curved reach, it can be expected that the cross waves do continue in the downstream reach. The distance between two maximum (or two minimum) will be in the order of : $L = 2B / tg\beta = (2 \times 9) / tg(35.8) \cong 25$ [m]. This situation is schematically represented on the figure.

3.7.2 Problems, unsolved

Ex. 3.1
The width at the bed of a trapezoidal channel is b = 2.30 [m] and the side slopes are at an angle of 50° with the horizontal plane. The bed drops by 150 [cm] over a distance of 1.2 [km]. Determine the average velocity and the discharge for a flow depth of h = 1.60 [m] and a coefficient of friction after Bazin of $m_B = 0.46$ [$m^{1/2}$]. Calculate also the shear stress on the bed.

Ex. 3.2
Calculate the discharge for the channel as given below. The bed slope was measured, being $S_f = 0.09\%$. Consider the flow as steady and uniform.



Ex. 3.3
The circular pipe of a sewage system has a coefficient of Manning of n = 0.015 [$m^{-1/3}s$] and is put at an inclination of S_f = 0.0002. When the flow depth attains 0.9 of its diameter, the pipe must still transport a discharge of Q = 2.5 [m^3/s]. What should be the diameter ? Try to make the same calculations using the coefficient of Weisbach-Darcy.

Ex. 3.4
Water at a temperature of 20 [°C] flows in a large rectangular channel, whose coefficient of Manning was determined previously as being n = 0.011 [$m^{-1/3}s$]. The channel slope is $S_f = 0.0004$ and the flow depth is put at h_n = 1.20 [m]. Calculate the value of the corresponding coefficient of Chezy, C, using the formula of Strickler and the logarithmic expression of $U = 2.5u_* \ln(41.2\, R_h/\delta)$; compare these two resulting values. The viscous sublayer is given by $\delta = 11.6\, \nu/u_*$.

Ex. 3.5
A semi-circular canal is made of smooth metal, with n_1 = 0.012 [$m^{-1/3}s$]; the diameter is D = 2 [m] and the bed slope is S_f = 0.005. What diameter should the canal have if it will be reconstructed of corrugated metal, with n_2 = 0.022 [$m^{-1/3}s$] ?

Ex. 3.6
A trapezoidal channel must convey a discharge of Q = 5.25 [m^3/s] at a flow velocity of U = 1 [m/s]. This channel is made of brickwork (dry, rough stones); the lateral walls are inclined at 45° and the channel slope is S_f = 0.0005. Determine the flow depth and the width of the channel.

Ex. 3.7
A stream should be channelised between the sections 15.2 [km] and 17.5 [km] with a circular pipe having a diameter of 1.5 [m]; the flow drops 4.6 [m] between these two sections. This pipe should transport a maximum discharge of Q = 0.8 [m^3/s] at a flow depth of 1/3 of its diameter. What will be the coefficient of Manning, n , for this flow and what pipe material should be selected ?

Ex. 3.8
A rectangular channel being B = 3.6 [m] wide conveys a discharge of Q = 5.50 [m^3/s]. What will be the critical flow depth and the corresponding velocity ? For what slope, S_f , will the velocity be critical, if a coefficient of Manning of n = 0.02 [$m^{-1/3}$s] is assumed ?

Ex. 3.9
Find the critical flow depth in a trapezoidal canal, whose width at the bottom is b = 4.0 [m] and whose side slopes are inclined by 1 : 2. The discharge is given as Q = 90 [m^3/s].

Ex. 3.10
In a trapezoidal channel the bottom width is b = 6.10 [m] and the side slopes are inclined at m = 2. The elevation of the bed at km 35.0 is 681.30 [m] and the one at km 48.7 is 659.38 [m]. Determine the flow depth for a discharge of Q = 11.33 [m^3/s]. A coefficient of Manning of n = 0.025 [$m^{-1/3}$s] was previously determined. Subsequently determine the flow regime, using the average depth as the characteristic length.

Ex. 3.11
A trapezoidal canal must convey a discharge of 16.70 [m^3/s] at a flow depth of 1.05 [m] over a distance of 5 [km]. The side slopes are 2 horizontal to 1 vertical; the total drop is given as 8.5 [m]. What will be the bottom width, b, if the flow velocity is supposed to be critical (for the characteristic length use the ratio of the transversal section to the surface width). Give the corresponding coefficient of Manning.

Ex. 3.12
A rectangular channel having a width of B = 6.5 [m] conveys a water discharge of Q = 18 [m^3/s]. Establish the specific-energy curve, $H_s = f(h)$, in the range of $0 < h$ [m] < 8. What is the critical depth ? What would be the specific energy, H_s , if the flow depth is $h = 2h_{cr}$. What will be the flow depth of uniform, supercritical flow having the *same* specific energy, H_s ?

Ex. 3.13
A rectangular channel, made of smooth concrete and having a width of $B = 20$ [m], conveys a discharge of $Q = 200$ [m^3/s] at a specific energy of $H_s = 3.75$ [m]. Determine the flow depth, h_n , and the bed slope, S_f , for uniform, steady flow. Is this flow a supercritical one ?

Ex. 3.14
A trapezoidal channel, made of rather rough concrete and having a side slope of $m = 2$, is envisioned to transport a discharge of $Q = 17$ [m^3/s] at an average velocity of $U = 1.2$ [m/s]. Determine the width at the bed, the flow depth, and the bed slope for the hydraulically optimal cross section.

Ex. 3.15
A trapezoidal channel – the coefficient of Strickler is estimated to be $K_s = 40$ [m$^{1/3}$s^{-1}] – should be designed having a cross section of maximum discharge. The bottom width is $b = 2$ [m] and the side slopes are of $m = 3$. The average velocity is fixed at $U = 1.98$ [m/s]. What will be the geometry of this section, the discharge, Q, and the channel slope, S_f ?

Ex. 3.16
A drainage channel on a highway, running on a slope of $S_f = 0.0001$, has a triangular section whose side slopes are $m = 4$ and $m = 2$. The coefficient of Manning is $n = 0.02$ [m$^{-1/3}$s]. If flow in the channel is uniform, what will be the flow depth for a discharge of $Q = 0.1$ [m^3/s] ? By how much can the wetted surface be reduced if the channel is made semi-circular ?

Ex. 3.17
A channel of a trapezoidal cross section is built in an alluvium whose granulate is $d_{50} = 1$ [mm]. The flow depth is $h_n = 3$ [m] and the width at the bed is $b = 4$ [m]. The bed slope is $S_f = 0.001$ and the side slopes of worked stone are 45° inclined. What will be the corresponding velocity, U, and the discharge, Q. Check if the bed will be subject of erosion. May one expect the formation of dunes ?

Ex. 3.18
A very large canal in an alluvium, whose granulate of quartz is $d_{50} = 1$ [mm], has a bed slope of $S_f = 10^{-4}$. At what flow depth, h , will erosion commence ? What is the velocity which corresponds to this critical condition ?

Ex. 3.19
One envisions the construction of a non-erodible canal having an ideal, stable cross section. The bed slope is $S_f = 10^{-3}$. The analysis of the granulometry gave : $\rho_s/\rho = 2.65$, $d_{50} = 6.5$ [mm], and an angle of repose of $\varphi = 30°$. Establish the design dimensions for the following discharges : $Q_1 = 1.5$ [m^3/s] and $Q_2 = 4$ [m^3/s].

Ex. 3.20

Calculate the profile of a trapezoidal channel for a discharge of $Q = 12$ [m^3/s] on a bed slope of $S_f = 0.0016$. This channel should be excavated in an alluvium of large gravel. For the side slopes it is recommended to take $m = 2$.

Ex. 3.21

In the town of Ste-Justice, the construction of a road along the river Happy is envisioned. This project can be realised if the river's width is reduced to 67.5 [m]. All other parameters are the same as the ones in Ex. 3.B.

i) Establish the rating curve, $Q = f(h_n)$, assuming the flow is rough and turbulent.

ii) At which flow depth should one expect the commencement of erosion and deposition ?

iii) Compare these results with the ones of Ex. 3.B. Remark on the consequence of such a reduction in the river width.

Ex. 3.22

The downstream slope of the back of a large weir is 30°; it is long enough such that its flow can be considered uniform. The coefficient of Manning is assumed to be $n = 0.0149$ [$m^{-1/3}s$]. The width of the weir is $B = 7$ [m] and the flood discharge is set at $Q = 0.3$ [m^3/s]. Determine if one may expect air-entrainment and calculate the flow depth of the mixture as well as the pressure on the back of the weir.

Ex. 3.23

On a slope of 20°, a canal in concrete was projected to evacuate a unit discharge of $q = 35$ [m^2/s]. Calculate the flow depth and the pressure on the floor. Will air-entrainment take place ?

Ex. 3.24

A rectangular canal made of wood has a width of $B = 8$ [m] and should evacuate the flood discharge. The flow depth is $h = 1.0$ [m] and the flow velocity should be $U = 10$ [m/s]. Determine the radius of the curve which should be foreseen, under the condition that the maximum flow depth does not exceed a height of $h_{max} = 2.0$ [m].

Ex. 3.25

A rectangular channel, having a bed of sand and side walls of concrete, should be designed to have a cross section of maximum discharge :

i) Knowing that the flow in the channel has a depth of $h = 1.0$ [m] and runs at a Froude number of $Fr = 0.7$, what will be the discharge ?

ii) A change in the flow direction of $\alpha = 30°$ should be envisioned. Determine the radius of the curve for the case that the maximum super-elevation is not larger than 15% of the normal depth. What is the head loss in this curve ?

RANSPORT OF EDIMENTS

flow in a watercourse is a particularly difficult task, since the channel bed is ɔrm which varies in space and time. The movement of the sediments, which ›ed, represents a rather complex phenomenon.

ter, the hydrodynamic equations of the flow over a mobile bed are ›ome solutions are given. The different modes of the transport of (non-iments as bed load and as suspended load are presented. The formulae for ns of the transport of the total load will be exposed, as well as their domain .

; OF CONTENTS

6.1 GENERALITIES

6.1.1 Notions

1° The flow of water over a mobile bed has the ability to entrain the sediments (solid particles) ; a water-sediment mixture will consequently displace itself in the water-course. The movement of the sediments — *erosion, transport, deposition* — will modify the flow, but also the channel bed, thus its elevation, its slope and its roughness. The interaction between the water and the sediments makes the problem a coupled one.

2° When the bed is a *mobile* one, the fluvial hydraulics must concern itself with both the flow of the liquid phase, namely the mixture, and the movement of the solid phase, namely the sediments in the mixture.

3° A characterisation of the liquid and the solid phase of a water-sediment mixture is a difficult task.

The *liquid* phase is rather well described by :

i) its density, ρ ,
ii) its viscosity, μ ,
iii) the average velocity of the flow, U , and
iv) the friction velocity, u_*.

The *solid* phase is more difficult to characterise ; considered should be :

i) the size of the solid particles, given by its granulometric curve, which includes different types of diameters such as d_{50} , d_{90} , d_{35} , etc.,
ii) the form of these particles,
iii) the density of the particles, ρ_s ,
iv) together, these parameters can be defined by the settling velocity of the particles, v_{ss} , and
v) possibly, the cohesion between the particles.

All these parameters could vary along the watercourse. Furthermore, they will depend on the way the bed samples (in the nature) are taken and are analysed.

4° The dimensions of the sediments are relatively small compared to the ones of the flow ; thus the *turbulence* will play an essential role in all flows of a water-sediment mixture.

5° The transport of these sediments plays an important, if not the most important role in all problems of fluvial hydraulics. This phenomenon is very complex and consequently a theoretical study can only be performed in simple or simplified cases. The formulae, developed for the quantitative determination of the transport of sediments, are based on experimental results, being often limited, and thus should be used with much caution. Such formulae are of great value for the hydraulic engineer, but must be applied within hydraulic conditions under which they have been established.

6.1.2 Flow of a Mixture

1° For gravitational flow of a water-sediment mixture, one may distinguish three types of movement (see Table 6.1). :

i) The mixture may be considered Newtonian, if the volumic concentration of the particles is very small, $C_s << 1$ %. The difference between the density of the mixture and of the water, $\Delta\rho = (\rho_m - \rho) = (\rho_s - \rho)\ C_s$ (see eq. 7.2), remains also small, $\Delta\rho << 16$ [kg/m^3].

The *transport of sediments* (see chap. 6), as bed load and as suspended load, falls into this category. It is this type of transport of solid particles, which is most often encountered in watercourses.

ii) The mixture behaves quasi-Newtonian, if the volumic concentration remains small, $C_s < 8\%$. The difference between the density of the mixture and of the water becomes important, $\Delta\rho < 130$ [kg/m^3].

The transport of sediments as *concentrated suspension* (see *Graf*, 1971, p. 182-186) notably close to the bed, as well as the *turbidity currents* (see chap. 7) fall into this category.

Table 6.1 Classification of flows of a mixture.

Gravitational flow of a water-sediment mixture

non-Newtonian mixture	quasi-Newtonian mixture	Newtonian mixture
$C_s > 8$ %	$C_s < 8$ %	$C_s << 1$ %
* transport of sediments as hyperconcentrated suspension	* transport of sediments as concentrated suspension	* transport of sediments as bed load and suspended load
* debris flow	* turbidity currents	

iii) The mixture behaves non-Newtonian, if the volumic concentration becomes of importance, $C_s > 8\%$. The difference between the density of the mixture and of the water is also very large, $\Delta\rho > 130$ [kg/m^3].

The flow of a non-Newtonian fluid modifies all concepts of Newtonian hydraulics, such as the resistance to the flow, as well as the distribution of velocity and of concentration ; the settling velocity is also influenced and the solid particles stay longer in suspension.

The transport of sediments as *hyperconcentrated suspension* (see *Wan* et *Wang*, 1994), the *debris flow* (see *Takahashi*, 1991), as well as *hyperconcentrated turbidity currents* (see *Wan* et *Wang*, 1994) fall into this category.

- The transport of sediments as an hyperconcentrated suspension is encountered in watercourses of small slopes. Usually enormous quantities of sediments — being of small sizes — enter the channel due to surface erosion caused by extensive rainfalls in the catchment basin. These solid particles stay usually for long time periods in suspension, as wash load.

- Torrential flows of debris may establish themselves at rather steep slopes, $\alpha > 15°$. All kinds of particles, from the finest (having cohesion) to the largest (blocks of 1 [m^3]), participate in the movement, which is rather rare in occurrence and of short duration, and is usually caused by severe rainfalls.

2° It should be stressed, that the above schematic classification (see Table 6.1) is a simplification of the reality, where limits can often not readily be defined and where the different cases can coexist.

6.1.3 Modes of Transport (see Fig. 6.1)

1° The (total) transport of sediments by flow of water is the entire solid transport (of the particles) which passes through a cross section of a watercourse.

Traditionally (but a bit artificially) the transport of sediments is classified in different modes of transport (see Table 6.2) which correspond to distinctly different physical mechanisms.

Table 6.2 Modes of transport of sediments (T.S.).

- total T.S.
  - T.S. of bed material, q_s
    - T.S. as bed load, q_{sb}
    - T.S. as suspended load, q_{ss}
  - T.S. as wash load, q_{sw}

2° In a watercourse the sediments, namely the solid phase, are transported :

i) as *bed load*, q_{sb}, — volumic solid discharge per unit width [m^3/sm] — when the particles stay in close contact with the bed ; the particles displace themselves by gliding, rolling or (shortly) jumping ; this type of transport concerns the relatively larger particles ;

ii) as *suspended load*, q_{ss}, when the particles stay occasionally in contact with the bed ; the particles displace themselves by making more or less large jumps and remain often surrounded by water ; this type of transport concerns the relatively smaller particles ;

iii) as bed load + suspended load, being the (total) *bed-material load*, $q_s = q_{sb} + q_{ss}$, when the particles stay more or less in continuous contact with the bed.

iv) as *wash load*, q_{sw}, when the particles are almost never in contact with the bed ; the particles are washed through the cross section by the flow ; this type of transport concerns the relatively finest particles.

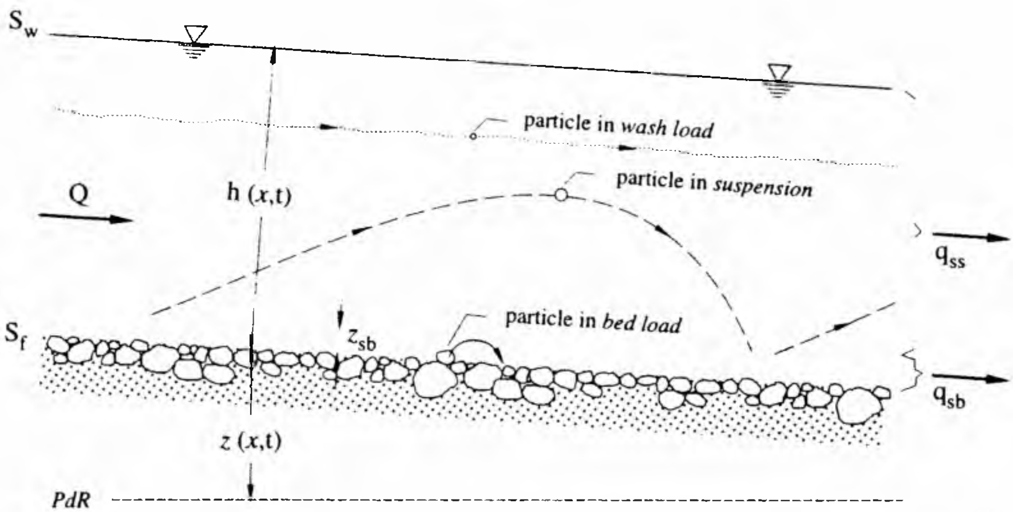


Fig. 6.1 Scheme of the modes of transport.

3° The transport of sediments , namely the erosion of the bed (see sect. 3.4.2), commences upon attainment of a certain critical value, which can be parametrised, for example, by the critical shear stress, τ_{ocr}.

4° It will be useful, but it is also rather imprecise, to give limiting values for the separation of the different modes of transport. Given here are purely indicative values, which use the ratio of the shear velocity of the flow, u_* , and the settling velocity of the particles, v_{ss} (see *Graf*, 1971) :

- $\dfrac{u_*}{v_{ss}} > 0.10$ beginning of bed-load transport,

- $\dfrac{u_*}{v_{ss}} > 0.40$ beginning of suspended load transport.

5° To determine quantitatively the transport of sediments, there are three possibilities available, namely :

- using existing formulae (see sects. 6.3, 6.4 and 6.5),
- obtaining field measurements with adequate instruments (see *Graf*, 1971, chap. 13),
- performing physical models (see *Graf*, 1971, chap. 14).

6.1.4 Types of Problems

1° Many of the hydraulic problems, which require a knowledge of the transport of sediments, can readily be put into one of the following categories :

- determination of a sedimentological rating curve, $q_s = f(q)$, for a given cross section of the channel (see Fig. 6.2);
- determination of the stability of the bed in a given cross section (see sect. 3.4.4);
- determination of the stability of the channel slope (aggradation and degradation) in a given reach of the channel (see sect. 6.2.4).

2° The different modes of transport of sediments, quantified in form of solid discharge, q_{sb} , q_{ss} and q_s, should be related to the liquid discharge, q. This will give the relation of the "sedimentological" rating curve (see Fig. 6.2) for a given cross section of the channel. This curve together with the "liquid" rating curve (see Fig. 3.8) give a rather complete hydraulic description for a given cross section of a channel having a mobile bed.

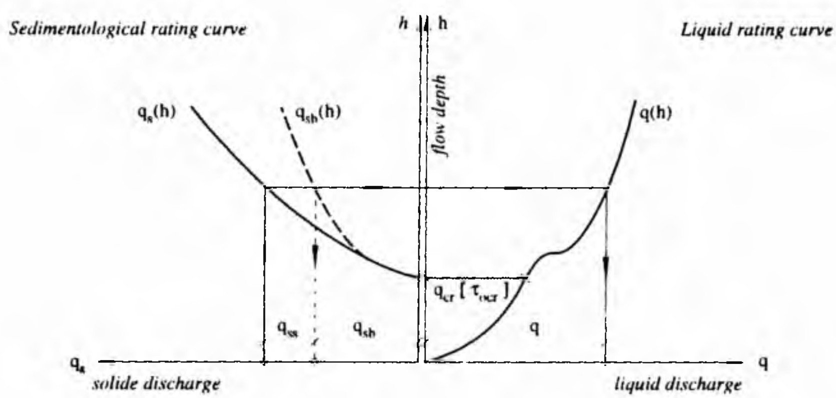


Fig. 6.2 Rating curves for the liquid discharge and the solid discharge.

3° The formulae, which are used to calculate the solid discharge, q_s , allow to know the *capacity* of the transport of sediments for a given flow. Under such conditions, the transport of sediments is said to be in equilibrium.

However it could happen, that the supply of solid discharge is not equal to the capacity of the transport. The transport of sediments is then not in equilibrium :

- if the capacity is larger than the supply, erosion and transport occurs,
- if the supply is larger than the capacity, deposition and transport occurs,
- if the supply is equal to the capacity, transport without erosion or deposition occurs,
- if the bed is armoured, the capacity may not be satisfied (see sect. 6.3.4).

One sees here the complexity of the problem, where along a watercourse the different scenarios can coexist or overlap.

6.2 HYDRODYNAMIC EQUATIONS

Presented will be the hydrodynamic equations, and some solutions, for flow in an open channel over a mobile bed, when entrainment of sediments is possible.

6.2.1 Equations of Saint-Venant - Exner

1° The equations of Saint-Venant (see sect. 5.11) for unsteady and non-uniform flow over a *fixed bed* in a prismatic open channel with a small bed slope (see Fig. 5.1), have been given before (see eq. 5.2 and eq. 5.3) ; for flow over a *mobile bed* they can be written as :

$$\frac{\partial h}{\partial t} + h\frac{\partial U}{\partial x} + U\frac{\partial h}{\partial x} = 0 \qquad B = Cte \tag{6.1}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial h}{\partial x} + g\frac{\partial z}{\partial x} = - g\, S_e \tag{6.2}$$

The energy slope, S_e, shall be expressed with a relationship established for uniform flow, by using a friction coefficient, f, for a mobile bed (see sect. 3.2.6), or :

$$S_e = f(f, U, h) \tag{6.3}$$

where h is the flow depth, U is the average velocity of the flow and $z(x,t)$ gives the elevation of the channel bed.

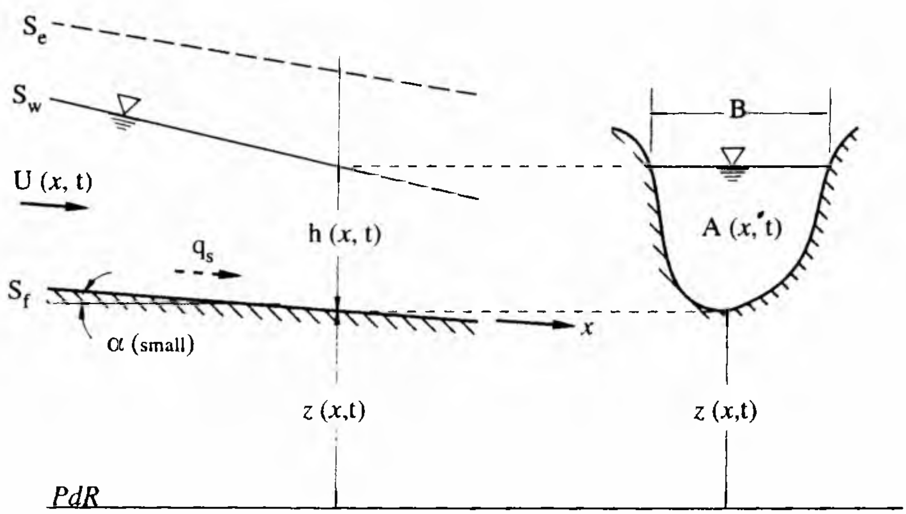


Fig. 6.3 Scheme of unsteady and non-uniform flow over a mobile bed, $z(x,t)$.

2° For flow over a *mobile bed* (see Fig. 6.3), the elevation (level) of the channel bed, $z(x,t)$, may vary. According to the relation of *Exner* (see *Exner*, 1925, and *Graf*, 1971, p. 288), such a variation can be expressed by :

$$\frac{\partial z}{\partial t} = - a_E \frac{\partial U}{\partial x} \tag{6.4}$$

where a_E is an erosion coefficient. This relationship, eq. 6.4, can be written (see *Graf*, 1971, p. 152 and *Krishnappan*, 1981, p. 91) in form of a continuity equation for the solid phase, namely :

$$\frac{\partial z}{\partial t} + \left(\frac{1}{1-p}\right)\left[\frac{\partial}{\partial t}(\tilde{C}_s h) + \frac{\partial}{\partial x}(C_s U h)\right] \equiv \frac{\partial z}{\partial t} + \left(\frac{1}{1-p}\right)\frac{\partial q_s}{\partial x} = 0 \tag{6.4a}$$

where p is the porosity of the sediments of the bed, being defined as the ratio of the volume of empty space (occupied by water) and of the total volume. $q_s = C_s U h$ is the volumic solid discharge per unit width and C_s ($\tilde{C}_s$) is the volumic concentration (in the cross section) of the solid phase, being defined by the ratio of the volume of the sediments and of the volume of the mixture. In general one admits that the solid discharge, q_s , is a function — still to be determined (see sect. 6.3 to sect. 6.5) — of the liquid discharge, $q = Uh$, or :

$$q_s = f(U, h \text{ ; sediment}) \tag{6.5}$$

3° The three basic (differential) equations, eqs. 6.1, 6.2 and 6.4a, contain three unknowns, $U(x, t)$, $h(x, t)$ and $z(x,t)$, with their independent variables, x and t . U and h are the average velocity and the flow depth of the water-sediment mixture (the liquid phase), or of the water only, if the concentration of the sediments, C_s, is negligible.

The two other unknowns, S_e (see eq. 6.3) and q_s (see eq. 6.5), have to be expressed with semi-empirical relationships.

4° The five relations, eqs. 6.1, 6.2 and 6.4a together with eq. 6.3 and eq. 6.5, are the *equations of Saint-Venant - Exner.*

The three relations, eqs. 6.1, 6.2 and 6.3, describe the flow of the liquid phase over a mobile bed ; the two other relations, eqs. 6.4a and 6.5, describe the transport (erosion and deposition) of the solid phase.

5° The liquid and the solid phase are *implicitly* coupled by the semi-empirical relations, eqs. 6.3 and 6.5. After the solution for the liquid phase, eqs. 6.1 and 6.2, a solution for the solid phase, eq. 6.4a, can be obtained, giving the variation of the bed elevation, $z(x,t)$.

The equations of Saint-Venant - Exner can be *explicitly* coupled, if the equation of continuity for the liquid phase, eq. 6.1, is expressed (see *Krishnappan*, 1981, p. 93) as follows :

$$\frac{\partial h}{\partial t} + \frac{\partial z}{\partial t} + \frac{\partial}{\partial x}(Uh) = 0 \qquad (6.1a)$$

A direct coupling is thus achieved (see *Correia* et *al.*, 1992), since the term, $\partial z/\partial t$ — which is however often rather small — exists now in both eq. 6.1a and eq. 6.4a. One looks now for a solution by solving simultaneously the equations for the liquid and for the solid phase.

6° To obtain solutions to the equations of Saint-Venant - Exner, use can be made of :

i) analytical methods (see sect. 6.2.3) for simple problems, and

ii) numerical methods (see sect. 6.2.5) for complex problems.

6.2.2 Propagation of Perturbations

1° The propagation of a perturbation, being a wave of small amplitude on the mobile bed, can now be investigated by using the equations of Saint-Venant - Exner, eq. 6.1 to eq. 6.5.

2° For a rectangular channel, these equations — see also the system of equations, eq. 5.2 to eq. 5.10 — are written as six equations of partial derivatives :

$$\frac{\partial h}{\partial t} + h\frac{\partial U}{\partial x} + U\frac{\partial h}{\partial x} = 0 \qquad (6.1)$$

$$\frac{1}{g}\frac{\partial U}{\partial t} + \frac{U}{g}\frac{\partial U}{\partial x} + \frac{\partial h}{\partial x} + \frac{\partial z}{\partial x} = -S_e \qquad (6.2)$$

$$(1-p)\frac{\partial z}{\partial t} + \frac{\partial q_s}{\partial U}\frac{\partial U}{\partial x} = 0 \qquad (6.4b)$$

with :

$$\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial t}dt = dh$$

$$\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial t}dt = dU$$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial t}dt = dz$$

The three last equations are expressions for the total derivatives of the three dependent variables, $h(x,t)$, $U(x,t)$ and $z(x,t)$.

In writing eq. 6.4b, it was assumed that the solid discharge is only a function of the flow velocity, $q_s = f(U)$, thus :

$$\frac{\partial q_s}{\partial x} = \frac{\partial q_s}{\partial U}\,\frac{\partial U}{\partial x}$$

3° Upon mathematical manipulations of these six equations — the determinant of the matrix of the coefficients must become zero — the following cubic relationship (see *de Vries*, 1965 and 1973, p. 2) is obtained :

$$-c_w^{\,3} + 2\,U\,c_w^{\,2} + \left(gh - U^2 + g\frac{\partial q_s}{\partial U} \right) c_w - gU\frac{\partial q_s}{\partial U} = 0 \qquad (6.6)$$

where the absolute celerity (the characteristic) is defined by :

$$c_w = \frac{dx}{dt} \qquad (2.34)$$

4° This equation, eq. 6.6, has evidently three real roots, thus three characteristics :

i) Two roots, c_{w_1} and c_{w_2} , are an expression of the celerity of the perturbation (wave) on the water surface ; the third root, c_{w_3} , gives the celerity of a perturbation (undulation) on the mobile bed.

ii) c_{w_3} is positive for subcritical flow, $U < \sqrt{gh}$; the form (undulation) of the bed, usually called dunes (see sect. 3.2.5), displaces itself in the same direction as the flow (see *de Vries*, 1973, p. 3).

c_{w_3} is negative for supercritical flow, $U > \sqrt{gh}$; the form (undulation) of the bed, usually called antidunes (see sect. 3.2.5), displaces itself in the opposite direction of the flow.

iii) For a fixed bed without solid discharge, $q_s = 0$, one gets :

$$c_{w_3} = 0$$

$$c_{w_1} = U + \sqrt{gh} \qquad \text{and} \qquad c_{w_2} = U - \sqrt{gh} \qquad (5.14a)$$

This solution has already been presented in sect. 5.2.1 and sect. 2.4.3 (see Fig. 2.9).

5° It seems reasonable (see *de Vries*, 1973, p. 4, and *Jansen* et *al.*, 1979, p. 96) to assume that for $Fr = U/\sqrt{gh} \neq 1$ the celerities, c_{w_1} and c_{w_2}, of the waves on the surface are much larger than the celerity, c_{w_3}, of the undulations on the bed, or :

$$c_{w_1} \text{ and } c_{w_2} >> c_{w_3}$$

When studying the perturbations on the bed having a weak celerity, c_{w_3}, it is now possible to consider the flow of the liquid phase as quasi-steady ; thus :

$$\partial U/\partial t = 0 \qquad \text{and} \qquad \partial h/\partial t = 0$$

Consequently, combining eq. 6.1 with eq. 6.2, one can write a single differential equation, which is the equation of the free-surface flow (see eq. 4.7), or :

$$\frac{\partial U}{\partial x}\left(U - \frac{h}{U} g\right) + g \frac{\partial z}{\partial x} = - g S_e \tag{6.7}$$

By eliminating $\partial U/\partial x$ between eq. 6.7 and eq. 6.4b, one obtains :

$$\frac{\partial z}{\partial t} + c_{w_3} \frac{\partial z}{\partial x} = - c_{w_3} S_e = \mathbf{F}(U) \tag{6.8}$$

where $\mathbf{F}(U)$ is a friction (roughness) term — being responsible for the decay of the perturbation on the bed — and

$$c_{w_3} = \frac{g}{(1-p)} \frac{(\partial q_s/\partial U)}{(gh/U - U)} = \frac{1}{(1-p)} \frac{U(\partial q_s/\partial U)}{h(1-Fr^2)} \tag{6.9}$$

where $Fr^2 = U^2/gh$. For subcritical flow, when $Fr^2 << 1$, the following approximation is possible :

$$c_{w_3} \cong \frac{1}{(1-p)} \frac{U}{h} \frac{\partial q_s}{\partial U} \tag{6.9a}$$

If the solid discharge is expressed by a power law of the form :

$$q_s = a_s U^{b_s} \qquad \text{and} \qquad \frac{dq_s}{dU} = b_s \left(\frac{q_s}{U}\right) \tag{6.5a}$$

one may write :

$$c_{w_3} = \frac{1}{(1-p)} b_s \frac{q_s}{h} \tag{6.9b}$$

It is to be noted (see eq. 6.9a), that the celerity of propagation of the undulations c_{w_3}, on the bed is usually rather small compared to the average velocity, U, of the flow itself.

6.2.3 Analytical Solutions

1° To obtain analytical solutions to the equations of Saint-Venant - Exner, which are non-linear and hyperbolic, is a very difficult and often impossible task. However simplifications are nevertheless possible, if one assumes that for flow at small Froude numbers, $Fr < 0.6$, a *quasi-steadiness* is maintained. This hypothesis of a steadiness of flow can be justified : in general a variation of liquid discharge, $\partial(Uh)/\partial t$, is a short-term phenomenon, while a variation of the bed elevation, $\partial z/\partial t$, is a long-term phenomenon, which produces itself when the variation of the discharge has already disappeared ; thus the flow may be considered reasonably constant, $q = Uh = Cte$. Under such conditions solutions are of great interest, notably if one studies the variation of the bed, $z(x,t)$, as a long-term phenomenon.

2° Using the hypothesis of quasi-steadiness of the flow, a system of two differential equations can be written as :

$$\frac{\partial U}{\partial x}\left(U - g\frac{h}{U}\right) + g\frac{\partial z}{\partial x} = -\,g\,S_e \tag{6.7}$$

$$(1-p)\frac{\partial z}{\partial t} + \frac{\partial q_s}{\partial U}\frac{\partial U}{\partial x} = 0 \tag{6.4b}$$

These two equations are non-linear ones ; only numerical solutions are possible. For certain special cases, analytical solutions (after linearisation) can be of help to understand the problem, notably the relative importance of the different parameters.

3° If one further assumes (see *Vreugdenhil* et *de Vries*, 1973, p. 8) that the quasi-steady flow is also quasi-uniform, $\partial U/\partial x = 0$, the above equation, eq. 6.7, becomes :

$$0 + g\frac{\partial z}{\partial x} = -\,g\,S_e = -\,g\frac{U^2}{C^2 h} = -\,g\frac{U^3}{C^2 q} \tag{6.10}$$

where C is the coefficient of Chézy and $q = Uh$ is the unit discharge.

By eliminating $\partial U/\partial x$ after differentiation of eq. 6.10 with respect to x, one obtains for the above equation, eq. 6.4b :

$$\frac{\partial z}{\partial t} - K(t)\frac{\partial^2 z}{\partial x^2} = 0 \tag{6.11}$$

where the coefficient (of diffusion), $K(t)$, being a function of time, is given by :

$$K = \frac{1}{3}\,\frac{\partial q_s}{\partial U}\,\frac{1}{(1-p)}\,\frac{C^2 h}{U} \tag{6.12}$$

This model, eq. 6.11, is a *parabolic* one and is limited to large values of x and o namely for $x > 3h/S_e$ (see *de Vries*, 1973, p. 9). The expression for the coefficie eq. 6.12, can also be written (see *de Vries*, 1973, p. 6) in the following way :

$$K = \frac{1}{3} \frac{\partial q_s}{\partial U} \frac{1}{(1-p)} \frac{U}{S_{e_o}} \left(\frac{U_o}{U}\right)^2 \qquad (6.12$$

and upon linearisation (possible for $U \simeq U_o$), one obtains :

$$K \equiv K_o \approx \frac{1}{3} \frac{\partial q_s}{\partial U} \frac{1}{(1-p)} \frac{U_o}{S_{e_o}} \qquad (6.12$$

where the index, o , refers to the uniform (initial) condition. Using the power-l expression for the solid discharge, namely :

$$q_s = a_s U^{b_s} \qquad (6.5$$

one obtains :

$$K \cong \frac{1}{3} b_s q_s \frac{1}{(1-p)} \frac{1}{S_{e_o}} \qquad (6.12$$

The parabolic model — obtained by using the different important assumptions is of interest, since it allows to obtain analytical solutions in certain well-defin cases.

Depending on the applied mathematical techniques and on the hypothesis us some solutions — which often are rather similar — have been communicated *Vreugdenhil* et *de Vries* (1973, p. 9), *Ashida* et *Michiue* (1971), *Jaramillo* et *J* (1984), *Ribberink* et *Sande* (1985) and *Gill* (1987).

4° A *hyperbolic* model for quasi-steady flow, but being non-uniform, has be proposed (see *Vreugdenhil* et *de Vries*, 1973, p. 5 and *Exner*, 1925) :

$$\frac{\partial z}{\partial t} - K \frac{\partial^2 z}{\partial x^2} - \frac{K}{c_{w_3}} \frac{\partial^2 z}{\partial x \, \partial t} = 0 \qquad (6.1$$

where K and c_{w_3} are respectively given by eq. 6.12 and eq. 6.9. Solving t equation, one fixes K and c_{w_3} at their initial values, K_o and c_{w_3o}. However, sin an analytical solution to eq. 6.13 is rarely possible (see *de Vries*, 1985, *Ribberi* et *Sande*, 1985 and *Lenau* et *Hjelmfeld*, 1992), this model turns out to be not that useful.

5° A model of a *simple wave* (see *Vreugdenhil* et *de Vries* , 1973, and *Exner*, 1925) is obtained by reduction of eq. 6.8, or :

$$\frac{\partial z}{\partial t} + c_{w_3} \frac{\partial z}{\partial x} = 0 \tag{6.14}$$

where c_{w_3} is given by eq. 6.9. Since the friction term, $\mathbf{F}(U)$, is now neglected, the application of this model remains limited (see *Ribberink* et *Sande*, 1985) to small values of x and of t, namely for $x << 3h/S_{e_o}$.

6.2.4 Degradation and Aggradation

1° A *degradation* (or *aggradation*) in a reach of a watercourse is encountered if the entering solid discharge is smaller (or larger) than the capacity of the transport of sediments. The sediments of the bed will be eroded (or deposited) and as a consequence the elevation of the channel bed decreases (or increases).

Degradation (erosion) and aggradation (deposition) are long-term processes of the evolution of the channel bed, $z(x,t)$.

The flow, being steady and uniform at the beginning, will also be steady and uniform at the end of the process ; in between the flow becomes non-uniform and quasi-steady. If one assumes, that during this transition the flow can be considered as being quasi-uniform, $\partial U/\partial x = 0$, one may make use of the *parabolic model.*

2° The equation for this parabolic model was given as :

$$\frac{\partial z}{\partial t} - K \frac{\partial^2 z}{\partial x^2} = 0 \tag{6.11}$$

where K is a function of time ; the other variables (see eq. 6.12) must be kept constant to facilitate a resolution of eq. 6.11.

Note that this model is limited to large values of x and of t, namely for $x > 3h/S_e$, and for Froude numbers of $Fr < 0.6$.

3° Analytical solutions to the parabolic model, eq. 6.11, can be obtained for cases, where the initial and boundary conditions are well specified. Such solutions will not only clarify a physical problem, but can often be considered as a first tentative for an understanding of the problem. Caution is however necessary since this model was established using different assumptions.

4° The analytical solutions will certainly help to explain the long-term evolution of th bed of the channel, when the variation of the liquid discharge can be readil neglected. The following examples can be cited (see Fig. 6.4) :

Degradation :
- the supply of solid discharge is reduced (interrupted) at the upstream ;
- the liquid discharge is increased ;
- a lowering of a fixed point on the channel bed at the downstream.

Aggradation :
- the supply of solid discharge is increased at the upstream ;
- the liquid discharge is decreased ;
- a mounting of a fixed point on the channel bed at the downstream.

Some applications of the parabolic model will be presented in the following pages.

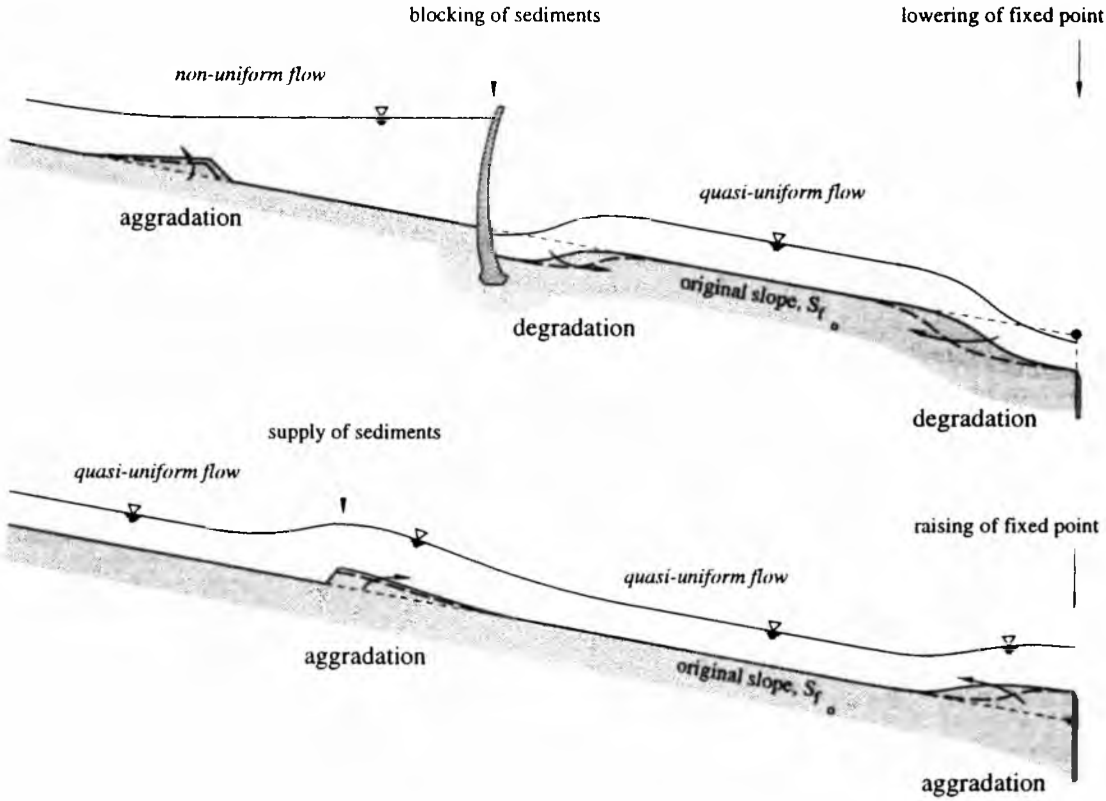


Fig. 6.4 Scheme of a degradation or an aggradation.

5° *Degrading channel* (see Fig. 6.5) :

i) Consider a channel with a mobile bed, having a uniform flow of constant unit discharge, q , at an initial time, t = 0 , and a flow depth of $h = h^o$. This discharge enters into a reservoir, whose water level is lowered by Δh_w, causing a local lowering of the fixed point of the bed of Δh. Consequently, a degradation of the bed is initiated and a long time after, $t = \infty$, one will notice throughout the reach of the channel a lowering of the bed and of the water surface ; the flow depth will then be again the initial depth, $h^o \equiv h^\infty$. During the period of degradation, t = t , the discharge, q , as well as the flow depth, h , remain quasi-constant.

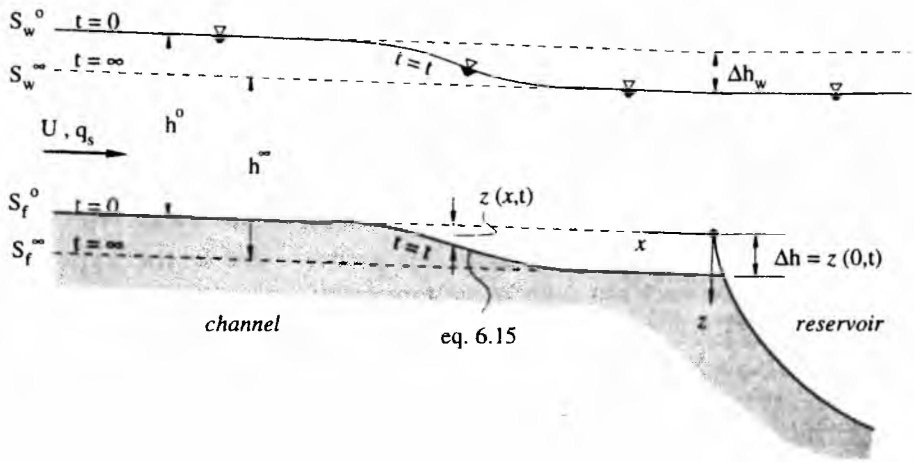


Fig. 6.5 Degradation due to lowering, Δh, of the fixed point on the bed.

ii) The flow is considered as being steady and quasi-uniform ; the use of the parabolic model, eq. 6.11, seems justified. Since the discharge stays constant, the coefficient K (see eq. 6.12) remains also constant.

iii) The problem has to be mathematically specified. The *x*-axis is put into the initial bed pointing upstream ; *z* stands now for the variation of the bed elevation with respect to the initial bed slope, S_f^o.

The initial and boundary conditions are :

$$z\,(x,0) = 0 \qquad ; \qquad \lim_{x \to \infty} z\,(x,t) = 0$$

$$z\,(0,t) = \Delta h$$

iv) The solution to eq. 6.11 — making use of Laplace transformations (see *Vreugdenhil* et *de Vries*, 1973, p. 9 and p. 11) — is :

$$z(x,t) = \Delta h \; erfc\left(\frac{x}{2\sqrt{Kt}}\right) \tag{6.15}$$

where the complementary error function is given as :

$$erfc\,(Y) = \frac{2}{\sqrt{\pi}} \int_{Y}^{\infty} e^{-\xi^2}\ d\xi \qquad (6.16)$$

which can be found in mathematical tables (see Ex.6.A).

v) One may now ask (see *de Vries*, 1973), after what time period, $t_{50\%}$, at a certain section, $x_{50\%}$, the bed elevation will be lowered by 50 % with respect to the final bed elevation, namely $z/\Delta h = 1/2$. Using eq. 6.15, one writes :

$$\frac{z(x,t)}{\Delta h} = \frac{1}{2} = erfc\left(\frac{x_{50\%}}{2\sqrt{Kt_{50\%}}}\right) = erfc\ (Y)$$

and, upon consulting the tables of the error function, one finds $Y \cong 0.48$, thus obtaining :

$$x_{50\%} = 0.48\,(2\sqrt{Kt_{50\%}}) \quad \text{where} \quad t_{50\%} \approx x^2_{50\%}\,/\,(0.96^2\,K) \qquad (6.17)$$

vi) It has been shown (see *Vreugdenhil* et *de Vries*, 1973, p. 11) that the parabolic model — which approaches itself to the hyperbolic one, being more correct but also more complex — is rather valuable, if the value of $(x^2/(2Kt)) < 1/2$ is small, or $x > 3h/S_{e_0}$, namely if the time, t, and the distance, x, are large and/or if the bed slope, S_{e_0}, is relatively large.

6° *Aggrading Channel* (see Fig. 6.6)

i) Consider a channel with a mobile bed, having a uniform flow. A particular cross section, till now in equilibrium, is overloaded, namely the supply of solid discharge, Δq_s, is increased (caused by an earth slide or by a mining operation). Aggradation on the channel bed will take place. Subsequently, after a lapse of time, Δt, the elevation of the bed as well as the water surface will increase by Δh. During the period of aggradation, t = t, the discharge, q, will remain essentially constant.

ii) The flow is considered as being steady and quasi-uniform ; the use of the parabolic model, eq. 6.11, seems justified. Since the discharge remains constant, the coefficient K (see eq. 6.12) stays also constant.

iii) The problem has to be mathematically specified. The x-axis is put into the initial bed, being positive towards the downstream ; z stands now for the variation of the bed elevation with respect to the initial bed slope, S_f^o.

The initial and boundary conditions are :

$$z\,(x,0) = 0 \quad ; \quad \lim_{x \to \infty} z\,(x,t) = 0$$

$$z\,(0,t) = \Delta h(t)$$

iv) The solution to eq. 6.11 is :

$$z(x,t) = \Delta h(t)\ erfc\left(\frac{x}{2\sqrt{Kt}}\right) \qquad (6.15a)$$

Evidently this solution is of the same type as the one in the preceding problem ; however now $\Delta h(t)$ is a function of time, to be determined. The coefficient $K \equiv K_0$, eq. 6.12, must be evaluated for the initial situation, thus not taking into account the overloading, Δq_s.

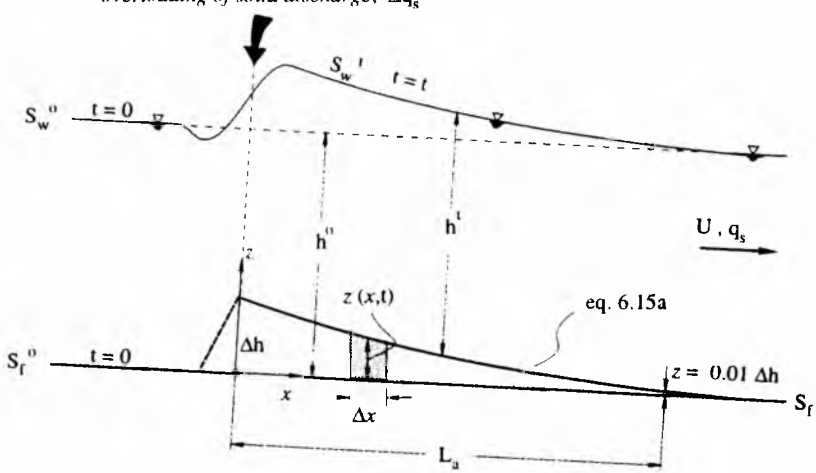


Fig. 6.6 Aggradation by overloading the supply of solid discharge, Δq_s .

v) It is necessary to define the length of the zone of aggradation, L_a, taken as being the one corresponding to a deposition of $z/\Delta h = 0.01$ ($Y \approx 1.80$); according to eq. 6.15a, one writes :

$$L_a = x_{1\%} \cong 3.65\sqrt{Kt_{1\%}} \qquad (6.19)$$

The volume of the supply of solid discharge, Δq_s, during a certain time, Δt, is given by $\Delta q_s \cdot \Delta t$; this quantity is distributed over the bed of the channel (see Fig. 6.6) as follows :

$$\Delta q_s \cdot \Delta t = (1 - p) \int_0^{L_a} z\ dx \qquad (6.18)$$

Subsequently, eq. 6.15a can be used to calculate (see *Soni* et *al.*, 1980, p. 122) the thickness of the layer, Δh , due to the aggradation :

$$\Delta h(t) = \frac{\Delta q_s \cdot \Delta t}{1.13\ (1-p)\ \sqrt{K \Delta t}} \tag{6.20}$$

where it becomes evident that Δh increases with time, t.

vi) Agreement of experimental work (in the laboratory) with eq. 6.15a has been communicated (see *Soni* et *al.*, 1980); but the coefficient of aggradation, K, had to be slightly adjusted. Also, it was remarked, that the parabolic model, eq. 6.11, remains valid over the entire region ; thus not limited to $x > 3h/S_f$.

7° Computation of a degradation and of an aggradation, such as was discussed in the present section, is only possible if the conditions assumed for a parabolic model are well fulfilled, namely :

i) quasi-steadiness of the flow (long-term variation of the bed) ;
ii) quasi-uniformity of the flow at Fr < 0.6 ;
iii) validity for $x > 3h/S_e$.

If these conditions (hypothesis) are not fulfilled, it is obviously necessary to solve the equations of Saint-Venant - Exner using numerical methods.

6.2.5 Numerical Solutions

1° Analytical solutions of the equations of Saint-Venant - Exner are only possible when the hypothesis of quasi-steadiness of the flow is justified. Furthermore, it is often necessary to assume also quasi-uniformity of the flow. However these assumptions are *no* more possible, if the temporal variation of the discharge, $\partial(Uh)/\partial t$, and the one of the elevation of the bed, $\partial z/\partial t$, are of the same order of magnitude, namely relatively rapid.

2° If the flow is unsteady and non-uniform (see chap. 5) or steady and non-uniform (see chap. 4), no analytical solutions, which are reasonably simple, are available.

The system of the equations of Saint-Venant - Exner, eq. 6.1b, eq. 6.2 and eq. 6.4a, together with eq. 6.3 and eq. 6.5, can be resolved — without making too severe assumptions — by numerical methods ; this may be well achieved with the use of computers.

3° The numerical methods are essentially the same which are used to solve the equations of Saint-Venant, namely for flow over a *fixed* bed (see sect. 5.2). They become however rather complicated, if they are applied for the modelisation of flow over *mobile* bed.

The *implicit* methods (see sect. 5.2.4) using finite differences are the ones which are at the present frequently used to solve the equations of Saint-Venant - Exner.

4° Here we shall only give reference to a selection of the existing literature, which employ numerical methods for the solutions of the equations of Saint-Venant - Exner for steady and unsteady flow over mobile bed : *Chen* et *al.* (1975) and *Cunge* et *al.* (1980, p. 271); *Yucel* et *Graf* (1971), *Krishnappan* (1981), *Holly* et *Rahuel* (1990) and *Correia* et *al.* (1992).

6.3 BED-LOAD TRANSPORT

6.3.1 Notions

1° Transport as bed-load is the mode of transport of sediments (see Fig. 6.1) where the solid particles glide, roll or (briefly) jump, but stay very close to the bed, $0 < z < z_{sb}$, which they may leave only temporarily. The displacement of the particles is intermittent ; the random concept of the turbulence plays an important role.

2° There exist a number of formulae, which can be used for the prediction of the bed-load transport (see *Graf*, 1971, chap. 7, *Yalin*, 1972, chap. 5, and *Raudkivi*,1976, chap. 7).

3° Many of theses formulae are of empirical nature, but often have incorporated dimensionless numbers. This allows to make experiments in the laboratory, where the hydraulic conditions can be well controlled ; subsequently it is possible to use such formulae for field conditions.

6.3.2 Theoretical Considerations

1° Considered will be that the bed of a channel (see Fig. 6.1) is plane but mobile, composed of solid particles of uniform size and being non-cohesive. These particles displace themselves under the action of the flow, which be uniform and steady.

For such simplified conditions — bed forms (see sect. 3.2.5) may form, the granulometric distribution may be non-uniform (see sect. 6.3.4) and cohesion may exist —, one tries to obtain functional relations, such as the ones given by eq. 3.40 and eq. 6.29. The form of such functions, being often rather complex, will be established by experiments, which more or less will take care of the reality of the problem.

2° The forces, which enter (see *Graf*, 19' 1, chap. 6) into the description of the uniform and steady motion of a single pa: cle, isolated and without cohesion, are :

- the hydrodynamic force : $F_H \propto f(\frac{u_* d}{\nu})\, \rho\, d^2\, u_*^{\,2}$

- the submerged weight of the particle : $W_p \propto g\,(\rho_s - \rho)\, d^3$

where u_* is the friction velocity, considered as being proportional to the velocity the particle.

3° The components of this two-phase flow are :

- the *fluid*, by its density, ρ , and its viscosity, ν ;
- The *solid material* , by its density, ρ_s , and a characteristic diameter, d ;
- the *flow*, by its flow depth, h or R_h, the slope, S_f, and the gravity, g ; th by the friction velocity, $u_* = \sqrt{\rho g R_h S_f}$, which characterises the turbulen (see sect. 2.6.4).

In all, there are thus 7 parameters.

4° A dimensionless analysis, using the Π-theorem (see *Yalin*, 1972, p. 61), shov that the arguments which quantify the two-phase flow, such as the bed-loa transport, can now be expressed by 4 dimensionless quantities, namely :

- a Reynolds number of the particle :

$$Re_* = \frac{u_* d}{\nu} \tag{6.21}$$

- a dimensionless shear stress (see eq. 3.38) :

$$\tau_* = \frac{\rho\, u_*^{\,2}}{(\gamma_s - \gamma) d} = \frac{\tau_0}{(\gamma_s - \gamma) d} = \frac{\gamma\, R_h\, S_f}{(\gamma_s - \gamma) d} \tag{6.22}$$

or a densimetric Froude number of the particle :

$$Fr_{*D} = \frac{u_*}{\sqrt{(s_s - 1)\, gd}} = \frac{\sqrt{\tau_0}}{\sqrt{(\gamma_s - \gamma) d}} = \sqrt{\tau_*} \tag{6.23}$$

- a relative depth :

$$\frac{h}{d} \text{ ou } \frac{R_h}{d} \tag{6.24}$$

- a relative density :

$$s_s = \frac{\rho_s}{\rho} \tag{6.25}$$

- In addition, a dimensionless particle diameter can be obtained by combining eq. 6.21 with eq. 6.22 (see point 3.4.2.7°), or :

$$d_* = d \left((s_s - 1) \frac{g}{\nu^2} \right)^{1/3} \tag{6.26}$$

5° Combining eq. 6.22 and eq. 6.21, a relation was proposed by *Shields* (see sect. 3.4.2 and Fig. 3.13), such as :

$$\tau_* = f(Re_*) \qquad \text{or} \qquad \tau_* = f(d_*) \tag{3.40}$$

for the study of the commencement of erosion, expressed by the dimensionless shear stress, τ_{*cr}. Furthermore, a relation of the form :

$$\tau_{*cr} = f(Re_*) \tag{6.27}$$

gives a delimitation of the zone of "motion" from the zone of "no motion" of the particles ; this was developed experimentally from laboratory date, showing a rather large spread. The function of Shields, eq. 6.27, is generally agreed upon as being valuable and useful, notably for the hydraulic engineers, if the granulometry is uniform or almost so.

6° The transport of sediments can be expressed as a function of these 4 dimensionless quantities, namely :

$$\Phi = f(d_*, \tau_*, R_h/d, \rho_s/\rho)$$

Utilising the Π-theorem, one obtains (see *Yalin*, 1972, p. 67) an expression for a dimensionless *intensity of the solid discharge* as the bed load, or :

$$\Phi \equiv q_{sb*} = \frac{q_{sb}}{\sqrt{(s_s - 1) g d^3}} \tag{6.28}$$

with q_{sb} [m²/s] as the volumic solid discharge per unit width.

Expressions, which are similar to the one of eq. 6.28, can be written (see *Yalin*, 1972, p. 65) as :

$$\Phi' = \frac{q_{sb}}{u_* d} \qquad \text{or} \qquad \Phi'' = \frac{q_{sb}}{Ud} \tag{6.28a}$$

Since some terms, R_h/d and ρ_s/ρ, are included in the term of τ_*, and taking $\tau_* = f(Re_*)$, one can formulate now a rather simple relationship :

$$\Phi = f(\tau_*) \qquad \text{or} \qquad \frac{q_{sb}}{\sqrt{(s_s-1)gd^3}} = f\left(\frac{\tau_o}{(\gamma_s-\gamma)d}\right) \qquad (6.2$$

which is often written as :

$$\Phi = f(\Psi) \qquad (6.29$$

where $\tau_* \equiv \Psi^{-1}$ and Ψ is called the dimensionless *intensity of shear stre:* applied upon the solid particles.

This expression, eq. 6.29, links the solid transport, q_{sb} (see eq. 6.28), to the she stress, τ_* (see eq. 6.22). Thus an increase in τ_* — passing by $\tau_{*_{cr}}$, where erosi begins — is responsible for an increase in q_{sb}.

The form of this function, eq. 6.29, must still be established ; it is given by t formulae of bed-load transport, which are established by experiments performed the laboratory and in the field.

7° One often assumes that this relation, eq. 6.29, can be expressed in form of a pow law, or :

$$\Phi = \alpha(\tau_*)^{\beta} \qquad (6.3$$

Making use of the ratio, which defines the coefficient of friction :

$$\frac{U}{\sqrt{\tau_o/\rho}} = \sqrt{\frac{8}{f}} \qquad (2.5$$

one can formulate the following proportionalities :

$$U^2 \propto \tau_o \propto \tau_*$$

Thus it is possible to express the above equatioin, eq. 6.29, by an approxim: relation (see *de Vries*, 1973) in the form of :

$$q_{sb} = a_s U^{b_s} \qquad (6.5$$

where a_s , α and $b_s = 2\beta$, β are the coefficients which depend essentially on t granulometry. This simple, but often useful relation, shows that the avera velocity, U , of the flow is the predominant parameter for the determination of t solid discharge, q_{sb}.

6.3.3 Bed-load Relations

1° At the present, the formulae for a determination of the solid discharge as bed load give only reasonably satisfying results within a domain of the parameters for which the chosen formula has been established. Consequently, the application and use of such formulae has to be done with great care.

Here will be given a selection (in chronological order) of some of the many available formulae ; their most characteristic hydraulic aspects will be pointed out.

2° From the different empirical formulae, proposed by *Schoklitsch* in 1934 and 1950 (see *Graf,* 1971, p. 133), the last one is presented, namely :

$$q_{sb} = \frac{2.5}{s_s} S_e^{3/2} (q - q_{cr}) \tag{6.31}$$

The critical liquid discharge, q_{cr}, characterises the commencement of erosion — usually expressed by τ_{cr} — ; it is given with the use of the Manning-Strickler formula, eq. 3.16 and eq. 3.18, such as :

$$q_{cr} = 0.26 \, (s_s-1)^{5/3} \frac{d^{3/2}}{S_e^{7/6}} \tag{6.31a}$$

valid for $d \geq 0.006$ [m] (see *Bathurst* et *al.*, 1987); for a non-uniform granulometric mixture one takes $d = d_{40}$, as the equivalent diameter.

This relation, eq. 6.31, is applicable for larger grain sizes, $d \geq 6$ [mm], being rather uniform and for bed slopes being moderate to strong (see Table 6.3).

3° From the different empirical formulae — using the condition of similitude of Froude — which *Meyer-Peter* et *al.* have developed in 1934 and 1948 (see *Graf,* 1971, p. 136 or *Yalin,* 1972, p. 112), the last one is presented, namely :

$$0.25 \, \rho^{1/3} \frac{(g_{sb}')^{2/3}}{(\gamma_s-\gamma)d} = \frac{\gamma R_{hb} \xi_M S_e}{(\gamma_s-\gamma)d} - 0.047 \tag{6.32}$$

where $g_{sb}' = g_{sb} (\gamma_s-\gamma)/\gamma_s$ is the solid discharge in weight under water and $g_{sb}/\gamma_s = q_{sb}$; R_{hb} is the hydraulic radius of the bed. For a non-uniform granulometry, the mean diameter, $d = d_{50}$, is taken as the equivalent diameter.

This relation can be written in the dimensionless form (see eq. 6.29) such as :

$$\Phi = 8 \, (\xi_M \tau_* - \tau_{*cr})^{3/2} \tag{6.32a}$$

where τ_{*cr} is the dimensionless critical shear stress (see eq. 6.27 and Fig. 3.13) ξ_M is a roughness parameter, given by :

$$\xi_M = \left(\frac{K_S}{K_S'}\right)^{3/2}$$

where K_S' is the roughness of the granulates, to be evaluated with the formula of Strickler, eq. 3.18, and K_S is the total roughness of the bed, evaluated with the formula of Manning-Strickler, eq. 3.16, or :

$$K_S = \frac{U}{R_{hb}^{2/3}\, S_e^{1/2}} \qquad \text{and} \qquad K_S' = \frac{26}{d_{90}^{1/6}}$$

In the absence of bed forms it is recommended to take $\xi_M = 1$; but $1 > \xi_M > 0.35$ if bed forms are present.

This relation, eq. 6.32, is applicable for rather large grain sizes, d> 2 [mm], being uniform as well as non-uniform, and for bed slopes, being moderate to strong (see Table 6.3).

4° Using extensively the concepts of hydrodynamics, *Einstein* has developed in 1942 and 1950 (see *Graf*, 1971, pp. 139-150) a probabilistic model for the transport of sediments as bed load.

i) *Determination of the probability of erosion*

The probability, p_e, of erosion of a particle at any time instant depends (see *Einstein*, 1950, p. 35) on the hydrodynamic force, F_H, here the lift force and the submerged weight of the particle, W_P :

$$p_e = f\left[\frac{W_P}{F_H}\right] = f\left[\frac{k_2\, g(\rho_s-\rho)\, d^3}{1/2\; k_1 C_L (1+\eta)\, \rho\, d^2\, (5.75^2 u_*'^2 \beta_x^{\,2})}\right] =$$

$$= f\left[\frac{1}{(1+\eta)}\,(B\beta_x^{-2}\Psi')\right] \tag{6.33}$$

k_1 and k_2 are form factors of the particle ; $C_L = 0.178$ is a lift coefficient and η is a random variable of lift, where $\eta_o = 0.5$ is the most probable value. According to eq. 6.29a and eq. 6.22, the intensity of shear stress, Ψ', is defined by :

$$\Psi' = \frac{(\gamma_s-\gamma)\,d}{\rho u_*'^2} = \frac{(\gamma_s-\gamma)}{\gamma}\frac{d}{R_{hb}'S_f} = \frac{1}{\tau_*'} \tag{6.34}$$

where R_{hb}' is the hydraulic radius of the bed due to the granulate (see sect. 3.2.5); for a non-uniform granulometry, $d = d_{35}$ is taken. $B = 2k_2/(5.75^2 C_L k_1)$ is a numerical constant and β_x is a relation which takes in account the logarithmic velocity distribution as well as the roughness, $k_s = d_{65}$. The following expression (see *Einstein*, 1950, p. 36), where $d = X$ is a characteristic diameter of the granulometry, was given :

$$\beta_x = \log(10.6\, X/\Delta) \tag{6.35a}$$

with $$X \begin{cases} = 0.77\Delta & \text{if} \quad \Delta/\delta > 1.80 \\ = 1.39\delta & \text{if} \quad \Delta/\delta < 1.80 \end{cases}$$

where $\Delta = f(k_s/\delta)$ according to Fig. 6.7a

$$\delta = 11.5\, \nu/u'_*$$

The above functional relation, eq. 6.33 , is valid for a uniform granulometry, but can be generalised for a non-uniform one, in the following way :

$$p_e = f\left[\zeta_H\, \zeta_P\, \frac{1}{(1+\eta)}\, (B\beta_x^{-2}\,\Psi')\right] \tag{6.35}$$

ζ_H is a hiding coefficient — the smaller particles hide behind the larger ones — ; it was obtained experimentally (see Fig. 6.7b). ζ_P is a lift-force correction coefficient, also obtained experimentally (see Fig. 6.7c). This expression, eq. 6.35, can also be written (see *Einstein*, 1950, p. 37) as :

$$p_e = f(B_*\Psi_*)$$

where Ψ_* is the intensity of shear stress after Einstein :

$$\Psi_* = \zeta_H\, \zeta_P\, (\beta^2/\beta_x^2)\, \Psi' \tag{6.36}$$

and B_* is a constant to be determined experimentally (see eq. 6.42b) :

$$B_* = \frac{B}{\beta^2 \eta_o} \qquad \text{with} \qquad \beta = \log(10.6) \tag{6.36a}$$

Note, that for a uniform granulometry, one takes (see *Einstein*, 195(p. 36) $\zeta_H = 1$, $\zeta_P = 1$ and $(\beta^2/\beta_x^2) = 1$; thus one writes :

$$\Psi_* = \Psi' \tag{6.36b}$$

In order to express the probability of motion, *Einstein* (1950, p. 37 postulated the following function, being rather similar to the normal function or :

$$p_e = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_*\Psi_* - 1/\eta_0}^{+B_*\Psi_* - 1/\eta_0} e^{-\xi^2} d\xi \tag{6.37}$$

where ξ is a variable of integration.

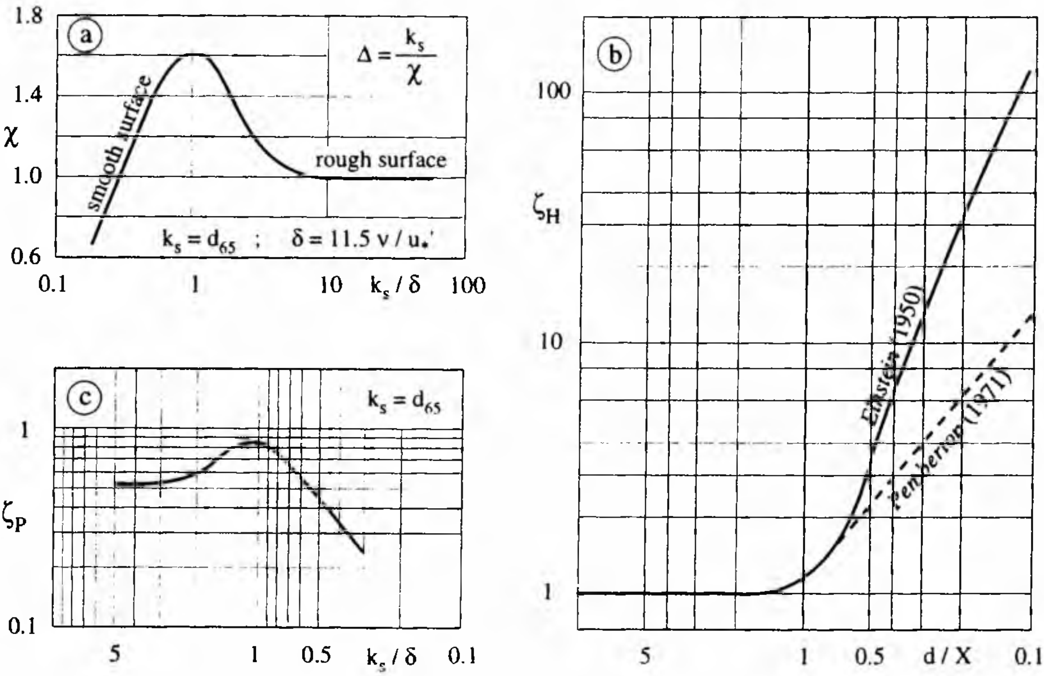


Fig. 6.7 Correction coefficients : (a) of velocity distribution, (b) of hiding and (c) of lift force (see *Graf*, 1971, p. 146).

ii) *Equation of bed load*

The number of particles, which are *deposited* per unity of time and of bed surface, $A_L d \cdot 1$, is given by :

$$N_D = \frac{g_{sb}\, i_{sb}}{(A_L d)(\gamma_s k_2 d^3)} \tag{6.38}$$

where $(\gamma_s k_2 d^3)$ is the weight of a particle and A_L is a constant. i_{sb} is a fraction (see Fig. 6.9) of the granulometric curve of the unit solid discharge, g_{sb} , in weight.

The number of particles, which are *eroded* per unity of time and of bed surface, is given by :

$$N_E = \frac{i_b}{k_1 d^2}\,(p_e/t_e) \tag{6.39}$$

where $(i_b/k_1 d^2)$ is the number of particles in a unit bed surface ; i_b is a fraction (see Fig. 6.9) of the granulometric curve of the bed material. p_e is the probability of erosion of a particle ; the exchange time, t_e , necessary for the replacement of a particle of the bed by another particle, is expressed by :

$$t_e \propto \frac{d}{v_{ss}} = k_3 \sqrt{\frac{\rho d}{g(\rho_s - \rho)}}$$

where v_{ss} is the settling velocity of the particle.

The *equation of bed load* after *Einstein* (1950) postulates that the rate of erosion, eq. 6.39, is equal to the rate of deposition, eq. 6.38 ; thus one takes $N_D = N_E$ and consequently :

$$\frac{g_{sb} i_{sb}}{(A_L d)(\gamma_s k_2 d^3)} = \frac{i_b p_e}{k_1 k_3 d^2} \sqrt{\frac{g(\rho_s - \rho)}{\rho d}} \tag{6.40}$$

Furthermore, one admits that a solid particle displaces itself by making jumps of a length of $A_L d$ (see eq. 6.38), which are linked to the exchange probability (see *Einstein*, 1950, p. 34) in the following way :

$$A_L d = \lambda d\,(\frac{1}{1 - p_e})$$

where λ is a constant of the jump of the particles. Introducing this expression into the above relation, eq. 6.40, yields :

$$(\frac{p_e}{1 - p_e}) = A_*\,(\frac{i_{sb}}{i_b})\,\Phi = A_* \Phi_* \tag{6.41}$$

Φ is the intensity of transport aft r Einstein, given by :

$$\Phi = \frac{q_{sb}}{\sqrt{(s_s-1)\, g\, d^3}} \qquad (6.28$$

where $q_{sb} = g_{sb}/\gamma_s$ is the volumic solid discharge per unit width and A_* is a empirical constant to be determined experimentally (see eq. 6.42b). For uniform granulometry, one takes (see *Einstein*, 1950, p. 36) simply :

$$\Phi_* = \Phi$$

The above relations, eq. 6.37 and eq.6.41 put together, give now the fin: form of the *equation of bed load of Einstein* (1950):

$$p_e = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_*\Psi_*-1/\eta_o}^{+B_*\Psi_*-1/\eta_o} e^{-\xi^2} d\xi = \frac{A_*\Phi_*}{1+A_*\Phi_*} \qquad (6.42$$

namely a functional relation (see eq. 6.29a), such as :

$$\Phi_* = f(\Psi_*) \qquad (6.42a$$

The (universal) constants have now to be determined experimentally both fc uniform and non-uniform granulometries (see *Einstein*, 1950, pp. 3 and 43) ; they are given (see *Graf*, 1971, p. 149) as being :

$$A_* = 43.6 \quad ; \quad B_* = 0.143 \quad ; \quad \eta_o = 0.5 \qquad (6.42b$$

This relation, eq. 6.42, is plotted in Fig. 6.8 — using the tables of the erroı function — together with the data of *Meyer-Peter* et *al.* and *Gilbert*. Th graphical representation facilitates the use of the above relation, eq. 6.42.

Since a non-uniform granulometry can be broken down into its fraction: i_{sb}/i_b , this relation is rather flexible. For a quasi-uniform granulometry, a equivalent diameter of $d = d_{35}$ can be taken.

It is interesting to remark, that the relation of *Einstein*, eq. 6.42, and the on of *Meyer-Peter* et *al.*, eq. 6.32, give rather similar results (see *Graf*, 1971, ꞩ 150), and this notably for $\Phi < 10$. Note also, that in the relation of *Einstei* the notion of a critical value (for erosion), $\Psi_{cr} = \tau_{*cr}^{-1}$, has nowhere bee used explicitly. Nevertheless, one may ask now, what numerical value for Ч one would get, if the value of Φ becomes very small ; for example :

$$\Phi \cong 0.0004 \quad \Rightarrow \quad \Psi \cong 25 \quad \Rightarrow \quad \tau_{*cr} \cong 0.04$$

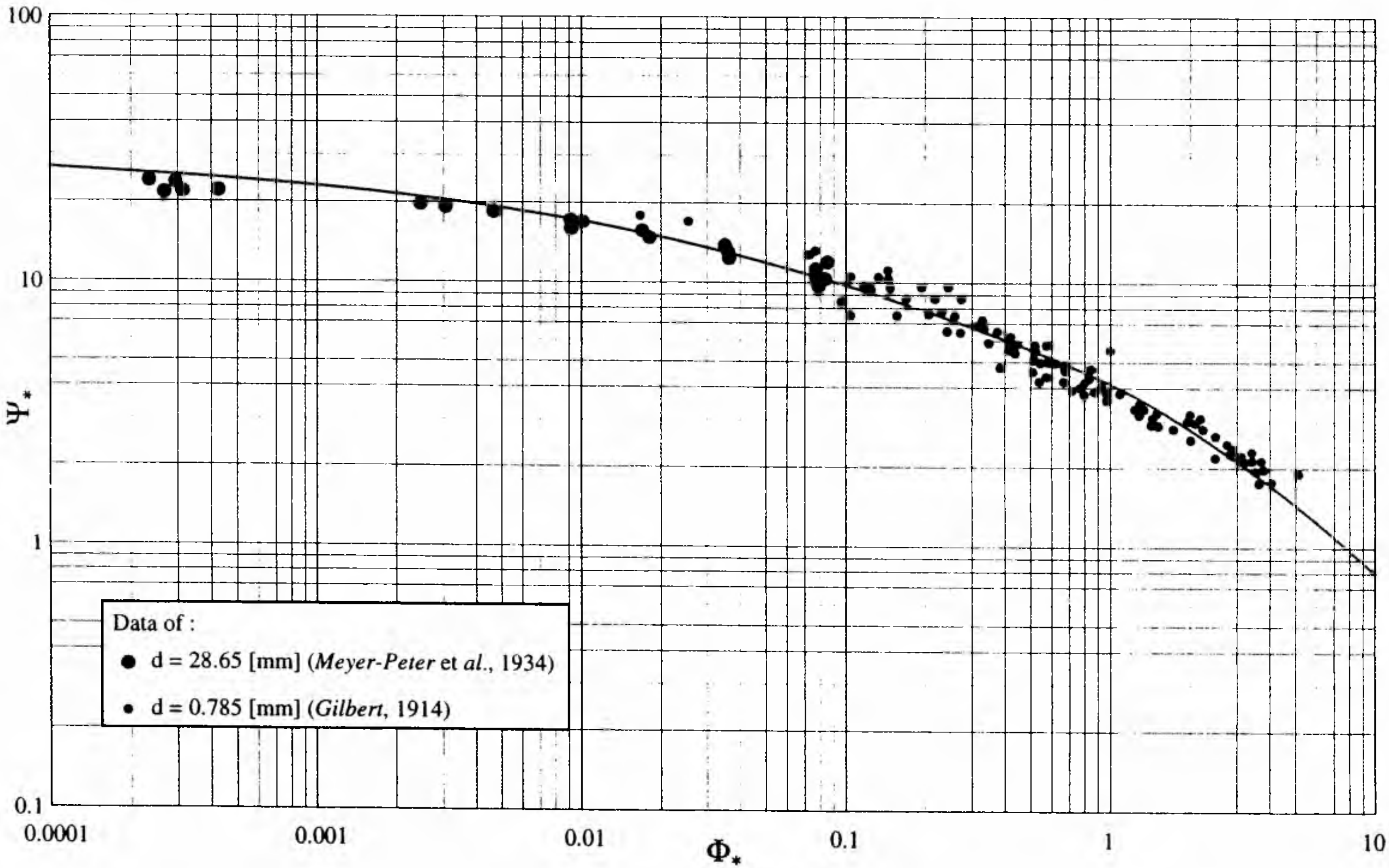


Fig. 6.8 Equation of bed load, $\Phi_* = f(\Psi_*)$, of Einstein (see *Graf,* 1971, p. 148).

Here one sees that there exists a rather good agreement with the critical valı τ_{*cr}, taken from the diagram of Shields (see Fig. 3.13).

The equation of Einstein, eq. 6.42 and Fig. 6.8, is well suitable for unifoı and non-uniform granulates over a large range of diameters, d > 0.7 [mr and of bed slopes (see Table 6.3). It is world-wide used with great success

6.3.4 Granulometry, Armouring

1° The non-cohesive sediments (solid particles), which make up the bed of watercourse, are in general of different sizes, being given by the granulomet curve of the bed material (see Fig. 6.9a).

This curve, which is in general half-logarithmic, can be divided into fractio (percentages), i_{b_i}, whose sum is :

$$1(100\ \%) = \Sigma\, i_{b_i} = i_{b_1} + i_{b_2} + \ldots + (5\ \% + 5\ \%)$$

Usually this curve is partitioned into 4 or 5 (unequal) fractions, after havi eliminated small fractions, namely ≈ 5 %, of the finest particles — they are part the wash load — and of the coarsest particles.

For each fraction the average diameter, d_i, is determined and the correspondi solid discharge, $i_{sb_i} q_{sb_i}$, is calculated, using one of the formulae for bed-lo transport. For the entire granulometric mixture, the solid discharge is now obtain as :

$$q_{sb} = \Sigma\, i_{sb_i}\, q_{sb_i} = i_{sb_1}\, q_{sb_1} + i_{sb_2}\, q_{sb_2} + \ldots$$

2° The granulometric curve of the bed material is in general different from the one the material moving as bed load or as suspended load (see Fig. 6.9l Consequently, for an average diameter of the granulate, d_i, the given fraction of t granulometric curve of the bed material, i_{b_i}, will be different from t corresponding fraction of the granulometric curve of the solid discharge, i_{sb_i}.

This subtlety was elaborated by *Einstein* (1950, p. 32) by introducing the ratio i_{sb}/i_b and the hiding factor, ζ_H, into the equation of bed-load transport, eq. 6.42.

For a very intensive sediment transport, all sizes (fractions) of particles will readi participate ; consequently $i_{b_i} \equiv i_{sb_i}$, since the curves L and C become identical (s Fig. 6.9b).

3° For cohesive material, the determination of the solid discharge represents a ve difficult task ; literature specialised on this topic should be consulted (see *Grı* 1971, chap. 12 and *Raudkivi*, 1976, chap. 9).

4° The granulometric curve of the bed material is obtained by taking samples from the bed of the channel. Recommended is (see *Einstein*, 1950, p. 48) to take many samples at different sections of the watercourse under study, and obtain an average granulometric curve. Each sample should be taken at (up to) a, to-be expected, maximal erosion depth, namely at a depth of 0.70 [m] below the bed surface.

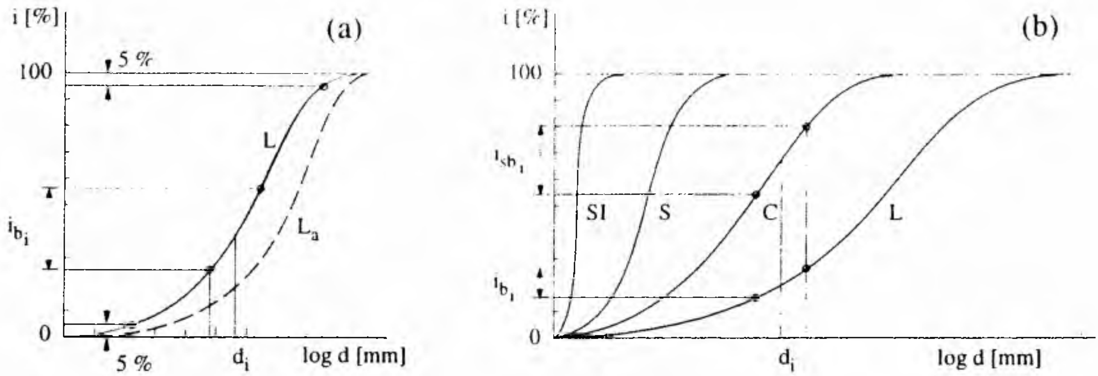


Fig. 6.9 Scheme of granulometric curves for the bed material, L , and the armoured bed material, L_a , the bed-load material, C , and the suspended (wash) load material, S (SI).

5° On a channel bed of non-uniform granulometry, the smaller particles are more easily eroded than the larger ones. a grain-size sorting takes place. An accumulation of the remaining larger particles results in an *armouring* of the bed, which subsequently protects the underlying "original" granulate (see *Graf*, 1971, p. 102).

It can thus happen, that erosion does not take place, if the bed becomes (naturally) *armoured* with the larger particles which remain at the bed surface after an important erosion process during a previous flood event. In such a case the flow cannot take its (full) capacity of sediments transport (see point 6.1.4.3°), and this until another exceptional flood will destroy the armour layer and the original granulate reappears to form once more another new armour layer.

The formulae developed for the capacity of transport are thus only valid for such watercourses, which pass through their own alluvium, namely in a bed being made up of material, which was also transported and can again be transported.

From above it becomes evident that granulometric samples taken in situ — if armouring takes place — have to be interpreted with great caution.

The development of an armour layer is an asymptotic process. When the friction velocity, u_* , increases, the smaller particles are eroded and the larger ones stay in place. The corresponding friction velocity is used to define the critical friction velocity for armouring, $u_{*a,cr}$. The (*maximum* possible) armoured bed will now be formed by the largest particles, d_{90} or larger, which are found in the granulometry of the original bed. For high discharges, when $u_* > u_{*a,cr}$, the armoured bed becomes unstable and will be destroyed. A bed, being composed of the sizes of the granulometric curve of the original composition, arrives at the surface and a extremely active erosion will take place.

There exists only limited conclusive information about the ratio of the origin. granulometry, d , and the granulometry of the armour layer, d_a. For some Swi: rivers, having large bed slopes, $S_f > 0.03$ [-], and large grain sizes, $d_{50} > 6$ [mm an indicative relationship of :

$$d_{50_a}/d_{50} \approx 1.4 \qquad ; \qquad d_{50_a}/d_{90} \leq 0.6$$

was developed by *Correia* et *Graf* (1988).

An empirical relationship for a prediction of the stability of the armour layer w: given by *Raudkivi* (1990, p. 113) as :

$$\tau_{*a,cr} = \frac{(u_{*a,cr})^2}{(s_s-1)g\, d_{50_{a,max}}} = \tau_{*cr}\left[0.4\,\left(\frac{d_{50}}{d_{50_{a,max}}}\right)^{1/2} + 0.6\right]^2$$

$$\text{with} \qquad d_{50_{a,max}} \lesssim 0.55\, d_{100}$$

where $d_{50_{a,max}}$ is the median diameter of the (maximal possible) armour and τ_{*cr} the dimensionless critical shear stress, taken as $\tau_{*cr} \approx 0.05$ (see Fig. 3.13 Evidently the armouring process is controlled by the largest fraction, d_{90} or large of the granulometric curve of the channel bed. No armouring takes place, if tl granulometry is uniform.

6.4 SUSPENDED-LOAD TRANSPORT

6.4.1 Notions

1° Transport of sediments in suspension is the mode of transport where the sol particles displace themselves by making large jumps, but remain (occasionally) contact with the bed load and also with the bed. The zone of suspension is delimit by : $z_{sb} < z < h$ (see Fig. 6.1).

2° Transport as suspended load could be considered as an advanced stage of transpc as bed load ; however the analytical methods do not allow a description of the two modes of transport with the same (or single) relationship.

6.4.2 Theoretical Considerations

1° The transport of sediments in suspension can be explained with the concept diffusion-convection, which gives the vertical distribution of the (loca concentration, $c_s(z)$, of the suspended particles.

2° For steady uniform flow, the vertical distribution of the concentration of the suspended particles, $c_s(z)$, in the fluid, can be obtained by using the equation of one-dimensional diffusion-convection (see sect. 8.4 or *Graf*, 1971, p. 166) :

$$0 = v_{ss} \frac{\partial c_s}{\partial z} + \frac{\partial}{\partial z} \left(\varepsilon_s \frac{\partial c_s}{\partial z} \right) \tag{6.43}$$

where $c_s(z)$ is the local volumic concentration, ε_s is the diffusivity of the suspended particles, whose units are $[L^2/T]$, and v_{ss} is the settling velocity of the particles.

This equation, eq. 6.43, relates the vertical exchange of solid particles due to the turbulence (upwards) with the gravitational motion (downwards), expressed with the settling velocity, v_{ss} ; it is valid only for weak concentrations, namely for $(1 - c_s) \cong 1$ or $c_s < 0.1$ [%].

3° Integration of the above equation, eq. 6.43 , yields:

$$v_{ss}\, c_s + \varepsilon_s \frac{dc_s}{dz} = \text{Cte} = 0 \tag{6.44}$$

where the constant of integration is taken to be Cte = 0 , implying that $c_s = 0$ at the water surface for $\varepsilon_s = 0$.

The above equation expresses that, at all levels, $z_{sb} < z < h$, there is a (vertical) equilibrium between the movement in the direction of gravity and the one due to the concentration gradient in the direction against gravity. In other words, the rate of sedimentation of particles per unit volume is equal to the rate of turbulent diffusion per unit volume.

For not so weak concentrations (see *Graf*, 1971, p. 185) the above equation, eq. 6.44, should be written as :

$$v_{ss}\, c_s\,(1 - c_s) + \varepsilon_s \frac{dc_s}{dz} = 0 \tag{6.44a}$$

4° The following remarks, concerning the diffusivity, should be made :

A relation between the diffusivity of suspended particles in the fluid, ε_s , and the turbulent diffusivity of a (soluble) substance in the fluid, ε_t , is in general admitted (see *Graf*, 1971, pp. 167 and 177) :

$$\varepsilon_s = \beta_s \varepsilon_t \tag{6.45}$$

where β_s is a factor of proportionality. For fine particles, which follow readily th fluid motion, one takes $\beta_s = 1$; for larger particles, one takes $\beta_s \leq 1$. Som researchers (see *Graf*, 1971, p. 178 and *Raudkivi*, 1990, p. 172) advance arguments to show that $\beta_s \geq 1$.

For weak concentrations it is usually assumed that :

$$\varepsilon_s \approx \varepsilon_t \tag{6.46}$$

thus one takes $\beta_s = 1$.

Furthermore, one may also postulate (see sect. 8.1.3) that :

- diffusion (per unity of surface) of matter, namely of a substance in the fluid, i given by :

$$\rho(\varepsilon_m + \varepsilon_t)\frac{\partial c}{\partial z} \approx \rho\varepsilon_t\frac{\partial c}{\partial z} = q_m$$

- diffusion (per unity of surface) of momentum is given (see eq. 2.49) by :

$$\rho(\nu + \nu_t)\frac{\partial u}{\partial z} \approx \rho\nu_t\frac{\partial u}{\partial z} = \tau_{zx}$$

Here is assumed that the turbulent diffusivity, ε_t , and the turbulent viscosity, ν_t are far more important than the molecular diffusivity, ε_m , and the viscosity, ν respectively (see *Graf*, 1971, p. 166).

According to the analogy of Reynolds (see *Taylor*, 1954, p. 451), the transfer o matter (as well as the one of heat) and the transfer of momentum by the turbulenc are analogous ; this is strictly correct close to a solid surface (the bed) Consequently, one may also take :

$$\varepsilon_t \equiv \nu_t \equiv \varepsilon_s \tag{6.47}$$

5° For the case, where the diffusivity is independent of the level, ε_s = Cte, the abov equation, eq. 6.44, can be integrated and yields :

$$\frac{c_s}{c_{sa}} = \exp\left[-\frac{v_{ss}}{\varepsilon_s}(z-a)\right] \tag{6.48}$$

where c_{sa} is the concentration at a reference level, a. This relation has bee experimentally verified (see *Graf*, 1971, p. 167).

6° In the open-channel flow, the turbulence and thus the diffusivity are vertically distributed, $\varepsilon_s(z)$, (see sect. 2.6). The distribution of the diffusivity, $\varepsilon_s \approx \nu_t$, is given (see *Graf*, 1971, p. 173) by :

$$\varepsilon_s = \kappa u_*' \frac{z}{h} (h - z) \tag{6.49}$$

This parabolic relation, established for unidirectional flow, has been obtained by assuming :

- the vertical distribution of the tangential turbulent shear stress :

$$\tau_{zx} = \tau_o \left(\frac{h-z}{h}\right) \tag{2.47b}$$

- the vertical distribution of the velocity (see sect. 2.5.2) :

$$\frac{du}{dz} = \frac{\sqrt{\tau_o/\rho}}{\kappa} \frac{1}{z}$$

where $\kappa = 0.4$ is the Karman constant, which is independent of the concentration (see *Colemann*, 1981) ;

- the expression of the Reynolds stress :

$$\tau_{zx} = \rho \, \nu_t \frac{du}{dz} \tag{2.49}$$

- the analogy of Reynolds :

$$\varepsilon_s \cong \varepsilon_t \cong \nu_t \tag{6.47}$$

This distribution of the diffusivity, eq. 6.49, has been experimentally verified (see *Raudkivi*, 1990, p. 170).

Substitution of eq. 6.49 into eq. 6.44, and separation of the variables, yields :

$$\frac{dc_s}{c_s} = -\frac{v_{ss}}{\kappa u_*'} \left(\frac{h}{h-z}\right) \frac{dz}{z} \tag{6.50}$$

where one defines the *Rouse* exponent as :

$$\mathfrak{z} = \frac{v_{ss}}{\kappa u_*'} \qquad \left(\text{or } \mathfrak{z}' = \frac{v_{ss}}{\beta_s \kappa u_*'} \right) \tag{6.50a}$$

This expression, eq. 6.50, can now be integrated by parts, within the limits $a < z < h$ (see *Rouse*, 1938, p. 341, and *Graf*, 1971, p. 173) and renders :

$$\frac{c_s}{c_{sa}} = \left(\frac{h - z}{z} \cdot \frac{a}{h - a} \right)^{\mathfrak{z}} \qquad (6.5$$

where c_{sa} is the concentration at a reference level, a. This equation, eq. 6.51, giv the distribution of the relative concentration, c_s/c_{sa} , for one single particle size, and $\mathfrak{z}$. Note that in the definition of the Rouse exponent, $\mathfrak{z}$, the friction velocit u_*', due to the granulate must be used.

In the Rouse exponent, eq. 6.50a, one should take the settling velocity, v_{ss} , of t particle in clear and quiescent water, thus being not influenced by turbulence or concentration. For *natural* particles of quartz, $s_s = 2.65$ [-], falling in quiesce water at $T = 20$ [C°], the settling velocity can be determined using the Fig. 6.10 (s *Graf*, 1971, p. 45).

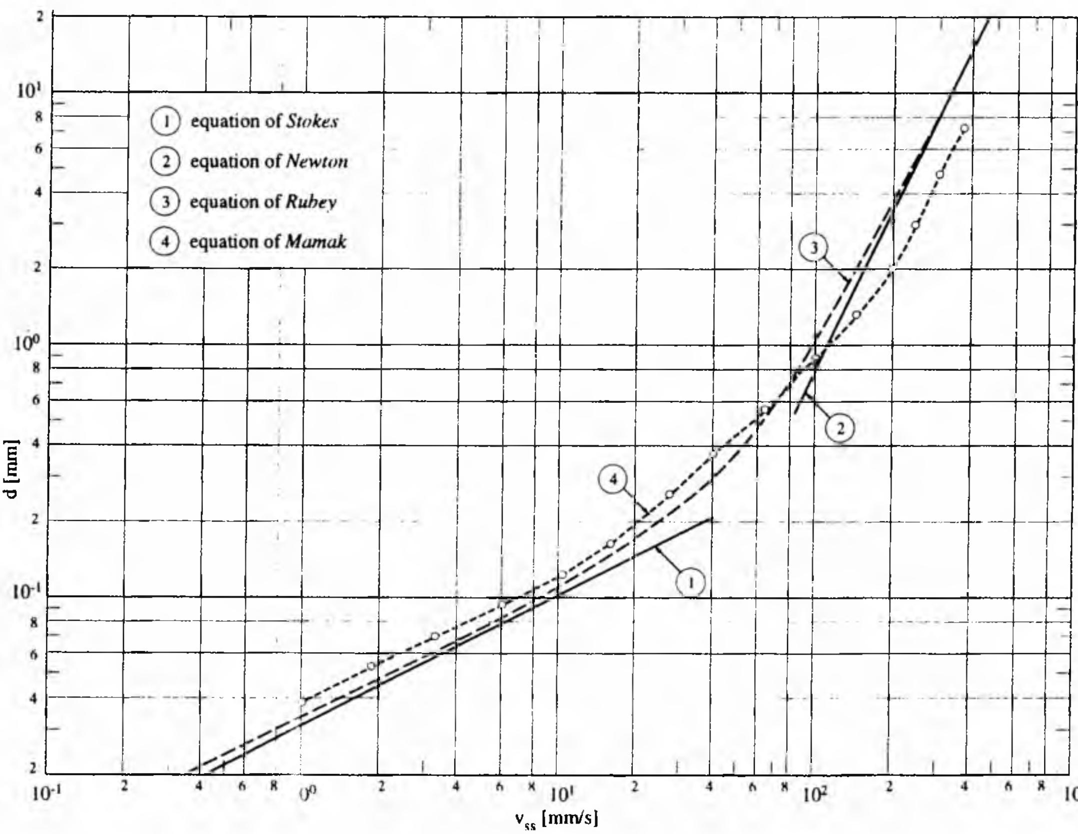


Fig. 6.10 Sett ng velocity , v_{ss} , as function of particle diameter, d.

The equation, giving the distribution of the relative concentration, eq. 6.51, for different values of the Rouse exponent, $\mathfrak{z}$, is shown in Fig. 6.11. The following is to be observed :

- For small $\mathfrak{z}$-values, the relative concentration is large and tends to become uniform over the entire flow depth, h.
- For large $\mathfrak{z}$-values, the relative concentration is small at the water surface and is large close to the bed.
- The size of the particles, expressed with the settling velocity, v_{ss} , is directly responsible for these distributions.
- Close to the bed, $z \cong 0$, the concentration goes towards infinity, $c_s = \infty$, thus to an impossible value. Thus one delimits this level usually by $a \equiv z_{sb} \cong 0.05h$ or by $z_{sb} = 2d$, below which there exists the bed load (see Fig. 6.1).
- The reference concentration, c_{sa} , is usually taken at a level of $a \equiv z_{sb}$; it will be calculated later (see eq. 6.57) with one of the bed-load formulae, q_{sb} .

Numerous are the investigations, both in laboratory and in situ, which give evidence of the validity of the above equation, eq. 6.51 (see *Graf*, 1971, p. 175).

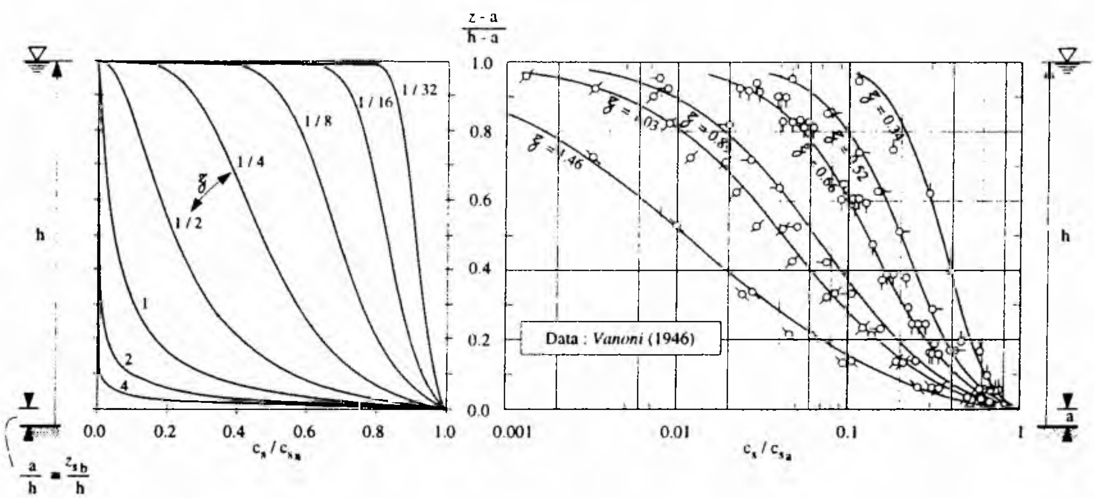


Fig. 6.11 Vertical distribution of the relative concentration, c_s/c_{sa} , in a suspension.

6.4.3 Suspended-Load Relation

1° The volumic solid discharge in suspension per unit width, in a region delimited by $z_{sb} < z < h$, is obtained by :

$$q_{ss} = \int_{z_{sb}}^{h} c_s \, u \, dz \tag{6.52}$$

where $c_s(z)$ is the local concentration, eq. 6.51, and $u(z)$ is the local velocity. Thi relation is valid for a single particle size, d or v_{ss}.

There exist different methods for the calculation of the suspended-load transpoı (see *Graf*, 1971, p. 189), but only the one of *Einstein* (1950) will be presentec being presently the most popular one.

2° The distribution of the velocity shall be given by a logarithmic relation (se *Einstein*, 1950, p. 17), of the form :

$$u(z) = u_*' \, 5.75 \log \left(30.2 \frac{z}{\Delta}\right) \qquad (6.53$$

where Δ is a correction term, given in Fig. 6.7a, and u_*' is the friction velocity du to the granulate.

3° Upon substitution of eq. 6.51 and of eq. 6.53 into the above equation, eq. 6.52 one obtains :

$$q_{ss} = \int_{z_{sb}}^{h} c_{sa} \left(\frac{h-z}{z} \cdot \frac{a}{h-a} \right)^{\mathfrak{z}} u_*' \, 5.75 \log \left(30.2 \frac{z}{\Delta}\right) dz \qquad (6.54$$

Replacing $a \equiv z_{sb}$ by a dimensionless expression, $z_{sb}/h = A_E$, and using h as th unity of z (see *Einstein*, 1950, p. 18), yields :

$$q_{ss} = \int_{z_{sb}}^{h} c_s \, u \, dz = \int_{A_E}^{1} c_s \, u \, h \, dz \qquad (6.52a$$

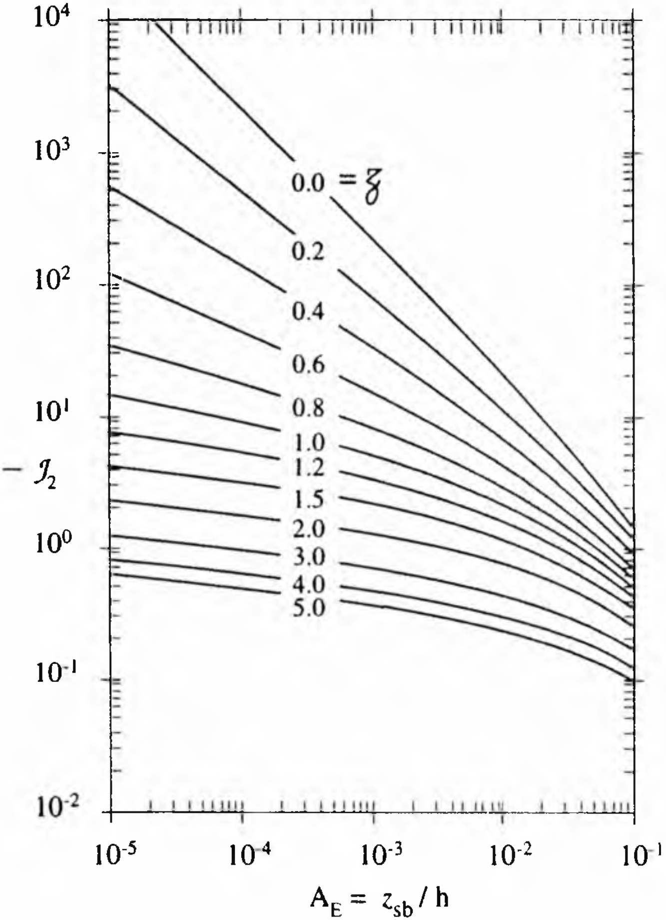
After some mathematical manipulations, one gets :

$$q_{ss} = c_{sa} \, u_*' \, 5.75 \, h \left(\frac{A_E}{1-A_E}\right)^{\mathfrak{z}} \cdot$$

$$\cdot \left\{ \log \left(30.2 \frac{h}{\Delta}\right) \int_{A_E}^{1} \left(\frac{1-z}{z} \right)^{\mathfrak{z}} dz + 0.434 \int_{A_E}^{1} \left(\frac{1-z}{z} \right)^{\mathfrak{z}} \ln z \, dz \right\} \qquad (6.55$$

The values of the following integrals :

$$\mathcal{J}_1 = 0.216 \frac{A_E^{\mathfrak{z}-1}}{(1-A_E)^{\mathfrak{z}}} \int_{A_E}^{1} \left(\frac{1-z}{z} \right)^{\mathfrak{z}} dz$$



, used in the method of *Einstein* (1950).

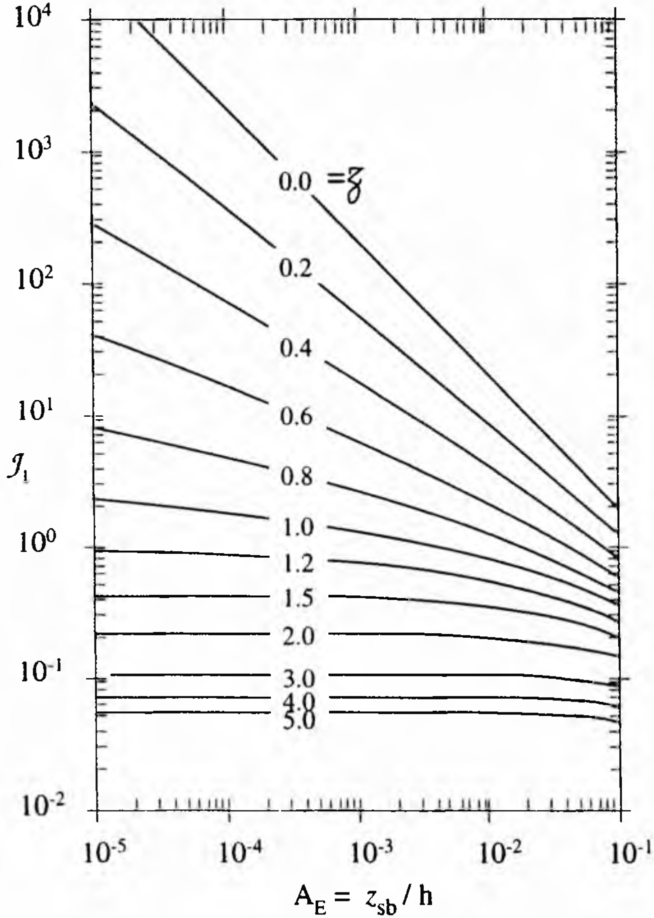


Fig. 6.12 The integrals, $\mathcal{J}_1(A_E, \zeta)$ and $\mathcal{J}_2(A_E, \zeta)$

$$\mathcal{J}_2 = 0.216 \frac{A_E^{\zeta-1}}{(1-A_E)^{\zeta}} \int_{A_E}^{1} \left(\frac{1-z}{z}\right)^{\zeta} \ln z \, dz$$

are numerically evaluated (see *Einstein*, 1950, p. 19-24) and this for diffe values of A_E and ζ ; they are given in Fig. 6.12.

Finally, the above equation, eq. 6.55, can be put into the following form :

$$q_{ss} = 11.6 \, c_{sa} \, u_*' \, z_{sb} \left[2.303 \log\left(30.2\frac{h}{\Delta}\right) \mathcal{J}_1 + \mathcal{J}_2\right] \qquad (6$$

where q_{ss} is the volumic solid discharge per unit width of the suspended load.

4° The *reference concentration*, c_{sa} , shall be taken there, where the concentra distribution, eq. 6.51, lacks any physical sense, namely very close to the be will thus be positioned within the layer where the bed load moves (see Fig. 6.1

One usually assumes (see *Graf*, 1972, p. 191), that the thickness of this la called *bed layer*, is twice the grain diameter, $z_{sb} \approx 2d$; for a granulometric mixt the bed layer takes different values for each granulometric fraction.

It is now of the foremost interest, to establish a relation between the bed-load suspended-load transport ; the reference concentration, c_{sa} , will make this link.

The formula of Einstein for bed-load transport, eq. 6.42, for one si granulometric fraction, $q_{sb} \, i_{sb}$, shall be used for the determination of the (aver concentration in this bed layer ; one writes :

$$c_{sa} = \frac{q_{sb} \, i_{sb}}{u_b \, z_{sb}} \qquad (6.$$

Exploiting experiments (see *Einstein*, 1950, p. 40) which rendered the velocit bed load as being $u_b = 11.6 \, u_*'$, one obtains an expression for the refere concentration, such as :

$$c_{sa} = \frac{q_{sb} \, i_{sb}}{11.6 \, u_*' \, z_{sb}} \qquad (6.5$$

5° Consequently, the solid discharge as suspended load per unit width — using expression, eq. 6.57a — is given by :

$$q_{ss} \, i_{ss} = q_{sb} \, i_{sb} \left[2.303 \log\left(30.2\frac{h}{\Delta}\right) \mathcal{J}_1 + \mathcal{J}_2\right] \qquad (6$$

$q_{ss}i_{ss}$ being the volumic solid discharge per unit width of the suspended load for one single granulometric fraction.

This relation, eq. 6.58, establishes the link between bed-load and suspended load transport for all particle sizes, which are found in the granulometric fraction of the bed-load.

6.5 TOTAL-LOAD TRANSPORT

6.5.1 Notions

1° Total-load transport of sediments — or better called *total bed-material load transport* — is made up of transport as bed load (see sect. 6.3) and of transport as suspended load :

$$q_s = q_{sb} + q_{ss} (+ q_{sw}) \tag{6.59}$$

Added should (possibly) be the transport as wash load, q_{sw} .

2° Different formulae (see *Graf*, 1971, chap. 9 or *White* et *al.*, 1973) exist, which can be used for the prediction of the bed-material load in a watercourse.

The formulae for determination of the total load — just as the ones of the bed load — give only reasonable results in the domain of their established parameters. Thus an application of any formula must be done with great care.

Here will be given a selection (in chronological order) of some existing formulae.

6.5.2 Total-Load Relations

1° The *indirect* methods determine the bed-material load by addition of the calculated bed load and the calculated suspended load. Thus these methods take into account that the hydromechanics of each mode of transport is not the same. However, a clear distinction between the two modes is not easily possible.

The *direct* methods determine the bed-material load directly, without making a distinction between the two modes of transport.

2° *Einstein* (1950, p. 40) proposed a formula for bed-load transport, eq. 6.42, and one for suspended-load transport, eq. 6.58 ; by combining up these two relations, it is possible to get a formula for the bed-material load transport :

$$q_s\, i_s = q_{sb}\, i_{sb} + q_{ss}\, i_{ss} = q_{sb}\, i_{sb} \left[1 + 2.303 \log \left(30.2 \frac{h}{\Delta}\right) \mathcal{I}_1 + \mathcal{I}_2 \right] \tag{6.60}$$

This relation gives the sediment-transport capacity, but does not, of course, include the wash-load transport.

This formula, eq. 6.60, can be used if the hydraulic and sedimentolog parameters are known in advance. If in addition a measurement of the susper load is also available, there exists a modified version (see *Graf*, 1971, p. 20 the above relation.

In many ways, the indirect method of *Einstein*, eq. 6.60, is hydraulically ra complete, but its application might seem laborious. Notably, the non-uniformit the granulometry is accounted for by using the ratio of i_s/i_{sb}. Furthermore, considers also the influence of the water temperature (see *Graf*, 1971, p. 238 the velocity distribution, eq. 6.35, and of the concentration distribution, eq. 6 using the exponent of Rouse, eq. 6.50a.

3° A relation for the direct prediction of the bed-material transport, valid for o channel flow [but also for flow in pipes], was developed by *Graf* et *Acar* (1968).

A *parameter of shear intensity* was elaborated as a criteria of solid transport *Graf*, 1971, pp. 218 et 443), such as :

$$\Psi_A = \frac{(s_s - 1)d}{S_e\, R_h} \qquad (6$$

which is the inverse of the dimensionless shear stress, given by eq. 6.22.

Applying the concept of power (work) of a flow system, a *parameter of trans* was proposed (see *Graf*, 1971, pp. 219 and 446), such as :

$$\Phi_A = \frac{C_s\, UR_h}{\sqrt{(s_s - 1)gd^3}} = \frac{(q_s/q)UR_h}{\sqrt{(s_s - 1)gd^3}} \qquad (6$$

which is similar to the dimensionless intensity of solid discharge, giver eq. 6.28.

Note, that the hydraulic radius, R_h , is here taken as the total one ; for a nar channel, the hydraulic radius of the bed, R_{hb} , should be taken. C_s is the volu concentration in the section and $d = d_{50}$ is the equivalent diameter.

It could be shown, that a functional relation between these parameters, Ψ_A and (see eq. 6.29a) is possible :

$$\Phi_A = f(\Psi_A)$$

whose form was experimentally determined. Using close to 800 experiments i the laboratory and close to 80 experiments in the field (see Table 6.3), all for i surface flow [and close to 300 experiments for pipe-line flow], the follov relationship (see *Graf*, 1971, pp. 220 and 448) was established :

$$\Phi_A = 10.39\,(\Psi_A)^{-2.52} \qquad (6$$

This relationship is found valid for $10^{-2} < \Phi_A < 10^3$ or for $\Psi_A \leq 14.6$. An extension of this work by *Graf* et *Acaroglu* (1968) has been done by *Graf* et *Suszka* (1987) ; it provided the following relationship :

$$\Phi_A = 10.4\ K\ (\Psi_A)^{-1.5} \tag{6.63a}$$

with
$$\begin{aligned} K &= \Psi_A^{-1} && \text{if} && \Psi_A \leq 14.6 \\ K &= (1 - 0.045\ \Psi_A)^{2.5} && \text{if} && 22.2 > \Psi_A > 14.6 \\ K &= 0 && \text{if} && \Psi_A > 22.2 \end{aligned}$$

The trend, for very weak solid transport, $10^{-5} < \Phi_A < 10^{-2}$, with $\Psi_A > 14.6$, is also evident in other experiments (see *Pazis* et *Graf*, 1977).

If one takes in the above relation, eq. 6.61, the energy slope, S_e, defined by eq. 6.2, the functional relation between Ψ_A and Φ_A, eq. 6.63, can also be used for the calculation of the sediment transport during unsteady flow (see *Graf* et *Song*, 1995).

The relations, eq. 6.63 and eq. 6.63a, are also valid when taking an equivalent diameter, $d \cong d_{50}$, if the granulometry is a non-uniform one.

4° For a direct determination of the total-load transport, q_s, *Ackers* et *White* (1973) proposed the use of some sedimentological parameters ; employed were hydraulic considerations and dimensional analysis.

A *parameter of mobility* of sediments was defined as :

$$F_{gr} = \frac{u_*^{\,n_w}}{\sqrt{(s_s - 1)gd}} \left[\frac{U}{\sqrt{32}\,\log(10h_m/d)}\right]^{(1-n_w)} \tag{6.64}$$

which becomes $F_{gr} = \sqrt{\tau_*}$ (see eq. 6.23) for very fine particles, where $n_w = 1$.

A *parameter of transport* of sediments was postulated as :

$$G_{gr} = C_w \left(\frac{F_{gr}}{A_w} - 1\right)^{m_w} \tag{6.65}$$

The total-load transport is calculated according to :

$$C_s = \frac{q_s}{q} = G_{gr}\frac{d}{h_m}\left(\frac{U}{u_*}\right)^{n_w} \tag{6.66}$$

where C_s is the volumic average concentration in a section and $h_m = A/B$ is the average flow depth.

The coefficients in the above relations were determined by regression an using close to 1000 experiments in the laboratory and close to 250 experim the field, with sediments having a uniform and a non-uniform granulo $0.04 < d_{50}$ [mm] < 4.0 and for flow at $Fr < 0.8$ (see Table 6.3). The res values of these coefficients are the following :

coefficient	$d_* > 60$ $d > 2.5$ [mm]	$1.0 < d_* \leq 60$	$d_* < 1$ $d < 0.04$ [
n_w	0.0	$(1.0 - 0.56 \log d_*)$	1.0
m_w	1.50	$(9.66/d_*) + 1.34$	
A_w	0.17	$(0.23/\sqrt{d_*}) + 0.14$	
C_w	0.025	$\log C_w = 2.86 \log d_* - (\log d_*)^2 - 3.53$	

Above, the dimensionless particle diameter, d_*, is used, defined as :

$$d_* = d \left[(s_s - 1) \frac{g}{\nu^2} \right]^{1/3} \qquad ($$

For a non-uniform granulometry, one takes $d = d_{35}$ as the equivalent diamete

6.5.3 Applications of Relations

1° Different formulae for the determination of the solid transport have been pres However, none of these relations can pretend to translate the intrinsic comple the transport of sediments.

2° Most of these formulae should not be used beyond the conditions within whic were established. Table 6.3 contains a summary of the range of the param d and S_f, investigated for the establishment of each formula by their auth other author(s) may have extended this range. Also listed are the recommen by the author(s) for the choice of the equivalent diameter, d_x, if the granulom quasi or non-uniform.

3° The formulae for the transport of sediments are often established, using labo data and less often using field data.

A verification of these formulae in watercourses is a very delicate task, sinc difficult to measure correctly the solid discharge in the field. Furthermore, it is a rather subjective evaluation, since the zones of the modes of transport c easily be separated.

4° Numerous studies have been reported, comparing measurements in waterco with the different existing formulae.

For a better appreciation of the validity of the above presented formulae, it wil be of interest to compare the computed results with the direct measurements solid discharge in the field.

Table 6.3 Parameters used for establishing the different formulae.

Formula	d [mm]	S_f [-]	d_τ [mm], equivalent diameter for a non-uniform granulate
Schoklitsch (eq. 6.31)	0.3 – 7.0 (44.0)	0.003 – 0.1	d_{40}
Meyer-Peter et *al.* (eq. 6.32)	3.1 – 28.6	0.0004 – 0.020	$d_m(d_{50})$
Einstein (eq. 6.42)	0.8 – 28.6	–	d_{35}
Graf et *Acaroglu* (eq. 6.63)	0.3 – 1.7 (23.5)	–	d_{50}
Ackers et *White* (eq. 6.66)	0.04 – 4.0	Fr < 0.8	d_{35}

Many (nineteen) of the existing formulae for the calculation of the total transport have been studied by *White* et *al.* (1973) and compared with experimental results. They evaluated almost 1000 laboratory experiments with uniform and non-uniform sediments of $0.04 < d_{50}$ [mm] < 4.9, at flow depth of $h < 0.4$ [m], and almost 270 experiments in watercourses with sediments of $0.1 < d$ [mm] < 68.0 and a width/depth ratio of $9 < B/h < 160$.

Each formula was applied to all the data of the solid-discharge measurements. Subsequently was established a ratio of the values calculated, C_{calc} , and the values observed, C_{obs}, where $C \equiv C_s$ is the total-load transport, expressed in concentration.

Some results of this investigation are given in Fig. 6.13, where one may see the success of a prediction (in percentage) for different ranges of the ratio, C_{calc}/C_{obs}. For the formulae, which are presented in this book — considering only the range of $1/2 < C_{calc}/C_{obs} < 2$ — it can be seen that the percentage is for the formula of

Einstein (1950), eq. 6.60 : 44 % of success

Graf et *Acaroglu* (1968), eq. 6.63 : 40 % of success

Ackers et *White* (1973), eq. 6.66 : 64 % of success

This implies that with the formula of *Ackers* et *White*, 64 % of the experimen data can be predicted in the above-mentioned range. This is usually considered a good (or a not-so-bad) result ; more than half of the studied (nineteen) formu give results which are less good, namely < 40 %. Also noticed is that with formula of *Einstein* there is a slight under-estimation of the solid discharge ; wh the one of *Graf* et *Acaroglu* gives a slight over-estimation.

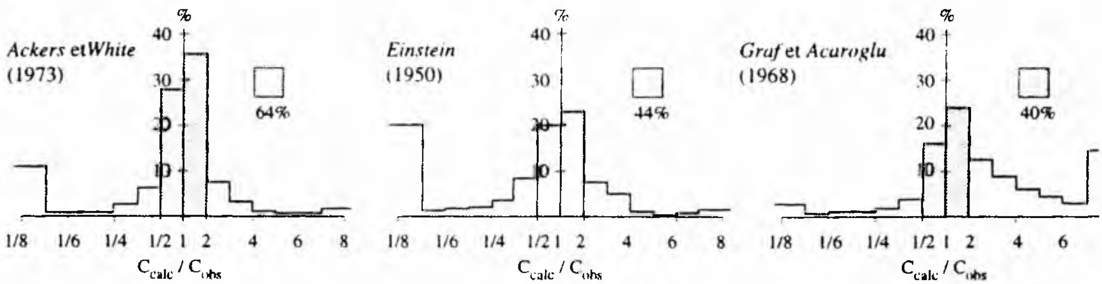


Fig. 6.13 Comparison, with respect to C_{calc}/C_{obs} , of the success of prediction for the presented formulae.

The comparative study of *White* et *al.* (1973) is reasonably objective, but certair not conclusive. Other studies exist (see *Raudkivi*, 1976, p. 227) which shc clearly that an objective validation is nearly impossible.

5° Amongst the different existing formulae for the determination of the total-lo transport, but equally for the ones of the bed-load and suspended-load transpo each one will give an answer, but none will be very precise nor very true.

Finally, it must be said, that the results obtained with these formulae give on valuable guide-lines for the engineer. For practical purposes, it is advised to consı more than one formula ; the obtained result may however render different valu (see *Graf*, 1971, p. 156).

6.5.4 Wash Load

1° The wash load, q_{sw} , contains all these particles which are never in contact with tl bed and displace themselves by being carried (washed) through the channel by tl flow (see Fig. 6.1).

This mode of the transport of sediments (see Table 6.2) is limited to the very fine particles which are rare in the granulometry of the bed material. The distribution these particles is rather uniform over the entire flow depth (see Fig. 6.11).

Einstein (1950, p. 7) has proposed that the granulometry of the wash load is the fraction of granulometry of the bed which is smaller than 10 %. It was also proposed (see *Raudkivi*, 1976, p. 220) that the wash load is composed of the fine particles having a diameter of $d < 0.06$ [mm].

2° Since there exists no physical relationship to the flow, it has been difficult to advance an analytical method for the determination of the wash load.

The wash load depends more on the hydrological, geomorphological and meteorological conditions within the drainage basin (see *Graf*, 1971, p. 232), namely on the overland surface erosion and less on the erosion in the stream bed.

3° Thus it is to be remarked, that at the present no methods exist for the prediction of the wash load.

In order to obtain a quantitative information on the wash load, measurements in the field must be performed. One measures thus the total suspended load, $q_{ss} + q_{sw}$. Subsequently is calculated the suspended load, q_{ss}, (see sect. 6.4) and consequently the suspended wash load, q_{sw}, can be obtained.

4° In some watercourses, the transport as wash load can be much more important than the bed-material load, $q_{sw} > q_s$. Obviously, this makes the problem of sediment transport hopelessly complicated.

If the total suspended-load transport, $q_{ss} + q_{sw}$, becomes very large, one may well imagine that this influences on the flow behaviour ; such a mixture of water-sediments is probably not anymore a Newtonian one (see Table 6.1). The flow of such a non-Newtonian mixture will modify the hydraulics, thus the distribution of the velocity and of the concentration, but also the flow resistance as well as the bed forms.

An early version of section 6.2 was published as :
Graf W.H. (1994) : *Les équations de Saint-Venant-Exner.*
Österr. Ing. und Arch. Zeitschrift, Jgg. 139, N°9, Wien, A

6.6 EXERCISES

6.6.1 Problems, solved

Ex. 6.A

A rectangular channel has a width of B = 5 [m]. At some point, the bed of the chann changes from a fixed bed to a mobile bed with a uniform sediment of d_{50} = 1 [mm] ar s_s = 2.6 [-]. The discharge of Q = 15 [m^3/s] remains constant and the water depth h = 2.2 [m]. In the fixed-bed reach of the channel there is no sediment transport. Th flow initiates however erosion in the mobile-bed reach of the channel, where the porosi of the bed material is p = 0.3 [-].

A *degradation* of the channel starts at the junction between the fixed bed and the mobi bed. Determine the time it will take to lower the bed level down to $z = 0.4\Delta h$ at a static located at $L = 6R_h/S_f$ downstream from the junction; subsequently draw the bed profi for this particular moment. Furthermore, show the temporal variation of the degradatic at this station. Calculate also the resulting bed profile if the mobile bed is limited to length of x_f = 90 [km].

SOLUTION :

i) The steady flow will be considered to be quasi-uniform during the phase of degradation (see Fig. Ex. 6.A.1); therefore the *parabolic model* can be used :

$$\frac{\partial z}{\partial t} - K\frac{\partial^2 z}{\partial x^2} = 0 \qquad (6.1$$

where x is positive towards the downstream and follows the initial bed profil z represents the bed-level variation with respect to the initial bed, S_f^o. Note that th use of the parabolic model is limited to : Fr < 0.6 and $x > 3R_h/S_e$.

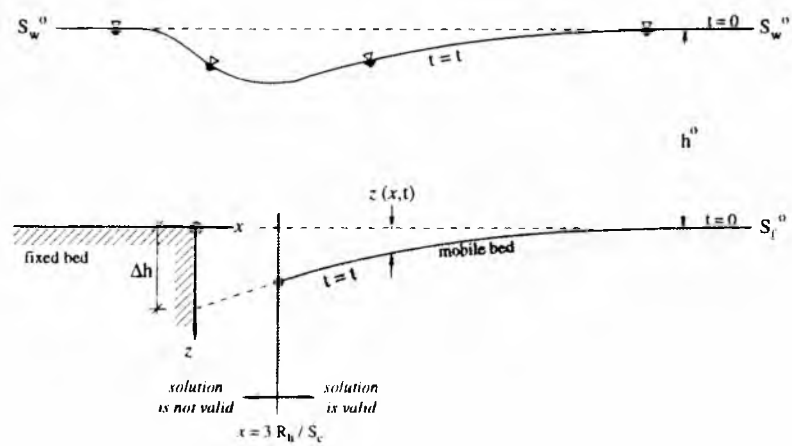


Fig. Ex.6.A.1 Scheme of the degradation.

The initial and boundary conditions are given as :

$$z\,(x,0) = 0 \quad ; \quad \lim_{x \to \infty} z\,(x,t) = 0$$

$$z\,(0,t) = \Delta h(t)$$

The solution to eq. 6.11 is given by :

$$z\,(x,t) = \Delta h \; erfc\left(\frac{x}{2\sqrt{Kt}}\right) \tag{6.15}$$

ii) Calculation of the quasi-uniform *flow* in the mobile-bed channel.

The slope of the energy line, S_e, is calculated using the Manning-Strickler formula:

$$U = \frac{Q}{Bh} = K_s\, R_h^{2/3}\, S_e^{1/2} \tag{3.16}$$

with
$$\begin{aligned} K_s &= 21.1/d_{50}^{1/6} = 66.7\ [m^{1/3}/s] \\ h &= 2.2\ [m] \quad , \quad B = 5.0\ [m] \quad , \quad R_h = 1.17\ [m] \\ Q &= 15.0\ [m^3/s] \quad , \quad q = Q/B = 3\ [m^2/s] \\ U &= q/h = 1.36\ [m/s] \end{aligned} \tag{3.18}$$

The slope of the energy line : $S_e = \mathit{0.00034}$ [-]

The Froud number is : $Fr = \dfrac{U}{\sqrt{gh}} = 0.29$ [-]

It should be emphasized that the Froude number has to be small, $Fr < 0.6$, being one of the conditions (see sect. 6.2.3) for the validity of the parabolic model, eq. 6.11.

iii) Calculation of the *solid discharge* in the mobile-bed channel.

The solid discharge, $q_s = C_s Uh$, is calculated using the *Graf* et *al.* (1968) formula :

$$\frac{C_s\, UR_h}{\sqrt{[(\rho_s-\rho)/\rho]\, g\, d_{50}^3}} = 10.39 \left\{ \frac{[(\rho_s-\rho)/\rho]\, d_{50}}{S_f\, R_h} \right\}^{-2.52} \tag{6.63}$$

with
$$\begin{aligned} (\rho_s-\rho)/\rho &= 1.6\ [-] \\ d_{50} &= 1\ [mm] \\ S_f \equiv S_e &= 0.00034\ [-] \end{aligned}$$

$$C_s\, UR_h = 3.9 \cdot 10^{-5}\ [m^2/s]$$

The solid discharge is : $q_s = C_s\, U\, h\, \dfrac{R_h}{R_h} = 3.9 \cdot 10^{-5}\, \dfrac{2.2}{1.17} = \mathit{7.3 \cdot 10^{-5}}\ [m^2/s]$

iv) The *coefficient*, K , in the parabolic model, eq. 6.11, is approximately given by :

$$K_o \equiv K \approx \frac{1}{3}\, b_s q_s \, \frac{1}{(1-p)} \, \frac{1}{S_e^{\,o}} \qquad (6.12$$

with $S_e^{\,o} = 0.00034$ [-]
$(1-p) = 0.7$ [-]
$b_s = 2\,(2.52) \cong 5$ (where $\beta = 2.52$ is the exponent in eq. 6.6 according to eq. 6.5a and eq. 6.30)

The coefficient is : K = *0.511* [m²/s]

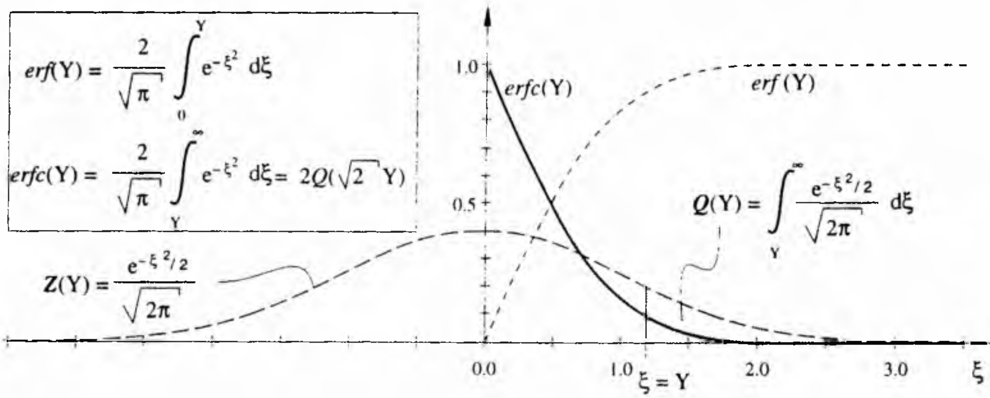
Table for the complementary error function

(see *Handbook of Mathematical Functions*, 1964, National Bureau of Standards, pp. 310-311, formula 7.1.28

Y	*erfc*(Y)	Y	*erfc*(Y)	Y	*erfc*(Y)
0.00	*1.00000*	1.20	*0.08969*	2.30	*0.00114*
0.10	*0.88754*	1.30	*0.06599*	2.40	*0.00069*
0.20	*0.77730*	1.40	*0.04772*	2.50	*0.00041*
0.30	*0.67137*	1.50	*0.03390*	2.60	*0.00024*
0.40	*0.57161*	1.60	*0.02365*	2.70	*0.00013*
0.50	*0.47950*	1.70	*0.01621*	2.80	*0.00008*
0.60	*0.39614*	1.80	*0.01091*	2.90	*0.00004*
0.70	*0.32220*	1.90	*0.00721*	3.00	*0.00002*
0.80	*0.25790*	2.00	*0.00468*	3.10	*0.00001*
0.90	*0.20309*	2.10	*0.00298*	3.20	*0.00001*
1.00	*0.15730*	2.20	*0.00186*	3.30	*0.00000*
1.10	*0.11979*				

$$erf(Y) = \frac{2}{\sqrt{\pi}} \int_0^Y e^{-\xi^2}\, d\xi$$

$$erfc(Y) = \frac{2}{\sqrt{\pi}} \int_Y^\infty e^{-\xi^2}\, d\xi = 2Q(\sqrt{2}\,Y)$$



The complementary error function can be calculated approximately using the following expression :

$erfc(Y) = 1 / (1 + a_1\, Y + a_2\, Y^2 + a_3\, Y^3 + a_4\, Y^4 + a_5\, Y^5 + a_6\, Y^6)^{16} + \varepsilon(Y)$ where $|\varepsilon(Y)| \leq 3 \cdot 10^{-7}$

$a_1 = 0.0705230784$; $a_2 = 0.0422820123$; $a_3 = 0.0092705272$

$a_4 = 0.0001520143$; $a_5 = 0.0002765672$; $a_6 = 0.0000430638$

v) In the present problem, it is asked to determine the time it takes to lower the bed level down to $z = 0.4\Delta h$, thus :

$$\frac{z(x,t)}{\Delta h} = \frac{0.4\Delta h}{\Delta h} = 0.4$$

The eq. 6.15 is now written as :

$$0.4 = erfc\left(\frac{x}{2\sqrt{Kt}}\right) = erfc\,(Y)$$

Using the table of the complementary error function yields :

$$Y \cong 0.6 = \left(\frac{x}{2\sqrt{Kt}}\right) \qquad \Rightarrow \qquad t \cong \frac{x^2}{4\,Y^2\,K} \cong \frac{x^2}{1.44\,K}$$

At the station $x \equiv L = 6R_h/S_e = 20.73$ [km], the lowering of the bed down to a level of $z = 0.4\Delta h$ occurs at the time :

$$t = \frac{(20.73 \cdot 10^3)^2}{(1.44)\,(0.511)} = 5.84 \cdot 10^8 \text{ [s]} = \mathit{1.62 \cdot 10^5} \text{ [h]} \cong \mathit{18.52} \text{ [years]}$$

To draw the bed profile for the entire channel at this particular moment, $t = 5.84 \cdot 10^8$ [s], the calculations are repeated for different values for the distance x (see following table).

Calculation of the bed profile				
$R_h = 1.17$ [m] ; $S_f = 0.00034$ [-] ; $K = 0.511$ [m^2/s]				
$\Delta h =$ **3.11** [m] ; $t = 5.84 \cdot 10^8$ [s]				
x [m]	$x\,(S_e / R_h)$ [-]	$Y = x/(2\sqrt{Kt})$ [-]	$z/\Delta h = erfc(Y)$ [-]	z [m]
10500	3.04	0.30	0.66735	2.073
11000	3.18	0.32	0.65253	2.027
13000	3.76	0.38	0.59465	1.847
15000	4.34	0.43	0.53923	1.675
20000	5.79	0.58	0.41299	1.283
20730	*6.00*	*0.60*	*0.39615*	*1.231*
30000	8.68	0.87	0.21946	0.682
40000	11.58	1.16	0.10157	0.316
50000	14.47	1.45	0.04070	0.126
60000	17.37	1.74	0.01405	0.044
70000	20.26	2.03	0.00417	0.013
80000	23.15	2.32	0.00106	0.003
90000	26.05	2.60	0.00023	0.001
100000	28.94	2.89	0.00004	0.000

The depth of degradation of the channel bed due to a solid discharge $q_s = 7.3 \cdot 10^{-5}$ [m²/s] , during a time period of $t = 5.84 \cdot 10^8$ [s] is given by :

$$\Delta h = \frac{q_s \cdot \Delta t}{1.13\,(1-p)\sqrt{K\ \Delta t}} = \frac{(7.3 \cdot 10^{-5})\sqrt{5.84 \cdot 10^8}}{(1.13)\,(0.7)\,\sqrt{0.511}} = \mathbf{3.11}\ [\mathrm{m}] \qquad (6.$$

and $z = 0.4\Delta h = 1.23$ [m] .

The bed profile, $z(x)$, for $t = 5.84 \cdot 10^8$ [s] = 18.52 [years], is plotted Fig. Ex. 6.A.2. This solution is valid only if $x > 3R_h/S_e$. For distances $x < 3R_h/S_e$, the solution is only an indicative one.

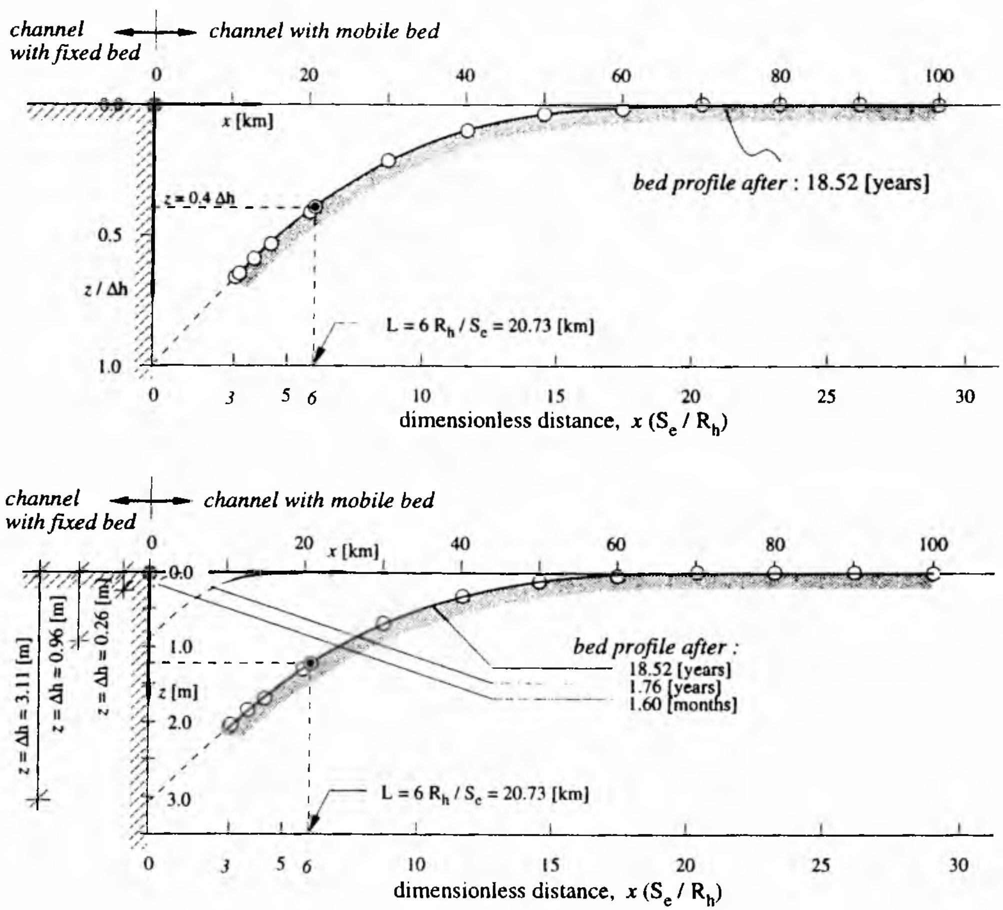


Fig. Ex.6.A.2 Bed profile after *18.52* [years] of degradation.

For sake of comparison, the bed profiles, $z(x)$, for $t = 1.76$ [year] and f $t = 1.6$ [month] are also plotted (without giving the calculations) in Fig. Ex. 6.A.

vi) The *temporal evolution of the degradation* at the station located at $x \equiv L = 6R_h/S_e = 20.73$ [km] is given by :

$$z(t) = \Delta h \; erfc\left(\frac{x}{2\sqrt{K\ \Delta t}}\right) = \Delta h \; erfc\left(\frac{20730}{2\sqrt{0.511\ \Delta t}}\right) \tag{6.15}$$

where, $\Delta h(t)$ can be evaluated by :

$$\Delta h = \frac{q_s \cdot \Delta t}{1.13\ (1-p)\sqrt{K\ \Delta t}} \tag{6.20}$$

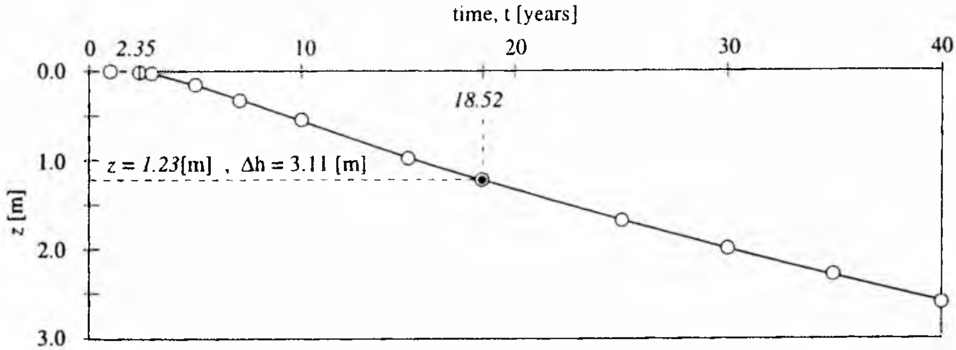


Fig. Ex.6.A.3 Evolution of the degradation at the station $x \equiv L = 6R_h/S_e = 20.73$ [km].

Calculation of the evolution of the degradation					
$R_h = 1.17$ [m] ; $S_f = 0.00034$ [-] ; $K = 0.511$ [m^2/s]					
$x \equiv L = 6R_h/S_{f_o} = 20730$ [m]					
t [years]	t [s]	$Y = x/(2\sqrt{Kt})$ [-]	$z/\Delta h = erfc(Y)$ [-]	Δh [m]	z [m]
1	3.15E+07	2.58	0.00026	0.72	0.0002
3	9.46E+07	1.49	0.03502	1.25	0.0438
5	1.58E+08	1.15	0.10248	1.61	0.1654
7	2.21E+08	0.98	0.16756	1.91	0.3201
10	3.15E+08	0.82	0.24823	2.28	0.5667
15	4.73E+08	0.67	0.34579	2.80	0.9669
18.52	*5.84E+08*	*0.60*	*0.39618*	*3.11*	*1.2309*
25	7.88E+08	0.52	0.46522	3.61	1.6794
30	9.46E+08	0.47	0.50500	3.95	1.9970
35	1.10E+09	0.44	0.53711	4.27	2.2941
40	1.26E+09	0.41	0.56371	4.57	2.5740
45	1.42E+09	0.38	0.58622	4.84	2.8391
50	1.58E+09	0.37	0.60559	5.11	3.0916

The evolution of the bed degradation can now be calculated by assuming diff values for $\Delta t \equiv t$. By using the approximate formula for the complementary function (see before), the calculation can easily be programmed on a spreads The table above summarizes these calculations; Fig. Ex. 6.A.3 shows the evol of the erosion, $z(t)$, at the station, $x \equiv L$.

This solution is however only valid (see *Ribberink* et *Sande*, 1984, p. 30) for :

$$t > \frac{40}{30} \frac{R_h^2}{S_f} \frac{1}{q_s} = \frac{40}{30} \frac{1.17^2}{0.00034} \frac{1}{7.3 \cdot 10^{-5}} = 7.42 \cdot 10^7 \text{ [s]} \cong 2.35 \text{ [years]}$$

vii) Calculation of the final bed profile if the channel reach with the mobile b *limited* to a length of $x_f = 90$ [km].

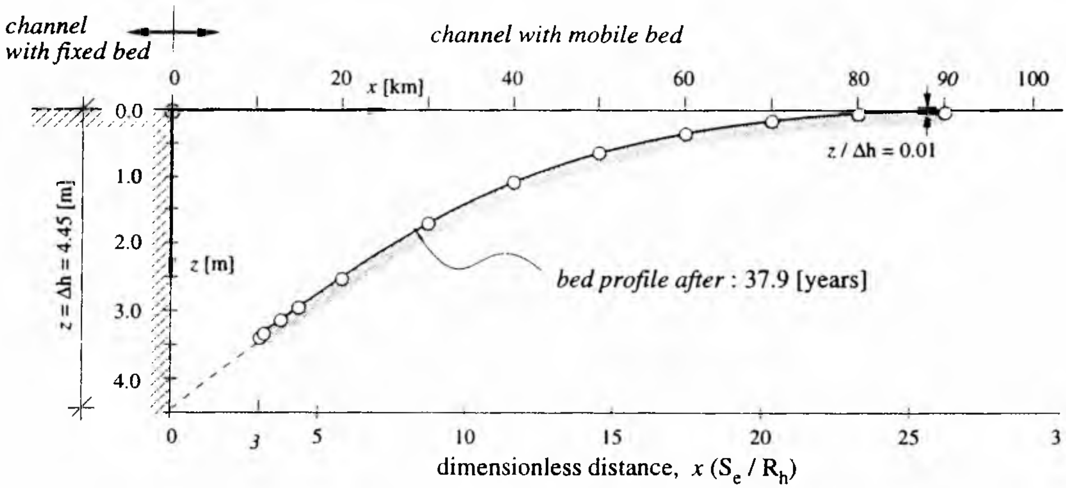


Fig. Ex.6.A.4 The channel-bed profile after *37.9* [years] of degradation

By assuming a very small amount of erosion, such as $z = 0.01\Delta h$, at the st $x_f = 90$ [km] , one can write :

$$\frac{z(x,t)}{\Delta h} = 0.01 = erfc\left(\frac{x_f}{2\sqrt{Kt}}\right) = erfc(Y)$$

Using the table of the complementary error function yields :

$$Y = 1.82 = \left(\frac{x}{2\sqrt{Kt}}\right) \qquad \Rightarrow \qquad t = \frac{x^2}{4\ Y^2\ K} \cong \frac{x^2}{13.25\ K}$$

and with $K = 0.511$ [m^2/s], one calculates :

$$t = \frac{(90\cdot 10^3)^2}{(13.25)\,(0.511)} = 1.2\cdot 10^9\ [s] = 3.3\cdot 10^5\ [h] \cong \mathit{37.93}\ [years]$$

To obtain the bed profile for the entire channel at this moment, $t = 1.2\cdot 10^9$ [s], the calculations for the degradation are repeated using different values for x (see the following table). The final bed profile, calculated in this way, is plotted in Fig. Ex. 6.A.4.

This solution is valid only if $x > 3R_h / S_e$.

The depth of the bed degradation due to a solid discharge, $q_s = 7.3\cdot 10^{-5}$ [m^2/s] , during a time period of $t = 1.2\cdot 10^9$ [s] , is given by the eq. 6.20 :

$$\Delta h = \frac{q_s\cdot \Delta t}{1.13\,(1-p)\sqrt{K\ \Delta t}} = \frac{(7.3\cdot 10^{-5})\,\sqrt{1.2\cdot 10^9}}{(1.13)\,(0.7)\,\sqrt{0.511}} = \mathbf{4.45}\ [m]$$

Calculation of the final bed profile				
$R_h = 1.17$ [m] ; $S_f = 0.00034$ [-] ; $K = 0.511$ [m^2/s]				
$\Delta h =$ **4.45** [m] ; $t = 1.2\cdot 10^9$ [s]				
x [m]	$x\ (S_e / R_h)$ [-]	$Y = x/(2\sqrt{Kt})$ [-]	$z/\Delta h = erfc(Y)$ [-]	z [m]
10500	3.04	0.21	0.76396	3.397
11000	3.18	0.22	0.75307	3.349
13000	3.76	0.26	0.71005	3.157
15000	4.34	0.30	0.66793	2.970
20000	5.79	0.40	0.56734	2.523
30000	8.68	0.61	0.39091	1.738
40000	11.58	0.81	0.25264	1.123
50000	14.47	1.01	0.15273	0.679
60000	17.37	1.21	0.08617	0.383
70000	20.26	1.42	0.04529	0.201
80000	23.15	1.62	0.02214	0.098
90000	*26.05*	*1.82*	*0.01006*	*0.045*

Ex. 6.B

A river on a bed slope of $S_f = 0.0005$ [-] conveys a unit discharge of $q = 1.5$ [m^2/s]. T river bed is made of granular material of uniform size of $d_{50} = 0.00032$ [m] witl specific gravity of $s_s = 2.6$ [-]; the porosity of the bed material is $p = 0.4$ [-]. There exi a weak transport of sediments.

At a certain station on this river, the solid discharge is locally increased $\Delta q_s = 0.0001$ [m^2/s] for a time period of $\Delta t = 50$ [h]. Determine the *aggradation* of t bed to be expected.

SOLUTION :

i) The flow is steady and is considered to be quasi-uniform during the period aggradation (see Fig. Ex. 6.B.1); thus the *parabolic model* can be used :

$$\frac{\partial z}{\partial t} - K\frac{\partial^2 z}{\partial x^2} = 0 \qquad (6.1$$

where x is positive towards the downstream and follows the initial bed profi z represents the bed-level variation with respect to the initial bed, S_{f_0}. Note that t use of the parabolic model is limited to : $Fr < 0.6$ and $x > 3R_h/S_e$.

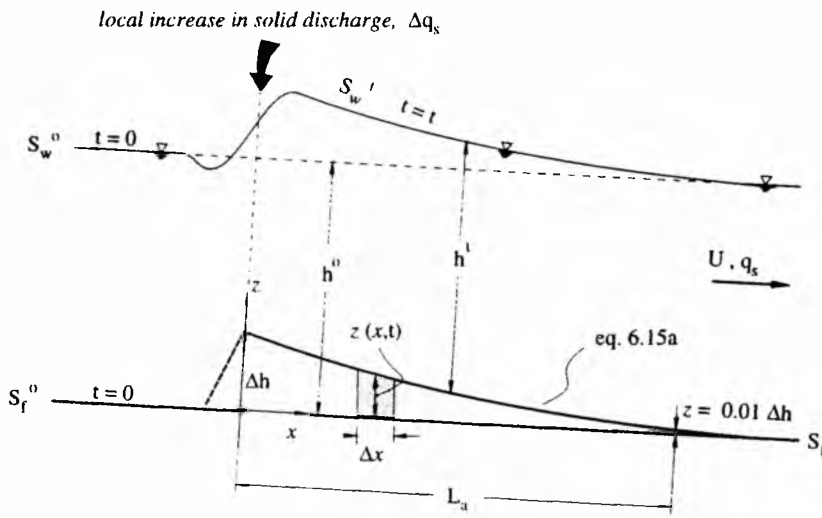


Fig. Ex.6.B.1 Sketch of the aggradation.

The initial and boundary conditions are given as :

$$z(x,0) = 0 \qquad ; \qquad \lim_{x \to \infty} z(x,t) = 0$$

$$z(0,t) = \Delta h(t)$$

The solution to eq. 6.11 is given by :

$$z\,(x,t) = \Delta h\ erfc\ \left(\frac{x}{2\sqrt{Kt}}\right) \tag{6.15}$$

ii) Calculation of the quasi-uniform *flow* in the river having a mobile bed.

The normal depth is calculated using the Manning-Strickler formula :

$$U = \frac{q}{h} = K_s\ h^{2/3}\ S_f^{\ 1/2} \tag{3.16}$$

with $K_s = 21.1/d_{50}^{\ 1/6} = 80.7\ [m^{1/3}/s]$ (3.18)
$q = 1.5\ [m^2/s]$
$S_f = 0.0005\ [-]$

The flow depth is : $h = \mathit{0.895}$ [m]
The average velocity is : $U = 1.676$ [m/s]
The Froude number is : $Fr = \dfrac{U}{\sqrt{gh}} = 0.566$

It should be remembered that the Froude number has to be small, namely $Fr < 0.6$.

iii) Calculation of the *solid discharge* in the river having a mobile bed.

The solid discharge, $q_s = C_s\,Uh$, is calculated using the relationship given by *Graf* et *al.* (1968) :

$$\frac{C_s\ UR_h}{\sqrt{[(\rho_s-\rho)/\rho]g\ d_{50}^3}} = 10.39\ \left\{\frac{[(\rho_s-\rho)/\rho]\ d_{50}}{S_f\ R_h}\right\}^{-2.52} \tag{6.63}$$

with $(\rho_s-\rho)/\rho = 1.6\ [-]$
$d_{50} = 0.32$ [mm]
$R_h \cong h = 0.895$ [m]

The solid discharge is : $q_s = \mathit{1.678 \cdot 10^{-4}}\ [m^2/s]$

iv) The *coefficient*, K , in the parabolic model, eq. 6.11, is approximately given by :

$$K_o \equiv K \approx \frac{1}{3}\ b_s q_s\ \frac{1}{(1-p)}\ \frac{1}{S_e^{\ o}} \tag{6.12c}$$

with $S_f^{\ o} \equiv S_e^{\ o} = 0.0005\ [-]$
$(1-p) = 0.6\ [-]$
$b_s = 2\ (2.52) \cong 5$ (where $\beta = 2.52$ is the exponent in eq. 6.63, according to eq. 6.5a and eq. 6.30)

The coefficient is : $K = \mathit{0.932}\ [m^2/s]$

v) The thickness of the aggradation of the bed (see Fig. Ex. 6.B.1) due to a loc increase in solid discharge, $\Delta q_s = 0.0001$ [m^2/s], during a time period ($\Delta t = 50$ [h] $= 1.8 \cdot 10^5$ [s] , is given by eq. 6.20, or :

$$\Delta h(t) = \frac{\Delta q_s \cdot \Delta t}{1.13\,(1-p)\sqrt{K\ \Delta t}} = \frac{(0.0001)\sqrt{1.8 \cdot 10^5}}{(1.13)\,(0.6)\,\sqrt{0.932}} = \mathbf{0.065}\ [\mathrm{m}]$$

The length of the zone of aggradation, L_a, can be calculated with eq. 6.15 b assuming, for example, a precision of $z/\Delta h = 0.01$:

$$\frac{z(x,t)}{\Delta h} = \frac{0.01\Delta h}{\Delta h} = 0.01 = erfc\left(\frac{x}{2\sqrt{K\ \Delta t}}\right) = erfc\,(Y)$$

Using the table of the complementary error function (see Ex. 6.A), yields :

$$Y = 1.821 = \left(\frac{x}{2\sqrt{K\ \Delta t}}\right)$$

The length of the zone of aggradation (see eq. 6.19) can now be calculated a follows :

$$L_a \equiv x_{1\%} = 2Y\sqrt{K\ \Delta t} = (2)\,(1.821)\ \sqrt{(0.932)\,(1.8 \cdot 10^5)} = \mathit{1492.3}\ [\mathrm{m}]$$

vi) To plot the bed profile after a time period of $\Delta t = 50$ [h] $= 1.8 \cdot 10^5$ [s], calculation are made using eq. 6.15 for different distances, x. (see the following table).

The resulting bed profile, $z(x)$, is plotted in Fig. Ex. 6.B.2.

The calculations, summarized in the following table, are valid only if $x > 3h/S_e$ In the present case, it can be shown that :

$$x = 3h/S_e = (3)\,(0.895)\,/\,(5 \cdot 10^{-4}) = 5370\ [\mathrm{m}] \gg L_a = 1492.3\ [\mathrm{m}]$$

However, experimental data (see *Soni* et *al.*, 1980), have shown that the calculate value is only indicative, but nevertheless acceptable.

Calculation of the bed profile due to aggradation				
$R_h = h = 0.895$ [m] ; $S_f = 0.0005$ [-] ; $K = 0.932$ [m²/s]				
$\Delta h = \mathbf{0.065}$ [m] ; $\Delta t = 1.8 \cdot 10^5$ [s]				
x [m]	$x\ (S_e / R_h)$ [-]	$Y = x / (2\sqrt{Kt})$ [-]	$z/\Delta h = erfc(Y)$ [-]	z [m]
10.0	0.01	0.01	0.98623	0.064
50.0	0.03	0.06	0.93123	0.060
100.0	0.06	0.12	0.86296	0.056
300.0	0.17	0.37	0.60459	0.039
500.0	0.28	0.61	0.38813	0.025
700.0	0.39	0.85	0.22696	0.015
900.0	0.50	1.10	0.12032	0.008
1000.0	0.56	1.22	0.08434	0.005
1100.0	0.61	1.34	0.05761	0.004
1300.0	0.73	1.59	0.02484	0.002
1492.3	*0.83*	*1.82*	*0.01000*	*0.001*
1500.0	0.84	1.83	0.00962	0.001
1600.0	0.89	1.95	0.00575	0.000

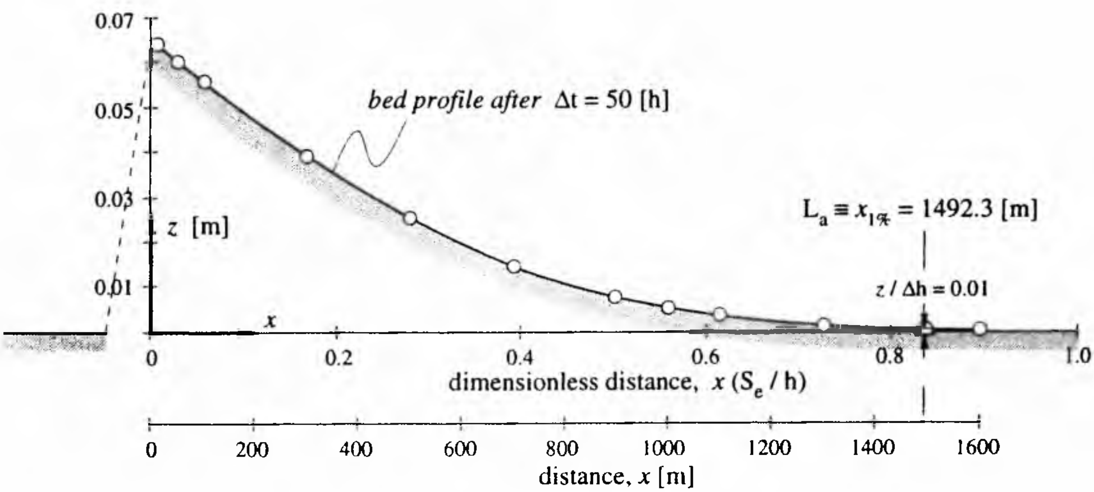


Fig. Ex.6.B.2 Bed profile after 50 [hours] of aggradation.

Ex. 6.C

The unit discharge of a river is kept constant at $q = 2.5$ [m^2/s]. The bed slope $S_f = 5.4 \cdot 10^{-4}$ [-]. The river bed is composed of quasi-uniform sediments ($s_s = 2.65$ [- with an average grain size of $d_{50} = 6$ [mm] and a porosity of $p = 0.3$ [-]. The Mannin coefficient of the bed was determined as being $n = 0.032$ [$m^{-1/3}s$].

This river enters into a reservoir, created by a dam which keeps the water at a height ($H = 23.5$ [m] at the immediate vicinity of the dam.

Determine the deposition pattern of bed-load material, which is carried by the river int the reservoir, after 20 [years] and 100 [years], respectively.

SOLUTION :

a) *General comments on the method of solution* :

Fig. Ex.6.C.1 shows the longitudinal profile of this river-reservoir system. The dai creates a backwater curve extending to a certain upstream distance. This curve can t calculated by using one of the methods presented in chap. 4. The backwater calculatic enables one to know the hydraulic parameters (average velocity, water depth, slope (energy grade line, etc.) for the entire length of the system.

Let there be two stations, (i) and (i+1), separated by a distance of Δx. If tl characteristics of the sediments at the bed are known, one can calculate the bed-loa discharge for these two stations, $q_{sb}(i)$ and $q_{sb}(i+1)$, by using one of the bed-loa formulae presented in sect. 6.3.3. It will then be seen that the bed-load transport at tl upstream station, (i+1) , is larger than the one at the downstream station, (i). In fact tl closer a station is to the dam, the larger is the water depth, resulting in a smaller averag velocity and as a consequence in a decrease in the bed-load transport capacity. Tl difference of the transport capacities between the two consecutive stations causes deposition (or erosion) of the sediments, which in turn modifies the bed level. Th modification of the bed level causes a change in the water-surface profile and therefoi modifies all hydraulic parameters. This cycle repeats itself.

To calculate the deposition of the sediments, namely the *delta* formation, one has t simulate the process described above. Such a simulation involves a large number (calculations and therefore is particularly well suited to treatment on a computer.

In this exercise, a computer program in FORTRAN IV language has been written to carı out this simulation. The program is written in standard FORTRAN and can be run c most of the personal computers. Although a basic knowledge of computers and (programming in languages like FORTRAN, BASIC or PASCAL will certainly be helpfı in understanding this exercise, it is not essential. Special care is taken to make the gener programming techniques understandable to everybody, even to those who do not hav any experience in programming.

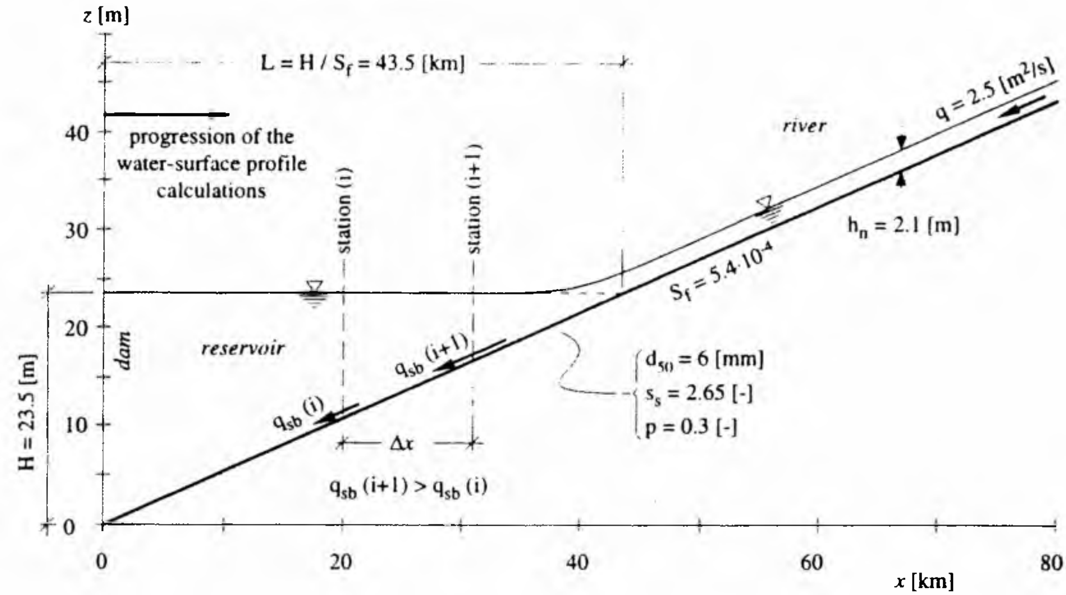


Fig. Ex.6.C.1 Modeling of the river-reservoir system.

b) *Definition of solution domain and boundary conditions* :

In reality, the cross sections of a natural river have complex forms. Nevertheless, by considering that the length of the river-reservoir system is much larger than the water depth, a simplified one-dimensional approach, where the hydrodynamic equations of the water flow and of the bed-load transport are expressed for a unit width, will be adopted.

The modeled river-reservoir system is presented in Fig. Ex.6.C.1. The origin of the coordinate system coincides with the dam location. The dam constitutes a control section and gives the boundary condition necessary for the water-surface profile calculations. Since the flow regime is subcritical, the calculations start at the dam, where the flow depth is known, and proceed towards the upstream.

The length of the reach to be modeled upstream of the dam can be decided by considering the boundary condition for the sediment transport at the upstream end. In fact it is necessary to extend the calculations up to a point where the river atteins its normal depth, h_n. It is even better to include in the calculations a certain length of the river with the normal depth. This insures a sufficiently long river reach at the upstream end, where the bed-load transport is in equilibrium, namely where the bed-load transport capacity at two consecutive sections will be the same.

For the calculation of the unit discharge, q, one can take $R_h = h$; the Manning-Strickler equation, eq. 3.16, can be written as :

$$q = \frac{Q}{B} = \frac{h}{n} R_h^{2/3} S_f^{1/2} = \frac{h^{5/3}}{n} S_f^{1/2}$$

The normal depth can now be calculated with the following expression :

$$h_n = \left(\frac{q\ n}{S_f^{1/2}}\right)^{3/5} = \left(\frac{(2.5)\ (0.032)}{\sqrt{S_f}}\right)^{3/5} = 2.1\ [\mathrm{m}]$$

The Froude number for the uniform flow is :

$$Fr = \frac{U}{\sqrt{gh_n}} = \frac{q}{h_n\sqrt{gh_n}} = \frac{q}{\sqrt{gh_n^3}} = \frac{2.5}{\sqrt{(9.81)\ (2.1)^3}} = 0.26\ [-]$$

The flow is therefore subcritical.

It is not necessary to compute an initial water-surface profile to guess the point where t river reaches its normal depth. By considering the known values of the water depth at t dam, H = 23.5 [m], and of the bed slope, $S_f = 5.4 \cdot 10^{-4}$ [-], the approximate length the reservoir can be calculated : $L = H / S_f \cong 43.5$ [km]. To be able to guarantee sufficiently long river reach at the upstream, where the flow remains uniform throughc the whole simulation period, namely 100 [years], a computational reach length of, f example, TL = 120 [km] shall be adopted. This total length of the system (TL) is divid in ND reaches having a length of DX; this yields NS = ND +1 stations. Starting from t dam location, the stations are numbered from the downstream to the upstream end.

c) *Structure of the program DELTA*:

A *decoupled* algorithm has been used in writing the program DELTA. The adjecti "decoupled" means that the calculations for the liquid and solid phases are carried c separately and successively (see Fig. Ex.6.C.2).

The calculations start at time t = 0, when the bed-level elevations are known. The wat surface profile is calculated without considering the sediment transport. Once the wat surface profile is calculated, the hydraulic parameters are known at all the stations. T bed-load transport rate is now calculated for all the stations. The balance of the sedimei entering and leaving is subsequently calculated for all reaches to find the volume deposition (or erosion). These volumes are then translated into a deposition heig Finally the bed levels are modified by using these deposition heights. This concludes t computational cycle for the time t = 0. The time is then advanced by Δt, and a new wat surface calculation is carried out with the new bed profile; and so on. It should be notic that during the calculations for one phase, the characteristics of the other phase are ke constant.

The program DELTA is written in a didactic style and does not have the pretention being optimized. The complete program code is presented Fig. Ex.6.C.11. Numerous comments inserted in the code explain the flow of t program almost step by step. As far as possible, the names of the variables are chosen recall the notation used in the text. An exhaustive list of variables together with the tyr of variables and explanations, are provided at the beginning of the main program and t related subprograms.

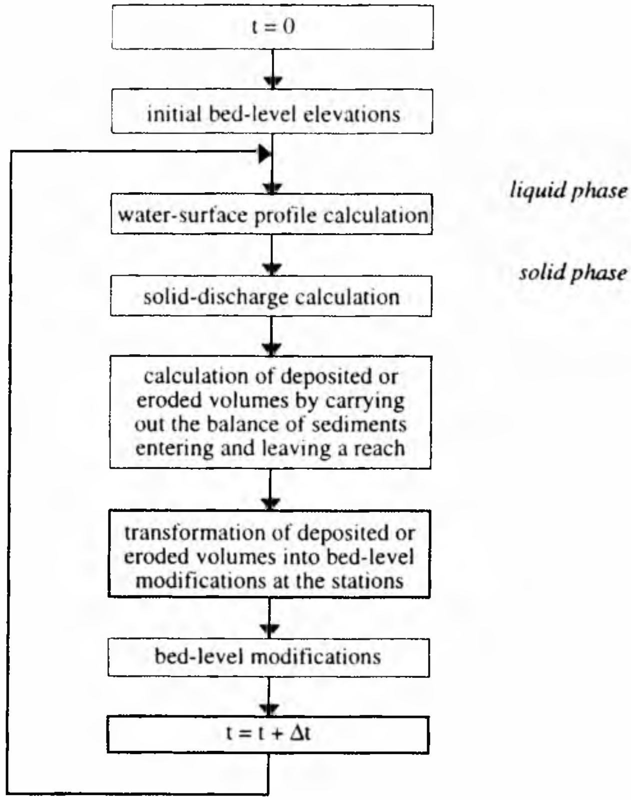


Fig. Ex.6.C.2 Decoupled simulation algorithm.

The program has a modular structure. It is composed of a main program DELTA and nine subprograms, each accomplishing a specific pre-defined task : DREAD; TITLES; RK4; DERIVE; SCHOKL; MEYPET; EINS42; FORMUL and DWRITE. The flowchart, given in Fig. Ex.6.C.3, shows not only the relations between different program units but also the calculation loops inside the principal program. However, it is important to note that the flowchart is somewhat simplified; it does not show all the details of the code. The specific tasks carried out by the main program and each subprogram are described below in detail (see Figs. Ex.6.C.3 and 11).

d) *Working principles of the program DELTA* :

The main program DELTA controls the flow of the entire program. The working principles of the program and the algorithms are described below step by step. The reader is advised to follow these explanations in parallel with the flowchart, given in Fig. Ex.6.C.3, and the program code presented in Fig. Ex.6.C.11.

- The main program first calls the subroutine subprogram DREAD to read the program data by questioning the user. The interactive dialog between the program and the user is presented in Fig. Ex.6.C.5. This dialog will be explained later in detail. The program data are read into the computer in 6 groups :

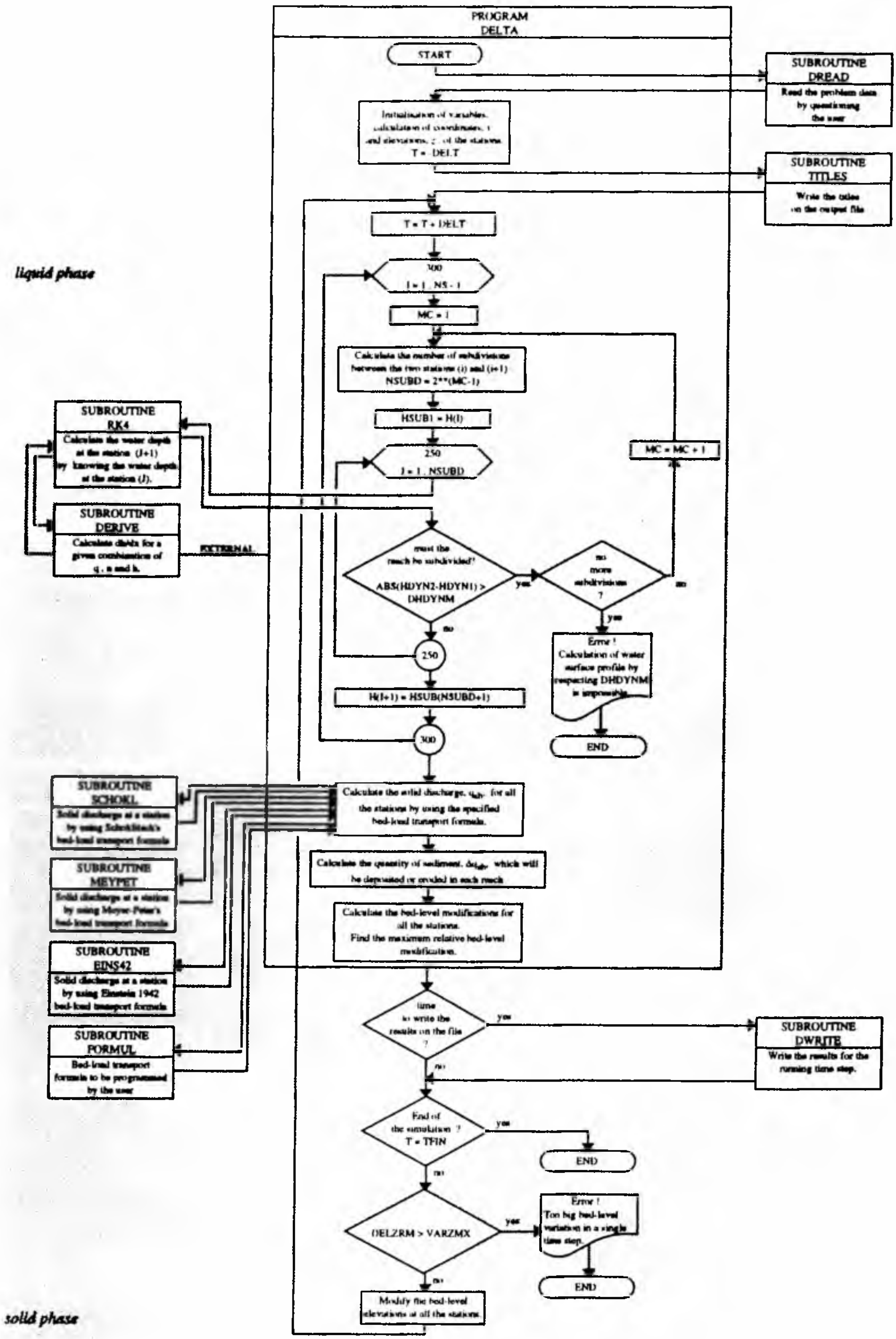


Fig. Ex.6.C.3 Flowchart of the program DELTA.

- *Physical characteristics* (initial bed slope, average sediment diameter, Manning-Strickler coefficient, densities of the water and of the sediments, discharge per unit width),
- *Choice of the bed-load transport formula* (number of the formula to be used),
- *Data concerning the modification of the bed profile* (maximum relative bed-level change tolerated, porosity of the sediments, the ratio of the upstream/downstream heights of the sediment deposition or erosion),
- *Information concerning the computational domain* (coordinates of the first and the last station, space-step length, maximum tolerated dynamic-head variation, maximum number of subdivisions which can be automatically created),
- *Boundary conditions* (water depth at the downstream end),
- *Simulation time and the printing of results* (time step, duration of the simulation, frequency of the printing of the results and the name of the output file).

- According to the data supplied by the user, the program calculates the coordinates of the stations and the initial bed level at these stations. It also initializes certain variables, such as the calculation-steps counter, eroded or deposited cumulative volumes, etc.

- The subprogram TITLES is called to echo-print of the program data on the output file.

- The time is advanced one time step. The calculation loop starts in fact at this point. It can be seen that before entering the calculation loop the time is initialized as T = – DELT; in this way the time for the first calculation step is correctly obtained as being T = 0.

- The calculation of the water-surface profile, using the running bed profile, is carried out using the 4th-order Runge-Kutta method. The differential equation for the free-surface flow in a rectangular channel is given by:

$$\frac{d}{dx}\left(\frac{Q^2}{2g(Bh)^2}\right) + \frac{dh}{dx} - S_f = -S_e \tag{4.5}$$

For a very wide river, $B >> h$, with a constant unit discharge of $q = Q/B$, this equation becomes :

$$\frac{q^2}{2g}\frac{d}{dx}\left(\frac{1}{h^2}\right) + \frac{dh}{dx} - S_f = -S_e$$

The slope of the energy-grade line can be expressed with the Manning-Strickler formula for uniform flow, eq. 3.16. Recalling that for a very wide river one can take $R_h = h$, the following is written :

$$S_e = \frac{q^2 n^2}{h^{10/3}} \tag{6.α}$$

Substituting this expression into the differential equation for the free-surface flow one obtains (see eq. 4.8a) :

$$\frac{dh}{dx} = -\frac{S_f - \dfrac{q^2 n^2}{h^{10/3}}}{1 - \dfrac{q^2}{gh^3}}$$

It is to be noted that a negative sign is added before the term on the right-hand side of the equation to take into account that the calculation progresses from downstream to upstream. This differential equation is solved using the 4th-order Runge-Kutta method (presented in detail in Ex.7.A.b) by calling the subroutine subprogram RK4. The differential equation is programmed in the subroutine subprogram DERIVE, whose name is passed to the subprogram RK4 in the argument list. For this reason, according to the rules of FORTRAN language, the subprogram DERIVE is declared as EXTERNAL in the main program.

- The calculations for the liquid phase are finished for this time step. The program now proceeds with the calculations for the solid phase. During these calculations it is assumed that the water surface does not vary. The bed-load transport at the stations is calculated by calling the subprogram corresponding to the method specified by the user :

  - The subprogram SCHOKL calculates the bed-load discharge with the method of *Schoklitsch* (1950), whose formula is given by eq. 6.31 :

$$q_{sb} = \frac{2.5}{s_s} S_e^{3/2} (q - q_{cr}) \quad \Rightarrow \quad QSU = \frac{2.5 * SEFF^{3/2} * (QU - QCRIT)}{SS}$$

  The critical liquid discharge is calculated using eq. 6.31a :

$$q_{cr} = 0.26 (s_s - 1)^{5/3} d_{40}^{3/2} S_e^{-7/6} \quad \Rightarrow \quad QCRIT = 0.26 * (SS - 1)^{5/3} * D50^{3/2} * SEFF^{-7/6}$$

  Since the grain-size distribution is uniform, one may take $d_{40} = d_{50}$. If at a station $q_{cr} > q$, the program will assume $q_{sb} = 0$. The slope of the energy-grade line, S_e (SEFF), is calculated by the main program using the Manning-Strickler formula (see the explanations for the water-surface profile calculation) and sent to the subprogram in the argument list.

  - The subprogram MEYPET calculates the bed-load discharge with the method of *Meyer-Peter* et *al.* (1948), whose formula is given by eq. 6.32 :

$$q_{sb} = \frac{1}{g(\rho_s - \rho)} \left(\frac{g \rho R_{hb} \xi_M S_e - 0.047 g (\rho_s - \rho) d_{50}}{0.25 \rho^{1/3}} \right)^{3/2}$$

  In the program this equation is written as:

$$QSU = \frac{1}{G*(ROS-ROE)} \left(\frac{G*ROE*RH*FCOR*SEFF - 0.047*G*(ROS-ROE)*D50}{0.25*ROE^{1/3}} \right)^{3/2}$$

Since the calculations are done for a unit width, the program uses R_{hb} = h (RH = H). It is assumed that $q_{sb} = 0$, if $(g\,\rho\,R_{hb}\,\xi_M\,S_e) < (0.047\,g\,(\rho_s - \rho)\,d_{50})$. The user may or may not use the roughness parameter, ξ_M (FCOR). This parameter is calculated in the subprogram DREAD, during data input, according to the expression :

$$\xi_M = \left(\frac{n'}{n}\right)^{3/2} \qquad \Rightarrow \qquad FCOR = \left(\frac{CN50}{CN}\right)^{3/2}$$

where n' (CN50) is the grain roughness calculated using the formula of Strickler, eq. 3.18 :

$$n' = \frac{d_{50}^{1/6}}{21.1} \qquad \Rightarrow \qquad CN50 = \frac{D50^{1/6}}{21.1}$$

and n (CN) represents the total bed roughness, which is introduced by the user during the data input. If the user chooses not to make any corrections, the program takes FCOR = 1.

- The subprogram EINS42 calculates the bed-load discharge with the method of *Einstein* (1942), described by *Graf* (1971, p. 145) :

$$q_{sb} = \frac{\sqrt{(s_s - 1)\,g\,d_{50}^3}}{0.465} \exp\left(\frac{-0.391\,(s_s - 1)\,d_{50}}{R_{hb}'\,S_e} \right)$$

In the program this equation is written as:

$$QSU = \frac{\sqrt{(SS - 1)*G*D50^3}}{0.465} \exp\left(\frac{-0.391*(SS - 1)*D50}{RH*SEFF} \right)$$

Since the calculations are done for a unit width, the program uses RH = H ($R_{hb}' \equiv R_{hb} \equiv h$).

- The subprogram FORMUL does not do any calculations, but simply displays a message, inviting the user to program a bed-load transport formula of his choice in this subprogram.

- Let us now go back to the main program DELTA. The quantity of the sediments to be deposited (or eroded) in a reach, Δq_{sb} , depends on the difference between the bed-load transport capacities at the upstream, $q_{sb}(i+1)$, and at the downstream, $q_{sb}(i)$, stations :

$$\Delta q_{sb}(i) = q_{sb}(i+1) - q_{sb}(i) \qquad \Rightarrow \qquad DELQS(I) = QSU(I+1) - QSU(I)$$

In the program the reaches between two consecutive stations are numbered frc downstream to upstream. The number of a reach is therefore the same as t number of the station at its downstream end (i.e.: the **reach** (i) is limited by t **station** (i) at the downstream and by the **station** (i+1) at the upstream). This fa is used in the program as a *programming trick*. The same variable I is used denote both a reach and the station. In this way, the solid discharges, QSU(I+ and QSU(I), refer to the **stations** (I+1) and (I), respectively, whereas DELQS(I) the difference in transport capacities for the **reach** (I). While reading the progra this double role of the variable I must be kept in mind.

- The procedure used by the program to translate the transport-capacity differenc between the upstream and the downstream ends of a reach into a deposition heig — this is Exner's relationship — is presented in Fig. Ex.6.C.4. The virtual volur of the sediments to be deposited at the **reach** (i), by taking into account the volur increase due to the porosity, p. is the following :

Deposition volume for the **reach** (i) = $\Delta q_{sb}(i)\ \Delta t\ \dfrac{1}{(1-p)}$

The program admits that this volume creates a trapezoidal deposit whose upstrea and downstream heights are $\delta z_{am}(i)$ and $\delta z_{av}(i)$, respectively. For the **reach** (one can therefore write :

$$\Delta q_{sb}(i)\ \Delta t\ \frac{1}{(1-p)} = \frac{\delta z_{am}(i) + \delta z_{av}(i)}{2}\ \Delta x \qquad (6.4$$

During data input the user specifies the ratio between the upstream and tl downstream heights of this trapezoidal deposit, $\lambda = \delta z_{am}(i)/\delta z_{av}(i)$ (in the progra $\lambda \equiv$ HAMHAV). By using this information one obtains :

$$\delta z_{av}(i) = \frac{2}{(1+\lambda)}\left(\frac{\Delta q_{sb}(i)\ \Delta t}{\Delta x\ (1-p)}\right) = \text{CRAV}\left(\frac{\Delta q_{sb}(i)\ \Delta t}{\Delta x\ (1-p)}\right)$$

$$\delta z_{am}(i) = \frac{2\lambda}{(1+\lambda)}\left(\frac{\Delta q_{sb}(i)\ \Delta t}{\Delta x\ (1-p)}\right) = \text{CRAM}\left(\frac{\Delta q_{sb}(i)\ \Delta t}{\Delta x\ (1-p)}\right)$$

For an internal **station** (i), the upstream (i–1) and the downstream (i) **reach** give two different deposition heights : $\delta z_{am}(i-1)$ and $\delta z_{av}(i)$, respectively. The fin value of the deposition height at such a **station** (i) corresponds therefore to tl average of these two values :

$$\Delta z(i) = \frac{1}{2}\left[\frac{2\lambda}{(1+\lambda)}\left(\frac{\Delta q_{sb}(i-1)\ \Delta t}{\Delta x\ (1-p)}\right) + \frac{2}{(1+\lambda)}\left(\frac{\Delta q_{sb}(i)\ \Delta t}{\Delta x\ (1-p)}\right)\right]$$

At the end of each time step the program searches for the maximum relati variation of the bed level, $\Delta z(i)/h_i$ (= DELZRM), and compares it with tl maximum tolerated value (VARZMX), as specified by the user during data input. DELZRM > VARZMX , the program displays an error message and stops.

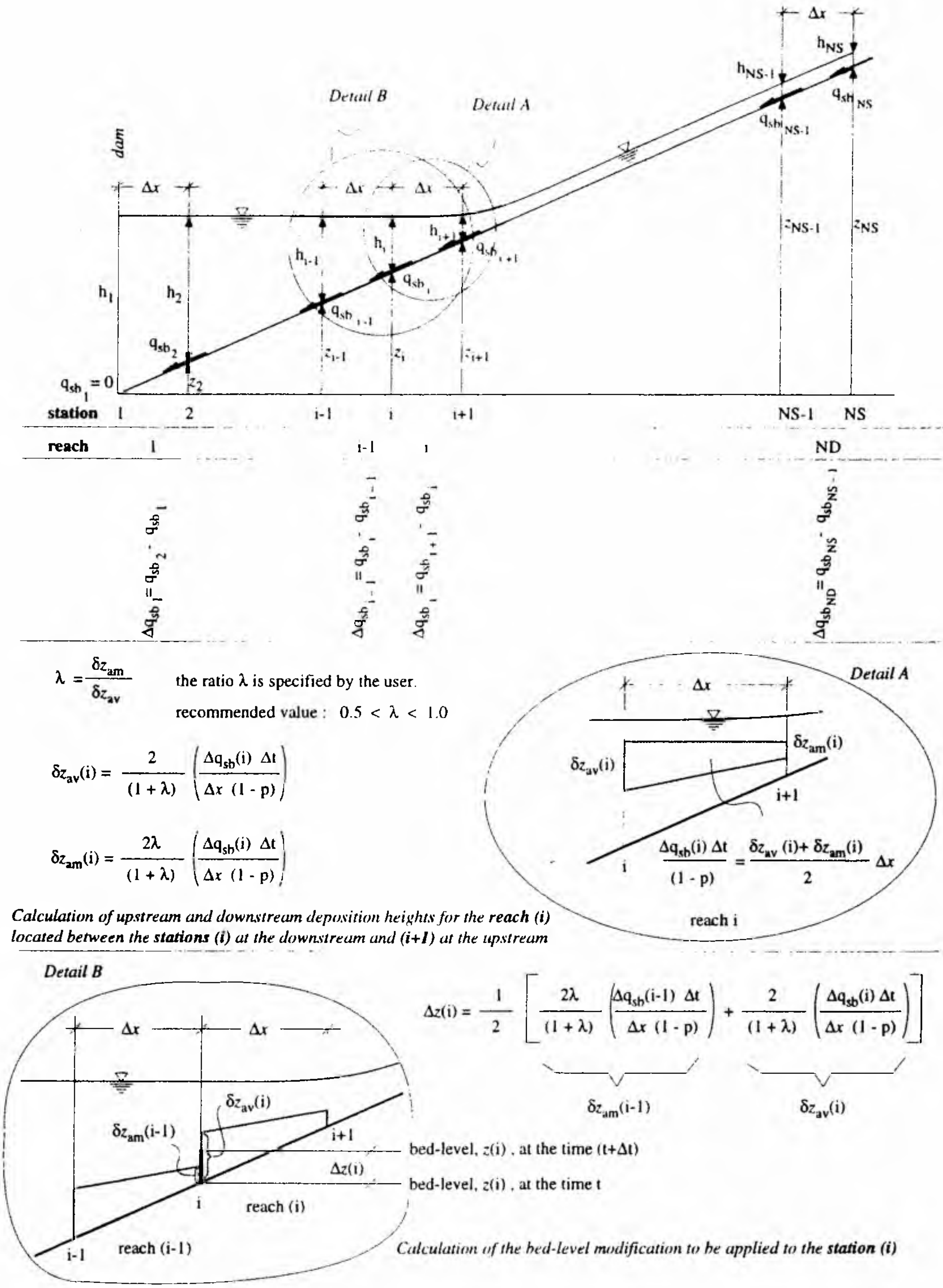


Fig. Ex.6.C.4 Procedure for calculating the volume of the sediments to be deposited at each reach and the modification of the bed level.

- At the end of a time step — if the printing time has come — the main program cal the subprogram DWRITE for printing the results for the current time step on th users specified output file as well as on a second file called GRAPH.DAT.
- If the simulation period specified by the user is not yet reached, the progra modifies the bed-level heights at the stations :

$$z_i^{t+\Delta t} = z_i^{t} + \Delta z_i$$

and then goes to the beginning of the calculation loop to start a new time step with water-surface profile calculation, using this new bed profile.

The program DELTA is a simplified way of solving numerically the equations of Sain Venant - Exner (see sect. 6.2.1). The equations of Saint-Venant, eq. 6.1 and eq. 6.2, ar written for a steady flow in the form of the equation for the free-surface flow, eq. 4.5; th slope of the energy-grade line, eq. 6.3, is expressed here by the Manning-Strickle formula, eq. 6.α; the relationship of Exner, eq. 6.4a, gives the volume of the deposit; th solid discharge, eq. 6.5, is expressed using the formula of *Schoklitsch* (1950), eq. 6.3 and the formula of *Meyer-Peter* et *al.* (1948), eq. 6.32.

e) *Use of the program DELTA for solving the problem* :

The source code of the program DELTA presented in the Fig. Ex.6.C.11 is first *compile* and *linked* to obtain an executable code. A FORTRAN compiler is of course necessary t do these operations on your computer. The reader should consult the manuals of hi computer to learn the exact procedure to follow.

The user feeds the program data into the computer interactively by answering th questions asked by the program. Numerous comments has been introduced into th conversation to remind the user of certain important points treated in chap. 6 and to guid the user in making his choices. The program also checks some of the likely errors in th data introduction and warns the user. In case of an error, the question is repeated until th user answers correctly.

The dialog between the user and the computer for solving the present problem i presented in Fig. Ex.6.C.5. The values typed in by the user are highlighted by a whit background. They are followed by a sign representing the RETURN (CR or ENTER o some computers) key on the keyboard.

The user begins the work by introducing the initial bed slope (SF) and the average grai diameter (D50) of the sediments. Using the average grain diameter the program calculate the Manning-Strickler coefficient due to grain roughness and displays it on the screen CN50 = 0.0202 ($m^{-1/3}s$). The program then asks the user to enter the total Manning Strickler coefficient (CN). It is the value of CN that is used later in all calculation (specifying a value of CN = CN50 will mean that the grain roughness is the only cause o the regular head loss). If the user has good reason to think that the head loss is larger the the one given by CN50 — since bed forms (such as dunes) or other irregularities ar present — the estimated total value should be entered. For the present problem the valu given in the problem will be entered, namely CN = 0.032 ($m^{-1/3}s$).

Supplying the density of the sediments (ROS) and of the water (ROE) as well as th discharge per unit width (QU) is sufficiently clear.

The program proposes now 4 formulae for solid discharge as bed-load transport and asks the user to select one of them. To select the formula of Meyer-Peter et al., one enters "2". Up to this point the dialog with the computer is the same for all cases. Now comes a short conversation with the computer which depends on the selected method. Some methods do not need supplementary data. The program only displays the method used. In case of the selection of the formula of Meyer-Peter et al., if CN > CN50, the program asks if the user wishes to use the roughness parameter, FCOR.

From here on the text of the dialog with the computer is again the same, regardless of the choice of the bed-load transport formula. The conversation continues with the introduction of the data concerning the modification of the bed profile. First, the desired value of the maximum relative variation of the bed level during a single time step, VARZMX = 0.1, must be entered. The program sees to it that at none of the stations, during a single time step, the bed-level modification, DZ, is higher than 10% of the water depth. The user enters also the porosity of the deposited sediments, p (POROS).

The procedure used by the program to transform the volume of the sediments deposited in reaches into upstream and downstream bed-level modification heights was described above in detail. For the ratio of the upstream/downstream heights a value of λ = HAMHAV = 0.75 is entered; this means that the deposition at the downstream end of the reach is higher than the one at the upstream end. It is important to note that this ratio must be between 0.5 and 1.0. In case of high bed-load transport rates, a uniform distribution of the sediments over the reach (HAMHAV = 1.0) may cause instabilities in the calculation of the bed levels at the stations.

The data input continues with the information on the calculation domain. The first station is located on the dam; by entering X1 = 0 [m], the origin of the coordinate system is placed at the dam. The calculations will be carried out up to an upstream distance of XF = 120000 [m].

The choice of the step length in the longitudinal direction, DX, is important. A very small step length necessitates a very small time step; this increases of course the calculation time. For the calculation of the deposition one can choose relatively large time steps. In the present case a step length of DX = 600 [m] is used. A too big step length may cause errors in the calculation of the water-surface profile (especially around the region where the river meets the reservoir) and may cause the program to stop the execution. To overcome this difficulty without loosing the advantages of working with long steps the program uses a clever programming trick.

The water-surface profile is calculated using the 4th-order Runge-Kutta method. The calculations start at the station located at the dam where the water depth is known. By sending this value to the subprogram RK4, the main program obtains in return the water depth at a station immediately upstream. To be able to guarantee a sufficient precision in the water-surface profile calculations, the program checks if the difference of the dynamic heads, $U^2/2g$, between these two successive stations is less than a value specified by the user, in the present case DHDYNM = 0.01 [m], or not. If in a reach this value is not respected, the program divides this reach into $2^1 = 2$ *sub-reaches*, and redoes the calculation in two steps. If the criteria is still not respected the program tries this time $2^2 = 4$ sub-reaches and so on. It is the user who specifies up to which power of 2 the

```
                PROGRAM FOR THE BED-LOAD TRANSPORT CALCULATION
               BY TAKING INTO ACCOUNT A BED-LEVEL MODIFICATION.

NOTES :
- UNIT SYSTEM = SI
- NUM. METHOD FOR WATER-SURFACE PROFILE CALCULATION = 4th ORDER RUNGE-KUTTA
- THE FLOW IS SUBCRITICAL (Fr < 1 ). THE WATER-SURFACE PROFILE CALCULATION
  STARTS AT THE DOWNSTREAM END AND PROGRESSES TOWARDS UPSTREAM
- SEDIMENT TRANSPORT CALCULATIONS ARE CARRIED OUT FROM UPSTREAM TO DOWNSTREAM
- THE CALCULATIONS ARE MADE FOR A UNIT WIDTH


PHYSICAL CHARACTERISTICS DATA :
Initial bed slope,                 SF (-)         ? = 5.4e-4      RETURN

Average diameter of sediments,     D50 (mm)       ? = 6           RETURN


According to eq. 3.18, the Manning coefficient
due to the grain roughness is :

CN50 = d50^(1/6) / 21.1 =   .0202(s/m^1/3)

Total Manning-Strickler coeff.,    CN (s/m^1/3) ? = 0.032       RETURN

Density of sediments,              ROS (kg/m3)  ? = 2650        RETURN
Density of water,                  ROE (kg/m3)  ? = 1000        RETURN

Unit discharge,                    QU (m2/s)    ? = 2.5         RETURN


CHOICE OF THE BED-LOAD FORMULA :
The program allows the use of one of the
following 4 bed-load formulae :
1- Schocklitsch        (1950)
2- Meyer-Peter et al.  (1948)
3- Einstein            (1942)
4- Your formula (to be programmed !)
Which formula do you choose ?      (1 a 4)       = 2          RETURN

MEYER-PETER (1934) bed-load formula is chosen.

Do you want to use the roughness parameter, FCOR ?
Answer Y(es) or N(o)/CR ? = Y

The roughness parameter is then,   FCOR = (CN50 / CN)**1.5 =    .502


BED-LEVEL MODIFICATION DATA :
The bed-level variation during a single time step
must not be too big; otherwise instabilities of
the bed profile may appear. At the end of each time
step, the program calculates the maximum relative bed-
level variation, DELZ/H , (bed-level variation divided
by the water depth) and checks that it is not bigger
than a value specified by the user.

Max. rel. bed-level variation,    VARZMX (-)    ? = 0.1        RETURN

Porosity of deposited sediments,  POROS (-)     ? = 0.3        RETURN

The volume of sediments deposited in a reach is
transformed into a bed-level variation height at the
downstream and upstream, by defining a trapezoid.
The user controls the sediment distribution by choosing
the ratio of upstream/downstream heights of the trapezoid.
We recommend to use the values : 0.5 < HAMHAV < 1.

Ratio of upstr./downstr. heights, HAMHAV (-)   ? = 0.75       RETURN
```

Fig. Ex.6.C.5 Interactive dialog with the computer for the data input (continuec

```
INFORMATION ON COMPUTATIONAL DOMAIN :
x-coordinate of first station,       X1  (m)     ? = 0           (RETURN)
x-coordinate of last station,        XF  (m)     ? = 120000      (RETURN)

Total reach length is therefore,     TL  (m)       =  120000.00

Now you have to specify the step length in x-direction.
If the step length is too long to guarantee a correct
prediction of the water-surface profile, the program
will automatically add some intermediate stations. The
results, however, are only printed at the stations
specified by the user; others remain invisible. The step
length for the space must therefore be specified to
guarantee a correct representation of physical processes
involved in the bed-load transport. In case of doubt, you
can repeat the simulation with different step lengths and
compare the results.

Step length in x-direction,          DX  (m)     ? = 600         (RETURN)

Number of reaches,                   ND            =   200
Number of stations,                  NS            =   201

Max. tolerated var. of dyn. head,   DHDYNM (m) ? = 0.01          (RETURN)

In case this value is exceeded the reach will be subdivided
in order to refine the calculations.
The number of divisions is specified in powers of 2.
The maximum value is       MCMAX =    7
Which corresponds to   2.0^MCMAX = 128 subdivisions.

Maximum number of subdiv. in powers of 2, NMC  ? = 7             (RETURN)


INFORMATION ON BOUNDARY CONDITIONS :
Note that the sediment transport at downstream end
is automatically taken as zero,      QSU(1)       = 0.0
Water depth at the downstream end,   H1  (m)     ? = 23.5        (RETURN)


PARAMETERS RELATED TO TIME AND PRINTING OF RESULTS :
Time step,                   DELT (days)      ? = 10             (RETURN)

Duration of simulation,      TFIN (days)      ? = 36500          (RETURN)

Results will be printed every NPP step        ? = 730            (RETURN)

Name of output file (max. 40 char.)           ? = MEYPET.OUT     (RETURN)

TIME STEP      =              1.
TIME (days)    =          .000

TIME STEP      =              2.
TIME (days)    =        10.000
....
....
TIME STEP      =           3651.
TIME (days)    =     36500.000

  2 subdivisions between stations       57 and    58

  4 subdivisions between stations       57 and    58

  8 subdivisions between stations       57 and    58

 16 subdivisions between stations       57 and    58

NORMAL END OF PROGRAM

PRESS ON RETURN KEY TO EXIT                                      (RETURN)
```

Fig. Ex.6.C.5 Interactive dialog with the computer for the data input (end).

program should continue to subdivide the reach. The maximum number of subdivis provided in the program is $2^7 = 128$, but this can be modified. In the present cas maximum number of subdivisions is chosen as NMC ≤ 7. While running the progra case of subdivision of a reach, a message is displayed on the screen indicating the nu of the reach and the number subdivisions applied (as a power of 2). In this way the can follow the calculations.

The program needs two boundary conditions to solve the problem. The boun condition for the calculation of sediment deposition is implicit. The program automati considers a zero solid discharge at the dam section (station). For the water-su calculations the user enters the water depth at the dam, H1 = 23.5 [m].

The last group of data concerns the time and the printing of the results. The choice o time step depends of course on the step length in the longitudinal direction. In the pr case a time step of DT = 10 [days] is chosen. The calculations are done for a simul period of hundred years (TFIN = 36500 [days]). The results are written in the outpu MEYPET.OUT every 730 steps; this corresponds to a period of 20 [years].

The program creates two other files in the current directory (the directory in whic program is run) :

DIALOG.DAT : contains the text of the dialog with the computer to enter the data.
GRAPH.DAT : contains the station numbers, the bed level and the water-su elevation separated by commas. This file can be easily read by commercial spreadsheet program to draw the longitudinal profile.

Before running the program on a micro-computer, the user should make sure tha current directory does not contain files with these two names; in such a case the pro will refuse to start. If you do not want to erase these files you must either rename the move them to another directory. You may also try to run the program in anc directory.

f) *Results of the calculations with the program DELTA* :

The formation and the advance of the delta in the river-reservoir system were simul for a period of 100 [years] with the program DELTA, using the methods of *Meyer-I* et *al.* (1948) and of *Schoklitsch* (1950).

The output file MEYPET.OUT containing the results of the simulation of the delta u the method of *Meyer-Peter* et *al.* (1948) is partially, namely for T = 0, 20 100 [years], presented in the Fig. Ex.6.C.6. Due to lack of space the results for T = 60 and 80 [years] are omitted. For the same reason the output file for the simulation u the method of *Schoklitsch* (1950) is not presented either. The results for these simulations are presented in a graphical form in Figs. Ex.6.C.7 and 8, respectively.

At the beginning of the output file one finds the problem title and a summary of al data fed in by the user as well as some useful parameters calculated by the program results at different time steps follow. Since a time step of 10 [days] was used, and it decided that the results are to be printed at every 730 steps (see Fig. Ex.6.C.5) results are printed with a time interval of 20 [years], starting with the initial t T = 0 [years].

The results for each time step are preceded by a header, indicating the time step-number as well as the time itself in seconds, hours, days and years. The total volumes of the deposited (DQSDET) and eroded (DQSERT) sediments refer to the cumulative sums of the positive and negative values of the variable DELQS, since the beginning of the simulation (T = 0). The absolute (DELZMX) and relative (DELZRM) maximum bed-level variations refer to the bed-level variations (DELZ) for the runnung time step; being the one printed at the top right. The explanations for the different columns of the output file are given below :

STATION NO		:	Station number. There are 201 numbered stations, starting with station 1 (the dam at the downstream end) up to the station 201 (at the upstream end).
X	(m)	:	Distance between the station and the downstream end (station no 1).
ZF	(m)	:	Bed-level elevation with respect to the first station. The bed-level elevations, ZF, at time T = 0 are also stored as initial bed-level elevations in the variable ZFI.
ZF – ZFI	(m)	:	Difference between the running bed-level elevation, ZF (at the time indicated on the top left) and the initial bed-level elevation, ZFI (at time T = 0). It is this column which shows the *height of the deposition* (or the erosion). On Fig. Ex.6.C.6, the region of the delta formation is highlighted by a thin frame.
H	(m)	:	Running actual water depth.
ZF + H	(m)	:	Water-surface elevation with respect to the bed-level elevation at the downstream end (at the dam).
U	(m/s)	:	Average velocity.
Fr	(-)	:	Froude number.
QSU	(m^3/s/m)	:	Solid discharge at a station.
DELQS	(m^3/s/m)	:	Difference of solid discharge between the extremities of a reach. There are only 200 reaches for NS = 201 stations. This is the reason for having DELQS = 0 on the line 201; however, this value is not used in the calculations. To calculate the deposition height at the upstream end of the computational domain, DELZ(NS), the program assumes implicitly that DELQS(NS) = DELQS(NS–1).
DELZ	(m)	:	Modifications to be applied to the bed, before starting the calculations for the next time step.

The different stages of the formation and the progression of the delta and its influence on the water-surface profile can readily be observed in Figs. Ex.6.C.7 and 8. These figures have been prepared by plotting the information in columns ZF, bed-level elevation, and (ZF+H), water-surface elevation, as a function of the distance given in column X. By studying the contents of the output file, presented in Fig. Ex.6.C.6, and the profiles plotted in Figs. Ex.6.C.7 and 8, one can make several interesting observations.

First the column QSU for the time **T = 0 [years]** in Fig. Ex.6.C.6 will be considei From the dam up to the station 69 (x = 40.8 [km]), there is no solid dischai The velocity in this reach is equal to or smaller than 1 [m/s]. From station 71 (x = 42.0 [km]) onward the solid discharge increases towards the upstrean attain a constant value of QSU $\cong 0.16 \cdot 10^{-4}$ [m^3/s/m], which is the solid discharge for river cross-section. The column DELQS shows that the deposition of the sedime occurs between the station 69 (x = 40.8 [km]) and station 95 (x = 56.4 [kn Downstream of this reach there is no sediment transport whereas at the upstre constitutes the river reach where the bed-load transport is in equilibrium. The format of the delta starts therefore at the point where the river enters the reservoir. The colu DELZ gives the modifications to be applied to the bed elevation between the station and the station 95, for the next time step.

At time **T = 20 [years]**, as the column ZF–ZFI shows, the downstream end of the d is at the station 62 (x = 36.6 [km]). The maximum height of the delta is h_d = 1.23 [m the station 64 (x = 37.8 [km]). By integrating the data in this column, using trapezoidal rule, the total volume of deposited sediments is found as being 14396 [m^3/ This volume takes into account the porosity. In the header of the results T = 20 [years], the cumulative volumes of deposition and erosion given as : DQSDET = 12650.7 [m^3/m] and DQSERT = –2559.5 [m^3/ı The net volume of the deposited sediments is now calculated as be (DQSDET + DQSERT) = 10091.2 [m^3/m]. By multiplying this volume with coefficient of swelling CFOI = 1 / (1–p) = 1.4286, one gets the swelled volume deposited sediment of 14416 [m^3/m]; this value is very close to 14396 [m^3/m] calcul before. In Fig. Ex.6.C.6, the extent of the delta is highlighted by enclosing a portioi the column ZF–ZFI in a rectangle. The bed-level modifications near the upstream end in fact very small (see Fig. Ex.6.C.7).

```
              PROGRAM FOR THE BED-LOAD TRANSPORT CALCULATION
             BY TAKING INTO ACCOUNT THE BED-LEVEL MODIFICATION.

NOTES :
- UNIT SYSTEM = SI
- NUM. METHOD FOR WATER-SURFACE PROFILE CALCULATION = 4th ORDER RUNGE-KUTTA
- THE FLOW IS SUBCRITICAL (Fr < 1 ). THE WATER-SURFACE PROFILE CALCULATION
  STARTS AT THE DOWNSTREAM END AND PROGRESSES TOWARDS THE UPSTREAM END
- SEDIMENT TRANSPORT CALCULATIONS ARE CARRIED OUT FROM UPSTREAM TO DOWNSTREAM
- CALCULATIONS ARE MADE FOR A UNIT WIDTH

PHYSICAL CHARACTERISTICS DATA :
Initial bed slope,                     SF (-)        = .0005400

Average diameter of sediments,      D50 (mm)         = 6.00
Manning coeff. for sed. grains,    CN50 (s/m^1/3)    =  .0202
Manning-Strickler coefficient.       CN (s/m1/3)     =  0320
Density of sediments,               ROS (kg/m3)      = 2650.00
Density of water,                   ROE (kg/m3)      = 1000.00
Unit discharge,                      QU (m2/s)       =    2.50

INFORMATION RELATED TO WATER-SURFACE CALCULATIONS
Max. tolerated var. of dyn. head,   DHDYNM (m)      =  .010
Maximum number of subdiv. in powers of 2, NMC       = 7

INFORMATION ON CALCULATION DOMAIN
x-coordinate of first station,         X1 (m)       =       .00
x-coordinate of last station,          XF (m)       = 120000.00
Total reach length is therefore,       TL (m)       = 120000.00
Step length in x-direction,            DX (m)       =    600.00
Number of reaches,                     ND (m)       =   200
Number of stations,                    NS           =   201

PRINTING OF RESULTS :
Printing of results every NPP step                  =   730
Name of output file (max. 40 char.)                 = MEYPET.OUT

CHOICE OF BED-LOAD FORMULA:
Bed-load transport formula by Meyer-Peter et al. (1934) is used.

Roughness coefficient.  FCOR = (CN50 / CN)^(3/2) =    .502

INFORMATION RELATED TO CALCULATION OF DEPOSITION VOLUME :
Max. rel. bed-level variation, VARZMX (-)       =   .10
Coefficient of swelling,          CFOI (-)       = 1.4286
Ratio of upst./dwnst. heights, HAMHAV (-)       =   .750

BOUNDARY CONDITIONS :
Bed-load transport at the downstream end is zero.
Water depth at downstream end,   H1 (m)          =  23.500

PARAMETERS RELATED TO TIME :
Time step,                       DELT (days)     =     10.000
Duration of simulation,          TFIN (jours)    =  36500.000
```

Fig. Ex.6.C.6 Output file of the program DELTA, resulting from a simulation using method of *Meyer-Peter* et *al.* (1948) (continued): file header.

Time step = [illegible]
Time (s) = 0.000000000D+00
Time (hours) = 000
Time (days) = 000
Time (years) = 000

Total volume of deposited sediments DVDEP (m3/m) = 0.138047D+02
Total volume of eroded sediments DVERO (m3/m) = 0.000000D+00

Max. [illegible] DELZMX (m) = 0.759922D-02 at the station, IMAXZ = 71
Max. relative [illegible] DELZRM = [illegible] at the station, IMAXR = 71

STATION NO	X (m)	ZF (m)	ZF - ZFI (m)	[illegible]	[illegible]	[illegible]	[illegible]	QS (m3/s/m)	DELQS (m3/s/m)	DELZ (m)
1	[illegible]	0.000000D+00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
3	1200.00	0.648000D+00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
5	2400.00	0.129600D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
7	3600.00	0.194400D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
9	4800.00	0.259200D+01	[illegible]	[illegible]	[illegible]	[illegible]	.009	0.000000D+00	0.000000D+00	0.000000D+00
11	6000.00	0.324000D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
13	7200.00	0.388800D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
15	8400.00	0.453600D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
17	9600.00	0.518400D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
19	10800.00	0.583200D+01	[illegible]	[illegible]	[illegible]	[illegible]	.011	0.000000D+00	0.000000D+00	0.000000D+00
21	12000.00	0.648000D+01	[illegible]	[illegible]	[illegible]	[illegible]	.012	0.000000D+00	0.000000D+00	0.000000D+00
23	13200.00	0.712800D+01	[illegible]	[illegible]	[illegible]	[illegible]	.012	0.000000D+00	0.000000D+00	0.000000D+00
25	14400.00	0.777600D+01	[illegible]	15.724	[illegible]	.154	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
27	15600.00	0.842400D+01	0.000000D+00	[illegible]	[illegible]	.166	.014	0.000000D+00	0.000000D+00	0.000000D+00
29	16800.00	0.907200D+01	0.000000D+00	14.414	0.235059D+02	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
31	18000.00	0.972000D+01	0.000000D+00	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
33	19200.00	0.103680D+02	0.000000D+00	[illegible]	0.235080D+02	.190	.017	0.000000D+00	0.000000D+00	0.000000D+00
35	20400.00	0.110160D+02	0.000000D+00	12.493	0.235094D+02	.200	.018	0.000000D+00	0.000000D+00	0.000000D+00
37	21600.00	0.116640D+02	0.000000D+00	11.847	0.235110D+02	.211	.020	0.000000D+00	0.000000D+00	0.000000D+00
39	22800.00	0.123120D+02	0.000000D+00	[illegible]	0.235130D+02	.223	.021	0.000000D+00	0.000000D+00	0.000000D+00
41	24000.00	0.129600D+02	0.000000D+00	[illegible]	0.235153D+02	.237	.023	0.000000D+00	0.000000D+00	0.000000D+00
43	25200.00	0.136080D+02	0.000000D+00	[illegible]	[illegible]	.252	.026	0.000000D+00	0.000000D+00	0.000000D+00
45	26400.00	0.142560D+02	0.000000D+00	[illegible]	0.235219D+02	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
47	27600.00	0.149040D+02	0.000000D+00	[illegible]	0.235265D+02	[illegible]	.032	0.000000D+00	0.000000D+00	0.000000D+00
49	28800.00	0.155520D+02	0.000000D+00	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
51	30000.00	0.162000D+02	0.000000D+00	[illegible]	0.235402D+02	[illegible]	.040	0.000000D+00	0.000000D+00	0.000000D+00
53	31200.00	0.168480D+02	0.000000D+00	[illegible]	0.235507D+02	[illegible]	.046	0.000000D+00	0.000000D+00	0.000000D+00
55	32400.00	0.174960D+02	0.000000D+00	[illegible]	0.235651D+02	.412	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
57	33600.00	0.181440D+02	0.000000D+00	[illegible]	0.235857D+02	.454	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
59	34800.00	0.187920D+02	0.000000D+00	[illegible]	0.236160D+02	[illegible]	.075	0.000000D+00	0.000000D+00	0.000000D+00
61	36000.00	0.194400D+02	0.000000D+00	[illegible]	0.236626D+02	.592	.092	0.000000D+00	0.000000D+00	0.000000D+00
63	37200.00	0.200880D+02	0.000000D+00	[illegible]	0.237374D+02	.685	.114	0.000000D+00	0.000000D+00	0.000000D+00
65	38400.00	0.207360D+02	0.000000D+00	[illegible]	0.238633D+02	.799	.144	0.000000D+00	0.000000D+00	0.000000D+00
67	39600.00	0.213840D+02	0.000000D+00	[illegible]	0.240754D+02	[illegible]	.181	0.000000D+00	0.000000D+00	0.000000D+00
69	40800.00	0.220320D+02	0.000000D+00	[illegible]	0.244164D+02	1.048	.217	0.000000D+00	0.411567D-07	0.483801D-04
71	42000.00	0.226800D+02	0.000000D+00	[illegible]	[illegible]	1.124	.242	0.338982D-05	0.395311D-05	0.759922D-02
73	43200.00	0.233280D+02	0.000000D+00	[illegible]	[illegible]	1.164	.255	0.104630D-04	0.213861D-05	0.526467D-02
75	44400.00	0.239760D+02	0.000000D+00	2.114	[illegible]	[illegible]	.260	0.139524D-04	0.830167D-06	0.217562D-02
77	45600.00	0.246240D+02	0.000000D+00	[illegible]	[illegible]	1.188	.261	0.152869D-04	0.290155D-06	0.776868D-03
79	46800.00	0.252720D+02	0.000000D+00	[illegible]	[illegible]	1.190	.262	0.157460D-04	0.979595D-07	0.264109D-03
81	48000.00	0.259200D+02	0.000000D+00	[illegible]	[illegible]	[illegible]	.262	0.159005D-04	0.326765D-07	0.883289D-04
83	49200.00	0.265680D+02	0.000000D+00	[illegible]	[illegible]	1.191	.262	0.159521D-04	0.108409D-07	0.293921D-04
85	50400.00	0.272160D+02	0.000000D+00	[illegible]	[illegible]	1.191	.262	0.159692D-04	0.359736D-08	0.973265D-05
87	51600.00	0.278640D+02	0.000000D+00	[illegible]	[illegible]	1.191	.262	0.159749D-04	0.117617D-08	0.323533D-05
89	52800.00	0.285120D+02	0.000000D+00	2.100	0.306116D+02	1.191	.262	0.159768D-04	0.397835D-09	0.107764D-05
91	54000.00	0.291600D+02	0.000000D+00	2.100	0.312596D+02	1.191	.262	0.159774D-04	0.147028D-09	0.393955D-06
93	55200.00	0.298080D+02	0.000000D+00	2.100	0.319076D+02	1.191	.262	0.159776D-04	0.345948D-10	0.864164D-07
95	56400.00	0.304560D+02	0.000000D+00	2.100	0.325556D+02	1.191	.262	0.159777D-04	0.000000D+00	0.381249D-07
97	57600.00	0.311040D+02	0.000000D+00	2.100	0.332036D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
99	58800.00	0.317520D+02	0.000000D+00	2.100	0.338516D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
101	60000.00	0.324000D+02	0.000000D+00	2.100	0.344996D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
103	61200.00	0.330480D+02	0.000000D+00	2.100	0.351476D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
105	62400.00	0.336960D+02	0.000000D+00	2.100	0.357956D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
107	63600.00	0.343440D+02	0.000000D+00	2.100	0.364436D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
109	64800.00	0.349920D+02	0.000000D+00	2.100	0.370916D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
111	66000.00	0.356400D+02	0.000000D+00	2.100	0.377396D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
113	67200.00	0.362880D+02	0.000000D+00	2.100	0.383876D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
115	68400.00	0.369360D+02	0.000000D+00	2.100	0.390356D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
117	69600.00	0.375840D+02	0.000000D+00	2.100	0.396836D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
119	70800.00	0.382320D+02	0.000000D+00	2.100	0.403316D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
121	72000.00	0.388800D+02	0.000000D+00	2.100	0.409796D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
123	73200.00	0.395280D+02	0.000000D+00	2.100	0.416276D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
125	74400.00	0.401760D+02	0.000000D+00	2.100	0.422756D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
127	75600.00	0.408240D+02	0.000000D+00	2.100	0.429236D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
129	76800.00	0.414720D+02	0.000000D+00	2.100	0.435716D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
131	78000.00	0.421200D+02	0.000000D+00	2.100	0.442196D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
133	79200.00	0.427680D+02	0.000000D+00	2.100	0.448676D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
135	80400.00	0.434160D+02	0.000000D+00	2.100	0.455156D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
137	81600.00	0.440640D+02	0.000000D+00	2.100	0.461636D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
139	82800.00	0.447120D+02	0.000000D-00	2.100	0.468116D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
141	84000.00	0.453600D+02	0.000000D+00	2.100	0.474596D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
143	85200.00	0.460080D+02	0.000000D+00	2.100	0.481076D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
145	86400.00	0.466560D+02	0.000000D+00	2.100	0.487556D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
147	87600.00	0.473040D+02	0.000000D+00	2.100	0.494036D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
149	88800.00	0.479520D+02	0.000000D+00	2.100	0.500516D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
151	90000.00	0.486000D+02	0.000000D+00	2.100	0.506996D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
153	91200.00	0.492480D+02	0.000000D+00	2.100	0.513476D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
155	92400.00	0.498960D+02	0.000000D+00	2.100	0.519956D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
157	93600.00	0.505440D+02	0.000000D+00	2.100	0.526436D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
159	94800.00	0.511920D+02	0.000000D+00	2.100	0.532916D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
161	96000.00	0.518400D+02	0.000000D+00	2.100	0.539396D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
163	97200.00	0.524880D+02	0.000000D+00	2.100	0.545876D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
165	98400.00	0.531360D+02	0.000000D+00	2.100	0.552356D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
167	99600.00	0.537840D+02	0.000000D+00	2.100	0.558836D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
169	100800.00	0.544320D+02	0.000000D+00	2.100	0.565316D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
171	102000.00	0.550800D+02	0.000000D+00	2.100	0.571796D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
173	103200.00	0.557280D+02	0.000000D+00	2.100	0.578276D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
175	104400.00	0.563760D+02	0.000000D+00	2.100	0.584756D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
177	105600.00	0.570240D+02	0.000000D+00	2.100	0.591236D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
179	106800.00	0.576720D+02	0.000000D+00	2.100	0.597716D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
181	108000.00	0.583200D+02	0.000000D+00	2.100	0.604196D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
183	109200.00	0.589680D+02	0.000000D+00	2.100	0.610676D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
185	110400.00	0.596160D+02	0.000000D+00	2.100	0.617156D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
187	111600.00	0.602640D+02	0.000000D+00	2.100	0.623636D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
189	112800.00	0.609120D+02	0.000000D+00	2.100	0.630116D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
191	114000.00	0.615600D+02	0.000000D+00	2.100	0.636596D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
193	115200.00	0.622080D+02	0.000000D+00	2.100	0.643076D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
195	116400.00	0.628560D+02	0.000000D+00	2.100	0.649556D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
197	117600.00	0.635040D+02	0.000000D+00	2.100	0.656036D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
199	118800.00	0.641520D+02	0.000000D+00	2.100	0.662516D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
201	120000.00	0.648000D+02	0.000000D+00	2.100	0.668996D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00

Fig. Ex.6.C.6 Output file of the program DELTA, resulting from a simulation using the method of *Meyer-Peter* et *al.* (1948) (continued) : **T = 0 [years].**

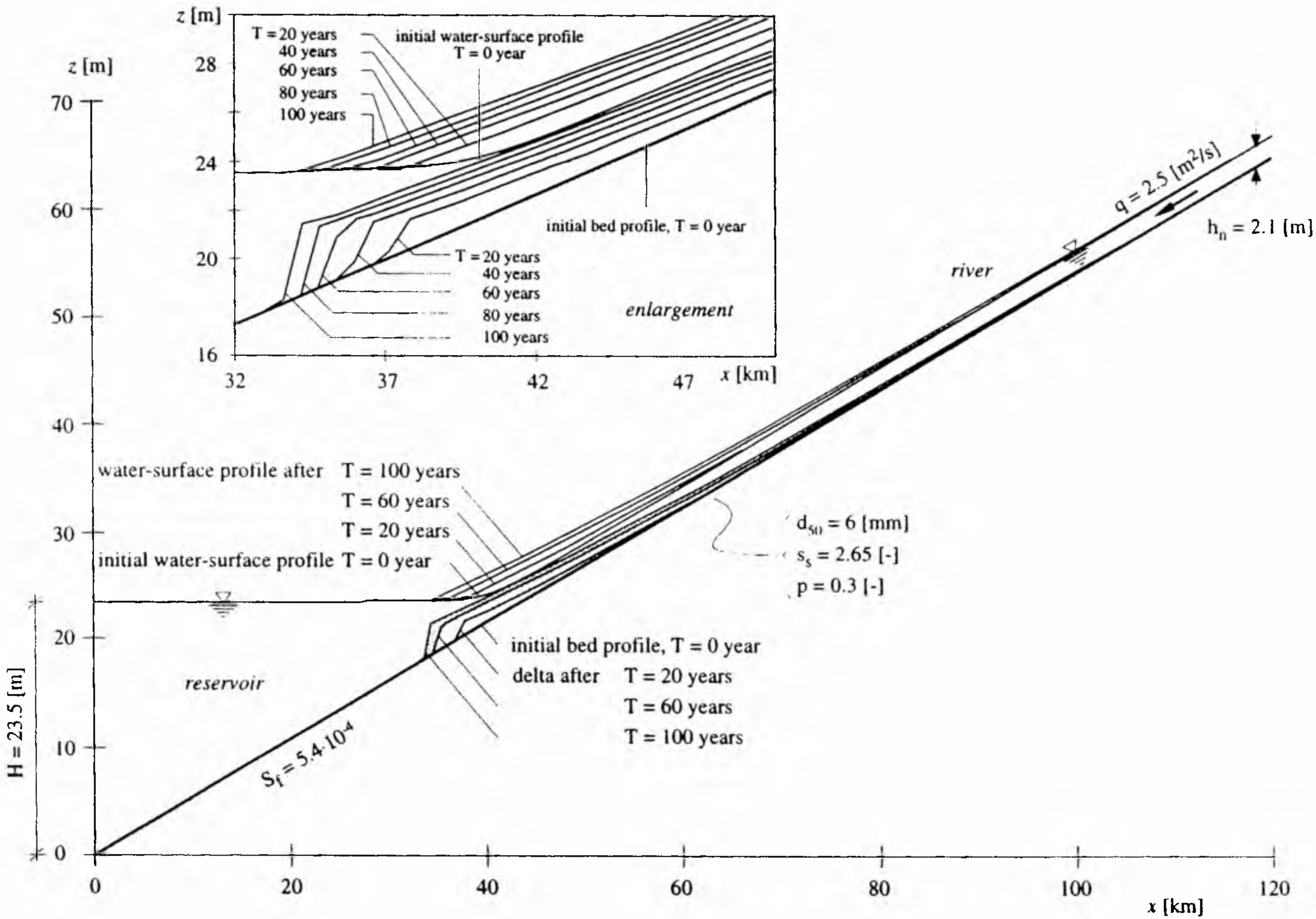


Fig. Ex.6.C.7 Time history of formation and advancement of the delta in a river-reservoir system, simulated for a period of 100 years with the program DELTA by using the bed-load formula of

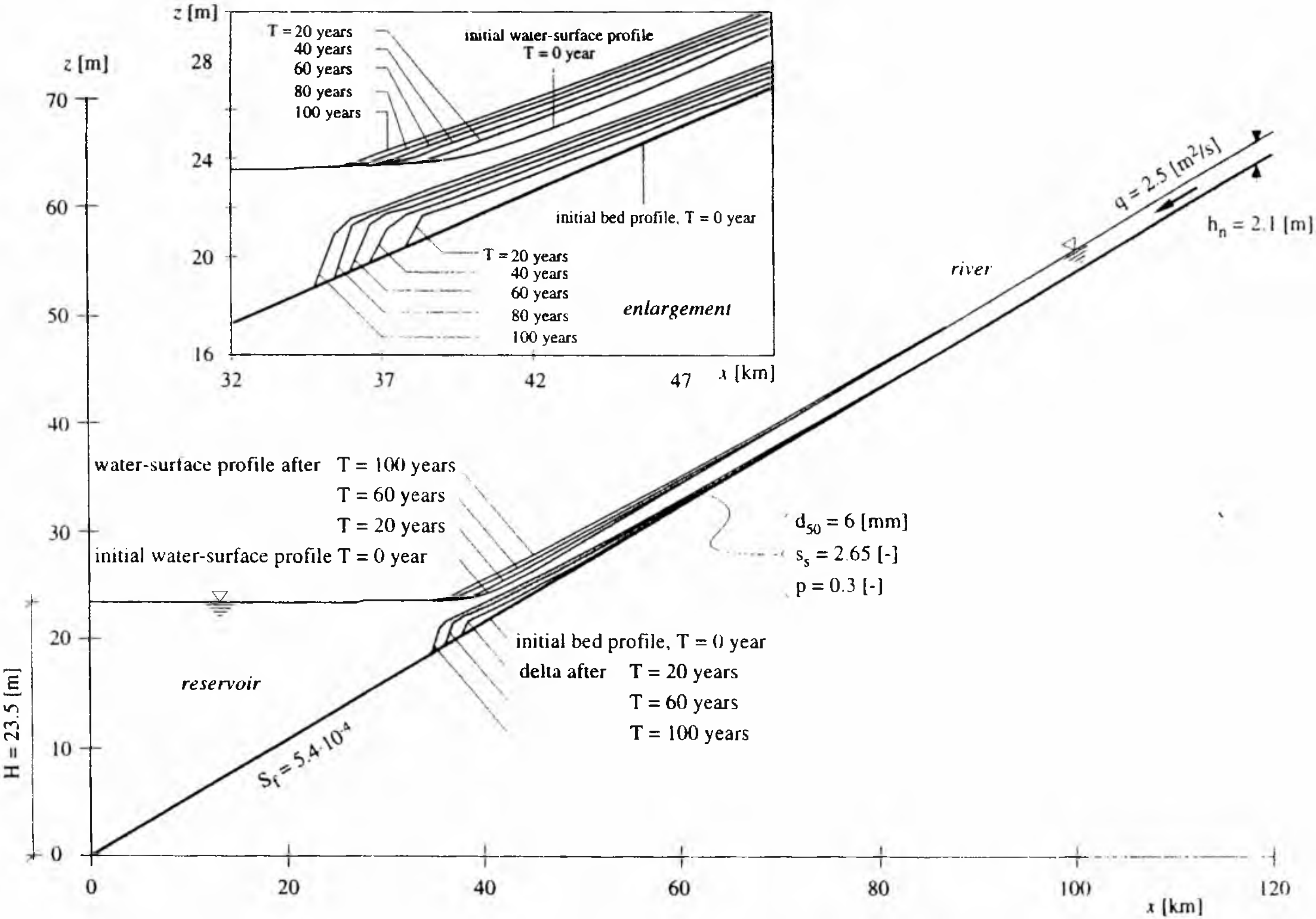


Fig. Ex.6.C.8 Time history of formation and advancement of the delta in a river-reservoir system, simulated for a period of 100 years with the program DELTA by using the bed-load formula of *Schoklitsch* (1950).

At time **T = 100 [years]**, the downstream end of the delta has reached the statio
(x = 33.3 [km]) (see column ZF–ZFI). The maximum height of the delta is 2.91
(station 58, x = 34.2 [km]). The total volume of deposited sediments is 71887 [m^3]
unit width.

On Figs. Ex.6.C.7 and 8, it is interesting to note that the influence of the delta can be
up to a quite long distance upstream.

The table below summarizes the most important characteristics of the deltas, calcul
using the two methods.

Results of the calculations, using the bed-load formula of *Meyer-Peter* et *al.* (1948

time [years]	Nose of delta		Maximum height of delta			Vol. o
	station no	x [m]	h_d (m)	station no	x [m]	delta [m^3.
20	62	36600	1.23	64	37800	14396
40	60	35400	1.78	62	36600	28789
60	59	34800	2.17	61	36000	43173
80	57	33600	2.56	59	34800	57539
100	56	33000	2.91	58	34200	71887

Results of the calculations, using the bed-load formula of *Schoklitsch* (1950)

time [years]	Nose of delta		Maximum height of delta			Vol. o
	station no	x [m]	h_d (m)	station no	x [m]	delta [m^3/
20	63	37200	0.91	65	38400	7395
40	62	36600	1.31	64	37800	14790
60	61	36000	1.62	63	37200	22185
80	60	35400	1.90	62	36600	29580
100	59	34800	2.14	61	36000	36973

The above table shows that the bed-load formula of *Meyer-Peter* et *al.* (1948) predic
delta which is *larger* than the one predicted by the bed-load formula of *Schokli*
(1950). After a period of 100 [years], although their height is almost the same, the d
obtained by using the method of *Meyer-Peter* et *al.* (1948) has a volume 1.94 times
volume of the delta obtained by using the method of *Schoklitsch* (1950). A graph
comparison of the results obtained using these two methods is presented
Fig. Ex.6.C.9. The difference between the two methods comes from the differe
between the predicted bed-load transport rates. For the same hydraulic conditions,
formula of *Meyer-Peter* et *al.* predicts a solid discharge which is larger than the
predicted by the formula of *Schoklitsch* (the same observation is also valid for Ex.6
In Fig. Ex.6.C.6., the bed-load formula of *Meyer-Peter* et *al.* predicts a solid disch
of $q_{sb} = 0.16 \cdot 10^{-4}$ [m^3/s/m] at time T = 0 for the initial river cross-section. The ou
file for the simulation using the method of *Schoklitsch* is not given here; the
can easily run the program himself and will obtain a solid discharge
$q_{sb} = 0.82 \cdot 10^{-5}$ [m^3/s/m]. The ratio between the two solid discharges is 1.95; this is
close to the ratio between the different values for the volume of the deltas.

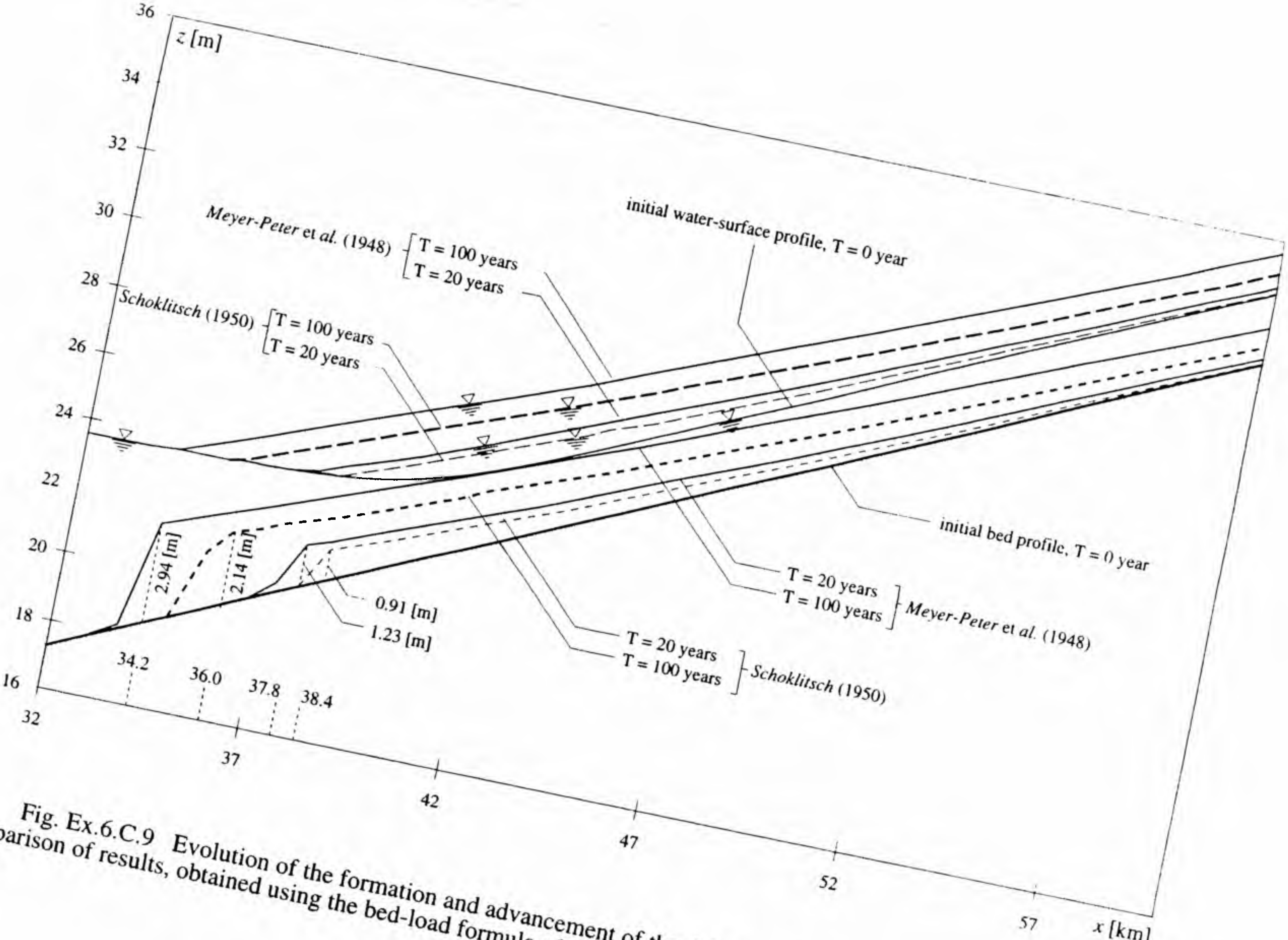


Fig. Ex.6.C.9 Evolution of the formation and advancement of the delta in the river-reservoir system; comparison of results, obtained using the bed-load formula of *Meyer-Peter* et *al.* (1948) and of *Schoklitsch* (1950).

Differences of this order of magnitude between the results obtained using different load formulae are not rare. In a real study of the sediment transport in a river or in a ı reservoir system, it is necessary to do the calculations using different methods aı compare the results against in-situ measurements, in order to determine the formula v is best suited to the particular case. Sometimes it may even be necessary to calibrat sediment-transport formula and/or the simulation program to adapt it to the parti problem in hand.

It is left to the user to run the program using the method of *Einstein* (1942) (see ; 6.3.3, 4 °) and to compare the results with those presented above. It is also importa recall that the option number 4 in choosing the bed-load formula is provided ı programmed by the reader. The user should select a bed-load formula of his choice program it into this subroutine.

g) *Remarks* :

Before finishing this exercise, it is in place to call attention of the reader to a few cr points :

- A computer cannot represent a number internally with an absolute precision.] computers on the market use a set of 32 bits (= 4 bytes), called *word*, for storing ε floating-point number of single precision (R*4). In such a computer a real numb stored according to a standard format as a combination of an integer number (23 l called *mantissa*, and an *exponent* of 2 (8 bits). The last remaining bit is reserve the *sign*. The relative precision that can be obtained with such a storage meth approximately $3 \cdot 10^{-8}$. In many cases this precision may appear to be suffic However, the arithmetic operations with real floating-point numbers reserve a surprises. When a very small number is added on a very large number or wheı difference between two numbers being nearly equal is calculated, the round-off e may become very important. In the present program, very often the difference bet two nearly equal numbers are calculated (for example for the calculation of DEL We also add very small values contained in the variable DELZ on the variabl containing much larger values. To avoid the errors caused by a round off, some o variables of the program are declared as "DOUBLE PRECISION" Fig. Ex.6.C.11). The computer uses then two words (= 8 octets = 64 bits) representing a real floating-point number (R*8). In this way a relative precision iı order of 10^{-15} is obtained.
- Since the program uses a decoupled algorithm, one cannot talk about a stab condition of the Courant type, for example (see sect. 5.2.3). The choice of the sp and time-step lengths remains nevertheless important from the point of view of al mentioned round-off errors. The step length in the longitudinal direction must be s enough to allow a correct representation of the form of the delta. A too short space and/or a too small time step may lead to deposition volumes and/or bed-l modification values close to the precision of the internal representation of the floaı point numbers in the computer. The reader is advised to try the program with diffe space and time step combinations.

- One of the most important variables controlling the delta formation is the HAMHAV-variable, which defines the way the deposited sediments will be distributed over a reach. It is recalled that a value of HAMHAV = 1, which means a uniform distribution of the sediments depositing in the reach, generates instabilities at the upstream and brings the program to a halt either because of an impossibility to calculate the water-surface profile or because of the detection of a deposition height being larger than the one specified by the user as the maximum allowable value. The reader is encouraged to try the program with different values of HAMHAV ≤ 1.

- A sufficiently long river reach, where the flow remains uniform during the whole simulation period, must be provided at the upstream end of the river-reservoir system to insure a correct functioning of the program DELTA. The upstream limit of the delta should not, therefore, touch the upstream end of the computational reach. To illustrate the influence of the computational domain on the results of the delta formation and progression, simulations were done by modeling three different lengths of the river-reservoir system, namely : TL = 60 [km], 80 [km] and 160 [km]. The form of the data for these three simulations are presented in Fig. Ex.6.C.10 (to be also compared with Fig. Ex.6.C.9). The results are summarized in the table below.

Delta after 100 [years]
Simulations using the bed-load formula of *Meyer-Peter* et *al.* (1948)
with different computational domain lengths, TL.

Modeled length	Space step	Nose of delta	Maximum height of delta		Apparent volume of delta after 100 [years] from the downstream end (dam) up to a distance of			
TL	DX	x	h_d	x				
[km]	[m]	[m]	[m]	[m]	60 [km]	80 [km]	120 [km]	160 [km]
60	400	32800	3.30	33600	69617	—	—	—
80	400	33600	2.85	34400	47576	57524	—	—
120	600	33000	2.91	34200	49205	63830	71887	—
160	400	34000	2.87	34400	49131	63789	71302	71977

The solution with TL = 60 [km] predicts a longer and higher delta (see also Fig. Ex.6.C.10). The other three simulations yield similar values. The difference between the simulation with TL = 120 [km] and the simulations with TL = 80 and 160 [km] is probably due to the difference in the space-step length.

Each simulation is done with a different computational domain length. In the first three simulations, the delta touches the most upstream station; it is therefore not simulated in its total length. Thus it is necessary to calculate the volume of the delta at intermediate intervals, as shown in the table above, for a comparison of deposition volumes obtained from different runs. The simulation with TL = 60 [km] overestimates considerably the volume of the delta up to that distance. It is also interesting to note that the last two simulations predict almost identical delta formations, despite the fact that only the solution with TL = 160 [km] has still a river reach with a uniform flow after 100 years of simulation. The solution with TL = 80 [km] gives a good prediction of the maximum height of the delta; the volume of deposition is, however, smaller.

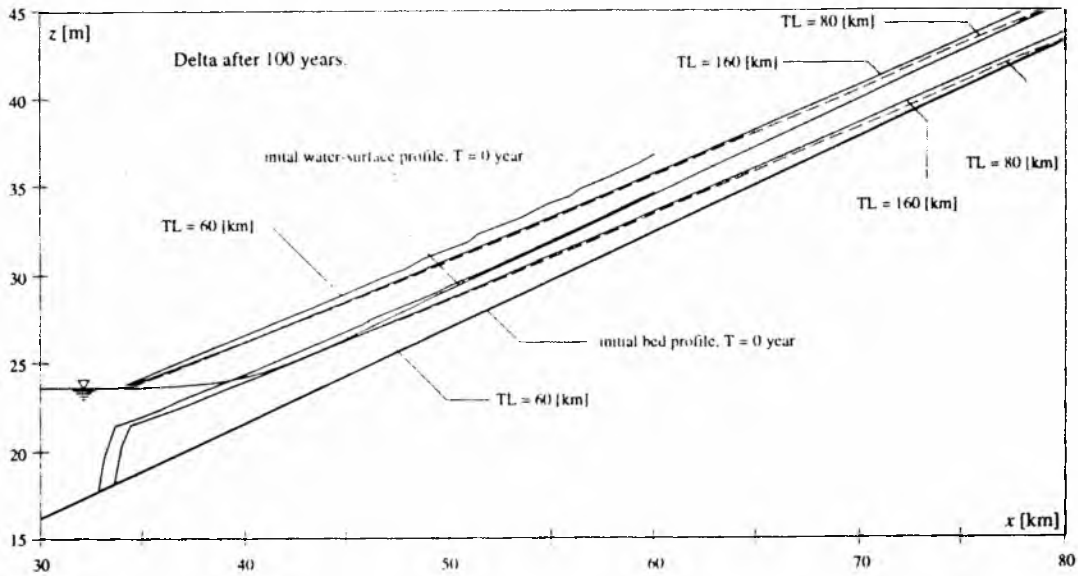


Fig.Ex.6.C.10 Comparison of deltas, simulated with differents lengths of the river reservoir system by using the bed-load transport formula of *Meyer-Peter* et *al.*

```
      PROGRAM DELTA
C*********************************************************************
C                                                                    *
C MAIN PROGRAM FOR CALCULATING THE BED-LOAD TRANSPORT IN A           *
C RIVER-RESERVOIR SYSTEM BY TAKING INTO ACCOUNT THE MODIFICATION     *
C OF THE BED LEVEL.                                                  *
C                                                                    *
C THE PROGRAM CONSIDERS A SUBCRITICAL FLOW WITH A CONTROL SECTION    *
C LOCATED AT THE DOWNSTREAM.                                         *
C                                                                    *
C*********************************************************************
C
C LIST OF VARIABLES DEFINED GLOBALLY FOR THE MAIN PROGRAM AND THE
C SUBPROGRAMS. VARIABLES DEFINED LOCALLY IN SUBPROGRAMS ARE LISTED AT
C THE BEGINNING OF EACH SUBPROGRAM.
C
C TYPE NAME   DIMEN      EXPLANATIONS
C ==== ====== =======    ============================================
C R*4  CFOI            = COEFFICIENT OF VOLUME INCREASE DUE TO POROSITY
C R*4  CN              = TOTAL MANNING COEFFICIENT
C R*4  CN50            = MANNING COEFFICIENT DUE TO GRAIN ROUGHNESS
C R*4  CPTT            = TIME-STEP COUNTER
C R*4  CRAM            = WEIGHTING COEFFICIENT FOR UPSTREAM STATION
C                        USED IN CALCULATION OF DEPOSITION
C R*4  CRAV            = WEIGHTING COEFFICIENT FOR DOWNSTREAM STATION
C                        USED IN CALCULATION OF DEPOSITION
C R*4  D50             = AVERAGE SEDIMENT DIAMETER
C R*4  DELQS (NSMAX-1)= SEDIMENTS DEPOSITED OR ERODED IN A REACH
C                        BOUNDED BY TWO PRINCIPAL SECTIONS
C R*8  DELT            = TIME STEP
C R*8  DELZ  (NSMAX)   = ARRAY CONTAINING THE BED LEVEL CHANGES AT ALL
C                        STATIONS
C R*8  DELZMX          = MAXIMUM BED-LEVEL CHANGE
C R*8  DELZRM          = DIMENSIONLESS MAXIMUM BED-LEVEL CHANGE
C R*4  DHDYNM          = MAXIMUM DIFFERENCE IN DYNAMIC HEAD (U2/2g)
C                        BETWEEN TWO CONSECUTIVE STATIONS
C R*8  DQSDET          = TOTAL VOLUME DEPOSITED SINCE T = 0
C R*8  DQSERT          = TOTAL VOLUME ERODED SINCE T = 0
C R*4  DX              = DISTANCE BETWEEN TWO PRINCIPAL STATIONS
C R*4  DXSUB           = DISTANCE BETWEEN INTERPOLATED STATIONS
C R*4  FCOR            = ROUGHNESS PARAMETER IN BED-LOAD FORMULA OF
C                        MEYER-PETER (1948) (K/K'=n'/n)
C C*40 FICHS           = NAME OF OUTPUT FILE
C R*4  FRNAM           = FROUDE NUMBER AT UPSTREAM STATION OF A REACH
C R*4  FRNAV           = FROUDE NUMBER AT DOWNSTREAM STATION OF A REACH
C R*4  G               = GRAVITATIONAL ACCELERATION
C R*4  H     (NSMAX)   = WATER DEPTH AT PRINCIPAL STATIONS
C R*4  HAMHAV          = RATIO OF UPSTREAM/DOWNSTREAM HEIGHTS OF TRAPE-
C                        ZOID FORMED BY SEDIMENTS DEPOSITED IN A REACH
C R*4  HDYN1           = DYNAMIC HEAD (U^2/2G) AT STATION 1
C R*4  HDYN2           = DYNAMIC HEAD (U^2/2G) AT STATION 2
C R*4  HSUB  (2**MCMAX+1) = WATER DEPTH AT INTERPOLATED STATIONS
C I*4  I               = DO-LOOP COUNTER VARIABLE
C I*4  IAM             = UPSTREAM STATION NUMBER FOR A GIVEN REACH
C I*4  IAV             = DOWNSTREAM STATION NUMBER FOR A GIVEN REACH
C I*4  II              = DO-LOOP COUNTER VARIABLE
C I*4  IMAXR           = NUMBER OF THE STATION WHERE WE HAVE 'DELZRM'
C I*4  IMAXZ           = NUMBER OF THE STATION WHERE WE HAVE 'DELZMX'
C J*4  J               = DO-LOOP COUNTER VARIABLE
C I*4  MC              = NUMBER OF SUBDIVISIONS (IN POWERS OF 2)
C I*4  MCMAX           = MAXIMUM NUMBER OF SUBDIVISIONS (IN POWERS OF 2)
C I*4  ND              = NUMBER OF PRINCIPAL REACHES
C I*4  NFTS            = NUMBER OF BED-LOAD FORMULA TO BE USED
C I*4  NMC             = NUMBER OF SUBDIVISIONS (IN POWERS OF 2)
C                        SPECIFIED BY USER
C I*4  NOUT            = UNIT NUMBER OF OUTPUT FILE
```

```
C I*4  NPP               = PRINTING FREQUENCY
C I*4  NPT               = PRINTING TIME COUNTER
C I*4  NS                = NUMBER OF PRINCIPAL STATIONS
C I*4  NSMAX             = NUMBER OF MAXIMUM STATIONS ALLOWED BY THE PROGRAM
C I*4  NSUBD             = NUMBER OF SUBDIVISIONS AT A GIVEN REACH
C R*4  PI                = PI NUMBER
C R*8  QCRIT             = CRITICAL DISCHARGE IN FORMULA OF SCHOKLITSCH
C R*8  QSU    (NSMAX)    = SEDIMENTS TRANSPORTED THROUGH A STATION
C R*4  QU                = UNIT DISCHARGE (WATER)
C C*1  QUIT              = READ FOR TERMINATING THE PROGRAM
C R*4  ROE               = DENSITY OF WATER
C R*4  ROS               = DENSITY OF SEDIMENTS
C R*4  SEFF              = SLOPE OF ENERGY GRADE LINE AT A STATION
C R*4  SF                = INITIAL BED SLOPE
C R*4  SFTR              = LOCAL BED SLOPE AT A GIVEN REACH
C R*4  SS                = SPECIFIC DENSITY OF THE SEDIMENTS
C R*8  T                 = TIME
C R*8  TFIN              = FINAL TIME
C R*4  TJ                = TIME (NUMBER OF DAYS)
C R*4  TL                = TOTAL LENGTH OF RIVER REACH STUDIED
C R*4  VARZMX            = MAXIMUM RELATIVE BED-LEVEL CHANGE ALLOWED BY
C                          THE PROGRAM IN A SINGLE TIME STEP
C R*4  X      (NSMAX)    = X COORDINATES OF PRINCIPAL STATIONS
C R*4  XI                = X COORDINATE OF MOST DOWNSTREAM STATION
C R*4  XF                = X COORDINATE OF MOST UPSTREAM STATION
C R*4  XSUB (2**MCMAX+1) = X-COORDINATES OF INTERPOLATED STATIONS
C R*8  ZF     (NSMAX)    = BED-LEVEL ELEVATIONS AT ANY TIME
C R*8  ZFI    (NSMAX)    = INITIAL BED-LEVEL ELEVATIONS
C
C
C***********************************************************************
C
C PARAMETERS
      PARAMETER ( NOUT = 10 , NSMAX = 1000 , MCMAX = 7 )
C
C LABELLED COMMON BLOCKS (SHARED VARIABLES)
      COMMON /DONNE1/ SFTR , QU , CN
C
C THE MAIN PROGRAM "DELTA" SENDS THE NAME OF THE SUBROUTINE SUBPROGRAM
C "DERIVE" TO THE SUBROUTINE SUBPROGRAM "RK4" IN THE LIST OF ARGUMENTS.
C ACCORDING TO THE FORTRAN PROGRAMMING RULES, THE SUBROUTINE SUBPROGRAM
C "DERIVE" SHOULD BE DECLARED AS "EXTERNAL".
      EXTERNAL DERIVE
C
C DECLARATION OF VARIABLES
      CHARACTER*1    QUIT
      CHARACTER*40   FICHS
      DOUBLE PRECISION  ZF (NSMAX) , ZFI (NSMAX)
      DOUBLE PRECISION  QSU(NSMAX) , DELZ(NSMAX) , DELQS(NSMAX-1)
      DOUBLE PRECISION  DQSDET , DQSERT , DELZMX , DELZRM
      DOUBLE PRECISION  DELT , TFIN , T
      DIMENSION         X(NSMAX) , H(NSMAX)
      DIMENSION         XSUB(2**MCMAX+1) , HSUB(2**MCMAX+1)
C
C
      OPEN( UNIT = 8 , FILE = 'GRAPH.DAT' , STATUS = 'NEW' )
C
C READ THE PROBLEM DATA
      CALL DREAD( NOUT , NSMAX , MCMAX ,
     1            QU , SF , CN  , D50 , ROS , ROE , SS , CN50 ,
     2            X(1) , XF , TL  , DX , ND  , NS , NMC ,
     3            H(1) , DHDYNM  , VARZMX ,
     4            DELT , TFIN ,
     5            NPTS , CFOI , FCOR , HAMHAV , CRAM , CRAV ,
     6            FICHS , NPP )
C
C INITIALISATIONS
      PI     = 3.1415927
      G      = 9.81
      T      = -DELT
      CPTT   = 0.0
      NPT    = -1
      DQSDET = 0.0
      DQSERT = 0.0
C
C CALCULATION OF X COORDINATES AND INITIAL BED LEVELS OF COMPUTATIONAL
C SECTIONS
      X(1)   = 0.0
      ZF(1)  = 0.0
      DO 10 I = 2 , NS
      X(I)      = X(I-1) + DX
      ZF(I)     = ZF(I-1) + SF * DX
      ZFI(I)    = ZF(I)
10    CONTINUE
C
C WRITE THE TITLES ON OUTPUT FILE
      CALL TITLES (NOUT , SF , D50 , CN50 , CN , ROS , ROE , QU ,
     1                 DHDYNM , NMC , VARZMX , CFOI , HAMHAV , FCOR ,
     2                 NPTS , XI , XF , TL , DX , ND , NS ,  H(1) ,
     3                 DELT , TFIN , NPP , FICHS)
C
C CALCULATION LOOP
100   T     = T + DELT
      CPTT = CPTT + 1.0
      NPT  = NPT + 1
      TJ   = T / 86400
      WRITE(*,110) CPTT , TJ
110   FORMAT(/' TIME STEP      = ',F15.0/
     1        ' TIME (days)    = ',F10.3)
C
C=======================================================================
C==============   BEGINNING OF LIQUID PHASE CALCULATIONS   =============
C=======================================================================
C
C BACKWATER CALCULATION
      DO 300 I = 1 , NS-1
      SFTR     = (ZF(I+1) - ZF(I)) / DX
      MC       = 0
200   NSUBD    = 2**MC
      DXSUB    = (X(I+1) - X(I)) / NSUBD
      HSUB(1) = H(I)
      XSUB(1) = X(I)
C
      DO 250 J = 1 , NSUBD
      XSUB(J+1)  = X(I) + DXSUB * NSUBD
      CALL RK4( G , HSUB(J) , DXSUB , HSUB(J+1) , DERIVE )
      HDYN1 = QU**2 / (2*G*HSUB(J)**2)
      HDYN2 = QU**2 / (2*G*HSUB(J+1)**2)
      IF(ABS(HDYN2-HDYN1).GT.DHDYNM)THEN
C
C INTERMEDIATE SECTIONS MUST BE GENERATED BY INTERPOLATION
        MC = MC+1
        WRITE(*,223) 2**MC , I , I+1
223     FORMAT(/1X,I2,' subdivisions between sections ',I5,' and ',I5)
        IF(MC.GT.NMC)THEN
          FRNAM = QU / (H(I) * SQRT(G * H(I)))
```

Fig.Ex.6.C.11   Program DELTA (continued).

Ex. 6.D

An artificial channel has been constructed to divert a certain discharge from a river. Thi channel has an approximately rectangular cross section with a width of $B = 46.5$ [m] an a bed slope of $S_f = 6.5 \cdot 10^{-4}$ [-]. The uniform flow is established when the flow dept is $h_n = 5.6$ [m]. The velocity-profile measurements carried out in this channel allowe to obtain the average velocity of $U = 1.8$ [m/s] for a friction coefficient c $n' = 0.0212$ [$m^{-1/3}s$]. The granulometry of the bed material has not been analyzed.

Estimate the bed-load transport in this channel. Subsequently, express the solid discharg as a concentration. Is suspended-load transport to be expected ?

SOLUTION :

i) First, preliminary calculations concerning the hydraulics of the channel and th sedimentology of the bed material should be carried out.

Unit discharge : $q = Uh = 1.8\,(5.6) = 10.08$ [m^3/sm]

Discharge : $Q = qB = 10.08\,(46.5) = 468.72$ [m^3/s]

Hydraulic radius : $R_h = \dfrac{Bh}{B + 2h} = \dfrac{(46.5)\,(5.6)}{46.5 + 2\,(5.6)} = 4.51$ [m]

Hydraulic radius of the channel bed (the channel banks are assumed to be smooth)

$$R_{hb} \cong h_n = 5.6 \text{ [m]}$$

The granulometry can be estimated using the calculated friction coefficient, n which, being obtained from a measured velocity profile, corresponds to the frictio coefficient due to grain roughness. By using the Strickler formula :

$$\frac{1}{n'} = K_s' = \frac{1}{0.0212} = \frac{26}{d_{90}^{1/6}} \qquad \text{and} \qquad K_s' = \frac{21.1}{d_{50}^{1/6}} \tag{3.18}$$

one obtains :

$$d_{90} = 0.0280 \text{ [m]} \quad ; \quad d_{50} = 0.0080 \text{ [m]}$$

By assuming a granulometric distribution to be logarithmic, one finds :

$d_{35} = 0.0055$ [m] ; $d_{40} = 0.0062$ [m] ; $d_{65} = 0.0117$ [m]

The total friction coefficient, n , which is due to the combined effect of grai roughness and bed forms, can now be obtained using the Manning-Strickle formula :

$$K_s = \frac{1}{n} = \frac{U}{R_{hb}^{2/3} S_f^{1/2}} = \frac{1.8}{(5.6)^{2/3}(0.00065)^{1/2}} = 22.41 \ [m^{1/3}/s] \qquad (3.16)$$

whereas :

$$K_s' = \frac{1}{n'} = \frac{1}{0.0212} = 47.17 \ [m^{1/3}/s].$$

The roughness parameter is therefore :

$$\xi_M = \left(\frac{K_s}{K_s'}\right)^{3/2} = \left(\frac{22.41}{47.17}\right)^{3/2} = 0.327$$

The settling velocity for d_{50} is (see Fig. 6.10) : $v_{ss} \cong 0.4$ [m/s]

ii) Three different bed-load equations will be used to estimate the solid discharge; namely the bed-load equation of *Schoklitsch*, eq. 6.31, of *Meyer-Peter* et *al.*, eq. 6.32, and of *Einstein*, eq. 6.42.

a) The bed-load equation of *Schoklitsch* is given by :

$$q_{sb} = \frac{2.5}{s_s} S_e^{3/2} (q - q_{cr}) \qquad (6.31)$$

The critical liquid discharge is calculated using :

$$q_{cr} = 0.26 (s_s - 1)^{5/3} \ d^{3/2} \ S_e^{-7/6} \qquad (6.31a)$$

where $d = d_{40} = 0.0062$ [m] for a non-uniform granulometry and assuming that the specific density of the bed material is $s_s = 2.65$ [-] :

$$q_{cr} = 0.26 (1.65)^{5/3} (0.0062)^{3/2} (0.00065)^{-7/6} = 1.53 \ [m^2/s]$$

The volumic solid discharge for a unit width is then :

$$q_{sb} = \frac{2.5}{2.65} (0.00065)^{3/2} (10.08 - 1.53) = \mathit{1.33 \cdot 10^{-4}} \ [m^2/s].$$

b) The bed-load equation of *Meyer-Peter* et *al.* is given by :

$$\frac{\gamma R_{hb} \xi_M S_e}{(\gamma_s - \gamma) d} - 0.047 = 0.25 \, \rho^{1/3} \frac{g'_{sb}{}^{2/3}}{(\gamma_s - \gamma) d} \qquad (6.32)$$

where $d = d_{50} = 0.008$ [m] for a non-uniform granulometry .

The solid discharge by submerged weight for a unit width can then calculated as :

$$g'^{\,2/3}_{sb} = \left(\frac{9.81\ (1650)\ (0.008)}{0.25\ (1000)^{1/3}}\right)\left(\frac{5.6\ (0.327)\ (0.00065)}{(2.65-1.0)\ 0.008} - 0.047\right)$$

$$g'_{sb} = [(51.91)\ (0.090-0.047)]^{3/2} = (2.23)^{3/2} = 3.35\ [\text{N/ms}]$$

The volumic solid discharge for a unit width is :

$$q_{sb} = \frac{g_{sb}}{\gamma_s} = \frac{g'_{sb}}{(\gamma_s-\gamma)} = \frac{3.35}{9.81\ (1650)} = 2.07{\cdot}10^{-4}\ [\text{m}^2/\text{s}]$$

c) The bed-load equation of *Einstein* is given by :

$$\Phi_* = f(\Psi'_*) \qquad (6.42$$

Considering the non-uniform granulometry with $d = d_{35} = 0.0055$ [m], o shall take :

$$\Phi = f(\Psi') \qquad \text{or} \qquad \frac{q_{sb}}{\sqrt{(s_s-1)gd^3_{35}}} = f\left(\frac{(\gamma_s-\gamma)\ d_{35}}{\tau_o'}\right)$$

First the shear-stress intensity parameter should be calculated :

$$\Psi' = \frac{(\gamma_s-\gamma)d_{35}}{\rho u_*'^2} = (s_s-1)\ \frac{d_{35}}{R_{hb}'S_f} \qquad (6.3$$

The friction velocity due to the grain roughness will be calculated using tl logarithmic velocity distribution :

$$\frac{U}{u_*'} = 5.75 \log\left(\frac{h}{k_s}\right) + 6.25 = 21.66 \qquad (3.131$$

with $k_s = d_{65} = 0.0117$ [m] and $h = 5.6$ [m]. It is found that :

$$u_*' = \frac{U}{21.66} = \frac{1.8}{21.66} = 0.083\ [\text{m/s}].$$

The value of Ψ' can then be calculated as follows :

$$\Psi' = \frac{g(\rho_s-\rho)d_{35}}{\rho\ u_*'^2} = \frac{9.81(2.65-1)0.0055}{(0.083)^2} = 12.92\ [-]$$

The hydraulic radius due to grain groughness is :

$$u_*' = \sqrt{g\ R_{hb}'\ S_f} \qquad \Rightarrow \qquad R_{hb}' = \frac{u_*'^2}{g\ S_f} = 1.08\ [\mathrm{m}]$$

whereas the hydraulic radius due to the bed forms is :

$$R_{hb}'' = R_{hb} - R_{hb}' = 5.6 - 1.08 = 4.52\ [\mathrm{m}]$$

This shows the importance of bed forms in this cross section of the channel. (If these values, $\Psi' = 12.92$ and $U/u_*'' = 10.60$, are compared with the relationship of Einstein-Barbarossa, represented in Fig. 3.6, a slight difference will be observed).

One can now either evaluate the function given by eq. 6.42, or read it directly on Fig. 6.8, to find :

$$\text{for} \quad \Psi' = 12.92 \qquad \Rightarrow \qquad \Phi \cong 0.033$$

The volumic solid discharge per unit width can then be calculated as :

$$q_{sb} = \Phi \sqrt{(s_s - 1)\ g d_{35}^3} = 0.033\ [(1.65)\ (9.81)\ (0.0055^3)\ ]^{1/2} \qquad (6.28)$$

$$= \mathit{5.41 \cdot 10^{-5}}\ [\mathrm{m^2/s}]$$

iii) The transport of sediments as bed load, obtained using these three bed-load equations, is presented in the table below, both by volume and by mass.

The difference between the values obtained using the different formulae is considerable but this is not surprising; one should in fact never expect to find exactly the same values using different formulae.

Formula	Solid discharge as bed load			
	q_{sb} [m²/s]	g_{sb} [kg/ms]	Q_{sb} [m³/s]	G_{sb} [kg/s]
Schoklitsch eq. 6.31	$1.33 \cdot 10^{-4}$	0.35	$6.18 \cdot 10^{-3}$	16.39
Meyer-Peter et *al.* eq. 6.32	$2.07 \cdot 10^{-4}$	0.55	$9.62 \cdot 10^{-3}$	25.50
Einstein eq. 6.42	$0.54 \cdot 10^{-4}$	0.14	$2.51 \cdot 10^{-3}$	6.65

iv) The liquid discharge of

$$Q = 468.72 \ [m^3/s] \qquad \text{or} \qquad G = 468.72 \cdot 10^3 \ [kg/s]$$

is responsible for the solid discharge — taking the average of the values listed ir above table — of :

$$Q_{sb} = 6.1 \cdot 10^{-3} \ [m^3/s] \qquad \text{or} \qquad G_{sb} = 16.2 \ [kg/s]$$

The average sediment *concentration*, C_s , can be expressed in different manr Here are the possible definitions :

concentration by volume : $C_s = \dfrac{\text{volume of sediments}}{\text{total volume}} \ \dfrac{[m^3/s]}{[m^3/s]}$

concentration by mass : $C_s' = \dfrac{\text{mass of sediments}}{\text{total mass}} \ \dfrac{[kg/s]}{[kg/s]}$

concentration by unit mass : $C_s'' = \dfrac{\text{mass of sediments}}{\text{total volume}} \ \dfrac{[kg/s]}{[m^3/s]}$

These definitions are related to one another in the following way :

$$C_s'' = \rho_s C_s \quad ; \qquad C_s' = \frac{\rho_s C_s}{\rho + (\rho_s - \rho) C_s} = \frac{\rho_s}{\rho_m} C_s$$

where ρ_m is the average density of the water-sediment mixture, defined by :

$$\rho_m = C_s \rho_s + (1 - C_s) \rho = \rho + (\rho_s - \rho) C_s$$

With above definitions, the concentrations can be obtained.

The density of the solid particles is : $\rho_s = 2650 \ [kg/m^3]$ or $2.65 \ [g/cm^3]$.

The concentrations can now be calculated :

$$C_s = \frac{6.1 \cdot 10^{-3}}{468.72} = 0.000013 \ [-]$$

$$C_s' = \frac{16.2}{468.72 \cdot 10^3} = 0.000035 \ [-]$$

$$C_s'' = \frac{16.2}{468.72} = 0.0345 \left[\frac{kg}{m^3}\right] \text{ or } \left[\frac{g}{l}\right] \text{ or } \frac{1}{1000}\left[\frac{g}{cm^3}\right]$$

The average density of the mixture is :

$$\rho_m = 1000.00 + 0.02 = \mathit{1000.02}\ [\text{kg/m}^3]$$

v) It is already shown that there is a strong bed-load transport in this channel. One can now ask, if there will be also suspended-load transport.

According to an indicative criteria, given in sect. 6.1.3, the suspended-load transport starts when :

$$\frac{u_*}{v_{ss}} > 0.40$$

For the present problem one obtains :

$$\frac{u_*}{v_{ss}} = \frac{0.189}{0.400} = 0.47$$

Therefore, a weak transport of sediments as suspended load is to be expected.

This expectation can still be controlled by determining the Rouse exponent, eq. 6.50a :

$$\mathfrak{z} = \frac{v_{ss}}{\kappa u_*'} = \frac{0.4}{0.4\ (0.083)} = 12.05$$

On Fig. 6.11, it can be seen that, for this $\mathfrak{z}$-value, the relative concentration distribution, consequently the suspended-load transport, will be indeed weak.

Ex. 6.E

A mountain river with a bottom slope of $S_f = 0.0062$ [-] has an approximately rectangula cross section, being B = 23.5 [m] wide. Analysis of the sediment samples taken fron well below the armour layer show that $d_{50} = 60$ [mm] and $d_{90} = 200$ [mm] and the density of sediments is $s_s = 2.65$ [-].

Determine the diameter, d_{50_a}, of the maximum possible armouring. At which flow deptł does the armour layer become unstable ?

SOLUTION :

i) The grain diameter of the armour layer is calculated (see point 6.3.4.5°) using the relationship :

$$d_{50_a} \cong 0.6\ d_{90}$$

$$d_{50_a} \cong 0.6\ (200) = \mathit{120}\ [\text{mm}]$$

ii) The stability of the armour layer (see point 6.3.4.5°) can be estimated using the expression :

$$\tau_{*a,cr} = \frac{u_{*a,cr}^2}{(s_s-1)\ g\ d_{50_a}} = \tau_{*cr}\left[0.4\ \left(\frac{d_{50}}{d_{50_a}}\right)^{1/2} + 0.6\right]^2$$

$$\tau_{*a,cr} = \tau_{*cr}\left[0.4\ \left(\frac{60}{120}\right)^{1/2} + 0.6\right]^2 = 0.05[0.88]^2 = \mathit{0.04}\ [\text{-}]$$

from which one obtains :

$$u_{*a,cr} = \sqrt{0.04\ [(s_s-1)\ g\ d_{50_a}]} = \sqrt{0.04\ [1.94]} = 0.28\ [\text{m/s}].$$

According to the definition of the friction velocity :

$$u_* = \sqrt{g\ R_h\ S_f} \quad \Rightarrow \quad R_h = u_*^2/g\ S_f$$

$$R_h = \frac{0.28^2}{9.81\ (0.0062)} = 1.25\ [\text{m}]$$

Now the limiting depth for the stability of the armour layer can be calculated as :

$$R_h = \frac{h\ B}{2h+B} \quad \text{or} \quad 1.25 = \frac{h\ (23.5)}{2h + 23.5} \quad \Rightarrow \quad h = \mathit{1.40}\ [\text{m}]$$

If the flow depth becomes larger than h = 1.40 [m], the armour layer is no longer stable and an important erosion of the bed material may be expected.

Ex. 6.F

The river *Happy* — whose *stage—water-discharge curve* was established in Ex. 3.B — has a variable discharge in the range of $10 < Q\ [m^3/s] < 1000$. The width of the river bed is b = 90 [m] and its non-erodible banks have a slope of 1:1. The topographical survey of the river showed that the bed slope is $S_f = 0.0005$ [-]. The sediment forming the bed has a specific density of $s_s = 2.652$ [-] and the grain-size analysis yielded : $d_{50} = 0.32$ [mm], $d_{35} = 0.29$ [mm] and $d_{90} = 0.48$ [mm]. The water temperature in the river is $T = 14$ [°C].

Determine the *stage—sediment-discharge curve*, $Q_s = f(h)$, for this river.

SOLUTION :

First the hydraulic calculations should be done to determine the *stage—water*-discharge curve, $Q = f(h)$, for the river. Once this is done, one can carry out the sediment-transport calculations to determine the *stage—solid*-discharge curve, $Q_s = f(h)$.

i) *Hydraulic calculations* :
The hydraulic calculations were presented and commented in Ex. 3.B. The calculation table, where each line represents the calculation of a discharge, Q , and other useful hydraulic parameters, R_h' , R_h'' , R_h , etc., is partially reprinted in Table Ex.6.F.1, together with the explanations for the columns. On every line, the calculations start by *assuming* a hydraulic radius due to grain roughness, R_h'. The corresponding discharge is obtained by following the procedure described in Ex. 3.B. The values for R_h' are selected such that the calculations cover the entire range of the desired water discharges in the river, $10 < Q\ [m^3/s] < 1000$.

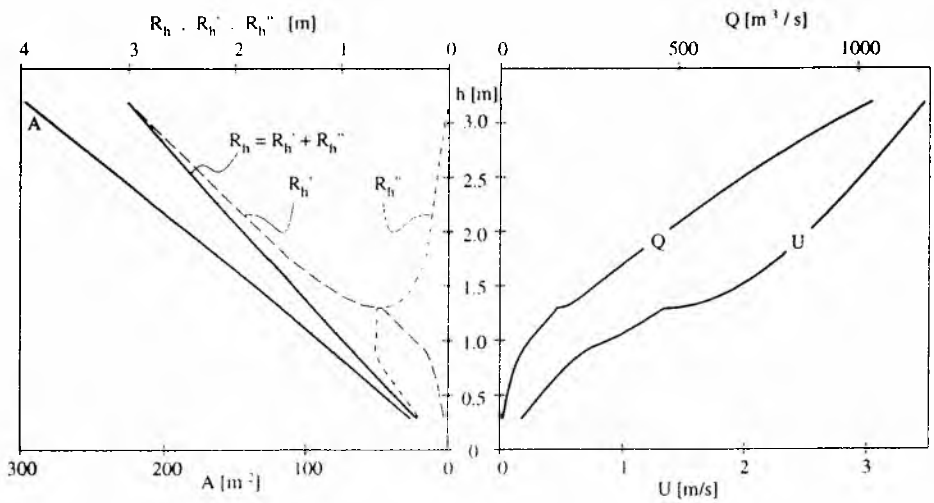


Fig. Ex. 6.F.1 *Stage—liquide*-discharge curve.

The *stage—water*-discharge curve, $Q = f(h)$, and the variation of other parameters, U, A, R_h', R_h'' et R_h , as a function of the flow depth, h, are presented in Fig. Ex. 6.F.1.

Table Ex.6.F.1

Computation sheet for determining the *stage*—**water**-discharge curve, using the method of *Einstein-Barbarossa* (1952)

$b = 90$ [m]	$T = 14$ [°C]	$\rho_s = 2650$ [kg/m³]
$m = 1$	$\rho = 999.1$ [kg / m³]	$d_{35} = 0.00029$ [m]
$S_f = 0.0005$ [-]	$\nu = 1.186 \times 10^{-6}$ [m²/s]	$k_s = d_{50} = 0.00032$ [m]

1	2	3	4	5	6	7	8	9	10	1
R_h'	u_*'	U	Ψ'	U / u_*''	u_*''	R_h''	R_h	u_*	h	(
[m]	[m/s]	[m/s]	[-]	[-]	[m/s]	[m]	[m]	[m/s]	[m]	[m
0.02	0.01	0.16	47.92	4.5	0.04	0.26	0.28	0.04	*0.29*	4
0.05	0.02	0.29	19.17	6.6	0.04	0.39	0.44	0.05	*0.44*	*1.*
0.10	0.02	0.45	9.58	8.7	0.05	0.55	0.65	0.06	*0.65*	2
0.15	0.03	0.58	6.39	10.4	0.06	0.63	0.78	0.06	*0.79*	4.
0.20	0.03	0.69	4.79	11.9	0.06	0.68	0.88	0.07	*0.89*	5
0.40	0.04	1.05	2.40	18.3	0.06	0.67	1.07	0.07	*1.09*	*10*
0.60	0.05	1.33	1.60	23.4	0.06	0.66	1.26	0.08	*1.29*	*15*
0.80	0.06	1.58	1.20	32.4	0.05	0.49	1.29	0.08	*1.32*	*19*
1.00	0.07	1.81	0.96	42.6	0.04	0.37	1.37	0.08	*1.41*	*23*
1.25	0.08	2.06	0.77	56.2	0.04	0.27	1.52	0.09	*1.57*	*29*
1.50	0.09	2.30	0.64	73.1	0.03	0.20	1.70	0.09	*1.76*	*37*
2.00	0.10	2.72	0.48	107.2	0.03	0.13	2.13	0.10	*2.23*	*55*
2.50	0.11	3.11	0.38	163.0	0.02	0.07	2.57	0.11	*2.71*	*78*
3.00	0.12	3.46	0.32	6844.0	0.00	0.00	3.00	0.12	*3.19*	*102*

<u>col.</u>	<u>symbol</u>	<u>explanations</u>	<u>expression</u>
1	R_h'	hydraulic radius due to grain roughness (*assumed* value)	
2	u_*'	friction velocity due to grain roughness, eq. 3.24,	$\sqrt{g\ R_h'}$
3	U	average velocity in the cross section	$u_*' \sqrt{8\ /}$
		with (see eq. 3.13b) : $\sqrt{8 / f'} = 5.6 \log (R_h' / k_s) +$ (	
4	Ψ'	parameter of Einstein-Barbarossa, eq. 3.31,	$\frac{(s_s - 1)}{R_h'\ S_f}$
5	U / u_*''	ratio of velocities corresponding to Ψ' (see eq. 3.31 and Fig. 3.6)	
6	u_*''	friction velocity due to bed forms	U / (U/u
7	R_h''	hydraulic radius due to bed forms	$(u_*'')^2$ / (g
8	R_h	total hydraulic radius, eq. 3.24,	R_h' + R
9	u_*	total friction velocity, eq. 3.7,	$\sqrt{g\ R_h}$
10	h	flow depth (see Tableau 1.1)	
11	Q	water discharge, eq. 3.2a,	Uh (b + n

ii) *Sediment-transport calculations* :
The sediment-transport calculations will be made for the representative grain diameter, using three different total-load relations, namely : (1) *Einstein*, (2) *Graf* and *Acaroglu* and also (3) *Ackers* et *White*.

1° The formula of *Einstein* (1950), which allows the calculation of the total load transported by the flow, is given by :

$$q_s = q_{sb} + q_{ss} = q_{sb}\left[1 + 2.303\ \log(30.2\,h/\Delta)\ \mathcal{J}_1 + \mathcal{J}_2\right] \tag{6.60}$$

Since the grain-size distribution is *quasi uniform*, the calculations can be done using an equivalent grain diameter (see Table 6.3) of :

$d = d_{35} = 0.00029$ [m]

The intensity of transport is :

$$\Phi_* \equiv \Phi = \frac{q_{sb}}{\sqrt{(s_s - 1)g d_{35}^3}} \tag{6.28}$$

where $q_{sb} = g_{sb}/\gamma_s$ is the volumic solid discharge for a unit width and g_{sb} the solid discharge by weight; both transported as bed load.

The intensity of shear is :

$$\Psi_* \equiv \Psi' = (s_s - 1)\ \frac{d_{35}}{R_{hb}' S_f} \tag{6.34}$$

where $R_{hb}' \equiv R_h'$ is the hydraulic radius of the bed due to the granulats.

With the functional relationship of :

$$\Phi_* = f(\Psi_*) \tag{6.42a}$$

given by eq. 6.42 and presented in Fig. 6.8, one can obtain the solid discharge, q_{sb}, transported as bed load.

Next, the integrals, $\mathcal{J}_1$ and $\mathcal{J}_2$, which appear in the suspended-load formula, eq. 6.56 (see also eq. 6.60), should be determined to calculate the solid discharge, q_{ss}, transported as suspended load.

The total load, $q_s = q_{sb} + q_{ss}$, transported by the river can then be calculated using eq. 6.60.

The calculations can be programmed on a microcomputer using a spreadsheet program. The table of calculations prepared in this way is presented in Table Ex. 6.F.2. Each line of this table gives the calculations of the solid discharge (by volume, mass and weight), as well as other useful parameters calculated for each flow depth, h. The detailed explanations on the contents of the columns are given below the table of calculations.

Table Ex.6.F.2

Computation sheet for determining the *stage*—**solid**-discharge curve, using the method of *Einstein* (1950)

$b = 90$ [m] $\rho = 999.1$ [kg/m³]

$S_f = 0.0005$ [-] $\nu = 1.186 \times 10^{-6}$ [m²/s]

1	2	3	4	5	6	7	8	9	10	11	12
h	R_h'	u_*'	δ	k_s/δ	χ	Δ	P_e	Ψ'	Φ	q_{sb}	Q_{st}
[m]	[m]	[m/s]	[m]	[-]	[-]	[m]	[-]	[-]	[-]	[m³/s/m]	[m³/
0.29	0.02	0.01	1.39E−03	0.256	0.91	3.92E−04	10.01	47.92	0.00	0.00E+00	*0.00E-*
0.44	0.05	0.02	8.81E−04	0.405	1.25	2.86E−04	10.76	19.17	0.00	7.96E−08	*7.16E-*
0.65	0.10	0.02	6.23E−04	0.573	1.48	2.42E−04	11.32	9.58	0.10	2.00E−06	*1.80E-*
0.79	0.15	0.03	5.09E−04	0.702	1.56	2.29E−04	11.56	6.39	0.35	6.91E−06	*6.22E-*
0.89	0.20	0.03	4.41E−04	0.811	1.60	2.24E−04	11.71	4.79	0.71	1.41E−05	*1.27E-*
1.09	0.40	0.04	3.12E−04	1.147	1.60	2.23E−04	11.91	2.40	2.51	5.00E−05	*4.50E-*
1.29	0.60	0.05	2.54E−04	1.404	1.57	2.28E−04	12.06	1.60	4.32	8.59E−05	*7.73E-*
1.32	0.80	0.06	2.20E−04	1.621	1.49	2.40E−04	12.03	1.20	6.06	1.21E−04	*1.08E-*
1.41	1.00	0.07	1.97E−04	1.813	1.42	2.51E−04	12.05	0.96	7.77	1.54E−04	*1.39E-*
1.57	1.25	0.08	1.76E−04	2.027	1.37	2.61E−04	12.12	0.77	9.86	1.96E−04	*1.76E-*
1.76	1.50	0.09	1.61E−04	2.220	1.31	2.72E−04	12.19	0.64	11.94	2.37E−04	*2.14E-*
2.23	2.00	0.10	1.39E−04	2.564	1.24	2.87E−04	12.37	0.48	16.06	3.19E−04	*2.87E-*
2.71	2.50	0.11	1.25E−04	2.866	1.19	2.99E−04	12.53	0.38	20.18	4.01E−04	*3.61E-*
3.19	3.00	0.12	1.14E−04	3.140	1.16	3.07E−04	12.66	0.32	24.28	4.83E−04	*4.34E-*

col.	symbol	explanations	expression
1	h	flow depth (see Table Ex.6.F.1)	
2	R_h'	hydraulic radius due to grain roughness (see Table Ex.6.F.1)	
3	u_*'	friction velocity due to grain roughness (see Table Ex.6.F.1)	
4	δ	thickness of viscous sublayer	$\delta = 11.5\ \nu/$
5	k_s/δ	relative roughness (see Fig. 6.7a)	$k_s/\delta = d_{65}/$
6	χ	correction term for logarithmic velocity distribution (see Fig. 6.7a)	
7	Δ	apparent roughness diameter (see Fig. 6.7a)	$\Delta = d_{65}/$
8	P_e	transport parameter (see eq. 6.58)	$P_e = 2.303 \log(30.2h/$
9	Ψ'	intensity of shear, eq. 6.34,	$\Psi' = (s_s - 1)\ \frac{d_{35}}{R_h'}$
10	Φ	intensity of transport, eq. 6.42,	$\Phi = f(\Psi$
11	q_{sb}	solid discharge, as bed load, by volume and by unit width, eq. 6.28,	$q_{sb} = \Phi \sqrt{(s_s - 1)\, g\, d_3}$
12	Q_{sb}	solid discharge, as bed load, by volume	$Q_{sb} = q_{sb}$

Table Ex.6.F.2 (suite)

Computation sheet for determining the *stage*—**solid**-discharge curve, using the method of *Einstein* (1950)

$d_{35} = 0.00029$ [m] $v_{ss}(d_{35}) = 0.0365$ [m/s]

$d_{65} = 0.00036$ [m] $\rho_s = 2650$ [kg/m³] $\kappa = 0.4$ [-]

13	14	15	16	17	18	19	20	21
A_E	z	J_1	J_2	q_{ss}	Q_{ss}	Q_s	G_s	G_s
[-]	[-]	[-]	[-]	[m³/s/m]	[m³/s]	[m³/s]	[kg/s]	[N/s]
2.03E−03	9.222	2.62E−02	−1.59E−01	0.00E+00	0.00E+00	*0.00E+00*	0	0
1.30E−03	5.832	4.46E−02	−2.87E−01	1.54E−08	1.38E−06	*8.54E−06*	0	0
8.87E−04	4.124	6.90E−02	−4.63E−01	6.36E−07	5.73E−05	*2.37E−04*	1	6
7.32E−04	3.367	9.11E−02	−6.19E−01	3.00E−06	2.70E−04	*8.92E−04*	2	23
6.48E−04	2.916	1.13E−01	−7.67E−01	7.77E−06	6.99E−04	*1.97E−03*	5	51
5.32E−04	2.062	2.02E−01	−1.34E+00	5.35E−05	4.81E−03	*9.31E−03*	25	242
4.49E−04	1.684	3.10E−01	−1.96E+00	1.52E−04	1.37E−02	*2.14E−02*	57	557
4.39E−04	1.458	4.43E−01	−2.62E+00	3.27E−04	2.94E−02	*4.03E−02*	107	1047
4.12E−04	1.304	6.10E−01	−3.37E+00	6.14E−04	5.52E−02	*6.91E−02*	183	1798
3.69E−04	1.166	8.74E−01	−4.47E+00	1.20E−03	1.08E−01	*1.26E−01*	333	3268
3.29E−04	1.065	1.21E+00	−5.73E+00	2.13E−03	1.92E−01	*2.13E−01*	566	5548
2.61E−04	0.922	2.14E+00	−8.90E+00	5.62E−03	5.05E−01	*5.34E−01*	1416	13888
2.14E−04	0.825	3.47E+00	−1.29E+01	1.23E−02	1.10E+00	*1.14E+00*	3019	29619
1.82E−04	0.753	5.25E+00	−1.77E+01	2.35E−02	2.12E+00	*2.16E+00*	5727	56183

col.	symbol	explanations	expression
13	A_E	dimensionless height, eq. 6.52a,	$A_E = \frac{z_{sb}}{h} = \frac{2d_{35}}{h}$
14	z	Rouse exponent, eq. 6.50a, v_{ss} : settling velocity (see Fig. 6.10)	$z = \frac{v_{ss}}{\kappa\, u_*'}$
15	J_1	Einstein's first integral (see Fig. 6.12)	
16	J_2	Einstein's second integral (see Fig. 6.12)	
17	q_{ss}	solid discharge, as suspended load, by volume and by unit width, eq. 6.58,	$q_{ss} = q_{sb}\,(P_e\, I_1 + I_2)$
18	Q_{ss}	solid discharge, as suspended load, by volume	$Q_{ss} = q_{ss}\, b$
19	Q_s	solid discharge, as total load, by volume	$Q_s = Q_{sb} + Q_{ss}$
20	G_s	solid discharge, as total load, by mass	$G_s = Q_s\, \rho_s$
21	G_s	solid discharge, as total load, by weight	$G_s = Q_s\, \rho_s\, g$

2° The formula of *Graf* et *Acaroglu* (1968), which allows the calculation of t total load transported by the flow, is given by :

$$\Phi_A = f(\Psi_A) \quad (6.($$

with the parameter of transport :

$$\Phi_A = \frac{C_s \, U R_h}{\sqrt{(s_s-1) g d_{50}^3}} \quad (6.($$

and the parameter of shear intensity :

$$\Psi_A = \frac{(s_s-1)\, d_{50}}{S_e \, R_h} \quad (6.($$

where the equivalent diameter is taken as (see Table 6.3) :

$d \equiv d_{50} = 0.00032$ [m].

It is to be noted, that R_h is the total hydraulic radius and $C_s = q_s/q$ is t average concentration by volume. The functional relationship is evaluat according to eq. 6.63.

As in the previous case, the calculations are programmed on a microcompu using a spreadsheet program. The computation sheet prepared in this way presented in Table Ex. 6.F.3.

3° The formula of *Ackers* et *White* (1973), which allows the calculation average concentration, C_s , by volume is given by :

$$C_s = G_{gr} \, \frac{d_{35}}{h_m} \left(\frac{U}{u_*}\right)^{n_w} \quad (6.($$

where the equivalent diameter is taken as (see Table 6.3) :

$d \equiv d_{35} = 0.00029$ [m]

The sediment-transport parameter is calculated as :

$$G_{gr} = C_w \left(\frac{F_{gr}}{A_w} - 1\right)^{m_w} \quad (6.($$

with the mobility parameter defined as :

$$F_{gr} = \frac{u_*^{\,n_w}}{\sqrt{(s_s-1)\, g \, d_{35}}} \left[\frac{U}{\sqrt{32}\log(10 h_m/d_{35})}\right]^{(1-n_w)} \quad (6.($$

The dimensionless diameter for $d \equiv d_{35}$ is determined using :

Table Ex.6.F.3

Computation sheet for determining the *stage*—**solid**-discharge curve, using the method of *Graf* et *Acaroglu* (1968)

$S_f = 0.0005$ [-] $d_{50} = 0.00032$ [m]

$\rho = 999.1$ [kg/m³] $\rho_s = 2650$ [kg/m³]

1	2	3	4	5	6	7	8	9	10
h	R_h	U	Q	Ψ_A	Φ_A	C_s	Q_s	G_s	G_s
[m]	[m]	[m/s]	[m³/s]	[-]	[-]	[-]	[m³/s]	[kg/s]	[N/s]
0.29	0.28	0.16	4.2	3.718	0.380	1.91E-04	*7.95E-04*	2	21
0.44	0.44	0.29	11.7	2.401	1.143	2.06E-04	*2.41E-03*	6	63
0.65	0.65	0.45	26.5	1.639	2.990	2.39E-04	*6.33E-03*	17	165
0.79	0.78	0.58	41.4	1.356	4.821	2.48E-04	*1.03E-02*	27	266
0.89	0.88	0.69	55.8	1.203	6.522	2.49E-04	*1.39E-02*	37	362
1.09	1.07	1.05	103.7	0.992	10.615	2.20E-04	*2.28E-02*	60	592
1.29	1.26	1.33	157.4	0.839	16.163	2.22E-04	*3.49E-02*	92	907
1.32	1.29	1.58	191.3	0.821	17.091	1.93E-04	*3.69E-02*	98	960
1.41	1.37	1.81	232.4	0.773	19.864	1.85E-04	*4.30E-02*	114	1118
1.57	1.52	2.06	297.3	0.694	26.121	1.91E-04	*5.69E-02*	151	1478
1.76	1.70	2.30	371.6	0.622	34.445	2.03E-04	*7.54E-02*	200	1960
2.23	2.13	2.72	559.3	0.496	60.791	2.41E-04	*1.35E-01*	358	3507
2.71	2.57	3.11	780.9	0.411	97.757	2.82E-04	*2.20E-01*	583	5721
3.19	3.00	3.46	1026.7	0.353	143.802	3.20E-04	*3.28E-01*	870	8531

col.	symbol	explanations	expression
1	h	flow depth (see Table Ex.6.F.1)	
2	R_h	total hydraulic radius (see Table Ex.6.F.1)	
3	U	average velocity (see Table Ex.6.F.1)	
4	Q	liquid discharge (voir Tableau Ex.6.F.1)	
5	Ψ_A	shear-stress intensity parameter, eq. 6.61,	$\Psi_A = \dfrac{(s_s - 1)\, d_{50}}{S_f\, R_h}$
6	Φ_A	transport parameter, eq. 6.63,	$\Phi_A = 10.39\, \Psi_A^{-2.52}$
7	C_s	concentration by volume in the section, eq. 6.62,	$C_s = \Phi_A \dfrac{\sqrt{(s_s - 1)\, g\, d_{50}^3}}{U\, R_h}$
8	Q_s	solid discharge, as total load, by volume	$Q_s = C_s\, Q$
9	G_s	solid discharge, as total load, by mass	$G_s = Q_s\, \rho_s$
10	G_s	solid discharge, as total load, by weight	$G_s = Q_s\, g\, \rho_s$

Table Ex.6.F.4

Computation sheet for determining the *stage*—**solid**-discharge curve, using the method of *Ackers* et *White* (1973)

$\rho = 999.1$ [kg/m³] $\nu = 1.186 \times 10^{-6}$ [m²/s]

$\rho_s = 2650$ [kg/m³] $d_{35} = 0.00029$ [m] $\Rightarrow$ $d_* = (g\,(s_s - 1)/\nu^2)^{1/3}\, d_{35} = 6.536$

$n_w = 1 - 0.56 \log d_* = 0.5434$ $m_w = 9.66 / d_* + 1.34 = 2.8180$

$A_w = 0.23 / \sqrt{d_*} + 0.14 = 0.2300$ $C_w = 10^{(2.86 \log d_* - (\log d_*)^2 - 3.53)} = 0.0137$

1	2	3	4	5	6	7	8	9	10
h	u_*	U	Q	F_{gr}	G_{gr}	C_s	Q_s	G_s	G_s
[m]	[m/s]	[m/s]	[m³/s]	[-]	[-]	[-]	[m³/s]	[kg/s]	[N/s]
0.29	0.04	0.16	4.2	0.256	2.979E−05	6.69E−08	*2.79E−07*	0	0
0.44	0.05	0.29	11.7	0.369	3.316E−03	5.85E−06	*6.83E−05*	0	2
0.65	0.06	0.45	26.5	0.490	1.937E−02	2.65E−05	*7.04E−04*	2	18
0.79	0.06	0.58	41.4	0.573	4.242E−02	5.22E−05	*2.16E−03*	6	56
0.89	0.07	0.69	55.8	0.638	6.923E−02	8.03E−05	*4.48E−03*	12	117
1.09	0.07	1.05	103.7	0.808	1.845E−01	2.10E−04	*2.17E−02*	58	565
1.29	0.08	1.33	157.4	0.939	3.268E−01	3.41E−04	*5.37E−02*	142	1397
1.32	0.08	1.58	191.3	1.020	4.446E−01	4.95E−04	*9.48E−02*	251	2463
1.41	0.08	1.81	232.4	1.099	5.808E−01	6.44E−04	*1.50E−01*	397	3891
1.57	0.09	2.06	297.3	1.197	7.845E−01	8.11E−04	*2.41E−01*	639	6269
1.76	0.09	2.30	371.6	1.289	1.014E+00	9.64E−04	*3.58E−01*	949	9314
2.23	0.10	2.72	559.3	1.467	1.570E+00	1.22E−03	*6.81E−01*	1805	17707
2.71	0.11	3.11	780.9	1.626	2.210E+00	1.44E−03	*1.12E+00*	2970	29133
3.19	0.12	3.46	1026.7	1.769	2.906E+00	1.63E−03	*1.68E+00*	4440	43559

col.	symbol	explanations	expression
1	h	flow depth (see Table Ex.6.F.1)	$h \cong h_m = S/B$
2	u_*	total shear velocity (see Table Ex.6.F.1)	
3	U	average velocity (see Table Ex.6.F.1)	
4	Q	liquid discharge (see Table Ex.6.F.1)	
5	F_{gr}	parameter of mobility, eq. 6.64,	$F_{gr} = \dfrac{u_*^{\,n_w}}{\sqrt{(s_s - 1)\, g d_{35}}} \left[\dfrac{U}{\sqrt{32}\log(10h/d_{35})}\right]^{(1-n_w)}$
6	G_{gr}	transport parameter, eq. 6.65,	$G_{gr} = C_w \left(\dfrac{F_{gr}}{A_w} - 1\right)^{m_w}$
7	C_s	concentration by volume, eq. 6.66,	$C_s = G_{gr}\, \dfrac{d_{35}}{h} \left(\dfrac{U}{u_*}\right)^{n_w}$
8	Q_s	solid discharge, as total load, by volume	$Q_s = C_s\, Q$
9	G_s	solid discharge, as total load, by mass	$G_s = Q_s\, \rho_s$
10	G_s	solid discharge, as total load, by weight	$G_s = Q_s\, g\, \rho_s$

$$d_* = d_{35}\left((s_s-1)\frac{g}{\nu^2}\right)^{1/3} \cong 6.5 \qquad (6.26)$$

which allows calculation of the coefficients, n_w, m_w, A_w and C_w (see point 6.5.2.4°).

Again, the calculations are programmed on a microcomputer using a spreadsheet program. The computation sheet is presented in Table Ex. 6.F.4.

The Fig. Ex. 6.F.2 gives the *stage*—**solid**-discharge curves — supplemented by the *stage*—**liquid**-discharge curve — for the **total load** calculated using the total-load relations of (1) *Einstein*, (2) *Graf* et *Acaroglu* et 3) *Ackers* et *White* as well as for the **bed load** calculated using the bed-load relation of (1) *Einstein*. (*Einstein*'s method, 1950, is an indirect method, thus it allows the evaluation of the bed load transport automatically.).

The Fig. Ex. 6.F.2 shows that the three methods used for the calculations do not give the same values for the solid discharge, Q_s. It is important however to remind that the formulae for the sediment transport can only give the engineer an idea about the order of magnitude of the solid discharge that one should reasonably expect in a particular flow situation. It should also be clear that the sediment-transport capacity (voir sect. 6.1.4) has been calculated.

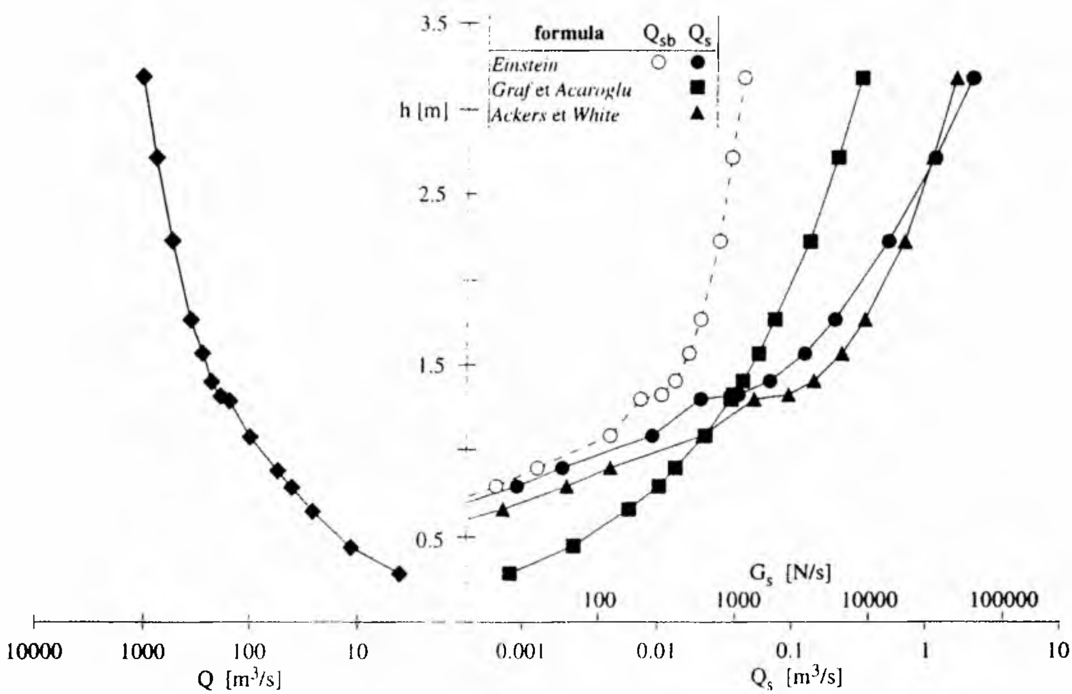


Fig. Ex.6.F.2 *Stage—liquid*-discharge and *stage—solid*-discharge curves.

6.6.2 Problems, unsolved

Ex. 6.1

A long channel of rectangular cross section has a bed slope of $S_f = 0.0004$ [-]. T channel bed is composed of a near-uniform granulate of $d_{50} = 0.5$ [mm] with a porosi of $p = 0.3$ [-]. The normal flow depth was measured as $h = h_n = 2.10$ [m]. This chanr enters a lake ; at the juncture the water levels of the channel and of the lake are the same

The water level in the lake is now *lowered* by $\Delta h = 0.10$ [m]. Determine, at what time t channel bed will be lowered by 90 % and by 50 % of Δh and this at two stations, situat at 1.5 [km] and at 20.0 [km] upstream of the juncture.

Ex. 6.2

In the channel, described in Ex. 6.1, the fixed point at the juncture will be ***raised*** l $\Delta h = 0.20$ [m]. Determine the temporal variation of the channel bed at a station, bei situated 20.0 [km] upstream of the juncture.

Ex. 6.3

The unit discharge in a river is $q = 5$ [m^2/s]. The bed has a slope of $S_f = 0.0005$ [-] ar the porosity is $p = 0$ [-] ; its granulate is given as $d_{50} = 0.4$ [mm].

Well downstream, this river enters a reservoir, where the water depth is kept constant at height of $H = 5$ [m]. Determine the aggradation one may expect in the reservoir after 2: [h] and 500 [h].

Ex. 6.4

The irrigation channel "Sivan" must be controlled for 81 days per year to deliver constant discharge of 10 [m^3/s] (during the remaining days of the year the discharge w be less). This rectangular channel, having a width of $B = 5.0$ [m], has a mobile bed with uniform granulate of $d_{50} = 5$ [mm] ; the bed slope is $S_f = 0.0003$ [-].

i) Calculate the solid discharge, which is annually transported.

ii) Study a proposal, which envisions that a spillway— blocking the flow at a wat depth being three times the normal flow depth — is installed at the downstream the channel. What will the sediment deposition amount to and where will it tal place ?

Ex. 6.5

A reach of a river of length $L = 38$ [km] — being indicative of the Bas-Rhône — conve a constant discharge of $Q = 4000$ [m^3/s]. In this reach the cross sections may t approximated by rectangular ones having a width of $B = 250$ [m] and having a consta bed slope of $S_f = 0.0007$ [-]. The roughness coefficient was estimated as being $K_s =$: [m$^{1/3}$/s] ; the diameter of the rather uniform granulate is $d_{50} = 27.4$ [mm].

It is envisioned to build a system of weirs, which would raise the water level by 10 [m], thus to H = 10 [m] + h_n. Study the evolution of bed level for a period of two years. Investigate the sediment-transport problem in the reach behind the weirs.

Ex. 6.6

The discharge in a channel, having a bed slope of $S_f = 0.00027$ [-] is $Q = 100$ [m^3/s] ; the width at the channel bed is b = 46 [m] and the side slopes are 2 / 1 . Flow in this channel is uniform and the temperature of the water is $T = 15$ [C°]. Samples of the bed material have been evaluated ; the granulometry is given in the following table and its density is $s_s = 2.65$ [-].

median diameter [mm]	granulometric fraction [%]
0.088	5
0.177	22
0.354	37
0.707	31
1.414	5

Calculate the total-load transport, Q_s, by making use of different available formulae. This is to be done :

i) for each individual granulometric fraction,
ii) for all fractions together, taking an equivalent diameter.

Ex. 6.7

For the channel studied in Ex. 6.6, verify if the transport of sediments is influenced by the water temperature ; the lowest and highest temperatures expected are respectively, $T = 10$ [C°] and $T = 20$ [C°]. Make the calculations using an equivalent diameter.

Ex. 6.8

For the channel studied in Ex. 6.6, determine the diameter of the armouring and the flow depth for which the armour layer turns unstable.

ABOUT THE AUTHORS

Professor at the Swiss Federal Institute of Technology in Lausanne (EPFL) since 1973, **Walter H. Graf** is in charge of its Hydraulic Research Laboratory, where M. S. Altinakar is one of his assistants. His research activities are concentrated on the hydrodynamics of free-surface flow, notably open-channel hydraulics and hydrodynamics of lakes and reservoirs. Previously W. H. Graf had teaching engagements at the University of California-Berkeley, at Cornell University and at Lehigh University in the USA. In 1995, he was named a "Guest Professor" at the prestigious Tsinghua University at Beijing, China. Over the years he served also as a visiting professor in Austria (1972, 1994), Republic of South Africa (1974), China (1984, 1994), Italy (1990, 1993), Japan (1995) and the Soviet Union (1975, 1982, 1989), where he lectured essentially on different topics of Fluvial Hydraulics. Amongst the different distinctions received could be mentioned the "Freeman Fellowship" from the American Society of Civil Engineers in 1972, the "IAHR Lecturer Award" given by the International Association of Hydraulics Research in 1988, the "guest scholarship" from the Kajima Foundation of Tokyo, Japan in 1995 and more recently, in 1997, the K. E. Hilgard Hydraulic Prize, awarded annually by the American Society of Civil Engineers.

Walter H. Graf is the author or editor of four books on *Hydraulics of Sediment Transport* (1971, McGraw Hill, USA; 1984, Water Res. Publ., USA), *Hydrodynamics of Lakes* (1979, Elsevier, NL), *Lake and Reservoir Hydraulics* (1987, Water Res. Publ., USA) and *Hydrodynamique* (1991, Ed. Eyrolles, F.; 1995, Presses polytechniques et universitaires romandes, CH). More recently the two volumes of *Fluvial Hydraulics* (1993, 1996, Presses polytechniques et universitaires romandes, CH) were produced; the last books were done with the collaboration of M. S. Altinakar.

In 1987, the XXII Congress of the International Association of Hydraulic Research (IAHR) was held in Lausanne, under the direction of W. H. Graf with the participation of M. S. Altinakar; together six volumes of proceedings were edited.

In addition to being senior assistant at the Hydraulic Research Laboratory, EPFL, Dr. **M. S. Altinakar** is principal Engineer in the hydraulics department of the Bonnard & Gardel Eng. Consulting firm headquartered in Lausanne. There, M. S. Altinakar was in charge of various large projects in hydraulic, hydropower and municipal engineering.

FLUVIAL HYDRAULICS

Flow and Transport Processes in Channels of Simple Geometry

Walter H. Graf
in collaboration with M. S. Altinakar

Laboratoire de recherches hydrauliques
Ecole polytechnique fédérale
Lausanne, Suisse

JOHN WILEY & SONS
Chichester · New York · Weinheim · Brisbane · Singapore · Toronto

Published 1998 by John Wiley & Sons Ltd,
Baffins Lane, Chichester,
West Sussex PO19 1UD, England

National 01243 779777
International (+44) 1243 779777
e-mail (for orders and customer service enquiries): cs-books@wiley.co.uk
Visit our Home Page on http://www.wiley.co.uk
or http://www.wiley.com

This book, translated by the authors, was originally published in French entitled Hydraulique Fluviale. Volume 1: Ecoulement permanent uniforme et non uniforme. Copyright © 1993 Presses polytechniques et universitaires romandes, Lausanne, Suisse. ISBN: 2-88074-261-7. Volume 2: Ecoulement non permanent et phénomènes de transport. Copyright © 1996 Presses polytechniques et universitaires romandes, Lausanne, Suisse. ISBN: 2-88074-300-1.

Other Wiley Editorial Offices

John Wiley & Sons, Inc., 605 Third Avenue,
New York, NY 10158-0012, USA

WILEY-VCH Verlag GmbH, Pappelallee 3,
D-69469 Weinheim, Germany

Jacaranda Wiley Ltd, 33 Park Road, Milton,
Queensland 4064, Australia

John Wiley & Sons (Asia) Pte Ltd, 2 Clementi Loop #02-01,
Jin Xing Distripark, Singapore 129809

John Wiley & Sons (Canada) Ltd, 22 Worcester Road,
Rexdale, Ontario M9W 1L1, Canada

British Library Cataloguing in Publication Data

A catalogue record for this book is available from the British Library

ISBN 0-471-97714-4

Printed from camera-ready-copy supplied by the authors
Printed and bound in Great Britain by Bookcraft (Bath) Ltd

This book is printed on acid-free paper responsibly manufactured from sustainable forestry, in which at least two trees are planted for each one used for paper production.

Contents

1. INTRODUCTION

In this book on *fluvial hydraulics* — here taken to be synonymous to open-channel hydraulics — we shall treat the flow and flow-related phenomena in artificial and natural channels with a free surface subjected to the atmospheric pressure.

This chapter of introduction begins with a presentation of the different types of channels as well as with the corresponding flow regimes. Subsequently, the notions of the distribution of velocity and of pressure are exposed.

A list of references as well as a list of symbols shall be presented in the final pages of this volume.

TABLE OF CONTENTS

1.1. CHANNELS

1° A channel is a transport system where water flows and where the free surface is subject to atmospheric pressure.

2° The hydraulic study of a channel often confronts the engineer with a question of the form :

> for a given longitudinal bed slope, a certain discharge must be conveyed; the form and the dimensions of the channel are to be determined.

1.1.1 Kinds of Channels (see Fig. 1.1)

1° Two categories of channels are to be distinguished :

i) natural channels,
ii) artificial channels.

2° *Natural* channels are watercourses, which exist naturally on (or under) the earth, such as gullies, brooks, torrents, rivers, streams and estuaries.

The geometric and hydraulic properties of such channels are generally rather irregular. The application of the hydraulic theory gives only approximate results, since numerous assumptions have to be made.

3° *Artificial* channels are watercourses developed by men on (or under) the earth, such as open channels (navigation channels, power canals, irrigation and drainage channels) or closed channels where flow does not fill the entire section (hydraulic tunnels, aqueducts, drains, sewage canals).

The geometric and hydraulic properties of such channels are generally rather regular. The application of the hydraulic theory gives reasonably realistic results.

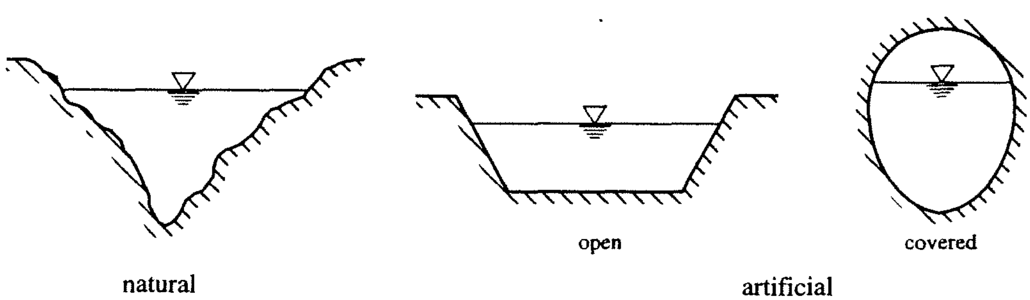


Fig. 1.1 Kinds of channels.

1.1.2 Geometry of Channels (see Fig. 1.2)

1° The (transversal) section of a channel is a section in the cross-sectional plane being normal to the direction of flow.

The section, or better the *wetted surface,* A, is the portion of the cross section occupied by the liquid.

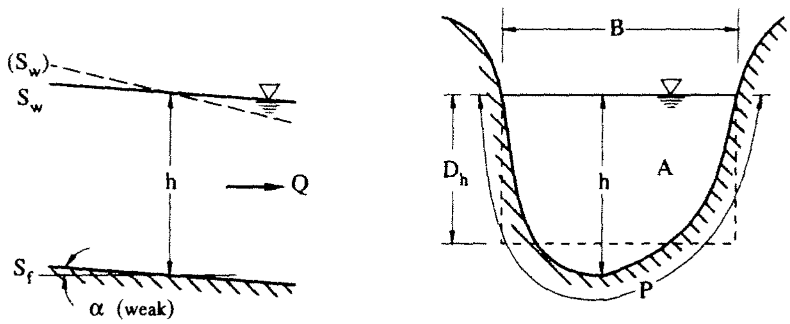


Fig. 1.2 Geometric elements of a channel section.

2° A channel, whose section does not vary and whose longitudinal slope and roughness remains constant — however the flow depth may vary — is called a *prismatic* channel; otherwise the channel is a non-prismatic one.

3° The geometric elements of a section or wetted surface, A, are the following :

i) The *wetted perimeter,* P, of the channel, being formed by the length of the line of contact between the wetted surface and the bed and the side walls, but does not include the free-water surface.

ii) The *hydraulic radius,* R_h, being the ratio of wetted surface, A, to its wetted perimeter, P , or :

$$R_h = \frac{A}{P} \tag{1.1}$$

being often used as a length of reference.

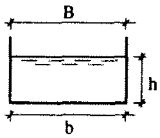
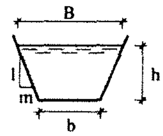
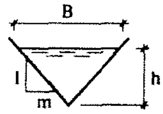
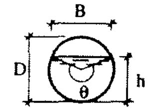
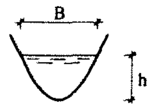
iii) The (top) *width,* B, of the channel being the width at the free surface.

iv) The *hydraulic depth,* D_h, of the channel being defined by :

$$D_h = \frac{A}{B} \tag{1.2}$$

v) The *flow depth,* h, or the water height — if not defined otherwise — is considered to be the maximum depth.

Table 1.1 Geometric elements for different sections of channels.

	 Rectangle	 Trapezoid	 Triangle	 Circle	 Parabola
Section A	$b\,h$	$(b+mh)h$	mh^2	$\frac{1}{8}(\theta-\sin\theta)\,D^2$	$\frac{2}{3}B\,h$
Wetted perimeter P	$b+2h$	$b+2h\sqrt{1+m^2}$	$2h\sqrt{1+m^2}$	$\frac{1}{2}\,\theta\,D$	$B+\frac{8}{3}\frac{h^2}{B}$ *
Hydraulic radius R_h	$\frac{b\,h}{b+2h}$	$\frac{(b+mh)\,h}{b+2h\sqrt{1+m^2}}$	$\frac{mh}{2\sqrt{1+m^2}}$	$\frac{1}{4}\left[1-\frac{\sin\theta}{\theta}\right]D$	$\frac{2B^2h}{3B^2+8h^2}$ *
Width B	b	$b+2mh$	$2mh$	$(\sin\theta/2)\,D$ or $2\sqrt{h\,(D-h)}$	$\frac{3}{2}\frac{A}{h}$
Hydraulic depth D_h	h	$\frac{(b+mh)\,h}{b+2mh}$	$\frac{1}{2}h$	$\left[\frac{\theta\angle\sin\theta}{\sin\theta/2}\right]\frac{D}{8}$	$\frac{2}{3}h$

* Valid for $0<\xi\le 1$, with $\xi = 4h/B$. If $\xi > 1$: $P = (B/2)\left[\sqrt{1+\xi^2} + 1/\xi\,\ln\left(\xi+\sqrt{1+\xi^2}\right)\right]$

4° Formulas for the geometric elements for five different types of channel sections (see *Chow*, 1959, p. 21) are given in Table 1.1. A natural watercourse might have a rather irregular geometric form, but often it can be rather well approximated by a trapezoidal or parabolic section.

5° Besides the geometric elements, the longitudinal slopes are also to be considered, namely the :

i) slope of the bed (bottom or floor), S_f,

ii) slope of the water surface (piezometric), S_w.

The value of the bottom slope depends essentially on the topography of the terrain; it is generally weak, thus may be expressed by : $S_f = \text{tg}\,\alpha \cong \sin\alpha$.

6° The wetted perimeter, P, can be composed of a fixed or immobile bed (concrete, rock) or of a mobile bed (granulates of sediments).

1.2 FLOW IN CHANNELS

1° Flow in natural or artificial channels is flow with a free surface, being the surface of separation between air and water; the pressure is there equal to the *atmospheric pressure.*

2° Flow in open channels is essentially due to the inclination (slope) of the bed, while flow in closed conduits (see *Graf & Altinakar*, 1991, chap. PP.1), is due to a difference in the charge between the sections.

1.2.1 Types of Flow

1° A classification of open-channel flow may be done according to the change of the flow depth, h or D_h, with respect to time and space :

$$D_h = f(t, x)$$

2° *Time variation* (see Fig. 1.3)

Flow is *steady* (stationary or permanent) if the average velocity of flow, U , and point velocity, u , but also the flow depth, h or D_h, remain invariable with time, in magnitude and in direction. Consequently the discharge remains constant :

$$U\,S = Q \tag{1.3}$$

between the different sections of the channel (see sect. 2.1 and eq. 2.6), supposing there is not lateral inflow or outflow.

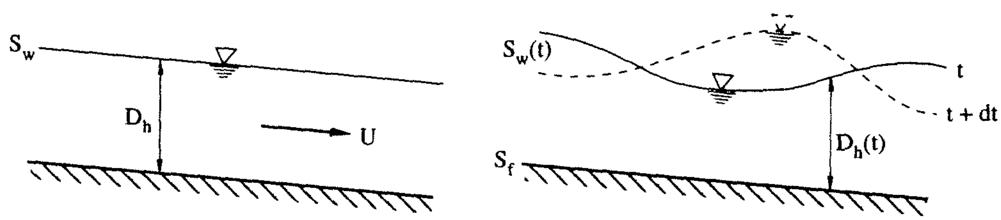


Fig. 1.3 Scheme of steady and unsteady flows.

Flow is *unsteady* if the flow depth, $D_h(t)$, as well as the other parameters vary with time. Consequently, the discharge is no more constant (see sect. 2.1 and eq. 2.1).

Strictly speaking, open-channel flow is rarely steady. However, the temporal variations are often sufficiently slow and the flow may be assumed to be steady and this at least for relatively short time intervals.

3° *Space variation* (see Fig. 1.4)

Flow is *uniform* if the flow depth, D_h , as well as the other parameters, remain unchanged at every section of the channel. The line of the bottom slope is thus parallel to the one of the free-water surface, or $S_f \equiv S_w$.

Flow is *non-uniform* or *varied* if the depth, $D_h(x)$, as well as the other parameters, vary along the length of the channel. The bottom slope is thus different from the slope of the water surface, or $S_f \neq S_w$.

Non-uniform flow can be steady or unsteady.

Varied flow can be accelerated, $dU/dx > 0$, or decelerated, $dU/dx < 0$, depending on the variation of velocity in the direction of flow.

If flow is a *gradually varied* one, the depth, $D_h(x) \cong D_h$, as well as the other parameters, vary slowly from one section to another. Over a small length of the channel, one may assume that the flow is quasi-uniform and the velocity, U, remains essentially constant.

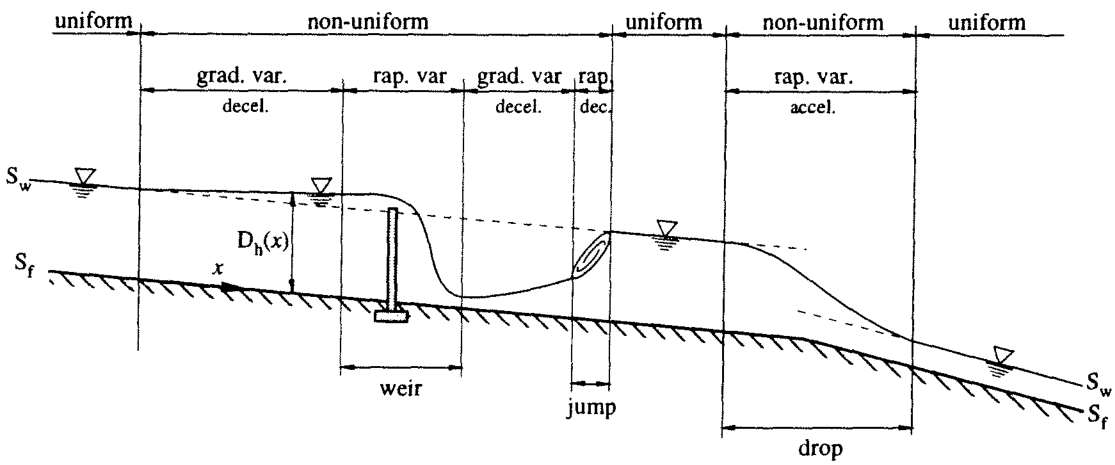


Fig. 1.4 Scheme of steady, uniform and non-uniform flows.

If flow is a *rapidly varied* one, the depth, $D_h(x)$, as well as the other parameters, change abruptly over a comparatively short distance, sometimes with a discontinuity. This happens generally in the neighbourhood of a singularity, such as at a weir or at a change of channel width, but also at an hydraulic jump or an hydraulic drop.

4° The kinds of flow one encounters in fluvial hydraulics (see Fig. 1.3 and Fig. 1.4) can be summarised as follows :

i) Steady flow
$$\begin{cases} \text{uniform} \\ \text{non-uniform} & \begin{cases} \text{gradually} \\ \text{rapidly} \end{cases} \end{cases}$$

ii) Unsteady flow
$$\begin{cases} \text{uniform } (\textit{rare}) \\ \text{non-uniform} & \begin{cases} \text{gradually} \\ \text{rapidly} \end{cases} \end{cases}$$

1.2.2 Flow Regimes

1° The physics of open-channel flow is governed basically by the interplay of the :

- inertia forces,
- gravity forces,
- friction (viscosity and roughness) forces.

2° The (reduced) equations of motion (see *Graf & Altinakar,* 1991, sect. FR.7.2) involve the following dimensionless numbers :

i) the *Froude number,* being the ratio of gravity to inertia forces, or :

$$\frac{\rho g}{\rho U_c^2 / L_c} = \frac{g L_c}{U_c^2} = Fr^{-2} \qquad \text{and} \qquad Fr = \frac{U_c}{\sqrt{g L_c}} \tag{1.4}$$

ii) the *Reynolds number,* being the ratio of friction to inertia forces, or :

$$\frac{\mu\, (U_c / L_c^2)}{\rho U_c^2 / L_c} = \frac{\nu}{U_c L_c} = Re^{-1} \qquad \text{and} \qquad Re = \frac{U_c L_c}{\nu} \tag{1.5}$$

Added to these two numbers is still :

iii) the *relative roughness,* being the ratio of the roughness height, k_s , to a characteristic length, or :

$$\frac{k_s}{L_c} \tag{1.6}$$

U_c and L_c are characteristic velocity and length; one takes often $U_c = U$ and $L_c = R_h$ or $L_c = D_h$.

In the hydraulics of open-channel flow, one generally defines these dimensionless numbers as :

$$Fr = \frac{U}{\sqrt{gD_h}} \quad ; \quad Re = \frac{4R_hU}{\nu} \quad \text{or} \quad Re' = \frac{R_hU}{\nu} \quad ; \quad \frac{k_s}{D_h} \tag{1.7}$$

3° The Reynolds number is used to classify the flow (see *Graf & Altinakar,* 1991, chap. FR.3) as follows :

- laminar flow $Re' < 500$
- turbulent flow $Re' > 2000$
- transition flow $500 < Re' < 2000$

From numerous experiments with different artificial channels (see *Chow*, 1959, p. 10) it results that flow is turbulent if the Reynolds number, Re', reaches a value of 2000.

In general, flow in open channels is a turbulent and often rough flow.

4° The Froude number is used to classify the flow (see sect. 2.3.3) as follows :

- subcritical (fluvial) flow $Fr < 1$
- supercritical (torrential) flow $Fr > 1$
- critical flow $Fr \equiv Fr_c = 1$

In general, flow in open channels can be of the three types.

5° Consequently, the combined effect of the Reynolds number, Re', and the Froude number, Fr , gives the following four regimes of flow :

- subcritical-laminar $Fr < 1$, $Re' < 500$
- subcritical-turbulent $Fr < 1$, $Re' > 2000$
- supercritical-laminar $Fr > 1$, $Re' < 500$
- supercritical-turbulent $Fr > 1$, $Re' > 2000$

A relationship depth/velocity, taken from the experiments by *Robertson et Rouse,* is given in Fig. 1.5; it is valid for very wide rectangular channels.

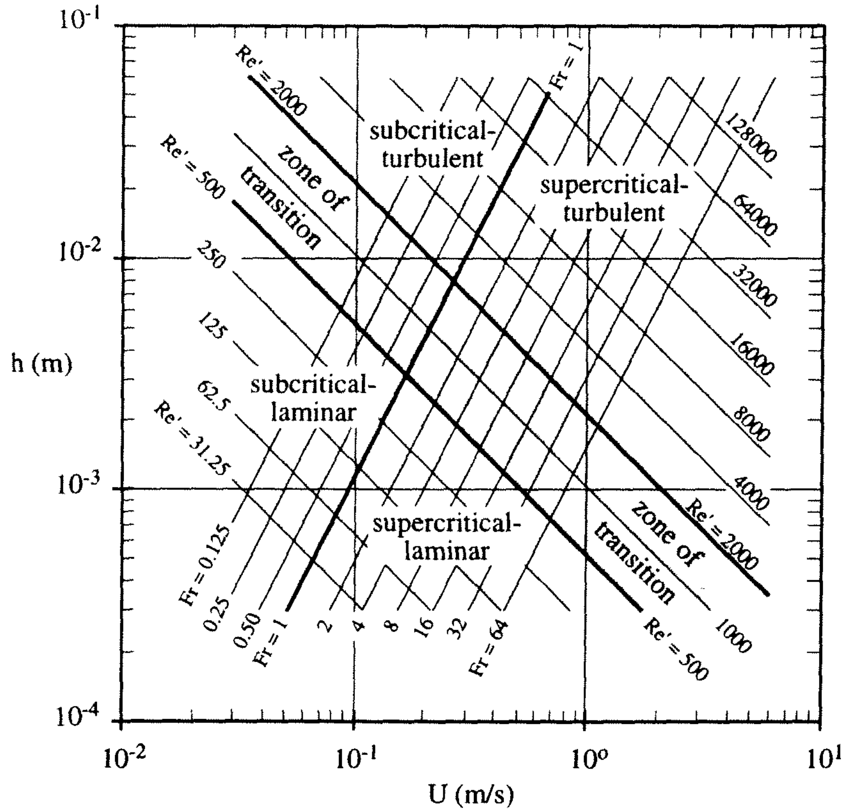


Fig. 1.5 The four regimes of open-channel flow.

1.3 DISTRIBUTION OF VELOCITY

1° In flow along a wall (the bottom of a channel), a distribution of velocity (see *Graf & Altinakar*, 1991, chap. FR. 6) is encountered. Being zero at the wall, the point velocity, u , increases rapidly towards the free surface; its maximum value is often found slightly below this free surface. The velocity profile is approximately logarithmic.

2° Steady flow depends in general on the three variables, x, y and z ; this is called *three-dimensional* flow. For a rectangular channel with a bed and vertical side walls, a schematic distribution of the point velocity, $u(x, y, z)$, is given in Fig. 1.6.

If such a channel has a large width, B — large in comparison with the depth, B > 5 h — flow is considered *two-dimensional*, with the exception of a small distance close to the vertical side walls.

Hydraulic calculations are considerably simplified, if one assumes the flow to be *one-dimensional*. The average velocity, U(x), in a vertical or in a section, is expressed by :

$$U = \frac{1}{h} \int_0^h u(z)\, dz \qquad \text{or} \qquad U = \frac{1}{A} \int_0^B \int_0^h u(z)\, dz dy \tag{1.8}$$

3° In open channels of simple geometry, one encounters generally turbulent flow where the point velocity, u(x, z), differs little from the average velocity, U(x). In the steady state, such an hypothesis allows to consider the flow as one-dimensional.

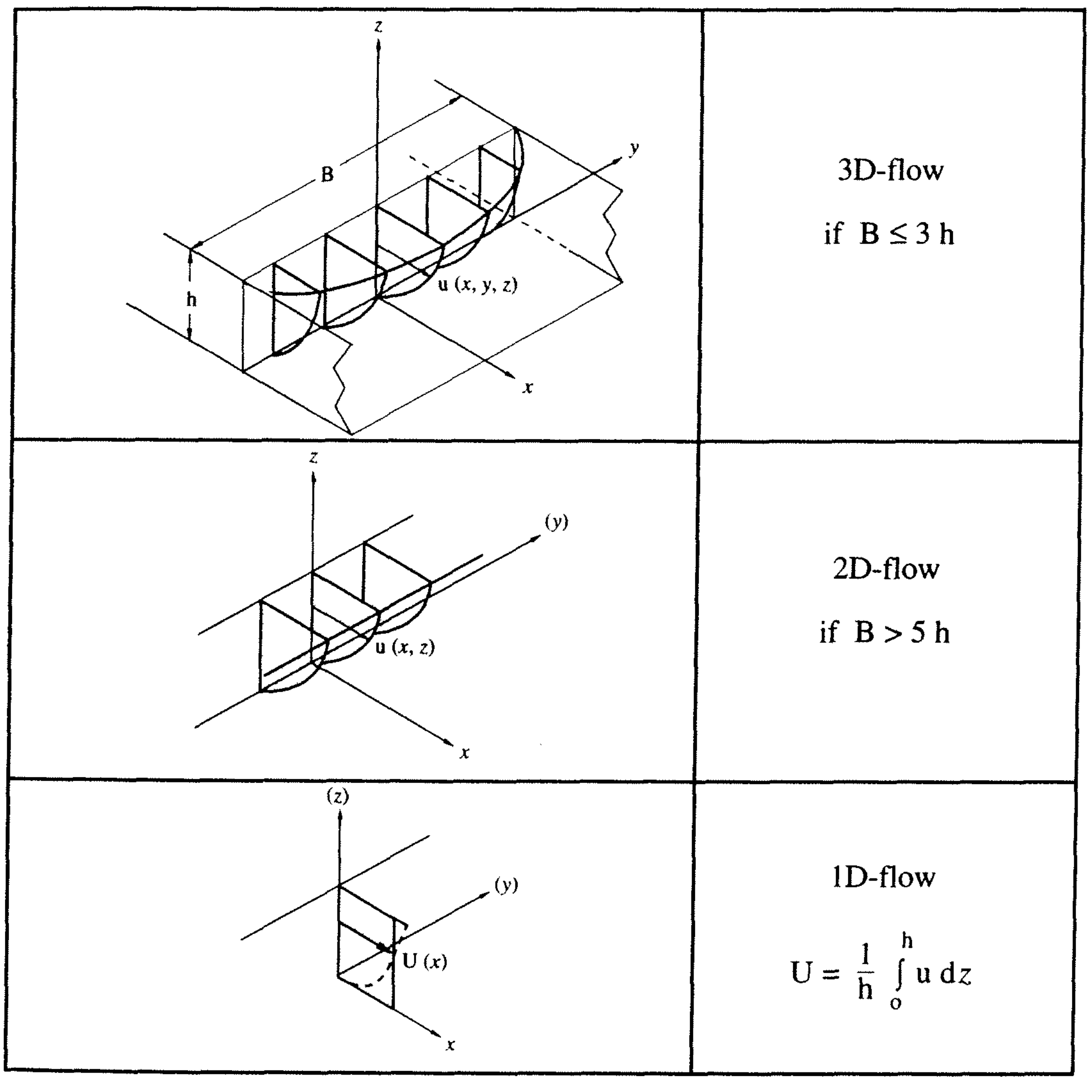


Fig. 1.6 Distribution of velocity.

4° For a determination of the average velocity, U , in a given section, the following approximate relations can be used (see Fig. 1.7) :

$$U \cong (0.8 \text{ à } 0.9)\, u_s \qquad \text{(formula of Prony)}$$
$$U \cong 0.5\,(u_{0.2} + u_{0.8}) \qquad \text{(formula of USGS)} \qquad (1.9)$$
$$U \cong u_{0.4}$$

where $u_{0.2}$, $u_{0.8}$, $u_{0.4}$ and u_s are the point velocities at given positions .

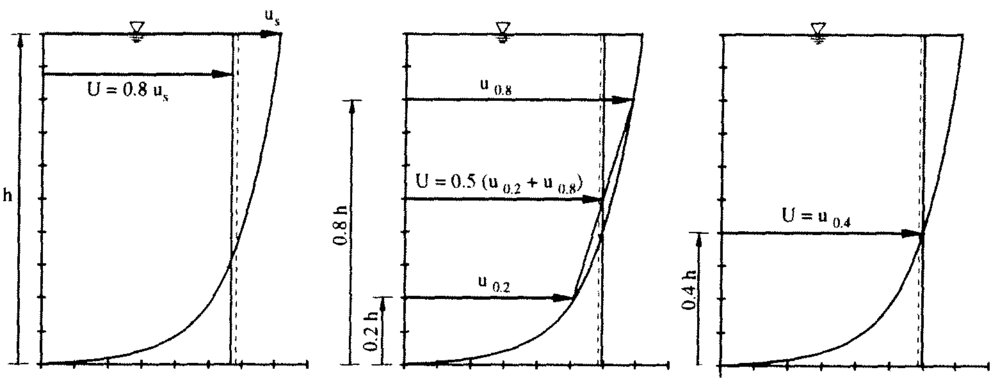


Fig. 1.7 Average velocity.

1.4 DISTRIBUTION OF PRESSURE

1° The equation of steady motion for an incompressible fluid (see *Graf & Altinakar*, 1991, p. 132), written for the normal component, n(≡ z), is :

$$U \frac{U}{r} = -\frac{1}{\rho}\frac{\partial}{\partial n}(p + \gamma z') \qquad (1.10)$$

where (U^2/r) is the centrifugal acceleration of a mass-fluid, which displaces itself on a curved line, r (see Fig. 1.8).

2° Assuming that U and r remain reasonably constant and after integration of eq. 1.10, one obtains :

$$(p + \gamma z') = -\rho \int_0^z \frac{U^2}{r} dn + Cte = -\rho \frac{U^2}{r} z + Cte \qquad (1.10a)$$

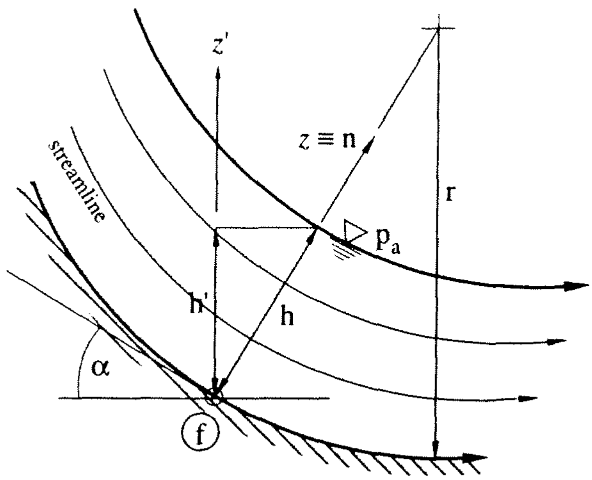


Fig. 1.8 Flow over a concave bottom.

3° Taking a point on the bottom of the channel and another one on the free surface, one respectively writes :

for $z = 0$ ($z' = 0$) : $p = p_f$ where : $p_f = Cte$

for $z = h$ ($z' = h'$) : $p = p_a$ where : $p_a + \gamma\, h' = -\rho \dfrac{U^2}{r}\, h + Cte$

An expression for the relative (with respect to the atmospheric pressure) pressure on the bottom of the channel, is given by :

$$p_f = \gamma\, h' + \rho \frac{U^2}{r}\, h + \overset{=0}{\cancel{p_a}} \tag{1.11}$$

having an hydrostatic and an accelerating contribution.

1.4.1 Uniform Current

1° For uniform flow, when the average velocity, U, remains constant and the streamlines are reasonable rectilinear (with $r \rightarrow \infty$), the distribution of pressure is *hydrostatic* in a section, normal to the bottom (see Fig. 1.9). Thus one may write, taking $z \equiv n$ (eq. 1.10), the following :

$$0 = \frac{\partial}{\partial z}\,(\gamma\, z' + p) \tag{1.12}$$

2° An expression for the pressure, relative to the bottom, can now be given as :

$$p_f = +\gamma h' \qquad (1.13)$$

which gives :

$$\left(\frac{p}{\gamma}\right)_f = h \cos\alpha \qquad (1.14)$$

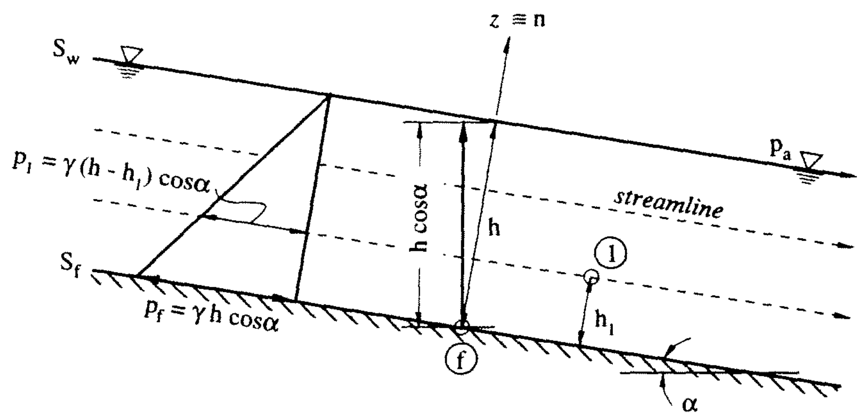


Fig. 1.9 Flow with a uniform current.

3° For the usually encountered open channels, the inclination, α , is rather weak, namely $\alpha < 6°$ or $J_f < 0.1$, implying that $\cos\alpha \simeq 1$. Consequently eq. 1.14 reduces to :

$$\left(\frac{p}{\gamma}\right)_f = h \qquad (1.15)$$

where h is the flow depth in the channel.

1.4.2 Curvilinear Current

1° For flow, being (slightly) non-uniform, thus having a curvilinear current of converging or diverging type, there exists an acceleration component caused by the inertia forces. As done above, one writes :

$$\frac{U^2}{r} = -\frac{1}{\rho}\frac{\partial}{\partial n}(p + \gamma z') \qquad (1.10)$$

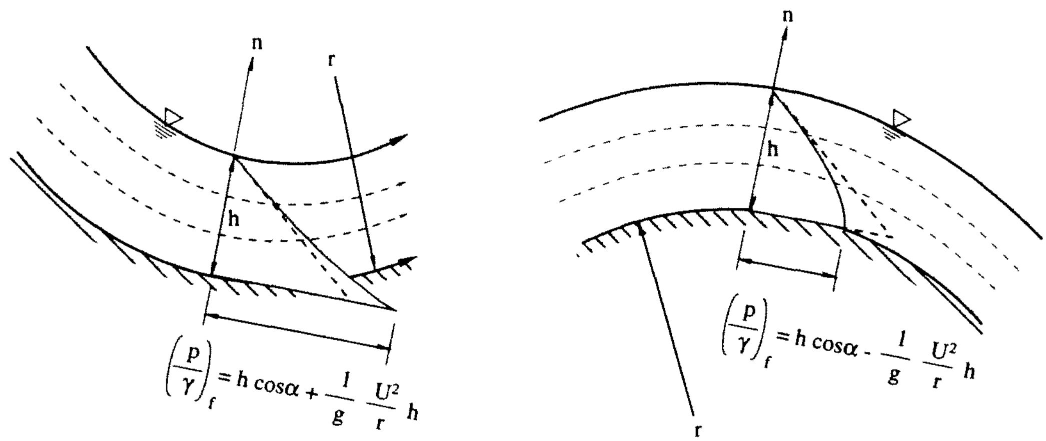


Fig. 1.10 Flow over a concave and a convex bottom.

and the expression for the pressure relative to the bottom is given by :

$$p_f = + \gamma h' \pm \rho \frac{U^2}{r} h \tag{1.11}$$

being (+) for a concave and (–) for a convex bottom.

Subsequently one obtains :

$$\left(\frac{p}{\gamma}\right)_f = h \cos \alpha \pm \frac{1}{g} \frac{U^2}{r} h \tag{1.16}$$

2° The distribution of pressure is no more hydrostatic (see Fig. 1.10). For an external concave current, the centrifugal force increases the pressure; while for a convex current, this force decreases the pressure. In the latter case, the pressure could get below the atmospheric pressure, thus causing separation of flow on the channel bed.

2. HYDRODYNAMIC CONSIDERATIONS

Some fundamental notions of hydrodynamics, being the basis of open-channel hydraulics, will be exposed in this chapter.

The equations of continuity and of energy will be developed for the general case. Subsequently, the specific energy, a concept useful for the understanding of different problems, will be introduced. Elementary knowledge of gravity waves is presented.

Finally the hydrodynamic equations are developed, as well as their applications to uniform and non-uniform flow. Experimental results, being a support to the theory, are presented, such as the distribution of velocity, the characteristics of turbulence and also the friction coefficients.

TABLE OF CONTENTS

2.1 EQUATION OF CONTINUITY

1° The equation of continuity, one of the basic equations of fluid mechanics, is an expression of the conservation of mass.

The variation of a mass fluid, contained in a given volume during a certain time, must equal the sum of the mass fluid which enters, diminished by the one which leaves.

2° Studied will be a flow of an incompressible fluid, being steady, uniform and almost rectilinear, in an open channel with a free surface and having a weak bed slope (see Fig. 2.1). Considered will be two channel sections; Q will be the entering discharge.

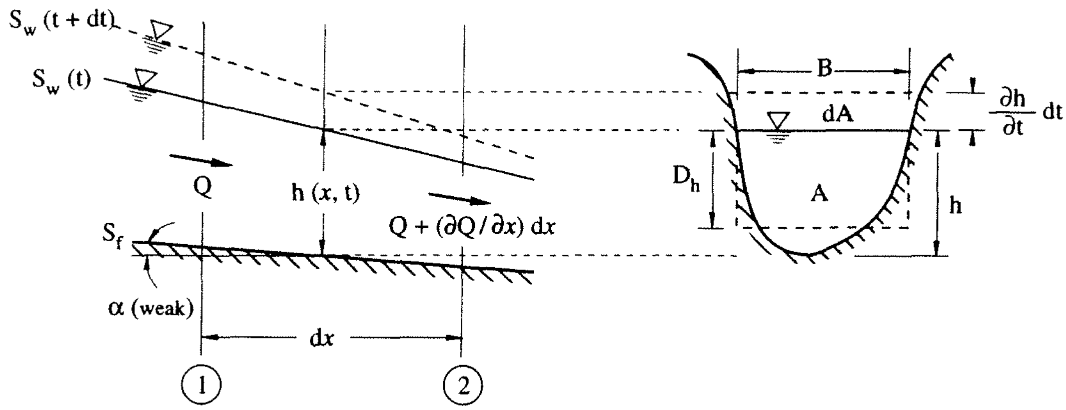


Fig. 2.1 Scheme for the equation of continuity.

The volume, entering by the first section is Qdt; the volume leaving by the second section, being at a distance, dx, from the first one, is $\left[Q + (\partial Q/\partial x)dx\right]dt$. The variation of the volume between these two sections during the time, dt, is consequently :

$$-\left(\frac{\partial Q}{\partial x}\right) dx\, dt$$

This variation of the volume is the result of a modification of the free surface, $\partial h/\partial t$, between the two sections during the time, dt; it is expressed by :

$$(B dx)\ \frac{\partial h}{\partial t}\ dt$$

where B(h) is the width of the channel at the free surface and $h(x,t)$ is the flow depth.

Assuming the fluid incompressible, the above two expressions are made equal (see *Chow*, 1959, p. 525) and one obtains:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \qquad (2.1)$$

where $dA = Bdh$.

3° For a given section, the following relation can be given :

$$Q = UA \qquad (2.2)$$

where U is the average velocity in the section, A. Thus eq. 2.1 can be expressed as :

$$\frac{\partial(UA)}{\partial x} + B\frac{\partial h}{\partial t} = A\frac{\partial U}{\partial x} + U\frac{\partial A}{\partial x} + B\frac{\partial h}{\partial t} = 0 \qquad (2.3)$$

Using the definition of the hydraulic depth, $D_h = A/B$, one can also write :

$$D_h\frac{\partial U}{\partial x} + U\frac{\partial D_h}{\partial x} + \frac{\partial h}{\partial t} = 0 \qquad (2.4)$$

The above equations represent different forms of the equation of continuity, valid for prismatic channels (see sect. 5.1.1).

4° For a rectangular channel, eq. 2.3 is given by :

$$\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = h\frac{\partial U}{\partial x} + U\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0 \qquad (2.5)$$

where $q = Q/B$ is the unit discharge.

5° For steady flow, $\partial A/\partial t = 0$, the equation of continuity, eq. 2.1, reduces to :

$$\frac{dQ}{dx} = 0 \qquad (2.6)$$

6° If a supplementary discharge leaves (or enters) the channel between the two sections, eq. 2.1 can be adapted, such as :

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} \overset{+}{(-)} q_\ell = 0 \qquad (2.7)$$

where q_ℓ is the supplementary discharge per unit length.

2.2 EQUATION OF ENERGY

1° The equation of energy is an expression of the first principle of thermodynamics.

2° The energy for an element of incompressible fluid, written in homogeneous quantities of length (see Fig. 2.2) — here as the height of a liquid with specific weight $\gamma = \rho g$ — in almost rectilinear flow, taken with respect to the plane of reference (PdR), is given by :

$$\frac{u^2}{2g} + \frac{p}{\gamma} + z_P = \frac{p_t}{\gamma} = Cte \tag{2.8}$$

The different terms represent :

$\dfrac{u^2}{2g}$ the velocity head

$\dfrac{p}{\gamma}$ the pressure head

z_P the elevation (position) of a point, P

$\dfrac{p_t}{\gamma} = H$ the (mechanical) energy head or the total head

$\dfrac{p}{\gamma} + z_P = \dfrac{p^*}{\gamma}$ the piezometric head

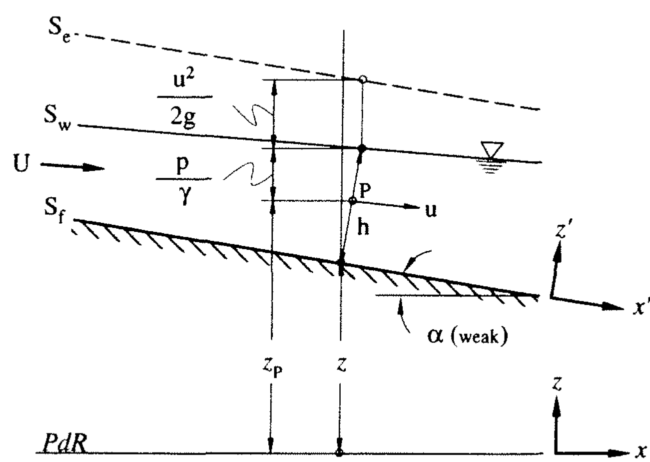


Fig. 2.2 Scheme for the equation of energy in a cross section.

3° The following assumptions shall be applied:

i) The piezometric head, p^*/γ, is supposed to be constant over a normal to the bed, implying that the distribution of pressure is hydrostatic.

ii) By considering that z gives the elevation of the bed, the slope (weak) of the channel, S_f, is given by :

$$S_f = \operatorname{tg} \alpha = -\frac{dz}{dx} \cong \sin \alpha$$

iii) If h is the flow depth, the pressure head at the bed of the channel (see eq. 1.14) is :

$$\left(\frac{p}{\gamma}\right)_f = h \cos \alpha$$

For weak slopes, $\alpha < 6°$, where $S_f < 0.1$, one may take $\cos \alpha \cong 1$. The system of the coordinates, xz, is thus almost identical with the one of the coordinates, $x'z'$, (see Fig. 2.2).

iv) In a perfect fluid, each fluid element moves with the same velocity, which is the average velocity in the section, U.

Making use of these reasonable assumptions, the total head in a section is now given by :

$$\frac{U^2}{2g} + h + z = H \tag{2.9}$$

The flow is here considered to be one-dimensional and rectilinear.

The equation of energy, eq. 2.9, is a manifestation of the principle of energy if the liquid is *perfect*. From one to another section, each of the three terms in eq. 2.9 can take a different value, but the sum, H, remains constant.

4° For flow of a *real* fluid with a free surface, being unsteady and non-uniform (gradually varied), the difference of the total head between two sections, separated by a distance, dx, (see Fig. 2.3) is given as :

$$\alpha_e \frac{U^2}{2g} + h + z = \left[\alpha_e \frac{U^2}{2g} + d\left(\alpha_e \frac{U^2}{2g}\right)\right] + [h + dh] + [z + dz] +$$

$$+ \frac{1}{g}\frac{\partial U}{\partial t} dx + \frac{1}{g}\frac{\tau_o}{\rho}\frac{dP}{dA} dx \tag{2.10}$$

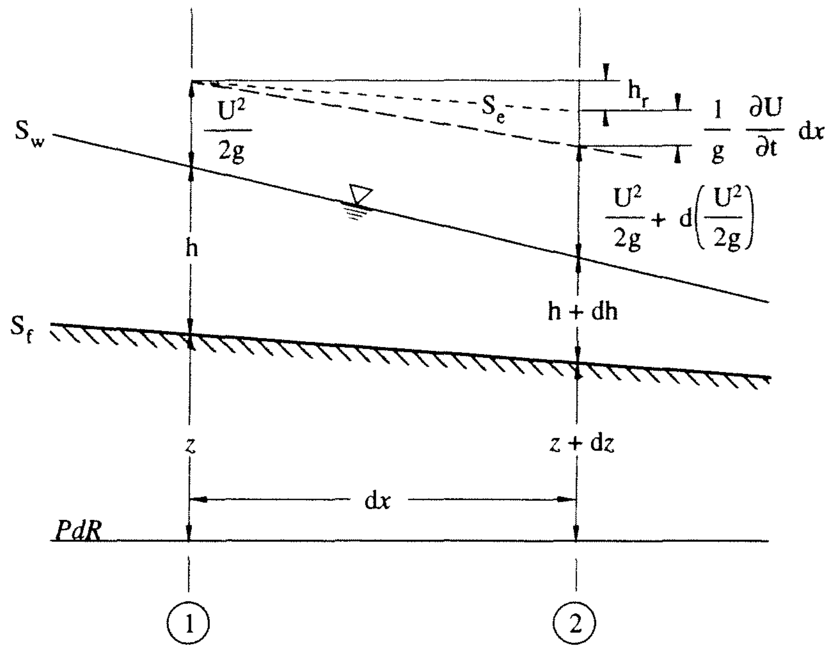


Fig. 2.3 Scheme of the equation of energy, between two sections.

i) $\frac{1}{g}\frac{\partial U}{\partial t}dx$ is the term of energy due to acceleration in the flow x-direction (see *Graf & Altinakar,* 1991, p. 137).

ii) $\frac{1}{g}\frac{\tau_o}{\rho}\frac{dP}{dA}dx = h_r$ is the term of energy or head loss due to friction (see *Graf & Altinakar,* 1991, p. 138);

The friction forces provoke a dissipation of mechanical (into thermal) energy. dP is the perimeter of an elementary surface, dA, and τ_o is the shear stress due to the frictional forces acting on the surface, $dPdx$. This term, representing the effect of friction, is usually written as h_r.

iii) The kinetic energy correction coefficient, α_e, results from the distribution of the velocity in the section (see Fig. 1.6). Its numerical values (see *Chow,* 1959, p. 28) notably for turbulent flow are very close to unity. In most common cases, the velocity head can thus be taken as :

$$\alpha_e \frac{U^2}{2g} \approx \frac{U^2}{2g}$$

where U is the average velocity in the section.

The equation of energy, eq. 2.10, can thus be given as :

$$d\left(\frac{U^2}{2g} + h + z\right) = -h_r - \frac{1}{g}\frac{\partial U}{\partial t}dx \tag{2.11}$$

Dividing by the distance, dx, and using partial differentials, one gets :

$$\frac{1}{g}\frac{\partial U}{\partial t} + \frac{U}{g}\frac{\partial U}{\partial x} + \frac{\partial h}{\partial x} - S_f = -S_e \tag{2.12}$$

steady, uniform

steady, non-uniform

unsteady, non-uniform

where $h_r = S_e\,dx$ and $S_f = -(dz/dx)$; S_e is the energy slope.

Eq. 2.12 is the dynamic equation for unsteady and non-uniform flow.

The head loss, h_r, must be evaluated with a formula such as the one of Weisbach-Darcy, eq. 3.10, of Chézy, eq. 3.11, or of other experimenters. Such relations are only valid for steady, uniform flow; however — for lack of better information — they are also used (see *Chow*, 1959, p. 217) for unsteady and non-uniform flow.

5° The equation of continuity, eq. 2.5, and the dynamic equation of motion, eq. 2.12, form together the *equations of Barré de Saint-Venant* (see *Chow* 1959, p. 528). Despite the various simplifications made to obtain the equations of Saint-Venant, their solutions are often rather complicated. In some physical cases, which are simple but still realistic, explicit solutions are possible.

6° For flow, which is steady but non-uniform, eq. 2.12 reduces to :

$$\frac{U}{g}\frac{\partial U}{\partial x} + \frac{\partial h}{\partial x} - S_f = -S_e \tag{2.13}$$

7° For flow, which is steady and uniform, eq. 2.12 reduces to :

$$S_f = S_e \tag{2.13a}$$

The bed slope, S_f , the energy slope, S_e , and the piezometric slope of the water surface, S_w , are identical. The average velocity, U , and the flow depth, h , are constant; the equation of continuity is given with eq. 2.6.

8° The dynamic equation of motion, eq. 2.12, can also be obtained by applying the momentum equation. The resulting equation is almost the same (see *Chow*, 1959, p. 51 and *Henderson*, 1966, p. 9), i.e. : eq. 2.12.

2.3 SPECIFIC ENERGY

1° Up till now, the total head, H , in a given cross section was defined with respect to an arbitrary horizontal plane (see Fig. 2.2); for a weak bed slope one writes :

$$\frac{U^2}{2g} + h + z = H \qquad (2.9)$$

If the plane of reference is now placed into the bed slope, S_f , a fraction of the total head, called the *specific energy*, H_s , is defined (see *Bakmeteff*, 1932, chap. 4); one writes now (see Fig. 2.4) :

$$\frac{U^2}{2g} + h = H_s \qquad (2.14)$$

Using the equation of continuity, Q = UA, one obtains :

$$\frac{Q^2/A^2}{2g} + h = H_s \qquad (2.14a)$$

2° The notion of the specific energy is often very useful; it helps to understand and to solve different problems of free-surface flow.

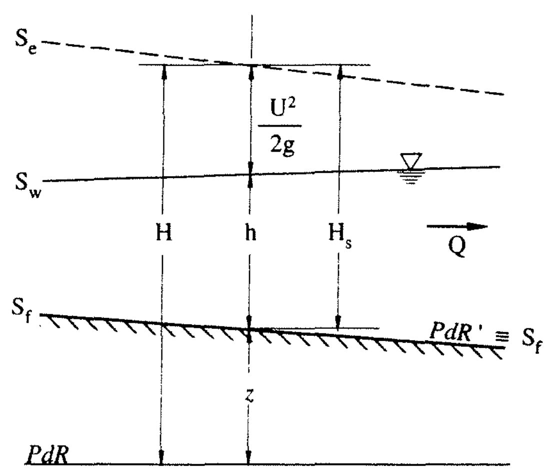


Fig. 2.4 Definition of the total head, H, and the specific energy, H_s .

3° For a given section in the channel, the area of flow, A, is a function of the flow depth, h , and eq. 2.14a gives a relation of the following form :

$$H_s = f(Q, h)$$

which allows a study of the variation of :

i) h with H_s , for $Q = Cte$
ii) h with Q , for $H_s = Cte.$

2.3.1 Specific-energy Curve

1° Eq. 2.14a gives the evolution of the specific energy, H_s, as a function of the flow depth, h, for a given discharge, Q = UA.

This curve (see Fig. 2.5) has two asymptotes :

i) for $h = 0$, a horizontal asymptote ,
ii) for $h = \infty$, the line $h = H_s$ is the other asymptote.

In addition, the curve has a minimum, H_{s_c} , for :

$$\frac{dH_s}{dh} = -\frac{Q^2}{gA^3}\frac{dA}{dh} + 1 = 0 \tag{2.15}$$

Since dA/dh is equal to the width of the channel, B, at the free surface and by using the definition of the hydraulic depth, $D_h = A/B$, one obtains :

$$\frac{Q^2}{g}\frac{B}{A^3} = \frac{U^2}{gD_h} = 1 \tag{2.15a}$$

2° For a channel with a rectangular cross section, one has $D_h = h$. The flow depth, h, which corresponds to the minimal specific energy, H_{s_c}, is called *critical depth*, h_c.

3° Following the curve, given with Fig. 2.5, one notices that for a given discharge Q = Cte , and for an arbitrary value of specific energy, H_s, — for the case when flow can take place — there are always two solutions for the flow depth, h_1 and h_2. They are called the corresponding (alternate) depths; one of which, h_1, is smaller and the other one, h_2, is larger than the critical depth, h_c. Both of these depths are indications of different regimes of flow, thus :

$h < h_c$ *supercritical* (torrential) regime

$h > h_c$ *subcritical* (fluvial) regime

$h \equiv h_c$ *critical* regime

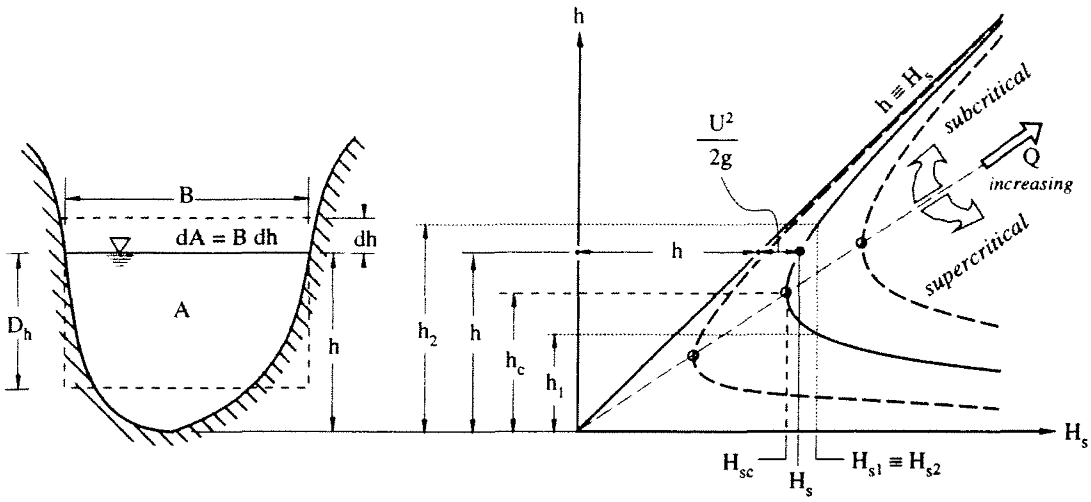


Fig. 2.5 Specific-energy curve, $H_s = f(h)$, for Q = Cte.

Each curve (see Fig. 2.5) has thus two branches. Consequently, a steady flow in a channel can exist in two different ways, both having the same specific energy, H_s :

i) in supercritical regime, where the flow depth is small and the velocity large,
ii) in subcritical regime, where the flow depth is large and the velocity small.

4° For a variation of the discharge, Q, the corresponding curves have the same form ; they follow each other for an increase in the discharge, starting at the origin, (see Fig. 2.5).

2.3.2 Discharge Curve

1° Eq. 2.14a gives also the evolution of the discharge, Q, as a function of the flow depth, h, for a given specific energy, H_s , such as :

$$Q = A \sqrt{2g\ (H_s - h)} \tag{2.16}$$

From this curve (see Fig. 2.6), one obtains :

i) for $h = 0 \quad , \quad Q = 0$

ii) for $h = H_s \quad , \quad Q = 0 \, .$

2° In addition, the curve has a maximum value, Q_{max} , for :

$$\frac{dQ}{dh} = \frac{2g\ (H_s - h)\ (dA/dh) - Ag}{\left[2g\ (H_s - h)\right]^{1/2}} = 0$$

Taking $dA/dh = B$ and $D_h = A/B$, one may write :

$$\frac{dQ}{dh} = \frac{gB\left[2\,(H_s - h) - D_h\right]}{\left[2g\,(H_s - h)\right]^{1/2}} = 0 \tag{2.17}$$

This derivative is zero, if :

$$2\,(H_s - h) - D_h = 0 \tag{2.18}$$

The values, h and D_h, which correspond to the maximum discharge, Q_{max}, represent the *critical depth*, h_c et D_{h_c}. For flows smaller than Q_{max}, one finds again the two different flow regimes (see Fig. 2.6 and also Fig. 2.5).

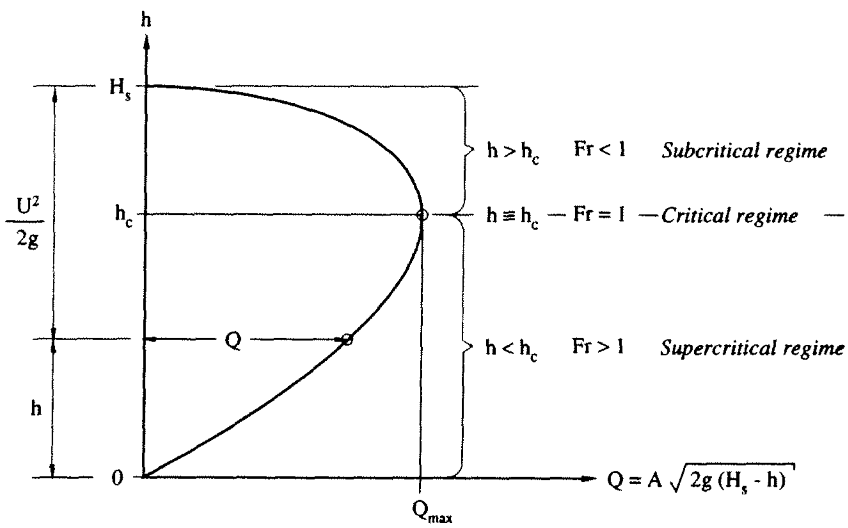


Fig. 2.6 Discharge curve, $Q = f(h)$, for H_s = Cte.

3° For a channel with a rectangular cross section, $D_h \equiv h$, eq. 2.18 becomes :

$$2\,(H_s - h) - h = 0$$

from where one obtains the critical depth ($h \equiv h_c$ and $H_s \equiv H_{s_c}$) :

$$h_c = \frac{2}{3}\,H_{s_c} \tag{2.19}$$

For a channel with a triangular or parabolic cross section, one obtains respectively :

$$h_c = \frac{4}{5}\,H_{s_c} \qquad \text{and} \qquad h_c = \frac{3}{4}\,H_{s_c} \tag{2.19a, b}$$

2.3.3 Critical Depth

1° The critical depth, h_c, in a channel is the flow depth at which :

i) the specific energy is minimal, H_{s_c}, for a given discharge (see Fig. 2.5),
ii) the discharge is maximal, Q_{max}, for a given specific energy (see Fig. 2.6).

2° It follows, that eq. 2.18 can be written as :

$$2\,(H_{s_c} - h_c) = D_{h_c}$$

and that, using eq. 2.16, the maximum discharge, Q_{max}, is given by :

$$Q_{max} = A\sqrt{gD_{h_c}} \tag{2.20}$$

The average velocity, which corresponds to the critical hydraulic depth, D_{h_c}, is :

$$U_c = \sqrt{gD_{h_c}} \qquad \text{or} \qquad \frac{U_c^{\,2}}{2g} = \frac{D_{h_c}}{2} \tag{2.21}$$

In critical regime, the velocity head is thus equal to half of the hydraulic depth.

3° Eq. 2.21 or eq. 2.15a could also be expressed as :

$$\frac{U_c}{\sqrt{gD_{h_c}}} = 1 \tag{2.22}$$

which is precisely the definition of the Froude number (see eq. 1.4) in critical regime ; here the Froude number, Fr, is equal to unity :

$$Fr_c = 1 \tag{2.22a}$$

Note that the Froude number, $Fr = U/\sqrt{gD_h}$, is the ratio of inertia to gravity forces per unit volume (see *Graf & Altinakar*, 1991, sect. FR. 7.3). Consequently, the Froude number classifies also the different flow regimes, such as :

$Fr > 1$	*supercritical* regime	$U > U_c$
$Fr < 1$	*subcritical* regime	$U < U_c$
$Fr = 1$	*critical* regime	$U \equiv U_c$

4° The critical velocity, U_c, is given by :

$$U_c = \sqrt{gD_{h_c}} = c \tag{2.21a}$$

This is equal to the celerity, c, of the propagation of (superficial) infinitesimal gravity waves in a channel of hydraulic depth, D_{h_c} (see eq. 2.27 for the general definition).

5° The critical depth for a rectangular channel, $D_h \equiv h$, has been given by :

$$h_c = \frac{2}{3} H_{s_c} \tag{2.19}$$

or equally by :

$$(H_{s_c} - h_c) = \frac{h_c}{2} = \frac{U_c^2}{2g}$$

Using the definition of the unit discharge, $q = Uh$, one obtains :

$$\frac{h_c}{2} = \frac{q^2}{2gh_c^2} \qquad \text{or} \qquad h_c = \sqrt[3]{\frac{q^2}{g}} \tag{2.23}$$

The maximum unit discharge, q, which may exist in a channel of rectangular section is equal to :

$$q = \sqrt{gh_c^3} = \sqrt{g\left(\frac{2}{3}H_{s_c}\right)^3} \tag{2.24}$$

6° Experience shows that flow at critical depth, h_c, is often *unstable*, presenting itself by a fluctuating water surface. This is rather evident when observing Fig. 2.5 : even small variations of energy close to the critical value, H_{s_c}, cause large variations in the flow depth, h.

7° According to eq. 2.20 and eq. 2.24, the critical hydraulic depth, D_{h_c}, or the critical flow depth, h_c, depend only on the discharge. Thus it is inviting to use this information for metering flow in open channels :

Here, two examples are given :

i) *Free Overfall* : Flow in an horizontal channel (or a broad-crested weir) discharges freely into the atmosphere; the critical section is found rather close to the brink (see sect. 4.4.2).

ii) *Venturi Canal* : An adequate reduction in the cross section of the channel is provided, where the critical regime (see sect. 4.4.2) takes place.

8° Flow goes through the critical depth, if the fluvial regime passes to the torrential one. Critical depth is also observed, if the fluvial regime is terminated by a free overfall.

2.4 GRAVITY WAVES

Flow in open channels, which is variable in time, is accompanied by gravity waves at the water surface.

2.4.1 Wave Celerity

1° Considered will be a periodic, simple wave, representing the propagation of an irrotational motion as satisfied by the equation of Laplace; the pressure at the free surface is constant and the wave amplitudes are small. A channel of rectangular cross section with uniform flow depth is filled with stagnant water; there is thus no flow.

2° The two-dimensional and progressive wave in the $^-x^+$ direction, will be given (see Fig. 2.7) by a periodic displacement of the free surface as a function of time, t , (see *Kinsman*, 1965, p. 117), such as:

$$\eta(x, t) = A \cos(2\pi\, x/L - 2\pi\, t/T) \tag{2.25}$$

where A is the amplitude, being half of the wave height, H = 2A; L is the wave length and T is the wave period. The wave celerity is defined by :

$$c = \frac{L}{T} \tag{2.25a}$$

3° The hydrodynamic theory for waves of *small amplitude* (see *Lamb*, 1945, pp. 254 and 366, or *Kinsman*, 1965, p. 125), i.e. : $H/L \ll 1$ and $H/h \ll 1$, gives for the apparent velocity of propagation, also called the *celerity* of a perturbation :

$$c^2 = \frac{gL}{2\pi} \tanh\left(\frac{2\pi h}{L}\right) \tag{2.26}$$

where h is the water depth. Note that the celerity does not depend on the wave height, H.

4° This expression, eq. 2.26, reduces :

i) for *short* waves or waves of large depth, if L/h < 1, to :

$$c^2 = \frac{gL}{2\pi} \tag{2.26a}$$

ii) for *long* waves or waves of small depth, if L/h >> 1, to :

$$c^2 = gh \tag{2.27}$$

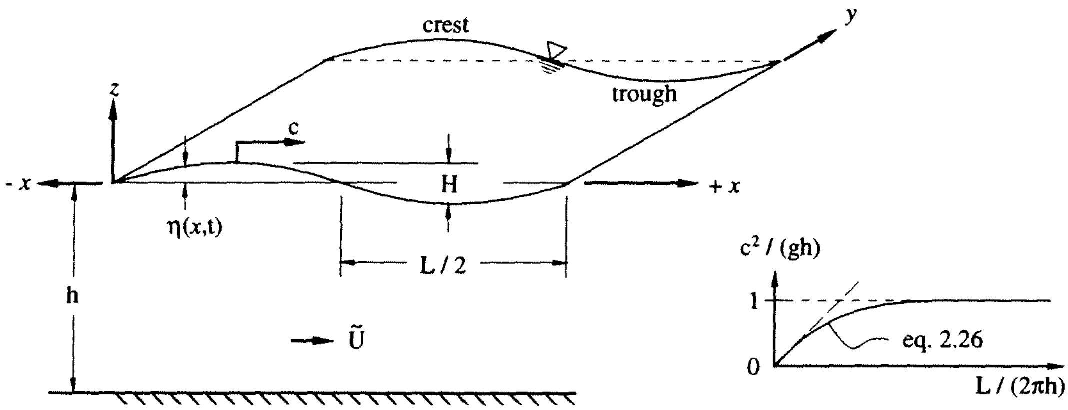


Fig. 2.7 Scheme of a surface wave.

5° If the long wave, where L/h >> 1, is not of small amplitude, thus H/h ≅ 1 , the wave celerity (see eq. 2.27) is given (see *Lamb*, 1945, p. 262) by :

$$c^2 = gh\left(1 + \frac{3}{2}\frac{A}{h}\right) \tag{2.28}$$

or also (see *Lamb*, 1945, p. 424) by :

$$c^2 = g\,(h + A) \tag{2.28a}$$

This last relation was experimentally obtained for a solitary wave.

6° The two signs, which are possible for the celerity, eq. 2.27 or eq. 2.28, show well that the wave can propagate in the direction of x^+ or x^- (see Fig. 2.7).

7° The relation, eq. 2.27, for the celerity, c , of *long* waves can also be obtained by application of the equation of continuity and of energy.

i) Consider the unsteady flow (see Fig. 2.8a) of a simple wave having an amplitude, $A \equiv \eta$. U is the liquid velocity in the section of the crest. By following the wave — one thus imagines the wave stays immobile — the flow becomes a steady flow (see Fig. 2.8b).

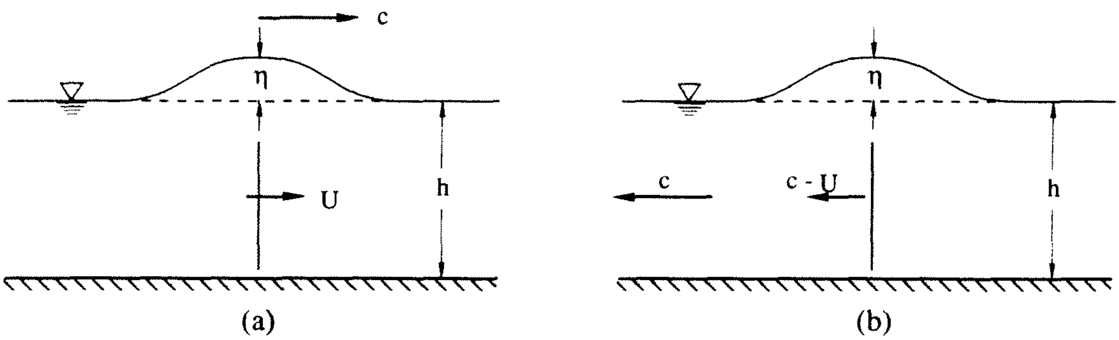


Fig. 2.8 Propagation of a wave.

ii) The equation of continuity reads now :

$$c\,h = (c - U)\,(h + \eta)$$

If $\eta \ll h$, the wave is thus infinitesimal and one may write:

$$U = c\,(\eta / h) \tag{2.29}$$

iii) The equation of energy reads :

$$h + \frac{c^2}{2g} = (h + \eta) + \frac{(c - U)^2}{2g} \tag{2.30}$$

or written otherwise :

$$\eta = \frac{(c\,U)}{g}\left(1 - \frac{U}{2c}\right) \tag{2.30a}$$

Neglecting the term (U / 2c) when compared to unity, one may write:

$$\eta = \frac{c\,U}{g} \tag{2.30b}$$

iv) Through substitution, the eqs. 2.29 and 2.30b give :

$$c^2 = gh \tag{2.27}$$

This is the celerity of wave having a small amplitude, η .

2.4.2 Wave Equation

1° In order to derive the wave equation, the equation of continuity and of motion will be applied to a situation of a wave of small amplitude, which propagates (see Fig. 2.7) in a stagnant liquid.

2° The equation of continuity, eq. 2.5, is expressed as :

$$(h+\eta)\frac{\partial \tilde{U}}{\partial x} + \tilde{U}\frac{\partial (h+\eta)}{\partial x} + \frac{\partial (h+\eta)}{\partial t} = 0$$

where $\tilde{U}$ is the velocity produced by the wave and averaged over the depth. By assuming that depth variation, $\partial h/\partial x = 0$ and $\partial h/\partial t = 0$, are negligible, one may write :

$$(h+\eta)\frac{\partial \tilde{U}}{\partial x} + \tilde{U}\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} = 0 .$$

If the wave is of small amplitude, $\eta /h << 1$, and assuming that $\partial\eta/\partial x << 1$, one obtains :

$$h\frac{\partial \tilde{U}}{\partial x} + \frac{\partial \eta}{\partial t} = 0 \tag{2.31}$$

3° The dynamic equation, eq. 2.12, is expressed as :

$$\frac{1}{g}\frac{\partial \tilde{U}}{\partial t} + \frac{\tilde{U}}{g}\frac{\partial \tilde{U}}{\partial x} + \frac{\partial (h+\eta)}{\partial x} - (S_f - S_e) = 0$$

While the last term shall be omitted, the second term is considered to be small compared to the first one. Since the depth variation is negligible, the above equation simplifies to :

$$\frac{1}{g}\frac{\partial \tilde{U}}{\partial t} + \frac{\partial \eta}{\partial x} = 0 \tag{2.32}$$

4° One sees immediately that these equations, eq. 2.32 and eq. 2.31, give the following relationship :

$$\frac{\partial^2 \tilde{U}}{\partial t^2} = gh\frac{\partial^2 \tilde{U}}{\partial x^2} \qquad \text{or} \qquad \frac{\partial^2 \eta}{\partial t^2} = gh\frac{\partial^2 \eta}{\partial x^2} \tag{2.33}$$

This is the classic equation for a progressive wave (see *Lamb*, 1945, p. 255), where $c^2 = gh$ is the celerity of a long wave, previously presented with eq. 2.27. A general solution to it was given with eq. 2.25.

2.4.3 Flow with a Wave

1° It was shown that the celerity, c , with which a gravity wave, being a long one and of small amplitude, propagates in a channel of rectangular section, is given with the relation of eq. 2.27. For a channel of an arbitrary section, one writes :

$$c = \pm\sqrt{gD_h} \tag{2.27a}$$

where D_h is the hydraulic depth.

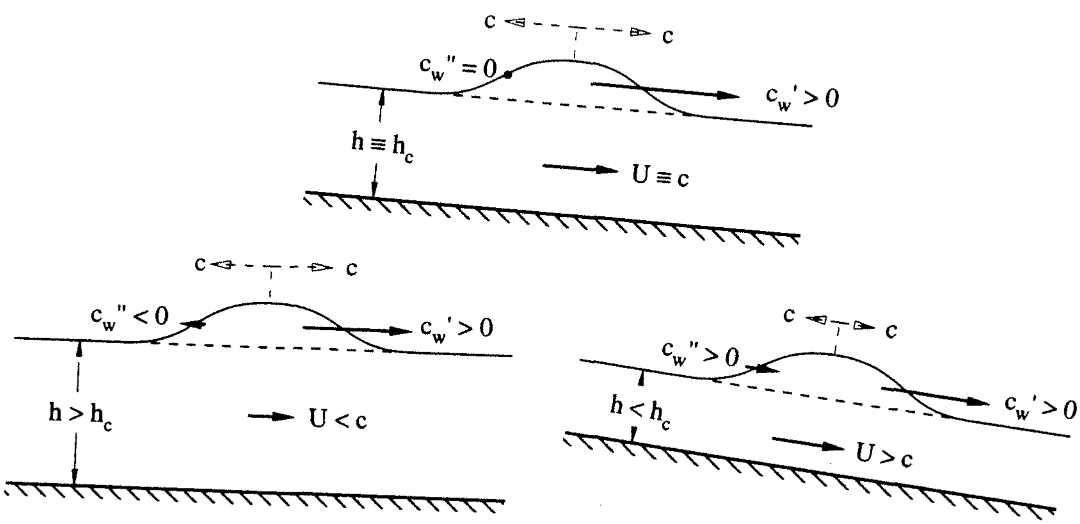


Fig. 2.9 Flow with a wave.

2° This relation, eq. 2.27a, was established for a channel where the liquid was stagnant. However the relation stays valid for the case where the liquid is in motion ; the wave superposes itself upon the flow in the channel. Consequently, the *absolute celerity*, c_w , of the wave for a channel having an average velocity, U, can be expressed as :

$$c_w = U \pm \sqrt{gD_h} \tag{2.34a}$$

and for a channel of rectangular section :

$$c_w = U \pm \sqrt{gh} = U \pm c \tag{2.34}$$

3° The absolute celerity, c_w, being the velocity with respect to the bed, has evidently two values :

$$c_w' = U + c \quad , \quad c_w'' = U - c \tag{2.34b}$$

Thus one may distinguish two plus one cases (see Fig. 2.9) :

i) $U < c$, where the wave with celerity, c_w', propagates downstream and where the wave with celerity, c_w'', propagates upstream; the flow regime is fluvial.

ii) $U > c$, where the wave with celerity, c_w', propagates downstream and where the wave with celerity, c_w'', propagates downstream as well; the flow regime is torrential.

iii) At a flow depth, at which the current velocity, U , and the wave celerity, c, are the same, thus :

$$U \equiv c = \sqrt{g h_c}$$

the flow is in critical regime (see sect. 2.3.3) ; h_c being the critical depth.

4° Flow with a gravity wave, which is long but not of small amplitude, will be treated later (see sect. 5.6).

2.5 HYDRODYNAMIC EQUATIONS

2.5.1 Equations of Motion

1° For flow (see Fig. 2.10), which is two-dimensional, plane, $\vec{V}(\bar{u}, 0, \bar{w})$, and turbulent, the equations of motion and of continuity (see *Graf & Altinakar*, 1991, p. 275 or *Rotta*, 1972, p. 129) can be written as :

$$\begin{aligned}
\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} &= -\frac{1}{\rho}\frac{\partial \bar{p}^*}{\partial x} + \\
&\quad + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) - \left[\frac{\partial}{\partial x}\left(\overline{u'^2}\right) + \frac{\partial}{\partial z}\left(\overline{u'w'}\right)\right] \\
\frac{\partial \bar{w}}{\partial t} + \bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{w}\frac{\partial \bar{w}}{\partial z} &= -\frac{1}{\rho}\frac{\partial \bar{p}^*}{\partial z} + \\
&\quad + \nu\left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial z^2}\right) - \left[\frac{\partial}{\partial x}\left(\overline{u'w'}\right) + \frac{\partial}{\partial z}\left(\overline{w'^2}\right)\right]
\end{aligned} \tag{2.35}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad (2.35a)$$

- $\bar{u}$ and $\bar{w}$ are the average point velocities in the x and z -direction ;
- u' and w' are the velocities due to fluctuations ;
- $\rho \overline{u'^2}$, $\rho \overline{u'w'}$, etc. are the supplementary (or Reynolds) stresses due to the turbulence;
- $\overline{p^*}$ is the average (point) driving pressure.

These equations, eqs. 2.35, are known as the *Reynolds equations*. In the absence of turbulence they reduce to the *Navier-Stokes* equations, valid notably for laminar flow.

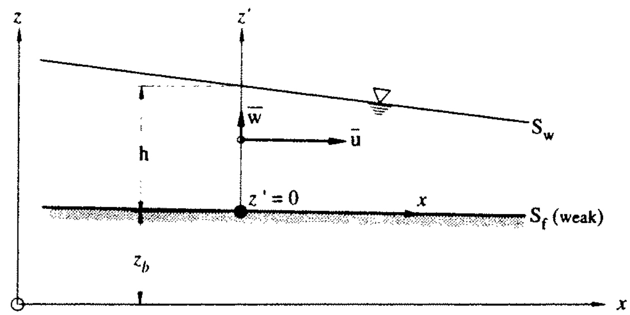


Fig. 2.10 Scheme for the equations of motion.

2° For free-surface flow on weak slopes, $S_f \ll 1$, the terms of the driving pressure, $\overline{p^*}(x,z) = \bar{p}(x,z) + g\rho z$, are expressed as :

$$\frac{\partial \overline{p^*}}{\partial x} = \frac{\partial \bar{p}}{\partial x} - \rho g\, S_f$$

$$\frac{\partial \overline{p^*}}{\partial z} = \frac{\partial \bar{p}}{\partial z} + \rho g \qquad (2.36)$$

where the bed slope is defined as : $S_f = -\dfrac{\partial z_b}{\partial x}$.

The bed of the channel is defined by z_b, but in the following it will be used without an index, thus written as $S_f = -\dfrac{\partial z}{\partial x} = -\dfrac{dz}{dx}$.

3° Steady free-surface flow may be considered as being flow at high Reynolds numbers. It is thus possible to use the approximations which are developed for boundary-layer flow (see *Graf & Altinakar*, 1991, sect. CL.1).

By considering the order of magnitude of each term in eqs. 2.35 and 2.35a, and by keeping only the terms of the highest order (see *Graf & Altinakar*, 1991, p. 351, or *Rotta*, 1972, p. 130), one has :

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{p}^*}{\partial x} + \nu\frac{\partial^2 \bar{u}}{\partial z^2} - \frac{\partial}{\partial x}(\overline{u'^2}) - \frac{\partial}{\partial z}(\overline{u'w'})$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}^*}{\partial z} - \frac{\partial}{\partial z}(\overline{w'^2}) \tag{2.37}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{2.35a}$$

4° In eqs. 2.37, the second one can be integrated and written as :

$$0 = -\frac{1}{\rho}\,\bar{p}^*(x, z') + \frac{1}{\rho}\,\bar{p}^*_f(x, z'=0) - \overline{w'^2}(z') + \overline{w'^2}_f(z'=0)$$

$\bar{p}^*_f$ being the driving pressure at the bed, $z' = 0$; due to the no-slip condition one takes $\overline{w'^2}_f = 0$. Thus one may write :

$$\bar{p}^*(x, z') = \bar{p}^*_f(x, 0) - \rho\overline{w'^2}(z') \tag{2.38}$$

or also :

$$\bar{p} + \rho g z' = \bar{p}_f + \rho g 0 - \rho\overline{w'^2}$$

With $\bar{p}_f = \rho g h$, an expression for the pressure is obtained :

$$\bar{p} = \rho g (h - z') - \rho\overline{w'^2} \tag{2.38a}$$

The driving pressure is consequently not constant over the flow depth, but will be slightly modified by the Reynolds stress.

The derivation of eq. 2.38 with respect to x, gives :

$$\frac{\partial \bar{p}^*}{\partial x} = \frac{\partial \bar{p}^*_f}{\partial x} - \rho\frac{\partial}{\partial x}(\overline{w'^2}) \tag{2.39}$$

where the last term is often neglected; using eq. 2.36, it can be written as :

$$\frac{\partial \overline{p}^*}{\partial x} \cong \frac{\partial \overline{p}^*_f}{\partial x} = \frac{\partial \overline{p}_f}{\partial x} - g\rho S_f = g\rho\left(\frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x}\right) \tag{2.40}$$

5° Upon substitution of eq. 2.39 and eq. 2.40 into the first of eqs. 2.37 one gets :

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial \overline{u}}{\partial z} - \overline{u'w'}\right) + \frac{\partial}{\partial x}\left(\overline{w'^2} - \overline{u'^2}\right) \tag{2.41}$$

The last term, which is due to the normal Reynolds stress, is also often neglected (see *Rotta,* 1972, p. 130). If one defines the total tangential stress by :

$$\tau_{zx} = \rho\left(\nu\frac{\partial \overline{u}}{\partial z} - \overline{u'w'}\right) \tag{2.42}$$

one can write eq. 2.41 as :

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x}\right) + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \tag{2.41a}$$

This equation is also valid for laminar flow, where :

$$\tau_{zx} = \mu\frac{\partial \overline{u}}{\partial z} \tag{2.42a}$$

6° Free-surface flow, being unsteady, can thus be represented (see *Grishanin,* 1969, p. 59) by a system of equations such as :

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -g\left(\frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x}\right) + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \tag{2.41b}$$

$$\overline{p} = \rho g(h - z') - \rho\overline{w'^2} \tag{2.38a}$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{w}}{\partial z} = 0 \tag{2.35a}$$

Note, that the pressure is not quite hydrostatic.

2.5.2 Uniform Flow

1° It will be assumed that flow is steady, unidirectional, $\vec{V}(\bar{u}, 0, 0)$, and uniform on the average.

The equations of motion, eq. 2.41a and eq. 2.38a, and of continuity, eq. 2.35a, reduce to :

$$0 = + g S_f + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \tag{2.43}$$

$$\bar{p} = \rho g (h - z') - \rho\overline{w'^2} \tag{2.38a}$$

$$\frac{\partial \bar{u}}{\partial x} = 0 \tag{2.44}$$

where the total tangential stress is expressed by :

$$\tau_{zx} = \mu \frac{\partial \bar{u}}{\partial z} - \rho\overline{u'w'} \tag{2.42}$$

2° After integration over the flow depth, h , one obtains :

i) the equation of motion, eq. 2.43, as being :

$$0 = g S_f \int_o^h dz + \frac{1}{\rho}\int_o^h \frac{\partial}{\partial z}\tau_{zx}\, dz$$

$$0 = + g S_f (h - 0) + \frac{1}{\rho}(0 - \tau_o)$$

and consequently one has :

$$\tau_o = \rho g S_f h \tag{2.45}$$

with τ_o as the stress due to friction, called the wall or bed-shear stress; the ratio, $\tau_o/\rho g h$, gives the energy slope, being now expressed by :

$$S_f = S_e \tag{2.45a}$$

In eq. 2.43 one notes that the longitudinal pressure gradient, namely the longitudinal component of the gravity (see eq. 2.36), provides the driving force of a uniform flow; the tangential stress (see eq. 2.42) is the dissipating force.

ii) the equation of continuity, eq. 2.44, as being :

$$\int_0^h \frac{\partial \bar{u}}{\partial x} dz = \frac{\partial}{\partial x} \int_0^h \bar{u}\, dz - \bar{u}_h \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial}{\partial x} (Uh) = 0 \qquad (2.46)$$

where $Uh = q$ is the unit discharge and U the average velocity; $\bar{u}_h$ is the velocity at the water surface.

For steady flow, one writes :

$$\frac{\partial q}{\partial x} = U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x} = 0 \qquad (2.46a)$$

The equation of motion, eq. 2.45a (see eq. 2.13a), and of continuity, eq. 2.46a (see eq. 2.6), in their integral form, form together the simplified equations of Saint-Venant for a steady and uniform flow (see sect. 2.2).

3° To obtain the distribution (see Fig. 2.11) of the total shear stress, $\tau_{zx}(z)$, the following equation must be integrated over the flow depth :

$$\frac{\partial \tau_{zx}}{\partial z} = \frac{\partial \bar{p}^*}{\partial x} = - \rho g\, S_f \qquad (2.43a)$$

As boundary conditions serve :

$z' \cong 0 \;(z'/h = 0.05) \quad \Rightarrow \quad \tau_{zx} = \tau_o$

$z' = h \quad \Rightarrow \quad \tau_{zx} = 0$

thus the following is obtained :

$$\tau_{zx}(z') = \rho g\, S_f (h - z') \qquad (2.47)$$

or written in dimensionless form :

$$\frac{\tau_{zx}(z')}{\tau_o} = \left(\frac{h - z'}{h}\right) \qquad (2.47a)$$

This gives (see *Monin* et *Yaglom*, 1971, p.268) a linear (triangular) distribution, being valid for turbulent flow with eq. 2.42, and for laminar flow with eq. 2.42a.

Nevertheless, for very small distances from the bed, $z'/h \lesssim 0.05$, the shear-stress distribution may be considered to be constant (see *Graf* & *Altinakar*, 1991, sect. 6.1) :

$$\frac{\tau_{zx}(z')}{\tau_o} = 1$$

The zone very close to the bed, $z'/h \lesssim 0.20$, where the shear stress is constant, is called the *inner region* (see *Hinze*, 1975, p.503 and *Monin* et *Yaglom*, 1971, p.311) where the total shear-stress variation becomes negligible.

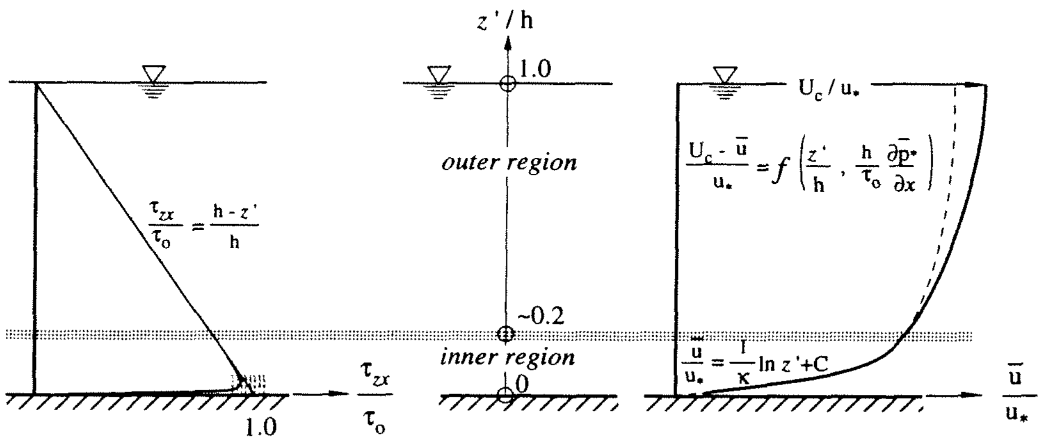


Fig. 2.11 Scheme of the distribution of shear stress, $\tau_{zx}(z)$, and of velocity, $\bar{u}(z)$; for uniform flow.

4° The system of equations, eq. 2.43, eq. 2.38a and eq. 2.44, cannot be used to obtain the distribution of the velocity, $\bar{u}(z)$, (see Fig. 2.11) since the Reynolds stresses are not known. Semi-empirical methods have to be exploited.

i) In the *inner region*, the pressure gradient, $(\partial \bar{p}^* / \partial x)$, — being very weak in uniform flow — in eq. 2.43a may be neglected (see *Monin* et *Yaglom*, 1971, p.268); one may write :

$$\frac{\partial \tau_{zx}}{\partial z} = 0 \tag{2.43b}$$

and consequently :

$$\tau_{zx} = \mu \frac{\partial \bar{u}}{\partial z} - \rho \overline{u'w'} = Cte \tag{2.42}$$

To find an expression for this equation, eq. 2.42, one may use the semi-empirical method of the mixing length, $l = \kappa z'$, proposed by *Prandtl* (see *Graf & Altinakar*, 1991, p. 280), such as :

$$\tau_{zx} = \mu \frac{d\bar{u}}{dz} + \rho \kappa^2 z'^2 \left(\frac{d\bar{u}}{dz}\right)^2 \tag{2.48}$$

where κ is Karman's universal constant.

Outside a very thin region — the viscous region — situated very close to the bed, the shear stress due to viscosity can be neglected, thus one has :

$$\tau_{zx} = \rho \kappa^2 z'^2 \left(\frac{d\bar{u}}{dz}\right)^2 \tag{2.49}$$

Since the shear stress, τ_{zx}, remains constant in the vertical — more or less correct in the inner region — and equal to the wall-shear stress, τ_o, one may write :

$$\tau_{zx} \equiv \tau_o = \rho \kappa^2 z'^2 \left(\frac{d\bar{u}}{dz}\right)^2$$

After separation of variables, the following differential equation is obtained :

$$d\bar{u} = \frac{\sqrt{\tau_o/\rho}}{\kappa} \frac{dz}{z'}$$

which, upon integration, renders :

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln z' + C \tag{2.50}$$

where $u_* = \sqrt{\tau_o/\rho}$ is the friction velocity. The value of the integration constant, C, must be determined experimentally; in this way the type of the surface (bed), being smooth or rough, will enter.

This *logarithmic law (of the wall)*, eq. 2.50, is only valid in the inner region, $z'/h \lesssim 0.2$, where the shear stress remains constant and the influence of a possible pressure gradient can be neglected (see *Rotta*, 1972, p. 153). The logarithmic law is universal (see *Monin* et *Yaglom*, 1971, p. 311), being the same for boundary-layer flow, as well as for flow in pipes and in open channels.

In the *inner* region, $z'/h \lesssim 0.2$, the velocity, $\bar{u}(z)$, whose variation is considerable, depends also on the wall-shear stress, on the fluid properties, on the type of the wall (bed) and on the distance from the wall; thus :

$$\bar{u} = f(\tau_0, \rho, \mu, k_s, z').$$

ii) In the *outer region*, $0.2 \lesssim z'/h \lesssim 1.0$, the velocity, $\bar{u}(z)$, whose variation is weak, depends also on the maximum velocity, on the flow depth and the driving pressure (see eq. 2.40), but does not depend on the viscosity or on the type of the wall (bed); thus :

$$\bar{u} = f(U_c, h, \frac{\partial \bar{p}^*}{\partial x} ; \tau_0, \rho, z').$$

In the outer region, a good agreement with experimental data is not possible, since $\tau_{zx} \neq \tau_0$. The logarithmic law, eq. 2.50, must be modified with a function which depends on the flow depth, h, and notably on an dimensionless pressure gradient, $(h / \tau_0)(\partial \bar{p}^* / \partial x)$. The velocity distribution is given here by the *law of velocity defect* (see *White*, 1974, p. 477), such as :

$$\frac{U_c - \bar{u}}{u_*} = f(\frac{z'}{h}, \frac{h}{\tau_0}\frac{\partial \bar{p}^*}{\partial x})$$

Amongst the different relations available (see *Hinze*, 1975, p. 630 et p. 697), the one of Coles shall here be used :

$$\frac{U_c - \bar{u}}{u_*} = \frac{1}{\kappa} \ln(\frac{\delta}{z'}) + \frac{\Pi}{\kappa}(2 - \tilde{\omega}) \qquad (2.51)$$

where the function, known as wake function, is defined by :

$$\tilde{\omega} = 2 \sin^2 (\frac{\pi}{2}\frac{z'}{\delta})$$

The wake parameter of Coles, Π, depends notably on the gradient of the longitudinal pressure :

$$\Pi = f(\beta)$$

where

$$\beta = \frac{h}{\tau_o} \frac{\partial \overline{p^*}}{\partial x} \tag{2.52}$$

Since this parameter, Π, must remain constant for flow in equilibrium, the β-value is an equilibrium parameter (see *White*, 1974, p. 477). The value of $(2\Pi/\kappa)$ represents the deviation from the logarithmic part of eq. 2.51 for $(z'/\delta) = 1$ (see *Graf & Altinakar*, 1991, p. 288).

The height, $z' = \delta\ (\leq h)$, is the position in the flow section (see Fig. 2.13) where the maximum velocity, U_c, is measured; if U_c is on the water surface, the flow depth, $\delta \equiv h$, is to be taken.

This empirical relation, eq. 2.51, whose validity is made evident by experiments, is valid in the outer region as well as in the inner one, but not in the viscous region. However this relation is valid for both smooth and rough surfaces.

Nevertheless, as a first (and often good) approximation, the logarithmic law can often be applied over the entire flow depth, h (see *Monin* et *Yaglom*, 1971, p. 298); this is especially so if the flow is uniform having a weak pressure gradient.

iii) The distribution of velocity — called universal since it is independent of the Reynolds number — as given with eq. 2.50 and eq. 2.51, is complete but also complex. For practical purpose, one may also use a simple empirical relation (see *Graf & Altinakar*, 1991, p. 289) of the type :

$$\frac{\bar{u}}{\bar{u}_R} = \left(\frac{z'}{z_R}\right)^q$$

where $\bar{u}_R(z_R)$ is a reference velocity, which was for example previously measured. The variation of $1/10 < q < 1/6$ depends on the Reynolds number; often $q = 1/7$ is taken.

2.5.3 Non-uniform Flow

1° It will be assumed that flow is steady, two-dimensional, $\vec{V}(\bar{u}, 0, \bar{w})$, and gradually varied (non-uniform).

The equations of motion, eq. 2.41a and eq. 2.38a, and of continuity, eq. 2.35a, reduce to :

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -g\left(\frac{\partial h}{\partial x} - S_f\right) + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \tag{2.53}$$

$$\bar{p} = \rho g(h - z') - \rho\overline{w'^2} \tag{2.38a}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{2.35a}$$

Flow, which is (gradually) non-uniform, may be considered unidirectional if the variation of flow depth, $\partial h/\partial x$, is weak (see *Grishanin*, 1969, p. 59 - 62).

2° After integration over the flow depth, h , (see *Grishanin*, 1969) one obtains :

i) the equation of motion, eq. 2.53, as being :

$$\int_0^h \bar{u}\frac{\partial \bar{u}}{\partial x}dz + \int_0^h \bar{w}\frac{\partial \bar{u}}{\partial z}dz = -g\int_0^h \frac{\partial h}{\partial x}dz + gS_f\int_0^h dz + \frac{1}{\rho}\int_0^h \frac{\partial \tau_{zx}}{\partial z}dz$$

$$\beta_u Uh\frac{\partial U}{\partial x} + U^2h\frac{\partial \beta_u}{\partial x} = -gh\frac{\partial h}{\partial x} + g\,S_f\,h + \frac{1}{\rho}(-\tau_o)$$

where $\beta_u = (1/U^2h)\int_0^h \bar{u}^2 dz$ is the correction coefficient (of Boussinesq) of the velocity distribution, which is usually taken as $\beta_u = \text{Cte} \cong 1$ for turbulent flow. Assuming that the energy slope is given by $S_e = \tau_o/\rho g h$, one may now write :

$$\frac{1}{g}U\frac{\partial U}{\partial x} + \frac{\partial h}{\partial x} - S_f = -S_e \tag{2.54}$$

ii) the equation of continuity, eq. 2.35a, as being :

$$\int_0^h \frac{\partial \bar{u}}{\partial x} dz + \int_0^h \frac{\partial \bar{w}}{\partial z} dz = \left(\frac{\partial}{\partial x} \int_0^h \bar{u}\, dz - \bar{u}_h \frac{\partial h}{\partial x} \right) + \bar{u}_h \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial}{\partial x} (Uh) = 0 \tag{2.46}$$

For steady, non-uniform (but also uniform) flow, one writes :

$$\frac{\underline{dq}}{dx} = U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x} = 0 \tag{2.46a}$$

Note, that the term, $\bar{u}_h (\partial h/\partial x) = \bar{w}_h$, gives the equation of the streamline at the water surface.

The equation of motion, eq. 2.54 (see eq. 2.13) and of continuity, eq. 2.46a (see eq. 2.6), form together the simplified equations of Saint-Venant for a steady and non-uniform flow (see sect. 2.2).

3 ° Furthermore, one may postulate :

i)
$$\frac{\partial h}{\partial x} - S_f = \frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x} = \frac{1}{\rho g} \frac{\partial \bar{p}^*}{\partial x} \tag{2.40}$$

ii) using the equation of continuity, eq. 2.46a :

$$\frac{\partial U}{\partial x} = - \frac{U}{h} \frac{\partial h}{\partial x}$$

iii) using the definition of head loss (see point 2.2, 4°) :

$$S_e = \frac{\tau_o}{\rho g\, h}$$

In this way, the equation of motion, eq. 2.54, can be expressed as:

$$\rho U^2 \frac{\partial h}{\partial x} = \tau_o \left(1 + \frac{h}{\tau_o} \frac{\partial \bar{p}^*}{\partial x} \right) \tag{2.54a}$$

This equation may be compared with the Karman equation (see *Graf & Altinakar*, 1991, sect. CL.4 : eq. CL.25a) for boundary-layer flow.

The dimensionless longitudinal gradient of the driving pressure :

$$\beta = \frac{h}{\tau_o} \frac{\partial \overline{p^*}}{\partial x} \tag{2.52}$$

which stands for the ratio of the forces due to the driving pressure and due to friction, defines the equilibrium parameter. This parameter can be used to classify non-uniform flow, being flow with a pressure gradient. Note, however, that also uniform flow (see eq. 2.43a) is flow with a (weak) pressure gradient, where $\beta = -1$.

4° To obtain the distribution (see Fig. 2.12) of velocity, $\bar{u}(z')$, one should distinguish two regions, as was also done with uniform flow.

i) In the *inner region*, one finds that the logarithmic law of the wall :

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln z' + C \tag{2.50}$$

remains valid (see *White*, 1974, p. 473), and this as long as the driving-pressure gradient is weak (see eq. 2.40), being either positive or negative, $\pm (\partial \overline{p^*} / \partial x)$. One can explain (see *Tennekes* et *Lumley*, 1972, p. 185) this, by assuming the inertia term in eq. 2.53 is negligible and that the term of the driving pressure is weak compared to the term of the Reynolds stress; a zone of quasi-constant stress is thus delimited.

Nevertheless the integration constant, C, may depend upon the pressure gradient (see *Tennekes* et *Lumley*, 1972, p. 186). The thickness of the inner region, $z'/h \le 0.2$, will now depend on the pressure gradient (see *White*, 1974, p. 473). The inner region can disappear in decelerating flow with strong positive pressure gradients, when flow separation occurs.

ii) In the *outer region*, one finds (experimentally) that the law of velocity defect, the one of Coles :

$$\frac{U_c - \bar{u}}{u_*} = \frac{1}{\kappa} \ln\left(\frac{\delta}{z'}\right) + \frac{\Pi}{\kappa}(2 - \tilde{\omega}) \tag{2.51}$$

remains valid. Depending on the gradient of the driving pressure (see eq. 2.40), one has to make the following distinction (see Fig. 2.12) :

- a positive (unfavourable) pressure gradient, $\partial \overline{p^*} / \partial x > 0$, being always accompanied by a decrease of the average velocity in the direction of the flow (*deceleration*); the velocity profiles get less uniformly distributed;

- a negative (favourable) pressure gradient, $\partial \overline{p^*}/\partial x < 0$, being usually accompanied by an increase of the average velocity in the direction of flow (*acceleration*); the velocity profiles get more uniformly distributed.

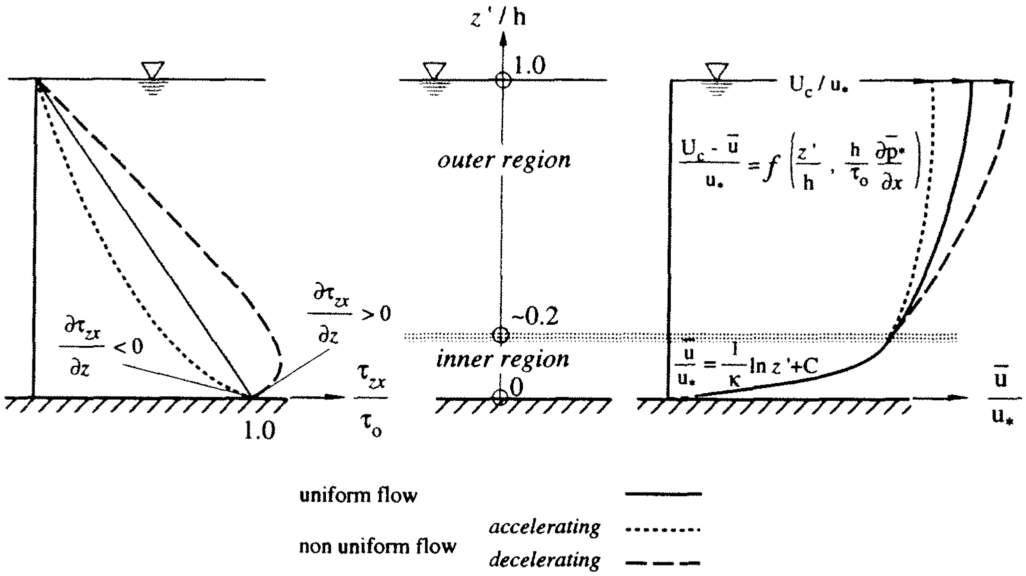


Fig. 2.12 Scheme of the distribution of the shear stress, $\tau_{zx}(z')$, and of the velocity, $\bar{u}(z')$, in non-uniform flow.

5° To obtain the distribution (see Fig. 2.12) of the total shear stress, $\tau_{zx}(z)$, the equation of motion, eq. 2.53, must be integrated (see *Rotta*, 1972, p. 240). The boundary conditions are the following :

$z' = 0 \quad \Rightarrow \quad \tau_{zx} = \tau_o$

$z' = h \quad \Rightarrow \quad \tau_{zx} = 0$

Consequently, one obtains :

i) Close to the wall, $z' \ll h$, where the non-slip conditions, $\bar{u} = 0$ and $\bar{w} = 0$, are valid, eq. 2.53 is written as :

$$\frac{\partial \tau_{zx}}{\partial z} = \rho g \left(\frac{\partial h}{\partial x} + \frac{\partial z_b}{\partial x}\right) = \frac{\partial \overline{p^*}}{\partial x} \qquad (2.53a)$$

which subsequently gives (see *White*, 1974, p. 474) :

$$\tau_{zx} = \tau_o + (\frac{\partial \overline{p^*}}{\partial x}) z$$

Depending on the pressure gradient, one has :

- for a positive pressure gradient, $\partial \overline{p^*}/\partial x > 0$, where the flow is *decelerating* :

  $$\frac{\partial \tau_{zx}}{\partial z} > 0 \qquad \text{and} \qquad \tau_{zx} > \tau_o$$

  the total stress has its maximum value, $\tau_{zx} \equiv \tau_{max}$, at a certain distance from the wall;

- for a negative pressure gradient, $\partial \overline{p^*}/\partial x < 0$, where the flow is generally *accelerating* :

  $$\frac{\partial \tau_{zx}}{\partial z} < 0 \qquad \text{and} \qquad \tau_{zx} < \tau_o$$

  the total stress has its maximum value, $\tau_{zx} \equiv \tau_{max}$, at the wall.

ii) Far from the wall, beyond the point where $\tau_{zx} \equiv \tau_{max}$, the distribution of total shear stress is monotone (see *Rotta*, 1972, p.240) and this up to the water surface, where $\tau_{zx} = 0$.

2.6 DISTRIBUTION OF VELOCITY

1° The experimental results, to support the theory developed in chap. 2.5, will now be presented.

It is taken that the flow of a real and incompressible fluid is completely developed along (the bed of) the channel. Assumed will be that the flow is two-dimensional, but unidirectional in the x-direction, being steady and uniform or non-uniform.

2° A direct consequence of a real-fluid flow is the manifestation of the (point) velocity profile, $u(z')$, where the z' is the distance measured from the bed of the channel.

By integration of the velocity profile the average velocity, U, across the flow section is obtained.

3° Between the dimensionless average velocity, U/u_*, and friction coefficient, f, there exists (see *Graf & Altinakar*, 1991, p. 433) the following relationship (see eq. 3.8) :

$$\frac{U}{u_*} = \sqrt{\frac{8}{f}} \tag{2.55}$$

where $u_* = \sqrt{\tau_0/\rho}$ is the friction velocity.

4° A summary of the velocity distribution, $u(z')$, of the average velocity, U, and of the friction coefficient, f, for uniform flow, both laminar and turbulent, is given in Table 2.1.

2.6.1 Laminar Flow

1° Uniform, steady and laminar flow in a channel of a large width, $R_h \equiv h$, has been studied in great detail (see *Graf & Altinakar*, 1991, p. 257); it is a special case of the *Couette* flow.

2° The distribution of the velocity, $u(z')$, for two-dimensional flow (see Fig. 1.6) is given by a parabolic relation :

$$\frac{u(z')}{u_*} = \frac{1}{2\mu\, u_*} \left(-\gamma \frac{dh}{dx}\right) (2hz' - z'^2) \tag{2.56}$$

where $h = h + z_b$ and (dh/dx) is the slope of the water surface, given herewith as $S_w \equiv S_f = -(dz_b/dx)$. Using the friction velocity, given as $u_*^2 = gh\, S_f$ (see eq. 3.7), this equation, eq. 2.56, becomes :

$$\frac{u(z')}{u_*} = \left(\frac{u_* z'}{\nu}\right) \left(1 - \frac{z'}{2h}\right) \tag{2.56a}$$

3° The average velocity, U , in the flow section, A, is given (see *Graf & Altinakar*, 1991, p. 257) by :

$$\frac{U}{u_*} = \frac{1}{u_*} \frac{g}{3\nu} S_f h^2 = \frac{1}{3} \left(\frac{u_* h}{\nu}\right) \tag{2.57}$$

a relationship which expresses a proportionality between the average velocity, U, and the bed slope, S_f.

4° Flow is considered to be laminar, if the Reynolds number is :

$$Re' = \frac{Uh}{\nu} \leq 500 \qquad \text{or} \qquad Re = \frac{4Uh}{\nu} \leq 2000$$

As long as flow stays laminar, the roughness of the bed of the channel is of no consequence.

5° The friction coefficient, f, is obtained by combining eq. 2.55 and eq. 2.57; that is :

$$\sqrt{\frac{8}{f}} = \frac{U}{u_*} = \frac{1}{3}\left(\frac{u_* h}{\nu}\right)\frac{U}{U} = \frac{1}{3}\,\mathrm{Re}'\sqrt{\frac{f}{8}}$$

or written also as :

$$f = \frac{24}{\mathrm{Re}'} \qquad \text{or} \qquad f = \frac{6}{\mathrm{Re}} \tag{2.58}$$

The coefficient, $B_l = 24$, is valid notably in two-dimensional flow, when the width of the channel is large and having an aspect ratio of $B/h > 5$. For channels which are less large, $B/h < 5$, or for channels being not rectangular, this coefficient may be smaller, $14 < B_l < 24$ (see *Chow*, 1959, p. 11).

2.6.2 Turbulent, smooth Flow

1° The universal velocity distribution for turbulent, smooth flow was developed using the concept of the mixing length (see point 2.5.2, 4°, or *Graf* & *Altinakar*, 1991, pp. 280 - 289).

2° The distribution of the velocity, $u(z')$, — one shall take now $u \equiv \bar{u}$ for the point-average velocity, or the bar will no more be used — is logarithmic (see eq. 2.50); it is given by :

$$\frac{u(z')}{u_*} = \frac{1}{\kappa}\ln\left(\frac{z' u_*}{\nu}\right) + B_s \tag{2.59}$$

The numerical constants, obtained from numerous experiments with *uniform* flow (see *Reynolds*, 1974, p. 187) are :

$$\kappa = 0.4 \qquad ; \qquad B_s = 5\,(\pm 25\%)$$

3° For *non-uniform* flows, the numerical constants are only slightly different (see *Reynolds*, 1974, p. 187 and *Cardoso* et *al.*, 1989).

4° This relation, eq. 2.59, is only valid close to the surface (bed), delimited by :

$$35 \le \frac{z' u_*}{\nu} \le 200 \qquad \text{or} \qquad \frac{z'}{h} \le 0.2$$

but experiments have shown good agreement over the entire flow depth, h. The region delimited by $(z'/h) \leq 0.2$, is the inner region (see Fig. 2.11) where the shear stress remains essentially constant.

5° Upon integration of eq. 2.59, one obtains an expression for the average velocity :

$$\frac{U}{u_*} = \frac{1}{\kappa} \ln \left(\frac{R_h u_*}{\nu}\right) + \overline{B}_s \qquad (2.60)$$

The constant of integration obtained from numerous experiments (see *Keulegan*, 1938) is given as :

$$\overline{B}_s = 3.5$$

but it depends slightly on the geometry of the cross section and on the Froude number (see *Chow*, 1959, p. 205).

6° The friction coefficient, f, can be now obtained in combining eq. 2.55 with eq. 2.60; this gives :

$$\sqrt{\frac{8}{f}} = \frac{U}{u_*} = \frac{1}{\kappa} \ln \left(\frac{R_h u_*}{\nu}\right) + \overline{B}_s$$

and for $\overline{B}_s = 3.5$, one has :

$$\sqrt{\frac{1}{f}} = 2.03 \log \left(Re' \sqrt{f} \right) + 0.32 \qquad (2.61)$$

or putting $Re' = Re/4$:

$$\sqrt{\frac{1}{f}} = 2.03 \log \left(Re \sqrt{f} \right) - 0.88 \cong 2 \log \left(\frac{Re}{3} \sqrt{f} \right) \qquad (2.61a)$$

The above relations, eq. 2.61, are valid for turbulent flow, $Re' > 500$, in a channel having smooth walls, $(u_* k_s/\nu) < 5$.

2.6.3 Turbulent, rough Flow

1° The universal velocity distribution for turbulent, rough flow was developed, using the concept of the mixing length (see point 2.5.2, 4 °, or *Graf & Altinakar*, 1991, pp. 280-289).

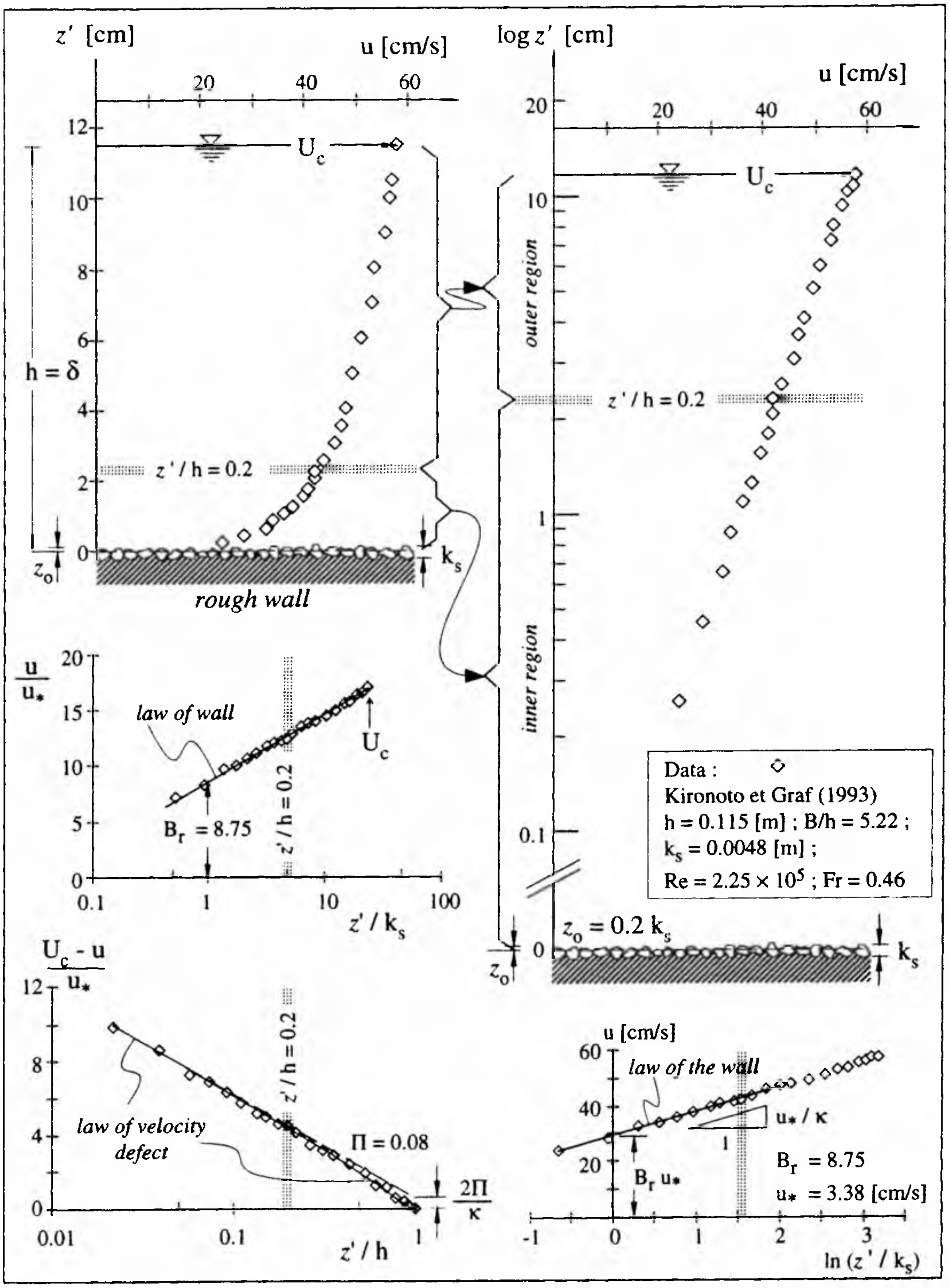


Fig. 2.13 Velocity profile, u(z'); *uniform*, rough flow.

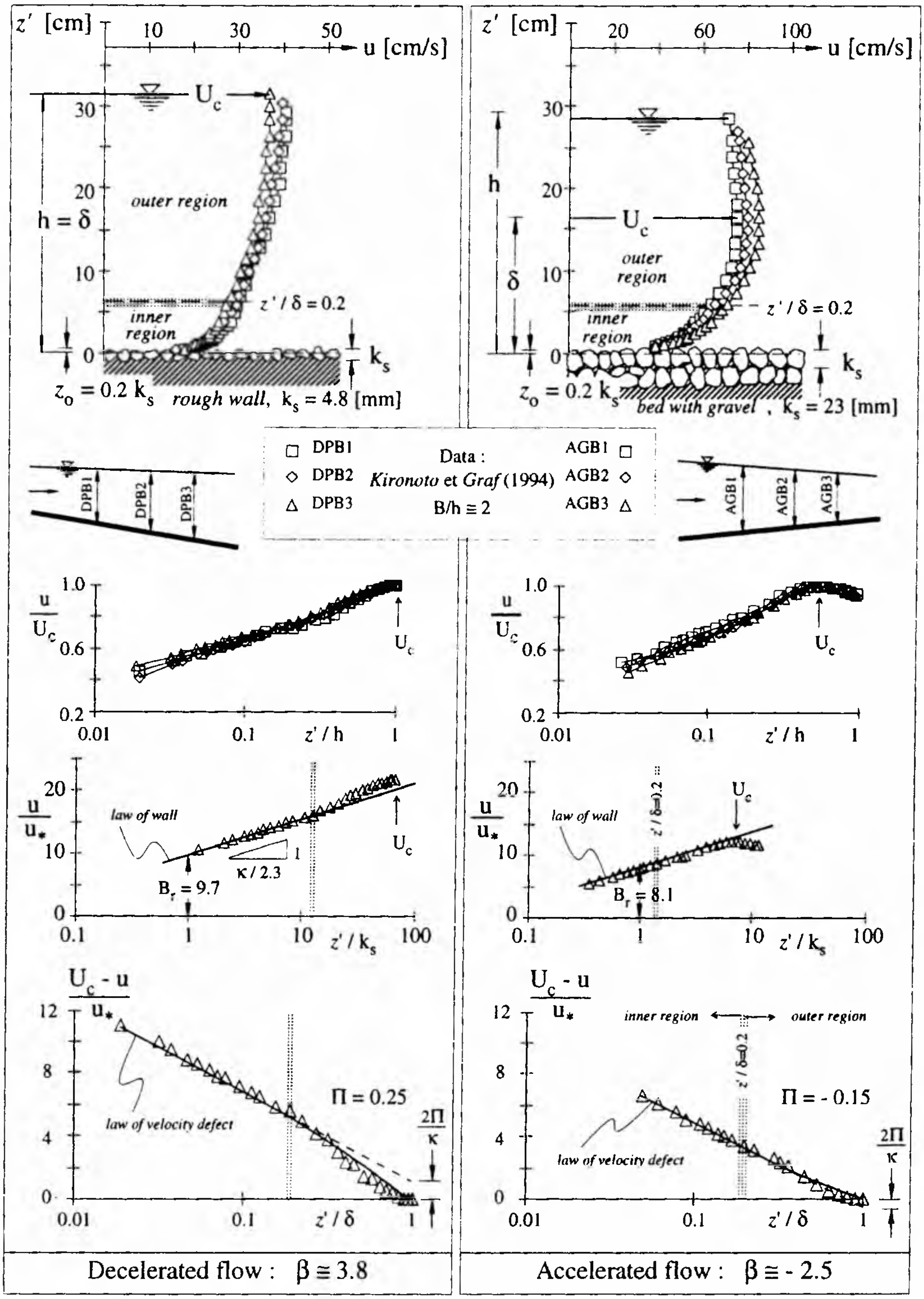


Fig. 2.14 Velocity profile, u(z'); *non-uniform*, rough flow.

2° As an example, the distribution of the velocity — measured in a laboratory flume by *Kironoto et Graf* (1993 et 1994) — is given with Fig. 2.13 for uniform and with Fig. 2.14 for non-uniform flow. Different coordinates are here used :

- $u \;\; vs \;\; z'$: is the original (measured) profile;
- $u \;\; vs \;\; \log z'$: shows the logarithmic form of the original profile;
- $\frac{u}{U_c} \;\; vs \;\; \frac{z'}{h}$: shows the auto-similarity, necessary for non-uniform flow in equilibrium;
- $u \;\; vs \;\; \ln\left(\frac{z'}{k_s}\right)$: used to determine the values of u_* and B_r, if $\kappa = 0.4$ is imposed;
- $\frac{u}{u_*} \;\; vs \;\; \frac{z'}{k_s}$: gives a dimensionless representation of the law of wall;
- $\frac{U_c - u}{u_*} \;\; vs \;\; \frac{z'}{\delta}$: gives the dimensionless representation of the law of velocity deficit.

3° The distribution of the velocity, $u(z')$ (see Fig. 2.13), is logarithmic (see eq. 2.50); it is given by :

$$\frac{u(z')}{u_*} = \frac{1}{\kappa} \ln\left(\frac{z'}{k_s}\right) + B_r \tag{2.62}$$

k_s being the equivalent or standard uniform roughness (see *Graf & Altinakar*, 1991, p. 287 et sect. 3.2.1). The numerical constants, which are obtained from numerous experiments (see *Reynolds*, 1974, p. 187 and *Kironoto* et *Graf*, 1993) for *uniform* flow, are given as :

$$\kappa = 0.4 \qquad ; \qquad B_r = 8.5\ (\pm 15\%)$$

The vertical distance, z', is measured from a level which passes slightly below the peaks of the roughness (see Fig. 2.13); in general one takes $z_0 \cong -0.2\ k_s$ (see *Graf*, 1991 or *Hinze*, 1975, p. 637). The relation of eq. 2.62, just as the one of eq. 2.59, is actually only valid within the inner region, $z'/h \leq 0.2$ (see Fig. 2.11 and Fig. 2.13), but an extension into the outer region is often possible.

4° For *non-uniform* flow, the constant, B_r, is slightly different (see *Kironoto* et *Graf*, 1994): it is larger in decelerating flow and smaller in accelerating flow (see Fig. 2.14). The same tendency is observed for *unsteady* flow (see *Tu* et *Graf*, 1992).

5° After integration of eq. 2.62, one obtains the following expression for the average velocity :

$$\frac{U}{u_*} = \frac{1}{\kappa} \ln \left(\frac{R_h}{k_s}\right) + \overline{B_r} \qquad (2.63)$$

The constant of integration, obtained from different experiments (see *Keulegan*, 1938, p. 722) is :

$$\overline{B_r} = 6.25$$

being almost independent of the geometrical form of the channel. For flow with large Froude numbers, $Fr > 1$, the value of $\overline{B_r}$ diminishes (see *Chow*, 1959, p. 205).

6° The friction coefficient, f, obtained by combining eq. 2.55 with eq. 2.63, is written as :

$$\sqrt{\frac{8}{f}} = \frac{U}{u_*} = \frac{1}{\kappa} \ln \left(\frac{R_h}{k_s}\right) + \overline{B_r}$$

substitution of $\overline{B_r} = 6.25$, gives :

$$\sqrt{\frac{1}{f}} = 2.03 \log \left(\frac{R_h}{k_s}\right) + 2.2 \qquad (2.64)$$

This relation is valid for turbulent flow, $Re' > 2 \cdot 10^4$, in channels with completely rough surfaces, $(k_s u_*/\nu) > 70$.

7° Between smooth surface flow, delimited by $(k_s u_*/\nu) < 5$, and rough surface flow, delimited by $(k_s u_*/\nu) > 70$, there exists the transition region, where the experiments of Nikuradse (see *Graf & Altinakar*, 1991, p. 427) are used to make the connection.

8° The friction coefficient, f, for flow over smooth, transition and rough surfaces, is given by the relation of Colebrook et White (see *Graf & Altinakar*, 1991, p. 436) which was adapted for channels by *Silberman* et *al.* (1963, p. 104), such as :

$$\sqrt{\frac{1}{f}} = -2.0 \log\left(\frac{k_s/R_h}{a_f} + \frac{b_f}{Re\sqrt{f}}\right) \tag{2.65}$$

where $12 < a_f < 15$ and $0 < b_f < 6$, being established for sections of different geometrical shapes and $Re = 4\,R_h U/\nu$. For very wide channels it is recommended to take : $a_f = 12$ and $b_f = 3.4$. If $k_s = 0$, the relation reduces to eq. 2.61a, valid for smooth surfaces; if $Re \to \infty$, it reduces to eq. 2.64, valid for completely rough surfaces.

9° Outside the inner region (see Fig. 2.11) and up to the entire flow depth, h, is the outer region, delimited by $0.2 < (z'/h) < 1.0$. In this region, the flow is conditioned by the maximum velocity, $u = U_c$, as well as by a possibly existing longitudinal pressure gradient. The distribution of the velocity deviates slightly from the logarithmic law. It is approximately given (see *Graf*, 1991 et *Hinze*, 1975) by a law of velocity deficit :

$$\frac{U_c - u}{u_*} = 9.6\left(1 - \frac{z'}{h}\right)^2 \tag{2.66}$$

being valid for turbulent flow, both for smooth and rough surfaces.

Nevertheless, the logarithmic law given by eq. 2.59 and eq. 2.62 can be used over the entire flow depth, if one does not desire a too high precision.

10° The distribution of the velocity over the entire flow depth — with exception of the viscous region — , thus in the zone of $0.01 < (z'/h) < 1.00$ (see Fig. 2.11 and Fig. 2.13), is given by the following law of velocity deficit (see point 2.5.2, 4 °) :

$$\frac{U_c - u}{u_*} = \frac{1}{\kappa} \ln\left(\frac{\delta}{z'}\right) + \frac{\Pi}{\kappa}\left(2 - \tilde{\omega}\right) \tag{2.51}$$

where Π is the wake parameter of Coles, which depends notably on the longitudinal pressure gradient, β, (see eq. 2.52). For *uniform* flow over smooth and rough surfaces (see *Kironoto* et *Graf*, 1993) in a channel having a weak pressure gradient, namely the bottom slope, one takes :

$$\Pi \cong 0.2$$

having a variation of $-0.1 \lesssim \Pi \lesssim 0.3$.

For flow (boundary-layer) without pressure gradient, $(\partial \overline{p^*} / \partial x) = 0$, one takes $\Pi \cong 0.55$ (see *Hinze*, 1975, p. 697).

Table 2.1

Summary of velocity profile, u, of average velocity, U, and of friction coefficient, f, for steady, *uniform* flow in channels.

$$\frac{u}{u_*} = \frac{1}{2\mu u_*} (\gamma S_f)(2hz' - z'^2)$$

$$\frac{U}{u_*} = \frac{1}{3}\left(\frac{h u_*}{\nu}\right)$$

$$f = 24/Re'$$

↑ LAMINAR / ↓ TURBULENT Flow — $Re' = \dfrac{Uh}{\nu} \cong 500$

smooth

$$\frac{u_* k_s}{\nu} < 5$$

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln\left(\frac{z' u_*}{\nu}\right) + 5.5$$

$$\frac{U}{u_*} = \frac{1}{\kappa} \ln\left(\frac{R_h u_*}{\nu}\right) + 3.5$$

$$\frac{1}{\sqrt{f}} = 2 \log\left(Re'\sqrt{f}\right) + 0.32$$

smooth
transition
rough

$$\frac{U_c - u}{u_*} = \frac{1}{\kappa} \ln\left(\frac{\delta}{z'}\right) + \frac{\Pi}{\kappa}(2 - \tilde{\omega})$$

$$\frac{1}{\sqrt{f}} = -2 \log\left(\frac{k_s/R_h}{a_f} + \frac{b_f}{4Re'\sqrt{f}}\right)$$

rough

$$\frac{u_* k_s}{\nu} > 70$$

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln\left(\frac{z'}{k_s}\right) + 8.5$$

$$\frac{U}{u_*} = \frac{1}{\kappa} \ln\left(\frac{R_h}{k_s}\right) + 6.25$$

$$\frac{1}{\sqrt{f}} = 2 \log\left(\frac{R_h}{k_s}\right) + 2.2$$

11 ° For *non-uniform* flow (see Fig. 2.12 and Fig. 2.14), eq. 2.51 remains still valid, but the wake parameter, Π , has no more a constant value. Such flows must remain in equilibrium, namely the velocity profile must stay auto-similar. An empirical relationship of the type :

$$\Pi = f(\beta)$$

is proposed, where β is the equilibrium parameter :

$$\beta = \frac{h}{\tau_o} \frac{\partial \overline{p^*}}{\partial x} \qquad (2.52)$$

which characterizes the longitudinal pressure gradient. Depending on this parameter, β , one finds (see *Kironoto* et *Graf*, 1994) that :

$\beta < -1$: flow is *accelerating,* and the wake parameter is $-1.0 < \Pi < 0.2$;

$\beta > -1$: flow is *decelerating*, and the wake parameter is $\Pi > 0.2$;

$\beta = -1$: flow is *uniform* having a weak pressure gradient, and the wake parameter is $\Pi \cong 0.2$.

The same tendency was observed (see *Tu* et *Graf*, 1992) for *unsteady* flow.

12 ° For two-dimensional flow, the maximum velocity, U_c . occurs at the water surface, $\delta \equiv h$. For three-dimensional flow, the maximum velocity, U_c , may occur below the water surface, $\delta < h$ (see Fig. 2.15); secondary flow is evident. To parametrise this, one may use a ratio of $(h - \delta)/h$ (see Fig. 2.15). The aspect ratio of $B/h \cong 5$ is the limiting value.

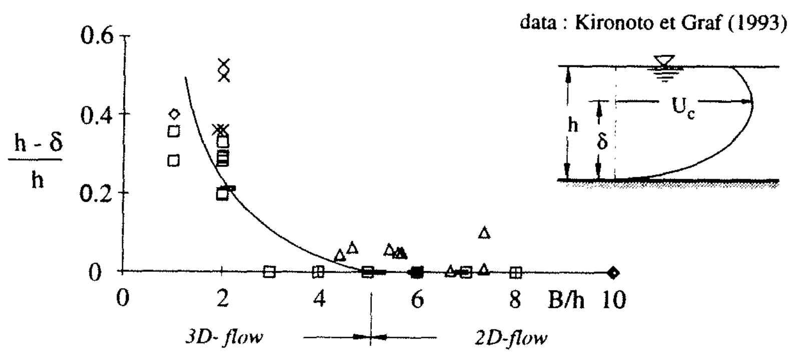


Fig. 2.15 The position of the maximum velocity, U_c ; in two- and three- dimensional flow.

2.6.4 Turbulence Characteristics

1° Flow at large Reynolds numbers ceases to be laminar; it becomes turbulent. In each point of the flow, the instantaneous velocity, u_i and w_i, is subject to variation in direction and in intensity. The velocity varies around a mean value defined by :

$$u_i = u + u' \quad , \qquad w_i = w + w'$$

The fluctuation components, u' and w', are by definition weak as compared to the respective mean values, u and w.

2° The equations of motion for laminar flow — the Navier-Stokes equation (see *Graf & Altinakar*, 1991, sect. FR.1) — are modified by the supplementary stresses due to the turbulence; these are the Reynolds equations, eq. 2.35, (see *Graf & Altinakar*, 1991, sect. FR.5). The supplementary stresses have the form of :

$$\rho\,\overline{u'^2} \quad , \quad \rho\,\overline{w'^2} \quad , \quad \rho\,\overline{u'w'} \quad , \quad \text{etc.}$$

Semi-empirical methods are used to express these stresses.

The characterization of the structure of turbulence is based here on experiments done in channels; some of these will be presented.

3° *Intensity of turbulence*

The temporal mean value of the velocity fluctuations are by definition zero :

$$\overline{u'} = 0 \ , \quad \overline{w'} = 0$$

This is however not the case for the mean quadratic values, $\overline{u'^2}$. The RMS-value (*Root-Mean-Square*), $\sqrt{\overline{u'^2}}$ and $\sqrt{\overline{w'^2}}$, is commonly used.

The ratio of the RMS-value and the friction velocity (see *Graf & Altinakar*, 1992, p. 267) is used to define the intensity of turbulence :

$$\frac{\sqrt{\overline{u'^2}}}{u_*} = f(z) \quad , \qquad \frac{\sqrt{\overline{w'^2}}}{u_*} = f(z) \tag{2.67}$$

which varies in space.

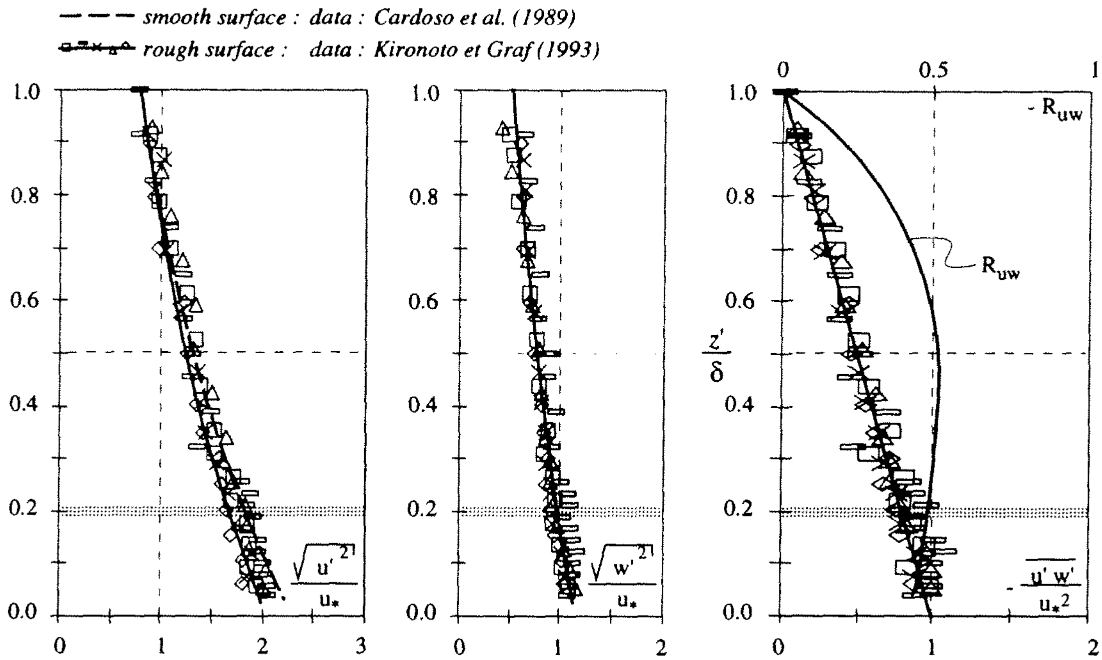


Fig. 2.16 Distribution of the normal and tangential stress of Reynolds; for *uniform* flow.

The vertical distribution of the normal stress, expressed as turbulence intensity, is given in Fig. 2.16 for *uniform* flow; the measurements were performed in the center of the channel having an aspect ratio of $2.1 < B/h < 6.9$. One notices that :

i) for channels with smooth (see *Cardoso* et *al.*, 1989) and rough surface (see *Kironoto* et *Graf*, 1993), the distribution are reasonably the same;

ii) close to the bed (surface), in the inner region, one has :

$$\sqrt{\overline{u'^2}} \cong 1.8\, u_* \qquad ; \qquad \sqrt{\overline{w'^2}} \cong 1.0\, u_*$$

iii) up to the flow depth, the distribution stays monotone; at the surface one has :

$$\sqrt{\overline{u'^2}} \cong \sqrt{\overline{w'^2}} \cong 0.6\, u_*$$

and the turbulence becomes isotropic.

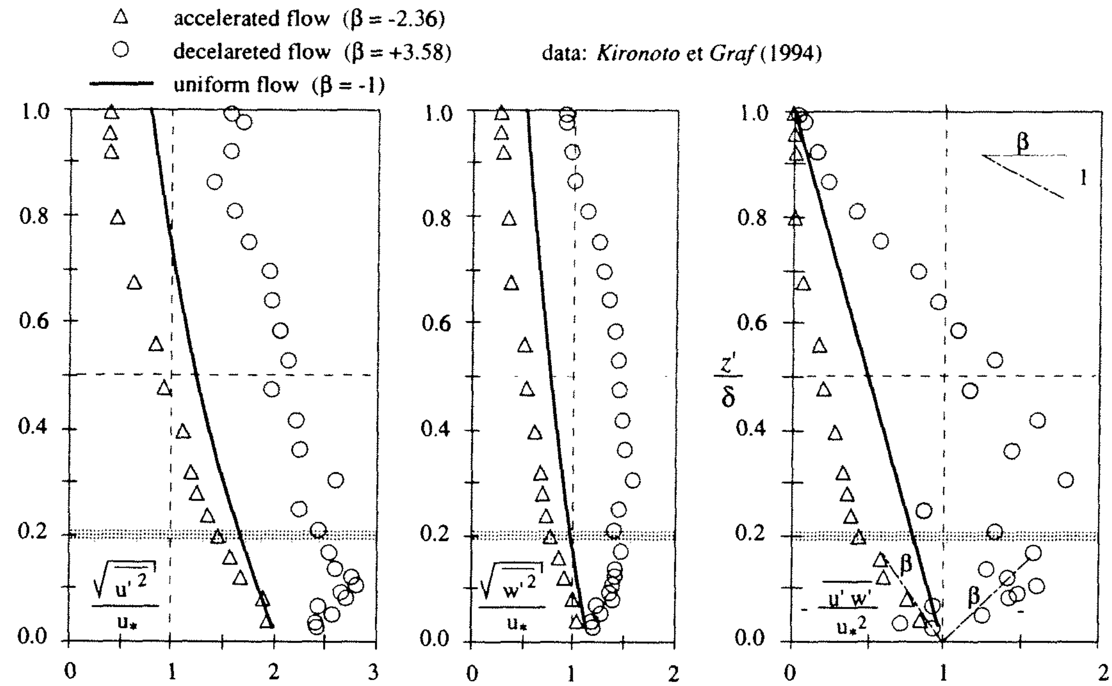


Fig. 2.17 Distribution of the normal and tangential stress of Reynolds; for *non-uniform* flow.

For *non-uniform* flow in equilibrium, the distribution of the normal stress is given in Fig. 2.17; the measurements (see *Kironoto* et *Graf*, 1994) are performed in the center of the channel having an aspect ratio of B/h ≅ 2. One notices that :

i) for *accelerating* flow, $\beta < -1$, the turbulence intensity is smaller than for uniform flow, $\beta = -1$. The maximum value is at the bed and diminishes towards the water surface. Consequently, in accelerating flow the turbulence is suppressed (see *Hinze*, 1959, p. 66);

ii) for *decelerating* flow, $\beta > -1$, the turbulence intensity is larger than for uniform flow, $\beta = -1$. The maximum value is above the bed and diminishes towards the water surface. Consequently, in decelerating flow the turbulence is enhanced.

Similar experimental observations have also been communicated (see *Bradshaw*, 1978, p. 68) for boundary-layer flow with pressure gradients.

4° *Reynolds stress*

The total tangential stress, eq. 2.42, are well represented by the supplementary stress, notably for flow at large Reynolds numbers; one writes :

$$\tau_{zx} = -\rho \, \overline{u'w'} \tag{2.42b}$$

The vertical distribution of the supplementary stress, in short the Reynolds stress, is given in Fig. 2.16 for experiments with *uniform* flow. One finds — as one has already seen in point 2.5.2, 3° — that :

i) the distribution is linear, or :

$$\frac{\tau_{zx}}{\tau_o} = \left(\frac{\delta - z'}{\delta}\right) \tag{2.47b}$$

ii) with the boundary conditions being :

$$z' \cong 0 \qquad \tau_{zx} \cong \tau_o$$
$$z' \cong \delta \qquad \tau_{zx} = 0$$

For *non-uniform* flow in equilibrium, the distribution of the Reynolds stress is given with Fig. 2.17. One finds — as one has already seen in point 2.5.3, 5° — that :

i) in *accelerating* flow, $\beta < -1$, the Reynolds stress, whose distribution is concave, diminishes;

ii) in *decelerating* flow, $\beta > -1$, the Reynolds stress, whose distribution is convex, increases;

iii) consequently, the energy dissipation, namely the head loss, is larger in decelerating than in accelerating flow;

iv) close to the bed, the gradient to the curve of distribution is given by :

$$\frac{\partial \tau_{zx}}{\partial z} = \frac{\partial \overline{p^*}}{\partial x} \approx \beta \left(\frac{\tau_o}{h}\right) \tag{2.53b}$$

which is in agreement with arguments advanced in point 2.5.3, 5° (see Fig. 2.12).

Similar observations have also been done for unsteady flow with a free surface (see *Tu* et *Graf*, 1992) as well as for boundary-layer flow having pressure gradients (see *Bradshaw*, 1978, p. 68).

A dependence between the velocity fluctuations, u' et w', is often given with a correlation coefficient :

$$-R_{uw} = \frac{\overline{u'w'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{w'^2}}} \tag{2.68}$$

The distribution of this coefficient for *uniform* flow — for *non-uniform* flow it is similar — is given with Fig. 2.16. One sees that :

i) over a large fraction of the flow depth, $0.1 < z'/\delta < 0.6$, the fluctuation are reasonably well correlated : $R_{uw} \cong -0.5$;

ii) this correlation diminishes close to the bed and close to the water surface.

5° *Mixing length*

The Reynolds stress, eq. 2.42b, can also be expressed by :

$$\tau_{zx} = \rho\, l^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2 = \rho\, \nu_t \left(\frac{\partial \bar{u}}{\partial z}\right) \tag{2.49}$$

where l, known as Prandtl's mixing length, is the distance over which the fluid mass displaces itself. ν_t is the mixing coefficient of Boussinesq which has dimensions of the kinematic viscosity, but its value is very large, such as $\nu_t >> \nu$.

The vertical distribution of the mixing length, l, in *uniform* flow — it is similar in *non-uniform* flow — is given with Fig. 2.18; one finds that :

i) close to the surface (bed), in the inner region, $z'/\delta \lesssim 0.2$, it is given as :

$l = \kappa z'$ where $\kappa = 0.4$

ii) in a large part of the outer region, $0.5 \lesssim z'/\delta \lesssim 1.0$, it is given as :

$l/\delta \cong 0.12$

but the data show a large spread.

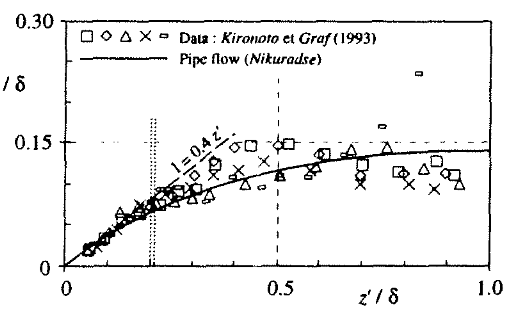


Fig. 2.18 Distribution of the mixing length.

6° *Energy spectrum*

The spectrum of kinetic energy (see *Graf & Altinakar*, 1991, p. 273) provides important information about the turbulence of the flow, namely about the energy distribution of the eddies having different sizes and frequencies.

The spectral function, E(n), gives the turbulent kinetic energy, $\overline{u'^2}(n)$ or $\overline{w'^2}(n)$, for a range of frequencies, dn ; or :

$$\overline{u'^2} = \int_0^\infty E(n)\, dn \tag{2.69}$$

or written in normalized form, $F(n) = E(n)/\overline{u'^2}$, as :

$$\int_0^\infty F(n)\, dn = 1 \tag{2.70}$$

F(n) has units of time, [t] , and n has units of frequency, $[t^{-1}]$.

The function, F(n) , — known as the *turbulence spectrum* — is a representation of the way the energy is distributed with the frequency, n. By a transformation — use is made of Taylor's hypothesis of frozen turbulence — of the frequency, n , into a wave number, $(2\pi/u)\, n = k$, the wave-number spectrum is obtained :

$$\int_0^\infty F(k)\, dk = 1 \tag{2.70a}$$

F(k) has units of length, [m], and k has units of length, $[m^{-1}]$.

With Fig. 2.19 is shown a one-dimensional frequency spectrum for longitudinal, u', and vertical, w', velocity fluctuations for uniform flow in an open channel with a free surface, at different levels, *z'*. These spectra are rather typical for *uniform* flow over smooth and rough surfaces, having $10^4 < Re < 10^6$ (see *Kironoto* et *Graf*, 1993) and also for *non-uniform* flow (see *Kironoto* et *Graf*, 1994).

The energy spectrum is in general rather wide and is usually delimited in three zones (see Fig. 2.19b):

i) The turbulence structure of the largest eddies has no universality; it is anisotropic and depends largely on the flow conditions, thus on the flow Reynolds number. The *macro scale* (see *Reynolds,* 1974, p. 79) of the turbulence, $k_1 = \Lambda^{-1}$, is the upper limit.

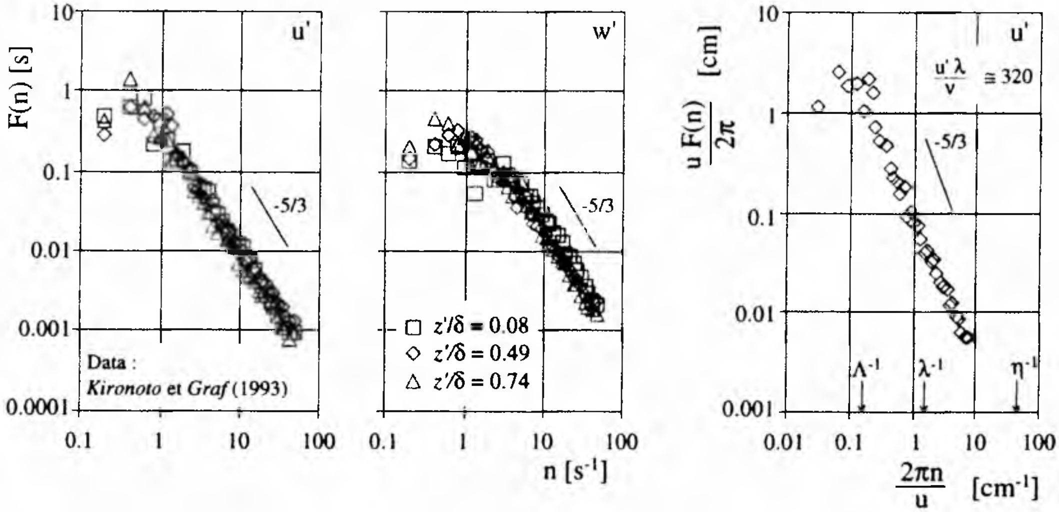


Fig. 2.19 Turbulence spectrum for different levels, z'/δ; in uniform flow.

ii) The turbulence structure of the very smallest eddies, where viscous dissipation dominates, has universality; the lower limit (see *Reynolds,* 1974, p. 99) is given by the Kolmogoroff scale, $k_3 = \eta^{-1} = (\nu^3/\varepsilon)^{1/4}$, where ν is the kinematic viscosity and ε is the dissipated turbulent energy, expressed by :

$$\varepsilon = 15\ \mu\ (\overline{u'^2}/\lambda^2).$$

iii) In the *inertial zone*, if the flow Reynolds number is high, the large eddies disintegrate into smaller eddies and so on (in cascades), and all this by action of the inertia forces. The resulting energy spectrum has universality, thus being independent of the Reynolds number. This part of the spectrum is described by *Kolmogoroff*'s hypothesis, which shows by way of a dimensional analysis, that the spectral function, F(k) , can be given by a law of the type (see *Reynolds,* 1974, p. 99) :

$$F(k) \propto \varepsilon^{2/3}\, k^{-5/3}$$

In this zone, $k_1 < k << k_3$, the turbulence is quasi isotropic and the spectrum is in equilibrium: the *micro-scale* (see *Reynolds,* 1974, p. 79) of the turbulence, $k_2 = \lambda^{-1}$, falls into this zone.

3. UNIFORM FLOW

Flow in a channel is considered as uniform and steady, if the flow depth remains invariable in the flow direction as well as in time. In fluvial hydraulics, uniform flow is taken as the base (reference) for all other considerations, and this despite the fact that truly uniform flow is rarely encountered in reality.

In this chapter, the equations of continuity and of motion will be developed. Subsequently are presented the different relationships for the determination of the coefficients of friction for fixed and mobile channel beds. Knowledge about this coefficient is paramount in all kinds of problems of fluvial hydraulics. The calculation of the discharge for flow over a fixed as well as a mobile bed is elaborated. Elementary knowledge about flow in curves as well as instabilities at the free-water surface will be exposed.

TABLE OF CONTENTS

3.1 HYDRODYNAMIC EQUATIONS

3.1.1 Notion of Uniformity

1° Flow is considered as uniform and steady (see sect. 1.2.1) if the flow depth, h or D_h, as well as other hydraulic parameters such as the average velocity, the discharge, the roughness and the channel slope, remain invariable in different cross sections of the channel along the axis of flow. The streamlines are rectilinear and parallel and the vertical pressure distribution is hydrostatic. The slope of the bed, S_f, of the water surface, S_w, and of the energy-grade line, S_e, are the same.

2° Truly uniform flow is rather rare in natural, but also in artificial channels. Uniform flow is only possible in very long prismatic channels and this far from the upstream and downstream boundary conditions (see Fig. 3.1).

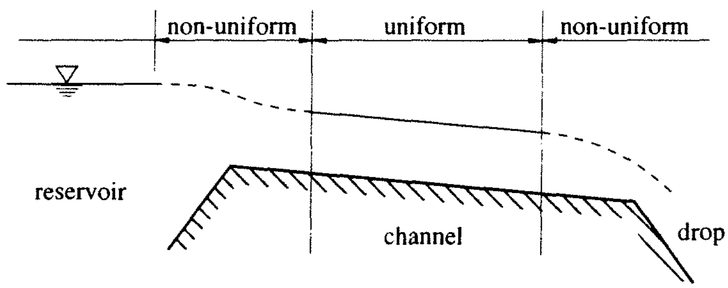


Fig. 3.1 Uniform flow between boundary conditions.

3° Despite the fact that uniform flow occurs rarely, this type of flow is usually taken as the standard (reference) flow for any theoretical and experimental study of other types of flow, but notably for the understanding of the flow resistance.

3.1.2 Equation of Continuity

1° As long as flow is uniform and steady, the cross section of the flow, A , remains the same in direction, x , and in time, t. The equation of continuity (see sect. 2.1) was given as :

$$\frac{\partial(UA)}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{2.1}$$

but becomes now :

$$\frac{\partial(UA)}{\partial x} = 0 \tag{3.1}$$

where Q = UA is the discharge and U is the average velocity.

2° Consequently, the discharge remains constant :

$$Q = Cte \tag{3.2}$$

Between two cross sections (see Fig. 3.2), one has :

$$A_1U_1 = Q = A_2U_2 \tag{3.2a}$$

and with $U_1 = U_2$ and $A_1 = A_2$, one writes : $Q = UA$.

3.1.3 Equation of Motion

1° Consider a prismatic channel (see Fig. 3.2). The liquid in motion provokes a friction force at the wetted perimeter :

$$F_F = \tau_o \, P \, dx$$

by an action of the longitudinal component of the gravity force :

$$F_G = \gamma A \, dx \, \sin\alpha = W \sin\alpha$$

In uniform flow, there exists an equilibrium between these forces :

$$\tau_o \, P \, dx = \gamma A \, dx \, \sin\alpha \tag{3.3}$$

Consequently, one obtains an expression for :

$$\tau_o = \gamma \frac{A}{P} \sin\alpha \tag{3.4}$$

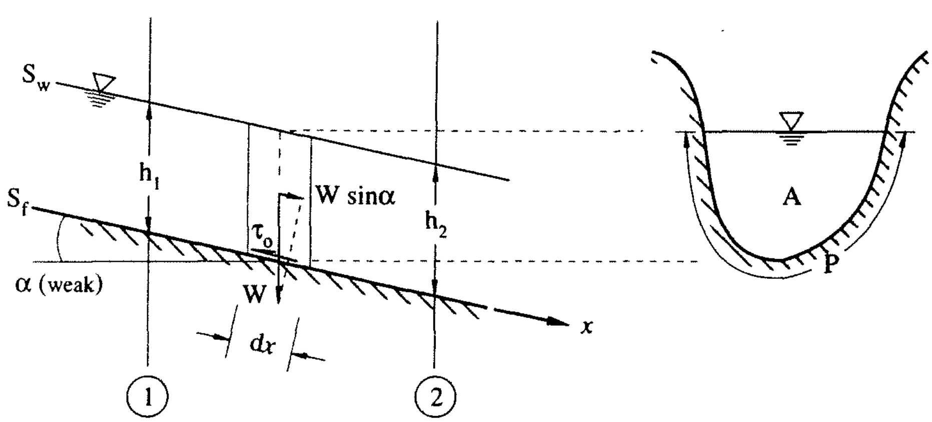


Fig. 3.2 Scheme of uniform flow.

The quotient of the wetted cross section , A , and its wetted perimeter, P , defines the hydraulic radius, R_h. The angle, α , is usually very small; thus one may write $\sin \alpha \cong \mathrm{tg}\ \alpha = S_f$. Above relation, eq. 3.4, now reads :

$$\tau_o = \gamma R_h S_f \qquad (3.5)$$

where τ_o is the tension due to the friction forces, called the shear stress, which acts on the wetted surface (wall and bed). Note that eq. 3.5 can be obtained directly from eq. 2.10 or eq. 2.12, by considering the uniformity of the flow.

2° In hydrodynamics, one defines :

$$\tau_o/\rho = u_*^2 \qquad (3.6)$$

where u_* is the friction velocity. Thus one can also write :

$$u_* = \sqrt{g R_h S_f} \qquad (3.7)$$

Instead of the shear stress, $\tau_o = \rho u_*^2$, one may also use the definition of the *friction coefficient* (see *Graf & Altinakar*, 1991, p. 433) which is given by :

$$f = \frac{\tau_o}{\rho U^2/8} = 8\left(\frac{u_*}{U}\right)^2 \qquad (3.8)$$

3° Upon substitution of eq. 3.8 into eq. 3.5, one obtains:

$$(f/8)\,\rho U^2 = \tau_o = \rho g R_h S_f$$

or, written otherwise :

$$S_f = f\frac{1}{4R_h}\frac{U^2}{2g} \qquad (3.9)$$

This relation is known as the equation of *Weisbach-Darcy* (see *Graf & Altinakar*, 1991, sect. FR. 2.1 and sect. PP. 2); it reveals itself as very useful for flow in pipes. The coefficient, f, of friction (head loss) depends on the Reynolds number and the relative roughness, but also on the form of the cross section.

The equation of Weisbach-Darcy can also be written as :

$$U = \sqrt{8g/f}\,\sqrt{R_h S_f} \qquad (3.10)$$

an expression which is frequently given in the form of :

$$U = C \sqrt{R_h S_f} \tag{3.11}$$

This is called the relationship of *Chézy*, where C is the resistance coefficient of Chézy.

4° In uniform regime (see eq. 3.10 and eq. 3.11), the flow depth, h, which corresponds to the hydraulic radius, R_h, is defined as being the *normal flow depth*, $h \equiv h_n$.

5° Different formulae have been elaborated over the years to render expressions for the friction (resistance) coefficients. Herewith some more common formulae will be presented, namely :

i) coefficient of Weisbach-Darcy (see sect. 3.2.1),
ii) coefficient of Chézy (see sect. 3.2.2),
iii) coefficient of Manning-Strickler (see sect. 3.2.3),
iv) coefficient of friction for mobile bed (see sect. 3.2.6).

3.2 COEFFICIENT OF FRICTION

1° It will certainly be useful, to express the friction coefficient, f, for laminar and turbulent flow with the equation of Weisbach-Darcy, eq. 3.10. However, the channel data presently available contain major limitations since channels should be of circular cross section and the roughness should be the standard one.

2° The relation of Chézy, eq. 3.11, is also rather useful as long as the flow in the channel is truly turbulent; this is often the case.

3° These two approaches, eq. 3.10 and eq. 3.11, give satisfactory results, notably for practical problems, if applied correctly and respecting their possible limitations. The ASCE (see *Silberman* et *al.*, 1963) recommended however the use of the equation of Weisbach-Darcy.

The precision which is obtained with these formulae, eq. 3.10 or eq. 3.11, is nevertheless strongly dependent upon the choice of the friction coefficient, f or C.

4° Artificial and particularly natural channels have all types of form of the cross section. No parameter exists which would well take care of the variability in form; the use of the hydraulic radius is often not sufficient.

5° An estimation of the friction coefficient for a fixed or immobile bed is already difficult; but still more difficult will be an estimation for a mobile bed.

3.2.1 Coefficient of Weisbach-Darcy

1° In above equations, eqs. 3.9 and 3.10, the definition of friction coefficient, f, is analogous to the one given for circular pipes. For pipes having an industrial roughness, a universal formulation is given (see *Graf & Altinakar*, 1991, sect. PP. 2) by :

i) the diagram of Moody-Stanton or

ii) the relation of Colebrook-White, for turbulent flow.

2° For cross sections, which are geometrically close to circular sections, one may readily use the experiments performed on pipes. However, some modifications are necessary; the hydraulic radius (see *Graf & Altinakar*, 1991, p. 439) should be written as follows :

$$4R_h = 4\frac{A}{P} \tag{3.12}$$

Thus $4R_h$ becomes the characteristic length, which is to be used in the definition of the Reynolds number, the relative roughness and the equation of Weisbach-Darcy, respectively :

$$Re = \frac{4R_h U}{\nu} \quad ; \quad \frac{k_s}{4R_h} \quad ; \quad S_f = f\frac{1}{4R_h}\frac{U^2}{2g}$$

For the roughness, k_s , in artificial channels, the equivalent roughness, established for industrial pipes, may be taken.

3° The use of the diagram of Moody-Stanton (see *Graf & Altinakar*, 1991, Fig. PP.9) with $Re = 4R_hU/\nu$ and $k_s/4R_h$ gives values for f for laminar and turbulent flow. Subsequently, one obtains the average velocity, U, using eq. 3.10, or the bed slope, $S_f = S_w$, of the channel, using eq. 3.9.

4° Instead of using the diagram of Moody-Stanton, one may also take the semi-empirical *relation of Colebrook-White* (see *Graf & Altinakar*, 1991, p. 436), valid only for turbulent flow, which is written for channels as follows (see eq. 2.65) :

$$\sqrt{\frac{1}{f}} = -2\log\left(\frac{k_s/R_h}{a_f} + \frac{b_f}{Re\sqrt{f}}\right) \tag{3.13}$$

with $12 < a_f < 15$ and $0 < b_f < 6$, established for different kinds of cross sections, as well as for different types of roughnesses (see *Silberman* et *al.*, 1963, p. 104).

The equivalent roughnesses, k_s, established for industrial pipes, but considered valid also for artificial channels, are given in Table 3.1. A more complete tabulation is given by *Wallisch* (1990, pp. 235-250).

For channels or watercourses whose bed is made up of a granulate, one generally takes $k_s \cong d_{50}$; d_{50} being the diameter equal to 50% of grains in the granulometric curve.

The importance of the form of the cross section can be somehow taken care of by a factor, which multiplies the hydraulic radius R_h ; thus (ϕR_h) and (ϕRe) replace R_h and Re in eq. 3.13. One takes (see *Ghetti* , 1981) :

- for a rectangular (B = 2 h) section $\phi = 0.95$
- for a large trapezoidal section $\phi = 0.80$
- for a triangular (equilateral) section $\phi = 1.25$

5° However, it must be pointed out that the results obtained with the diagram of Moody-Stanton or the *relation of Colebrook-White* will only be good approximations.

For channels, being very large and rectangular or very different from circular sections, above methods are less applicable.

Table 3.1 Equivalent roughness for industrial pipes.

Types of wall		*Uniform equivalent roughness* k_s [mm]
glass, copper, brass		< 0.001
lead		0.025
pipes, steel	new	0.03 à 0.1
	old	0.4
wrought iron	new	0.25
	old	1.0 à 1.5
	coated	0.1
concrete	smooth	0.3 à 0.8
	rough	< 3.0
wood		1.0 à 2.5
riveted steel		0.9 à 9
stone, worked rough		8 à 15
rock		90 à 600

6° Natural or artificial channels are usually of large dimensions. Consequently, the Reynolds number, $Re = 4R_hU/\nu$, and the roughness, k_s, have also large values. This implies that the turbulent flow is often also a rough one; the value of the friction coefficient, f, remains constant and is no more dependent on the Reynolds number.

This is a justification for using the relation of Chézy, eq. 3.10 or eq. 3.11, where the coefficient of Chézy depends only on the relative roughness, $C = f(k_s/R_h)$, (see eq. 3.13); thus one may write :

$$C = \sqrt{8g}\left(\frac{1}{\sqrt{f}}\right) = \sqrt{8g}\left[2\cdot\log\left(\frac{a_f}{k_s/R_h}\right)\right] \tag{3.13a}$$

Subsequently, taking $a_f = 12.7$, one obtains (see also eq. 2.63) :

$$\sqrt{\frac{8}{f}} = 5.6\log\left(\frac{R_h}{k_s}\right) + 6.25 \tag{3.13b}$$

For a roughness due to large grains, $R_h/d_{50} \leq 10$, one should take (see *Graf* et *al.*, 1987) :

$$\sqrt{\frac{8}{f}} = 5.75\log\left(\frac{R_h}{d_{50}}\right) + 3.25 \tag{3.13c}$$

where $k_s \equiv d_{50}$, with d_{50} as the median grain diameter.

7° In rough channels of large width, $R_h \approx h$, the friction coefficient, f, can be obtained making in-situ measurements of point velocities (see *Graf*, 1966) and assuming a logarithmic distribution (see sect. 2.63) :

$$\frac{u}{u_*} = 5.75\log\frac{30\,z}{k_s}$$

The point velocities, $u_{0.2}$ and $u_{0.8}$, at two elevations, $z' = 0.2\,h$ and $z' = 0.8\,h$, situated on the same vertical (see Fig. 1.7), are given by :

$$u_{0.8} = 5.75\,u_*\log(24h/k_s) \qquad u_{0.2} = 5.75\,u_*\log(6h/k_s)$$

By elimination of u_* in these two relations and putting $(u_{0.8}/u_{0.2}) = \zeta$, one gets :

$$\log\frac{h}{k_s} = \frac{0.78\zeta - 1.38}{1 - \zeta} \tag{3.19}$$

The average velocity, U , for a turbulent rough flow (see eq. 2.63) was given by :

$$\frac{U}{u_*} = 5.75 \log \frac{h}{k_s} + 6.25$$

A substitution of eq. 3.19 into this equation – making use of the definition of eq. 3.8 – renders :

$$\frac{U}{u_*} = \frac{1.78\,(\zeta + 0.95)}{(\zeta - 1)} \tag{3.20}$$

Subsequently one obtains for the coefficient of friction :

$$\sqrt{\frac{1}{f}} = \frac{1.78}{\sqrt{8}}\;\frac{(\zeta + 0.95)}{(\zeta - 1)} \tag{3.21}$$

and also (see eq. 3.13a) :

$$C = \sqrt{8g}\sqrt{\frac{1}{f}} = 1.78\sqrt{g}\;\frac{(\zeta + 0.95)}{(\zeta - 1)} \tag{3.21a}$$

The coefficients of friction, f or C , are thus obtained in an experimental way for very wide channels, where $h \cong D_h = R_h$, using the hypothesis of a logarithmic velocity distribution.

3.2.2 Coefficient of Chézy

1° For turbulent, rough flow the *formula of Chézy* :

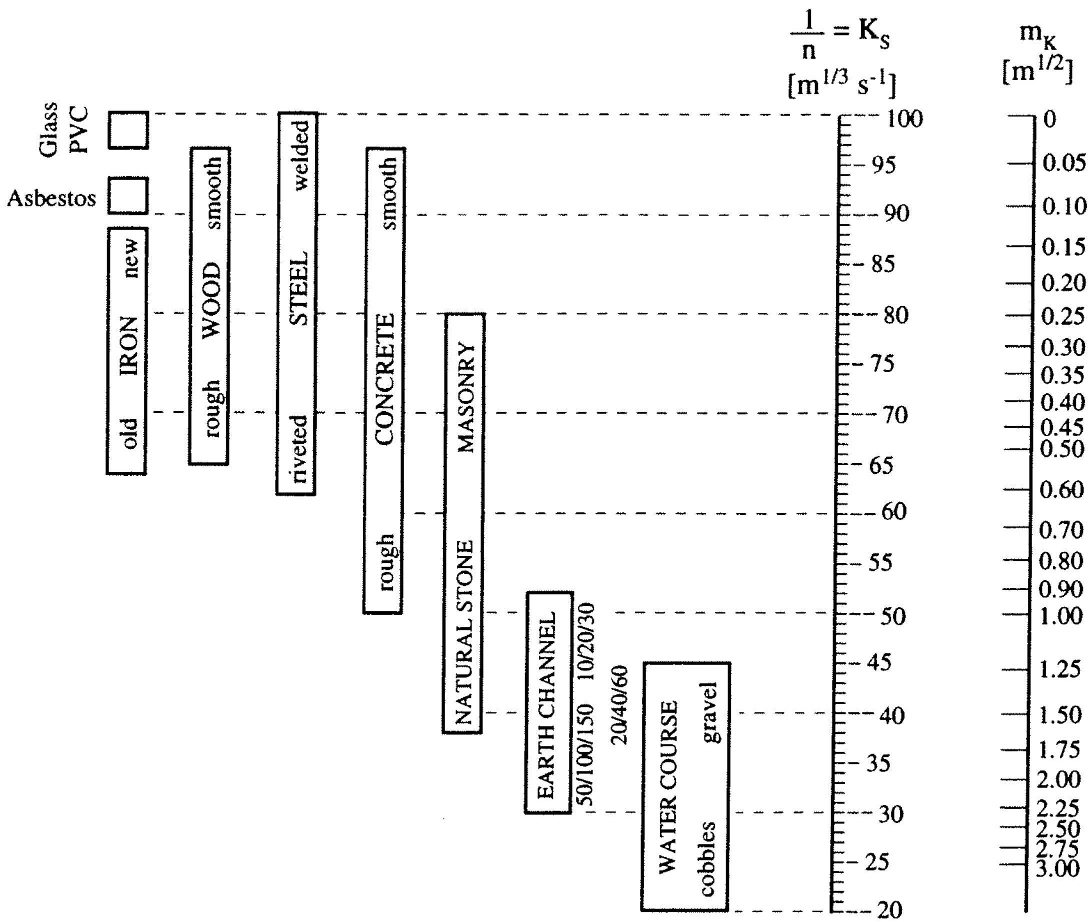
$$U = C\sqrt{R_h S_f} \tag{3.11}$$

can be used. However, it cannot be used for laminar or turbulent smooth flow.

The coefficient of Chézy, C $[m^{1/2}/s]$, is a dimensional expression; the numerical values use as unity the meter [m] and the second [s].

Different formulae, being all of empirical nature, have been advanced for the determination of the coefficient of Chézy, C ; all of which make use of the hydraulic radius, R_h .

Table 3.2 Coefficients of roughness of Manning, of Strickler and of Kutter.



2° The *formula of Bazin* considers C as being a function of the hydraulic radius, R_h [m], and of a coefficient, m_B $[m^{1/2}]$, which characterises the roughness of the walls and the bed. Established with data from small artificial channels, this relation reads :

$$C = \frac{87}{1 + \left(m_B / \sqrt{R_h} \right)} \tag{3.14}$$

The coefficient of Bazin varies from $m_B = 0.06$, for a smooth bed, to $m_B = 1.75$, for a bed made up of stones or covered with vegetation.

3° The *(simplified) formula of Kutter,* established with data from artificial channels as well as from larger rivers, has a similar form, being :

$$C = \frac{100}{1 + \left(m_K / \sqrt{R_h} \right)} \tag{3.15}$$

where m_K $[m^{1/2}]$ is the coefficient of Kutter. Some values for m_K are given in Table 3.2.

4° In the practice, one prefers presently the exponential relations and one uses commonly the *formula of Manning-Strickler* in the form of :

$$U = K_s R_h^{2/3} S_f^{1/2} \tag{3.16}$$

with

$$C = K_s R_h^{1/6} = \frac{1}{n} R_h^{1/6} \tag{3.17}$$

Here K_s $[m^{1/3}s^{-1}]$ is the coefficient of *Strickler* and n $[m^{-1/3}s^{1}]$ is the coefficient of *Manning*. Above relation, eq. 3.16, was elaborated using numerous measurements, performed in both natural and artificial channels. The values of n and K_s are given in Table 3.2. More detailed tables are available (see *Wallisch* 1990, pp. 252-267).

5° There exist other exponential relations, such as :

i) *formula of Forchheimer* : $C = \frac{1}{n} R_h^{1/5}$

ii) *formula of Pavloski* (see *Grishanin*, 1990, p. 45): $C = \frac{1}{n} R_h^{q}$

for $R_h \leq 1$ [m] : $q = 1.5\sqrt{n}$

$R_h > 1$ [m] : $q = 1.3\sqrt{n}$

3.2.3 Coefficient of Manning

1° The most popular formula is presently the one of *Manning-Strickler*, often called shortly the *formula of Manning* :

$$U = \frac{1}{n} R_h^{2/3} S_f^{1/2} \tag{3.16}$$

This is a rather simple relationship, but it must be used only for turbulent, rough flow, thus for flow at large Reynolds numbers. In such a case, the coefficient of Manning, n , stays constant for a given roughness, while the coefficient of Chézy, C , depends (see eq. 3.17) on the relative roughness, $(R_h^{1/6}/n)$.

2° Complete tabulations of the coefficient of Manning, n , have been presented by *Crause* (1951, p. 38), *Chow* (1959, pp. 110-113) and *Graf* (1984, pp. 306-309). Furthermore, *Chow* (1959, pp. 115-123) and *Barnes* (1967) provide photos of different natural and artificial channels as a visual support, to facilitate the choice of the coefficient of Manning in the range of 0.012 < n < 0.15.

Indicative values are summarised in Table 3.2.

It must be pointed out that the values of the coefficient of Manning are the same both in the metric and in the English system. In the latter case, one has to use the following relation :

$$C = \frac{1.48}{n} R_h^{1/6} \tag{3.17a}$$

3° For watercourses, where the bed and walls are made up of a non-cohesive granulate, the *formula of Strickler* (see *Strickler*, 1923, pp. 11-15) may be used :

$$K_s = \frac{21.1}{d_{50}^{1/6}} \qquad \text{or} \qquad K_s = \frac{26}{d_{90}^{1/6}} \tag{3.18}$$

where d_{50} or d_{90} [m] are the diameters, being equal to 50% or 90% of the grains in the granulometric curve.

4° The influence of vegetation on the coefficient of friction is extensively treated by *Chow* (1959, pp. 179-184) and *Wallisch* (1990, p. 229).

3.2.4 Composite Roughness

1° The coefficients of friction, f, n and C , are valid as long as the entire wetted perimeter has the same roughness; thus the wetted section is homogeneous.

2° In sections where the wetted perimeter is *not homogeneous*, the bed and the side walls have different roughnesses (see Fig. 3.3); thus it becomes necessary to compute an equivalent coefficient of friction.

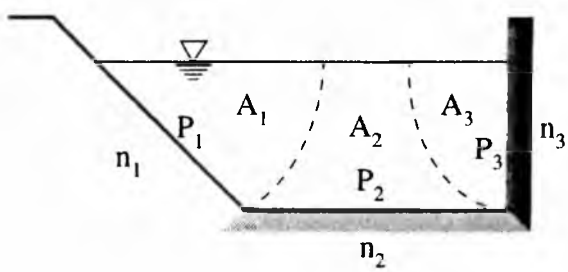


Fig. 3.3 Section of composite roughness.

3° According to *Einstein* (see *Chow*, 1959, p. 136), one divides – in a reasonable way – the wetted surface, A , in N parts, each one having its wetted perimeter, P_1 , P_2 ......P_N , and its coefficient of friction, n_1 , n_2 .....n_N. Furthermore, one assumes that the average velocity of each particular section, A_1 , A_2 .....A_N, is the same and thus also the same as the average velocity of the entire section, $U_1 = U_2 = \ldots\ldots\ldots = U_N \equiv U$.

4° Using, for example, the formula of Manning, eq. 3.16, on writes :

$$U = \frac{1}{n}\left(\frac{A}{P}\right)^{2/3} S_f^{\,1/2} = \frac{1}{n_1}\left(\frac{A_1}{P_1}\right)^{2/3} S_f^{\,1/2} = \ldots\ldots\ldots\ldots = \frac{1}{n_N}\left(\frac{A_N}{P_N}\right)^{2/3} S_f^{\,1/2}$$

If one assumes that $A^{2/3} = \sum_1^N A_N^{\,2/3}$, the equivalent coefficient of friction for a composite roughness can be computed, being :

$$n = \left[\frac{\sum_1^N \left(P_N\, n_N^{\,3/2}\right)}{P}\right]^{2/3} \qquad (3.22)$$

3.2.5 Bed Forms

1° Natural, but also artificial channels may have a *mobile bed*, defined as being a channel bed composed of solid particles (non-cohesive granulate, alluviums), which displace themselves by the action of the flow. The bed may become covered with *bed forms*, commonly called *dunes* (see Fig. 3.4). These solid particles are characterised by the density, ρ_s , the median diameter, $d \equiv d_{50}$, and their granulometric distribution.

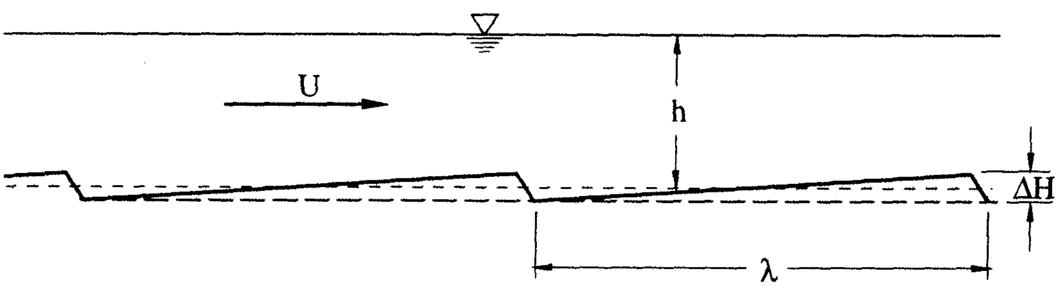


Fig. 3.4 Scheme of a channel bed with a series of dunes.

2° A mobile bed presents successively various aspects, which correspond to the different types of bed deformations. These are usually classified into three regimes, by using the Froude number, Fr (see Fig. 3.5) :

i) $Fr < 1$: The bed remains rather flat, and this till the velocity becomes critical (see sect. 3.4.2) and the sediment (solid) transport begins. Consequently, *mini-dunes* or ripples appear, followed by the *dunes* of growing dune length, λ .

Regime	Transport of sediments	Bed form
Fr < 1	no	flat
	yes	mini-dune
		dune
Fr ≅ 1	yes	flat
Fr > 1	yes	anti-dune

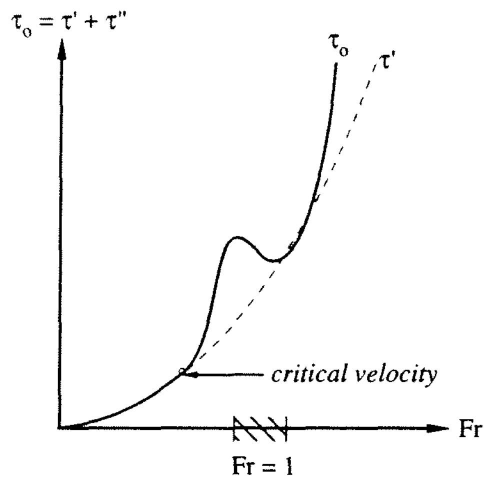


Fig. 3.5 Regime of flow over a mobile bed.

ii) Fr ≈ 1 : As the flow velocity increases, these dunes, already rather long, are washed out and tend to disappear. In this state of *transition*, the bed is once more a flat one.

iii) Fr > 1 : With a further increase of the flow velocity, another kind of dunes appear, commonly called *anti-dunes*, which, contrary to the dunes, travel usually into the upstream direction. The water surface becomes wavy and the sediment transport is very strong.

3° The geometry of a dune (idealised, since sometimes they are not well apparent) is approximated by a triangular form of length, λ, and of height, ΔH (see Fig. 3.4).

Indicatives relations (see *Graf*, 1984, p. 283), made dimensionless by the flow depth, are given as :

$$\frac{\Delta H}{h} < \frac{1}{6} \qquad ; \qquad \frac{\lambda}{h} \approx 5 \qquad (3.23)$$

4° The presence of bed forms will cause an increase in the flow resistance. For the calculation of the total shear stress on the bed, τ_o , one assumes (see *Graf*, 1984, p. 303) that the contribution of the roughness due to the particles, τ' , and the one due to the bed forms , τ'' , is additive, namely :

$$\tau_o = \tau' + \tau'' \qquad (3.24)$$

or (see eq. 3.5) :

$$\gamma R_h S_f = \gamma (R_h' + R_h'') S_f$$

where R_h' and R_h'' are the hydraulic radius due to the particle roughness and to the bed forms, respectively.

Using the definition of the friction velocity and of the coefficient of friction, eq. 3.6 and eq. 3.8, one writes :

$$u_*^2 = (u_*')^2 + (u_*'')^2$$
$$f = f' + f'' \qquad (3.25)$$

but also :

$$n = n' + n'' \quad ; \quad C = C' + C'' \qquad (3.26)$$

5° The total shear stress, τ_0 , (see eq. 3.24) varies as a function of the Froude number, Fr . This variation is schematically shown in Fig. 3.5.

3.2.6 Coefficient of Friction, mobile Bed

1° A quantification of friction coefficient for flow over a mobile bed has, up to now, not been very successful over a large range of flow parameters.

2° There exist methods where one determines directly the *entire* coefficient of friction, f or n .

There exist other methods, where one calculates the coefficient of friction due to the grain roughness, f' or n', using the formulae presented above (see sects. 3.2.1 to 3.2.3). Subsequently one determines the coefficient of friction due to the bed forms, f'' or n'', using other types of formulae.

3° A selection from the different existing formulae for a direct calculation is given in the following :

i) The determination of the entire coefficient of friction can be done using an exponential relation of the Chézy type (see eq. 3.11) :

$$U = K_T R_h^{x} S_f^{y} \qquad (3.27)$$

Sugio (1972) studied extensively watercourses, having $0.1 < d_{50}$ [mm] < 130 , and artificial channels, having $0.2 < d_{50}$ [mm] < 7.0 ; proposed was :

$$U = K_T R_h^{0.54} S_f^{0.27} \qquad (3.27a)$$

It should be noted that the exponent of $y = 0.27$ is very different from the one used in the relation of Chézy or of Manning, where $y = 0.50$ for channels with fixed beds.

The values for K_T are :	$K_1 = 54$	for mini-dunes
	$K_2 = 80$	for dunes
	$K_3 = 110$	for the upper regime
	$K_4 = 43$	for rivers with meanders.

This relation, eq. 3.27a, is simple to use and compares itself favourably with other formulae advanced for channels in *regime*, namely in equilibrium (see *Graf*, 1984, chap. 10), which are also presented by *Sugio* (1972, p. 24).

ii) The following relationship, presented by *Grishanin* (1990, p. 59), expresses the coefficient of Chézy, C , used in eq. 3.11, as being :

$$C = 5.25 \left(\frac{Ug}{\sqrt[3]{gv}}\right)^{1/2} \left(\frac{D_h}{B}\right)^{1/6} \tag{3.28}$$

It was established for Russian rivers, having $0.1 < d_{50}\,[\mathrm{mm}] < 0.44$ and $3 \times 10^{-6} < S_f < 2.2 \times 10^{-4}$.

iii) Yet another relationship, presented by *Grishanin* (1990, p. 69) and obtained from different (35) Russian rivers, was given as :

$$U = \frac{1}{M_G^2} \left(\frac{g}{B}\right)^{1/2} D_h \tag{3.29}$$

where M_G is a local non-dimensional invariant, $M_G = 0.91 \pm 0.12$, for channel beds of sand.

iv) Using a large series of artificial (laboratory) channels as well as natural watercourses, having grain diameters of $0.11 < d_{50}\,[\mathrm{mm}] < 1.35$ and bed slopes of $3 \times 10^{-6} < S_f < 3.7 \times 10^{-2}$, the following relation (see eq. 3.24) was proposed by *Brownlie* (1983, p. 975) :

$$\tau_* = \frac{\tau_0}{d_{50}\,(\gamma_s - \gamma)} = w\ (q_* S_f)^x\ S_f^y\ \sigma^z \left(\frac{\rho}{\rho_s - \rho}\right) \tag{3.30}$$

where $q_* = q/\sqrt{g d_{50}^3}$; q is the unit discharge, σ the standard deviation of the grains in the granulometric distribution and $\gamma_s = \rho_s g$ is the specific weight of the granulate. The coefficients were obtained by a statistical analysis, being :

- for channels with mobile beds, having mini-dunes and dunes :

  $w = 0.37 \quad , \quad x = 0.65 \quad , \quad y = 0.09 \quad , \quad z = 0.11$

- for channels with mobile beds, being flat (see Fig. 3.5) or having anti-dunes :

  $w = 0.28 \quad , \quad x = 0.62 \quad , \quad y = 0.09 \quad , \quad z = 0.08$

4° Different empirical relationships have been elaborated for the calculation of the friction velocity and of the coefficient of friction, u_*'' and f'', being due to bed forms. Here are given two relations :

i) The relation proposed by *Einstein-Barbarossa* is given usually in graphical form (see Fig. 3.6), where a large spread is evident. Used were many observations from American rivers, having $0.19 < d_{35}$ [mm] < 4.3 and $1.49 \times 10^{-4} < S_w < 1.72 \times 10^{-3}$. This relationship is expressed (see *Graf*, 1984, p. 310) by :

$$\frac{U}{u_*''} = f\left(\frac{\rho_s - \rho}{\rho}\,\frac{d_{35}}{R_h'\,S_f}\right) = f(\psi') \tag{3.31}$$

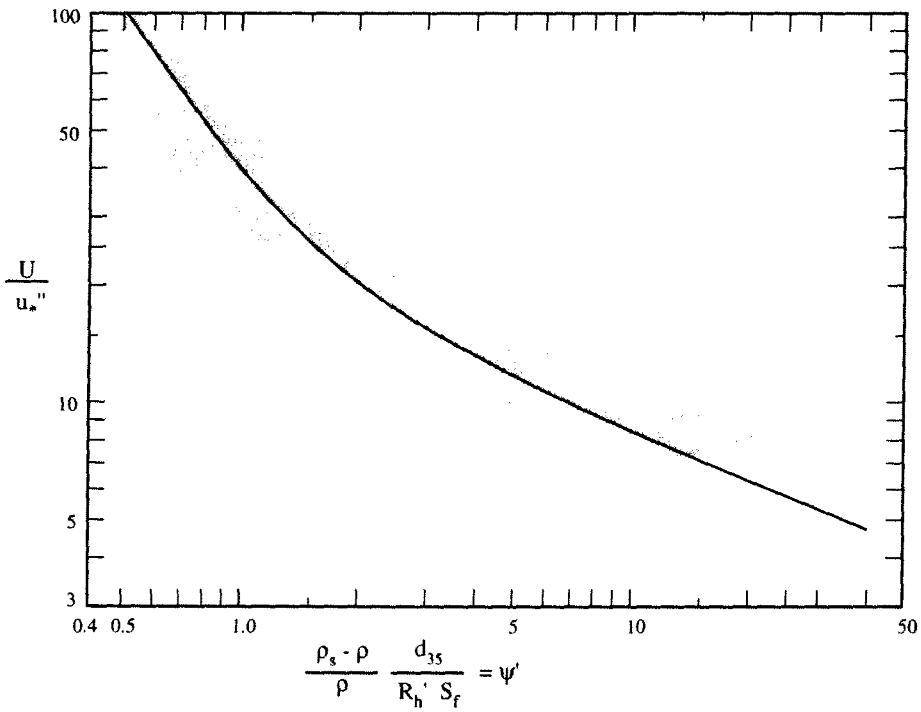


Fig. 3.6 Friction velocity, u_*'', due to bed forms for mobile bed ; after Einstein-Barbarossa.

ii) The relationship proposed by *Alam-Kennedy* is given as :

$$f'' = f\left(\frac{R_h}{d_{50}}\,,\ \frac{U}{\sqrt{g d_{50}}}\right) \tag{3.32}$$

Also this one is usually (see *Yalin*, 1972, p. 280) given in a graphical form (see Fig. 3.7), where a large spread (not shown) is evident. A great many data from artificial (laboratory) channels, having $0.04 < d_{50}[\mathrm{mm}] < 0.54$, and American rivers, having $0.08 < d_{50}[\mathrm{mm}] < 0.45$, have been used.

In this figure, Fig. 3.7, the relation of Einstein-Barbarossa corresponds to the region where the lines of the values of $U/\sqrt{g\,d_{50}}$ stay reasonably horizontal, namely for $R_h/d_{50} > 3 \times 10^3$.

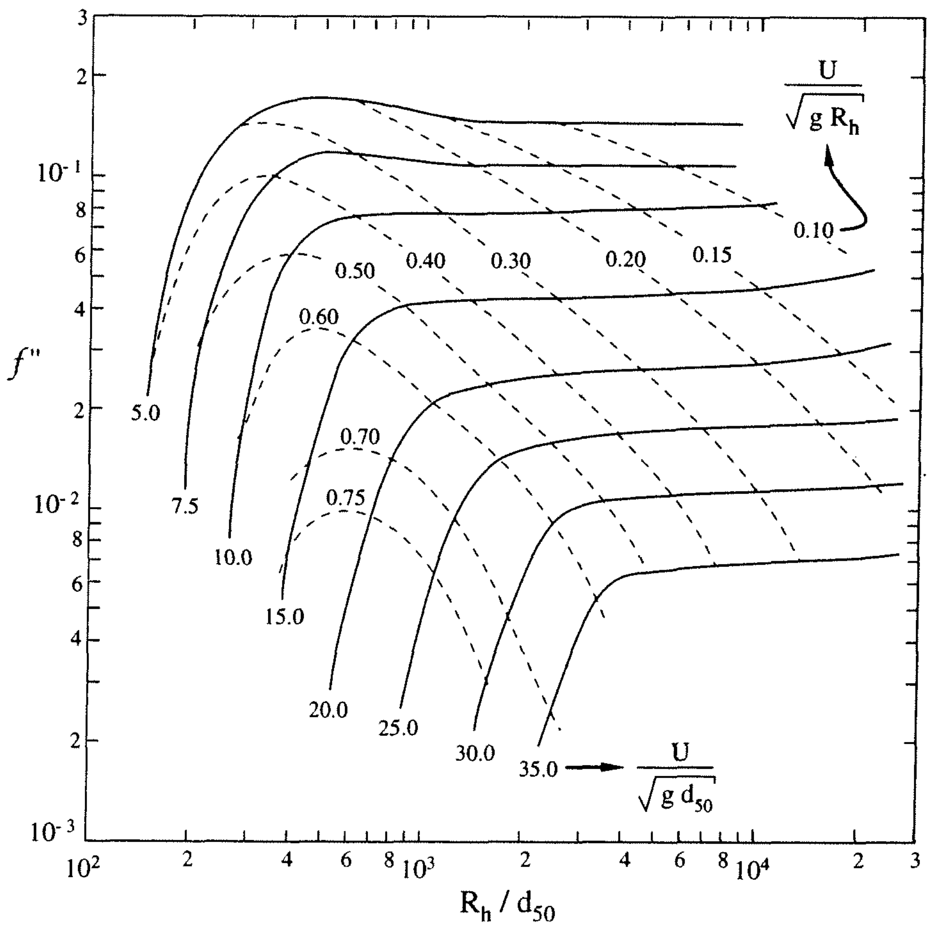


Fig. 3.7 Coefficient of friction, f'', due to bed forms for mobile bed; after Alam-Kennedy.

iii) Some more relations have been presented in *Graf* (1984, pp. 303-320) and

3.3 DISCHARGE CALCULATION, FIXED BED

1° The study of uniform flow in a watercourse is a common task for the hydraulic engineer.

2° Determination of the discharge, Q , in a channel with a fixed bed requires the knowledge of the channel geometry, of the roughness coefficient and of the bed slope.

3° Assumed will be that the walls (bed and side walls) of the channel are fixed or immobile, thus not subject to erosion.

3.3.1 Conveyance

1° The discharge, Q , at uniform flow, given by eq. 3.2a, using the corresponding velocity, given by eq. 3.16, can be expressed as :

$$Q = UA = \frac{1}{n} R_h^{2/3} S_f^{1/2} A \tag{3.33}$$

2° The values of the wetted section, A , and of the hydraulic radius, R_h , are determined and given by the flow depth, h. Furthermore, the nature of the wall roughness, n , is taken to be known. The following expression can be formed:

$$K(h) = \frac{1}{n} R_h^{2/3} A \tag{3.34}$$

known as the *conveyance* of the channel (see *Bakhmeteff*, 1932, p. 13), being only a function of the flow depth, $h \equiv h_n$. This depth is known as the *normal depth*, h_n , for the given discharge, Q . Thus, above expression, eq. 3.33, yields :

$$Q = K(h)\sqrt{S_f} \tag{3.35}$$

or

$$Q/\sqrt{S_f} = f(h) \tag{3.35a}$$

For a given form (shape) of the section, this relation can be obtained and plotted point by point (see Fig. 3.8). One can readily calculate the conveyance for geometrically simple sections; for complex ones, a graphical solution is necessary.

The normal depth , h_n , increases with the discharge, Q . For identical channels, but having different slopes, S_f , the normal depth increases if the bed slope decreases.

3° The conveyance, K , characterises the channel; it represents a measure of the capacity of water transport through the cross section.

The curve of the normal depths (see Fig. 3.8) will be found to be rather useful in solving different kinds of problems : if two of the three parameters, h_n , Q and S_f, (see eq. 3.35a) are known, the third one can be found; à priori, the roughness of the walls is taken to be known.

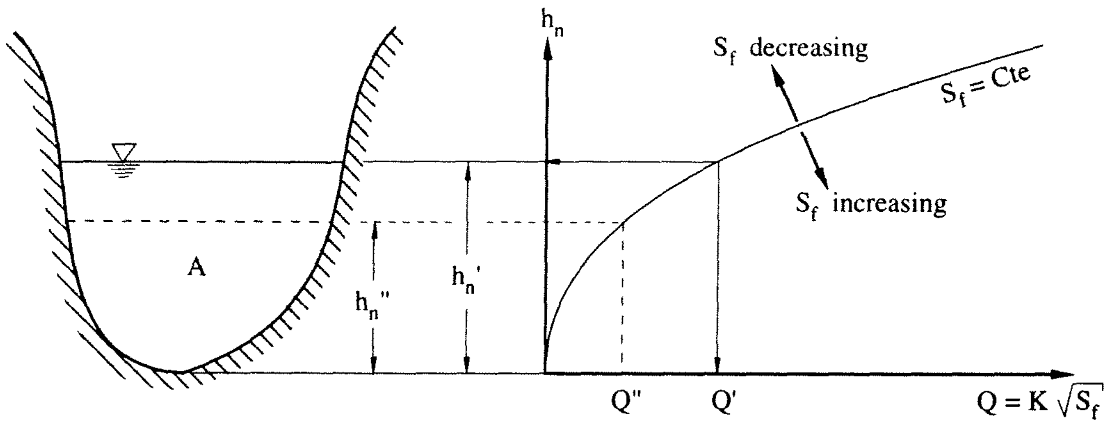


Fig. 3.8 Curve of conveyance or of normal depth.

3.3.2 Normal Depth

1° The normal depth, h_n , (see eq. 3.36) is the flow depth at uniform flow of discharge, Q , at a given bed slope, S_f . (All geometric elements of the cross section, which correspond to the normal depth, h_n, are known as normal elements, such as : R_{h_n} , A_n or P_n .)

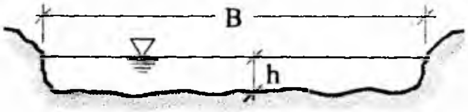
2° The normal depth of a channel of a given geometry is calculated using the relation for the discharge :

$$Q = UA = \frac{1}{n} R_h^{2/3} S_f^{1/2} A \qquad (3.33)$$

This relation shows that uniform flow is only possible in a channel whose bed slope is descending, $S_f > 0$. In a horizontal channel, $S_f = 0$, the normal depth would be infinite.

3° For a natural watercourse and for rectangular channels whose width, B, is very large (see Fig. 3.9), one takes $R_h \approx h$ as the hydraulic radius. The relation of the discharge, eq. 3.33, can now be written as :

$$Q = UA = (C\,h^{1/2} S_f^{1/2})\,(h\,B) \qquad (3.33a)$$



$$h = D_h$$
$$B >> h$$
$$R_h = \frac{B\,h}{B + 2h} \approx \frac{B\,h}{B} \approx h$$

Fig. 3.9 Section of a channel, having a large width.

For the normal or uniform depth, $h \equiv h_n$, one obtains :

$$h_n = \left(\frac{q^2}{C^2 S_f}\right)^{1/3} \qquad \text{where } q = Q/B \tag{3.36}$$

3.3.3 Composite Section

1° A cross section of a channel can be composed of different subsections (see Fig. 3.10), of which each one can have a different roughness and a different bed slope.

This is frequently the case during floods, when the flow leaves the channel and enters into the overflow section of the channel.

2° Such a case can be approximately treated by applying the formula of discharge for each subsection :

$$Q = Q_c + Q_o = \frac{1}{n_c} A_c R_{h_c}^{2/3} \left(\frac{\Delta h}{L_c}\right)^{1/2} + \frac{1}{n_o} A_o R_{h_o}^{2/3} \left(\frac{\Delta h}{L_o}\right)^{1/2} \tag{3.37}$$

Note that the wetted perimeters, P_c and P_o, should be calculated for the lines of contact between water and bed.

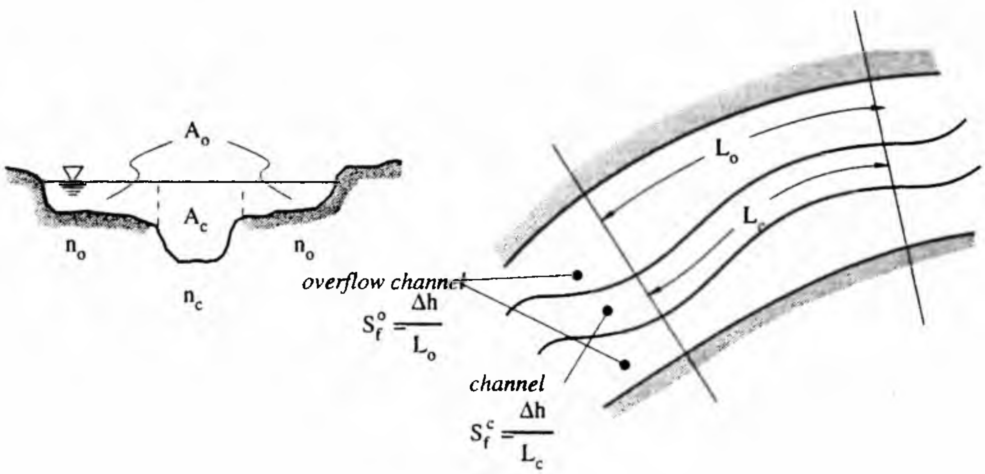


Fig. 3.10 Composite section.

3.3.4 Section of maximum Discharge

1° The construction of a channel with a given slope, S_f, and a given roughness, n, which should convey a certain discharge, Q, will be less expensive if the cross section, A, is the smallest possible.

2° Take the formula of discharge :

$$Q = UA = \frac{1}{n} R_h^{2/3} S_f^{1/2} A \tag{3.33}$$

where for $(S_f^{1/2}/n) = Cte$ one writes :

$$Q = Cte\,(A^{5/3}\,P^{-2/3})$$

For a wetted cross section, A , being constant, the above expressions show that the discharge will be maximal, $Q \Rightarrow Q_{max}$, if the hydraulic radius is maximal, $R_h \Rightarrow R_{hmax}$; thus if the wetted perimeter is minimal, $P \Rightarrow P_{min}$.

3° Amongst all geometrical forms possible, the cross section of a *semi-circular* form will give a P_{min} for a given constant cross section, A . This is given (see Fig. 3.11) by :

$$A = \frac{\pi r^2}{2} \quad , \qquad P = \pi r \quad , \qquad R_h = \frac{r}{2} = \frac{h}{2}$$

The semi-circular form can however be only realised (constructed) with artificial channels, made of metal, concrete or wood.

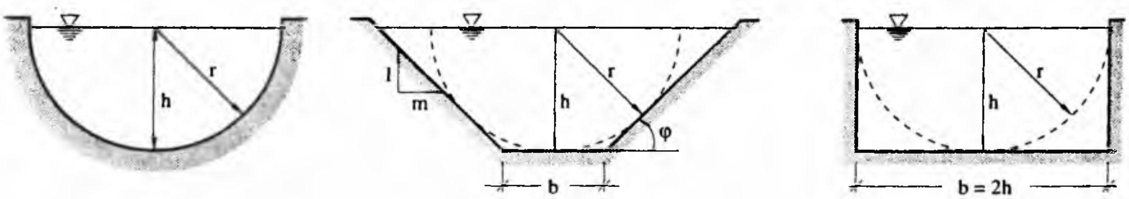


Fig. 3.11 Sections of maximum discharge.

4° For channels in an alluvium, one should also take into account the angle of repose, φ , as well as various constraints due to construction. Consequently, a *trapezoidal* form (see *Crausse*, 1951, p. 51) may be the most reasonable one (see Fig. 3.11) ; where one defines the wetted section, A, and the perimeter, P, such as (see Table 1.1) :

$$A = h\,(b + mh) \quad , \qquad P = b + 2h\sqrt{1 + m^2} \qquad \text{where } m = \text{ctg}\,\varphi \,.$$

Subsequently, one takes dA as being zero, since the section, A , remains constant :

$$dA = h\,db + (b + 2\,mh)\,dh = 0$$

If one puts the wetted perimeter, P , as being minimal, this yields :

$$dP = db + 2\sqrt{1 + m^2}\,dh = 0$$

By elimination of db and dh in above equations, one gets :

$$b = 2h\,(\sqrt{1 + m^2} - m)$$

This value, b, can be put in above relations for A and for P, and one obtains an expression for the hydraulic radius, or :

$$R_h = h/2$$

which remains independent from the angle of repose, φ .

5° It should be remarked that for m = 0 the trapeze becomes a rectangle (see Fig. 3.11) such as :

$$b = 2h$$
$$R_h = h/2$$

For a *rectangular* channel, where $b \equiv B$, the ratio width/depth must be $(B/h) = 2$.

3.4 DISCHARGE CALCULATION, MOBILE BED

1° Artificial and natural channels, whose flow moves in an alluvium, composed of a (non-cohesive) granulate, are channels of *mobile bed*. The discharge will be calculated using the coefficient of roughness for a mobile bed (see sect. 3.2.6).

2° In such channels, the velocity (in the vicinity of the bed) should :

i) not be superior to a certain critical value, otherwise there is a risk of erosion of the solid particles on the bed : this is the permissible maximum velocity or *velocity of erosion*, usually also called the *critical velocity;*

ii) not be inferior to a certain critical value, otherwise there is a risk of deposition or sedimentation of the solid particles which are possibly suspended in the flow : this is the permissible minimum velocity or *velocity of sedimentation.*

3° The flow velocity, U , to be selected for a 'good functioning' of the channel, must lie between the velocity of erosion, $U_E \equiv U_{cr}$, and the one of sedimentation, U_D :

$U_D < U < U_E$.

4° It is evident in Fig. 3.12, that the two velocities, U_D and U_E, will have distinctly different values .

3.4.1 Sedimentation Velocity

1° The allowable minimum velocity or velocity of sedimentation, U_D , is the minimum velocity which is necessary to transport the flow containing solid particles in suspension.

2° Recommended (see *Chow*, 1959, p. 158 and *Crausse*, 1951, p. 16) is to take the following approximate values :

$0.25 < U_D$ [m/s] < 0.9

depending on fine or very coarse material.

3° The diagram (see Fig. 3.12) which was established by *Hjulstrom* (see *Graf*, 1984, p. 88) delimits the zone of sedimentation as a function of the diameter of the (monodispersed) granulate.

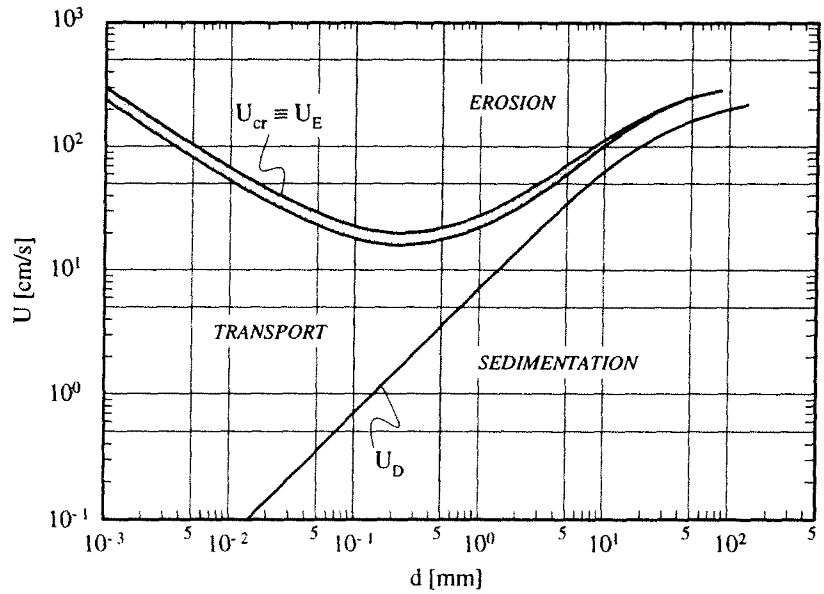


Fig. 3.12 Velocity of sedimentation and of erosion, U_D et U_{cr} , for a uniform granulate,

4° In an experimental study with granulates of $0.49 < d_{50}\,[\mathrm{mm}] < 3.02$, *Graf* et *Pazis* (1977) expressed the critical values by the shear stress, τ_o. It was found that

$$\tau_{o_D} < \tau_{o_E}$$

The difference was however shown to be negligible for this range of granulates; one may thus readily take $\tau_{o_D} \cong \tau_{o_E}$.

3.4.2 Critical Velocity

1° There will be erosion of the bed (and the walls) when one exceeds a certain critical value, expressed with :

i) the average critical velocity, U_{cr}, or the critical velocity, $u_{b_{cr}}$, at or close to the bed,

ii) the critical shear stress, $\tau_{o_{cr}}$.

2° From an hydraulic view point, it is more reasonable to use the shear stress, τ_o, as a criterion of erosion.

The shear stress was earlier defined as :

$$\tau_o = \gamma\, R_h\, S_f \tag{3.5}$$

and the average velocity as :

$$U = C\sqrt{R_h\, S_f} \tag{3.11}$$

This gives the following ratio, showing the relation between the shear stress, τ_o, and the velocity, U, or :

$$\frac{U}{\sqrt{\tau_o/\rho}} = \frac{C}{\sqrt{g}}$$

In fluvial hydraulics, one uses rather often (see *Graf*, 1984, p.91) a dimensionless form of the shear stress, τ_*, or :

$$\frac{\tau_o}{(\gamma_s-\gamma)d} \equiv \tau_* = \frac{\gamma\, R_h\, S_f}{(\gamma_s-\gamma)d} \tag{3.38}$$

where d is the diameter of the granulate (to be specified); γ_s and γ are the specific weight of the granulate and of water respectively. With this relation, one compares the flow parameters , R_h and S_f, with the granulometric parameters, d and (s_s-1).

3° Amongst the different formulae, which one finds in the literature (see *Graf*, 1984, chap. 6), only three will be presented herewith, namely the ones proposed by *Hjulstrom*, by *Neill* and by *Shields*.

4° In an analysis of available data from (monodispersed) uniform granulates, *Hjulstrom* used the average flow velocity, U , instead of the velocity close to the bed, u_b , by assuming that $u_b = 0.4U$. On Fig. 3.12, one can see the limiting zone, where erosion is encountered. This diagram of :

$$U_{cr} = f(d)$$

shows that fine sand ($d \cong 0.1$ mm) is rather easily eroded; the strong resistance to erosion for silt ($d \cong 0.01$ mm) is attributed to the cohesion between the particles.

5° For erosion of a bed composed of a uniform granulate of large diameter, *Neill* proposes the following relation :

$$\frac{\rho\, U_{cr}^{2}}{gd\,(\rho_s - \rho)} = 2.5\left(\frac{d}{D_h}\right)^{-0.2} \tag{3.39}$$

being valid for $0.01 < (d / D_h) < 1.0$.

6° Relying on some concepts of the hydrodynamics, *Shields* developed a relation between the dimensionless shear stress, τ_* (see eq.3.38), and the friction/particle Reynolds number, $Re_* = u_* d/\nu$, such as :

$$\tau_* \equiv \frac{\tau_o}{(\gamma_s - \gamma)\, d} = f\left(\frac{u_* d}{\nu}\right) \tag{3.40}$$

where $u_* = \sqrt{\tau_o/\rho}$. *Shields* has determined the form of this relation, using experimental data. An average curve (see *Graf*, 1984, p. 96), reasonably well defined (despite an important scattering), characterises the begin of erosion, expressed by τ_{*cr} . For the particle diameter, one takes usually $d \equiv d_{50}$. It is to be seen (see Fig. 3.13) that these critical values fall roughly in the range of :

$$0.03 < \tau_{*cr} < 0.06$$

The determination of τ_{*cr} is done using the above relation, eq. 3.40, by successive approximations. It must be underlined that the criterion of *Shields* is of great importance for the hydraulic engineer.

7° Since a direct use of the relation of *Shields*, eq. 3.40, is not a simple one, *Yalin* (1972, p. 82) has proposed an interesting combination of terms, such as :

$$\frac{Re_*^2}{\tau_*} = \frac{d^3 g}{\nu^2} \frac{(\rho_s - \rho)}{\rho}$$

Rather than using the Reynolds number, Re_* , it is now proposed to use a dimensionless diameter of the granulate, given by :

$$d_* = d \left(\frac{\rho_s - \rho}{\rho} \frac{g}{\nu^2} \right)^{1/3}$$

Consequently, above relation, eq. 3.40, can be expressed as :

$$\tau_* = f(d_*) \qquad (3.40a)$$

which is given with Fig. 3.13; one usually takes $d = d_{50}$.

If the properties of the fluid, ρ and ν , and of the granulate, ρ_s and d , are known, one can readily determine the corresponding value of τ_{*cr} and subsequently of $\tau_{o_{cr}}$.

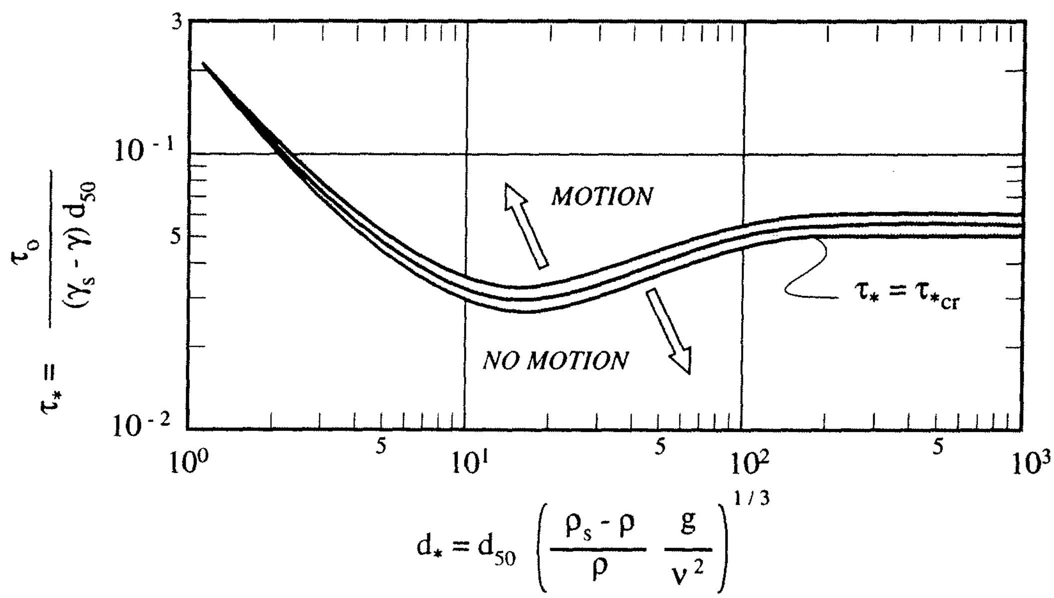


Fig. 3.13 Dimensionless shear stress, τ_* , as a function of the dimensionless diameter, d_* , after Shields-Yalin.

8° For cohesive material, the determination of the critical values, U_{cr} or τ_{*cr}, represents a difficult task; the specialised literature (see *Graf*, 1984, chap. 12, and *Raudkivi*, 1976, chap. 9) should be consulted.

3.4.3 Distribution of Shear Stress

1° The shear stress, τ_o, is given by :

$$\tau_o = \gamma S \frac{1}{P} S_f = \gamma R_h S_f \qquad (3.5)$$

For a channel of large width (see Fig. 3.9), when $R_h \cong h$, one writes :

$$\tau_o = \gamma h S_f \qquad (3.5a)$$

2° However, it must be remarked that the shear stress, τ_o, is distributed over the wetted perimeter, P. A typical distribution for a trapezoidal channel (see *Chow*, 1959, p. 169) is given with Fig. 3.14.

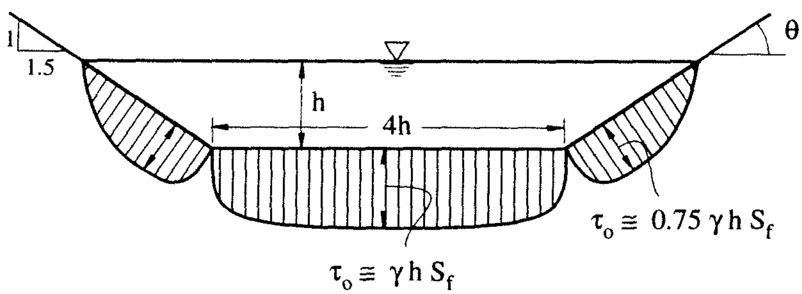


Fig. 3.14 Distribution of the shear stress in a trapezoidal channel.

3° An expression for the shear stress on the channel side walls, $(\tau_{o_{cr}})^w$, was proposed by *Forchheimer* and subsequently by *Lane* (see *Graf*, 1984, p. 116), being of the following form :

$$(\tau_{o_{cr}})^w = \tau_{o_{cr}} \left[\cos\theta \left(1 - tg^2\theta / tg^2\varphi\right)^{1/2}\right] \qquad (3.41)$$

$\tau_{o_{cr}}$ is the critical shear stress on the bed – given for example with Fig. 3.13 – , θ is the inclination of the side wall(s), and φ is the angle of repose. The latter depends on the granulometry and on the cohesion (see *Graf*, 1984, p. 115); it varies such as $20° < \varphi < 40°$. Evidently : $(\tau_{o_{cr}})^w < \tau_{o_{cr}}$, and for stable side walls : $\theta < \varphi$.

3.4.4 Stable Section

1° A *stable* cross section of a channel with a mobile bed, thus erodible, is a section where there is no erosion over the entire wetted perimeter, P.

2° An *ideal stable* cross section – with a maximal discharge and a minimal wetted perimeter – can be calculated with a method advanced by *Glover* et *Lane* (see *Graf*, 1984, p. 119). The form of such a section (see Fig. 3.15) can be determined as follows :

i) Assumed will be that the angle of the side wall at the water surface is identical to the angle of repose, $\theta_s \equiv \varphi$.

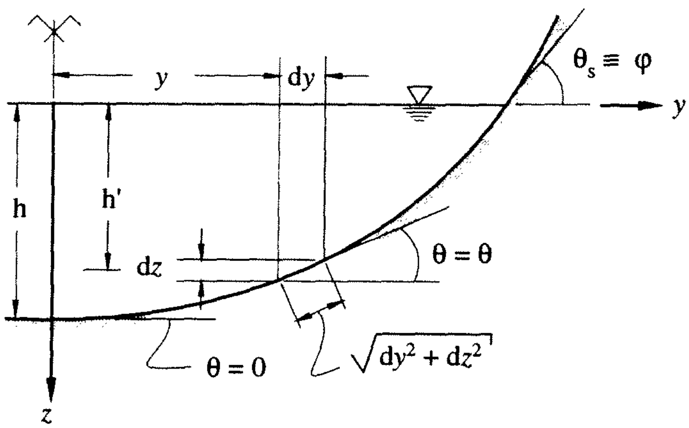


Fig. 3.15 Ideal stable section.

ii) The shear stress on an element on the bed situated at the side wall is :

$$(\tau_o{}^w) = \gamma\, h'\, S_f\, (dy / \sqrt{dy^2 + dz^2}) = \gamma\, h'\, S_f \cos\theta$$

iii) Subsequently, one assumes that this shear stress, $(\tau_o{}^w)$, be critical, $(\tau_{o_{cr}})^w$, over the entire wetted perimeter ; using now the expression of eq. 3.41, one writes :

$$\gamma\, h'\, S_f \cos\theta \Leftarrow (\tau_o{}^w) \equiv (\tau_{o_{cr}})^w \Rightarrow \gamma\, h\, S_f \left[\cos\theta(1 - tg^2\theta / tg^2\varphi)^{1/2}\right]$$

where h is the maximum water depth situated at $y = 0$.

iv) After mathematical manipulations and taking $(dz/dy) = tg\theta$, one obtains :

$$\left(\frac{dz}{dy}\right)^2 + \left(\frac{h'}{h}\right)^2 tg^2\varphi - tg^2\varphi = 0.$$

v) The solution of this differential equation is :

$$h' = h \cos\left(\frac{tg\varphi}{h}\,y\right) \tag{3.42}$$

which gives the geometry of the ideal stable cross section, being *sinusoidal.*

vi) The other hydraulic parameters of such a section are deduced as being :

$$\begin{aligned} A &= 2\,h^2 / tg\varphi \\ B &= \pi\, h / tg\varphi \\ U &= \frac{1}{n}\, S_f^{1/2}\,(h \cos\varphi / E)^{2/3} \end{aligned} \tag{3.42a}$$

with $h = \tau_o / \gamma\, S_f$ and $E(\sin\varphi)$ being an elliptic integral, approximated by $E \approx (\pi/2)\,(1 - \frac{1}{4}\sin^2\varphi)$.

vii) The discharge, Q_i , which *can* be conveyed through this *ideal stable* section, is evidently given by $Q_i = UA$.

3° If the discharge, Q , which *must* be conveyed through such a section is different from the ideal discharge , Q_i , thus $Q \neq Q_i$, a corrective calculation must be done :

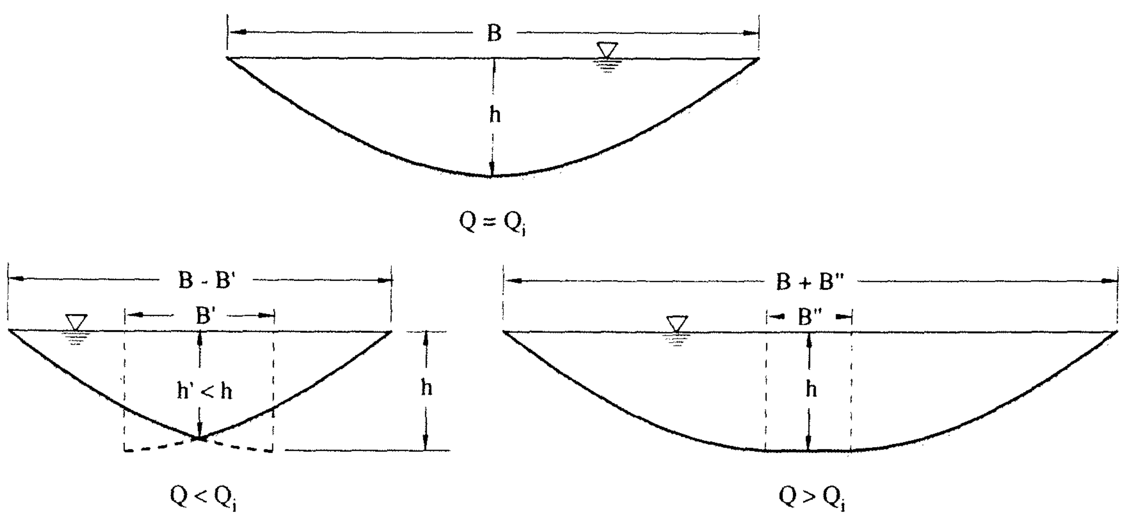


Fig. 3.16 Ideal stable section for different widths.

i) For $Q < Q_i$, the width, B , must be reduced by B', to be computed with :

$$B' = B\,(1 - \sqrt{Q/Q_i})$$

ii) For $Q > Q_i$, the width, B , must be increased by B", to be computed with :

$$B'' = n\,(Q - Q_i)\,/\,(h^{5/3} S_f^{\,1/2})$$

The effect of this change in width on the geometry of the channel section is shown in Fig. 3.16 (see *Chow*, 1958, p. 177).

3.5 FLOW IN CURVES

1° A curve or bend, positioned in a rectangular channel, causes a change in the flow direction.

2° If the discharge, Q , remains constant along the curve, the flow velocity , U , as well as the wetted section, A , remain also constant. The sectional distribution of the flow depth, h(*y*) , will be responsible for a transversal water slope and a super-elevation, Δz , at the outside of the curve.

3° The distribution of velocity in the curve can be approximated by the one of a free vortex (see *Graf & Altinakar*, 1991, p. 196). The velocity has a maximum at the inside of the curve (see Fig. 3.17).

3.5.1 Super-elevation

1° In a curve the streamlines will no longer stay parallel and the flow becomes three-dimensional. This is a complex physical phenomenon, for which an adequate analysis seems difficult.

2° The method proposed by *Kozeny* (1953, p. 223) puts forward the following arguments and this for turbulent flow (see Fig. 3.17) :

i) Assumed is that the head loss, h_r^c , – following a streamline, s , – can be expressed as :

$$S_e = \frac{h_r^c}{L_c} = \lambda\, u_s^2$$

where λ is a factor of proportionality and $L_c = \alpha r$ is the length of the curve; α being the angle and r the radius of the curve. Thus one may write :

$$u_s = \sqrt{\frac{h_r^c}{\lambda\ \alpha}} \cdot \sqrt{\frac{1}{r}} = \frac{\kappa}{\sqrt{r}}$$

ii) For $r = r_o$, one takes $u_s = u_a \equiv U$; this implies that the axial velocity, u_a, is identical to the average velocity, U , of the cross section. One obtains now $u_a \sqrt{r_o} = \kappa$ and an expression for the distribution of the velocity, such as :

$$\frac{u_s}{u_a} = \sqrt{\frac{r_o}{r}}$$

iii) If r_2 and r_1 are the outside and inside radius, the corresponding velocities are given by :

$$u_2 = u_a \sqrt{r_o/r_2} \qquad \text{and} \qquad u_1 = u_a \sqrt{r_o/r_1}$$

iv) The super-elevation (see Fig. 3.17) can now be calculated as being :

$$\Delta z = \frac{u_1^2}{2g} - \frac{u_2^2}{2g} = \frac{u_a^2}{2g}\left(\frac{r_o}{r_1} - \frac{r_o}{r_2}\right)$$

Since $B = (r_2 - r_1)$ is the width of the curve, one can also write :

$$\Delta z = \frac{B\ r_o}{r_1\ r_2}\ \frac{U^2}{2g} \qquad (3.43)$$

If the channel width, B , is small compared to the radius of the curve, r_o , one gets the simplified expression of :

$$\Delta z = \frac{B}{r_o}\ \frac{U^2}{2g} \qquad (3.43a)$$

v) The transversal water profile is convex; one can write :

$$\Delta z_2 = \frac{U^2}{2g}\left(1 - \frac{r_o}{r_2}\right) \qquad \text{and} \qquad \Delta z_1 = \frac{U^2}{2g}\left(\frac{r_o}{r_1} - 1\right)$$

The super-elevation, $\Delta z = \Delta z_2 + \Delta z_1$, given with eq. 3.43 has its maximal value, $\Delta z = \Delta z_{max}$, usually observed for fluvial flow, $Fr < 1$, at the entrance of the curve and for supercritical flow, $Fr > 1$, at the exit of the curve.

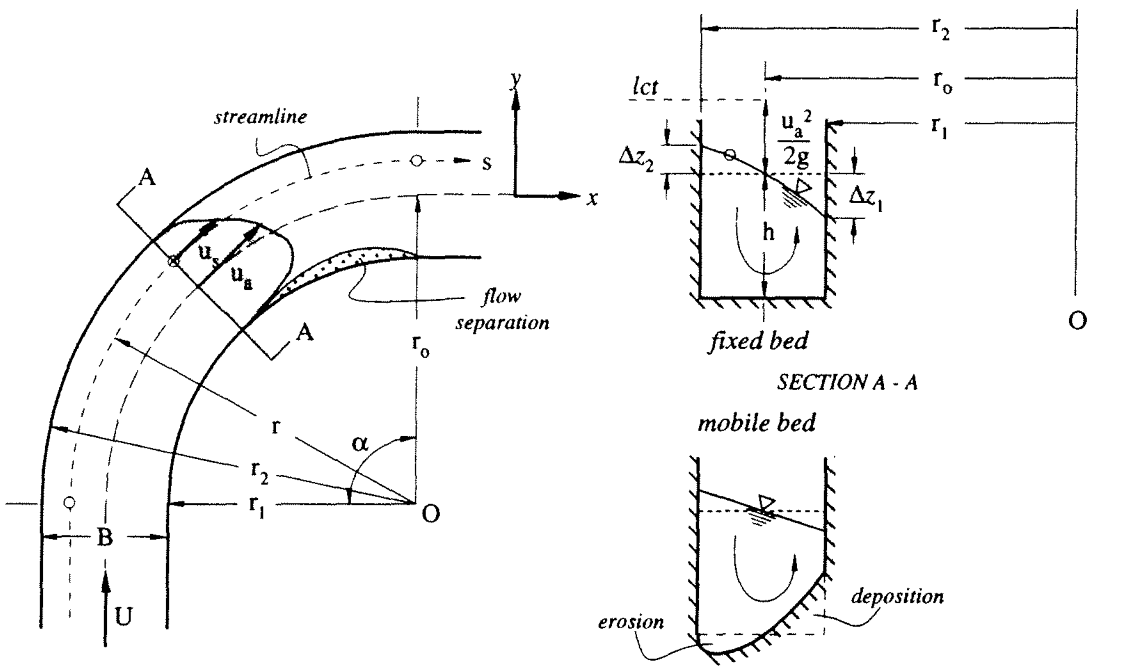


Fig. 3.17 Flow in a curve.

3° One can also define a coefficient of super-elevation by :

$$K^c = \Delta z \; / \, (U^2 \, / \, 2g) \tag{3.44}$$

which, taking eq. 3.43, is : $K^c = B \, r_o \, / \, (r_1 \, r_2)$.

Apmann (1973, p. 73) proposed the following empirical expression, after analysing flow in curves in artificial and natural channels, or :

$$K^c = \frac{5}{4} \; \mathrm{tgh} \left(\frac{r_o \, \alpha}{B} \right) \ln \left(\frac{r_2}{r_1} \right) \tag{3.44a}$$

4° The super-elevation, Δz , can readily be used for the determination of the discharge (see *Apmann*, 1973, p. 70) :

$$Q = A \sqrt{2g \; \Delta z \; / \; K^c}$$

This relation is of great use in the determination of flood discharges, which usually leave traces (marks) at their largest occurring flow depth, thus Δz .

5° The super-elevation at the outside of the curve (see Fig. 3.17) causes a vertical downward current, which comes back to the water surface at the inside of the curve. Such a secondary current will superpose itself on the primary flow and result in helicoidal flow over the entire reach of the curve.

6° If the outside side wall is of mobile material, erosion will take place; on the inside there will be deposition (see Fig. 3.17).

3.5.2 Supercritical Flow

1° Gravity waves (see sect. 2.4.3) will establish themselves in a curve (see Fig. 3.18), notably if the flow is supercritical, $Fr > 1$ (see *Ippen*, 1950, p.563). For a rectangular channel, the celerity of a gravity wave is given by :

$$c^2 = gh \tag{2.27a}$$

2° At the entrance of the curve at the points A and A', one observes under an angle, β:

i) positive perturbations (waves) along the line ABD ,

ii) negative perturbations (waves) along the line A'BC ,

ii) but no perturbations appear in the zone ABA' .

This angle, β , is approximately defined as being :

$$\sin\beta = \frac{c}{U} = \frac{\sqrt{gh}}{U} = \frac{1}{Fr} \tag{3.45}$$

where Fr is Froude number of the flow upstream of the curve.

3° Consequently, the flow depth varies :

i) increasingly along the line AC, having a maximum at C ,

ii) decreasingly along the line A'D, having a minimum at D .

The maximum (+) or minimum (-) flow depth can be calculated (see *Ippen* 1950, pp.551 and 564) as being :

$$h_{\substack{max \\ min}} = h\, Fr^2 \sin^2(\beta \pm \theta / 2) \tag{3.46}$$

The central angle, θ , is determined using geometrical considerations, or :

$$\mathrm{tg}\,\theta = \frac{B}{(r_o + B/2)\,\mathrm{tg}\,\beta} \tag{3.46a}$$

4° These maxima/minima are then reflected from one to the other side in the curve, such as is indicated in Fig. 3.18. The next maxima/minima appear after an interval of 2θ. This swinging, referred to as *cross waves*, can continue well beyond the end of the curve.

5° The maximum super-elevation, $(\Delta z' + \Delta z)$, due to the gravity waves, can be twice the super-elevation, Δz, obtained with eq. 3.43. It is calculated by :

$$\Delta z' = \frac{B}{r_o}\frac{U^2}{2g} \tag{3.47}$$

The water surface and the resulting transversal slopes are shown schematically with Fig. 3.18.

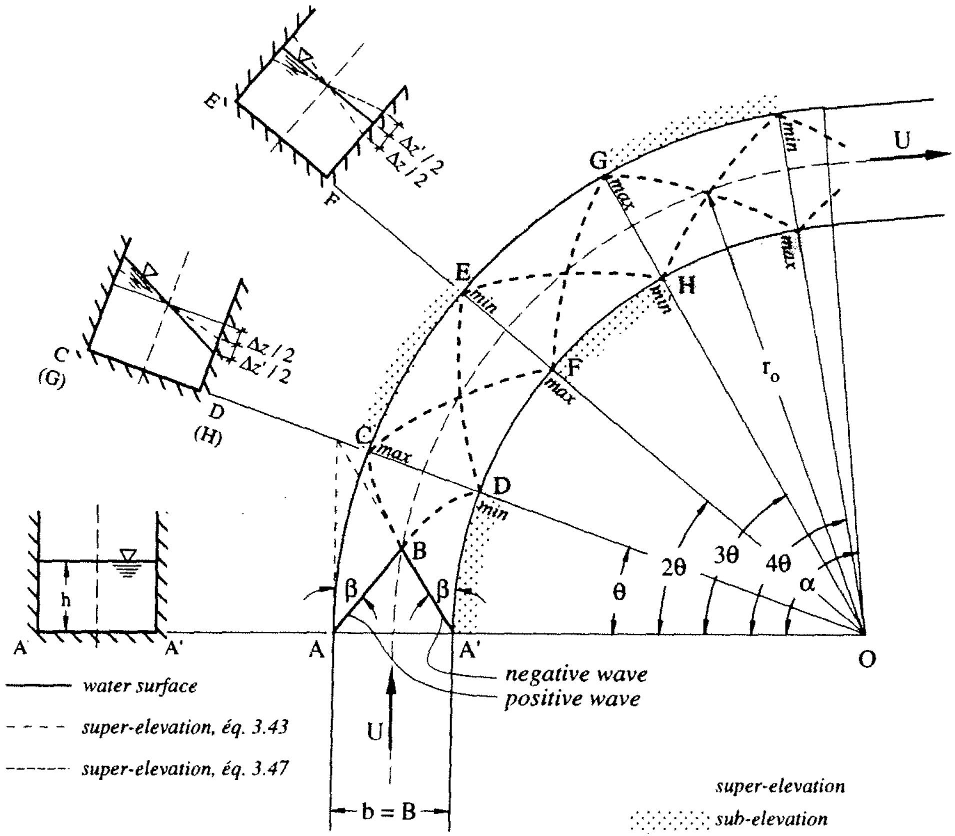


Fig. 3.18 Supercritical flow in a curve.

6° If the super-elevations get very large, methods are available to suppress it (see *Naudascher*, 1987, p.206), like the construction of transition curves or the installation of steps on the channel bed.

3.5.3 Head Loss

1° In flow over a curve, one encounters not only a head loss due to friction, h_r, but also one due to the curvilinear flow, h_r^c (see sect. 3.5.1).

2° This additional head loss is usually expressed by :

$$h_r^c = \zeta_c \frac{U^2}{2g} \tag{3.48}$$

where ζ_c is a coefficient which depends on :

$$\zeta_c = f(\text{Fr}, \ \text{Re}, r_0 / B, h / B, \alpha)$$

α being the angle of the curve, Fr and Re are the number of Froude and of Reynolds, respectively. According to numerous experiments, one takes (see *Chow*, 1958, p. 443) :

$$0.1 \leq \zeta_c \leq 1.1$$

where the larger values of ζ_c are for curves of $r_o / B = 0.5$.

3.6 INSTABILITY AT SURFACE

1° If the channel slope is very high and/or if the flow is supercritical, the water surface can become unstable. The normal flow depth, h_n, must now be considered as an average value.

2° Such an instability is characterised by :

i) a series of gravity waves of small flow depth, eq. 2.27, called *roll waves*, progressing downstream, and

ii) a breaking of these waves, causing an *air entrainment*.

3.6.1 Roll Waves

1° An instability at the water surface is evidenced by the formation of roll waves. The uniform steady flow becomes locally an unsteady one.

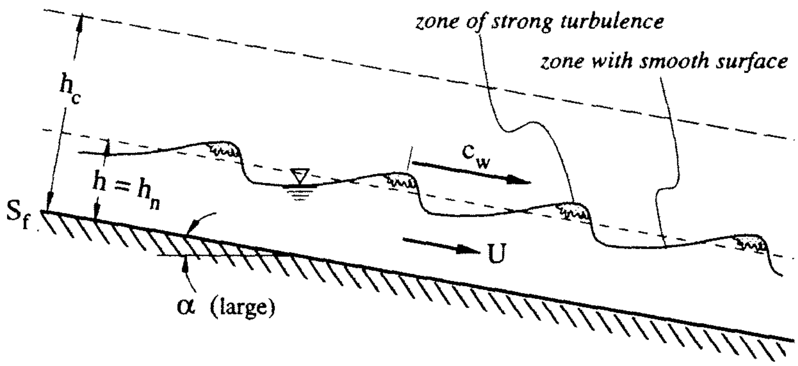


Fig. 3.19 Roll waves.

2° The roll waves are superposed on the uniform flow (see Fig. 3.19). They displace themselves towards the downstream – increasing in height and then collapsing – with an absolute celerity, c_w, being larger than the flow velocity, U, or :

$$c_w = U + \sqrt{gh} \quad > \quad U$$

3° There is no simple criteria available to determine the geometrical dimensions of this type of waves. Their height can, however, attain dimensions of the order of magnitude of the prevailing flow depth (see *French*, 1986, p. 625).

The crests of the roll waves are zones of strong turbulence, while the rest of the waves remains remarkably smooth.

4° Some theoretical considerations for a determination of the (in)stability of uniform flow are presented in *Liggett* (1975, chap. 6).

5° However, there exist useful practical criteria to determine the instability, which is responsible for the creation of roll waves. The geometry of the channel and the type of flow are taken in consideration.

i) Taken is the Froude number, $Fr = U / \sqrt{gh}$; the flow of a large channel is unstable (see *Albertson* et *al.*, 1960, p.355), if :

$Fr > 2$ for turbulent, rough flow,

$Fr \geq 1.5$ for turbulent, smooth flow,

$Fr \geq 0.5$ for laminar flow.

ii) Taken is the number of Vedernikov (see *Chow*, 1959, p. 210), defined as :

$$Ve = x\, f_g\, Fr \tag{3.49}$$

where x is an exponent of the hydraulic radius, R_h, in eq. 3.11 ($x = 2$ for laminar flow; $x = 2/3$ or $x = 1/2$ for turbulent flow, taking the relation of Manning or of Chézy, respectively). f_g is a shape factor given by :

$$f_g = 1 - R_h \frac{dP}{dS}$$

being $f_g = 1$ for very wide channels and $f_g = 0$ for very narrow channels. If :

$$Ve < 1$$

flow can remain stable and any wave at the surface will be depressed. If :

$$Ve \geq 1$$

stable flow is impossible; unsteady flow will prevail and existing waves will amplify and form the roll waves.

6° Roll waves form themselves not only in uniform flow, but are also encountered in non-uniform flow.

3.6.2 Air Entrainment

1° For large channel slopes, S_f, – such as exist also on the downstream face of a weir – the flow is usually supercritical and gravity waves appear at the water surface. These waves will break and entrain air into the water. The turbulence will diffuse (mix) the air bubbles across the entire flow depth; and water droplets will escape into the air.

2° In flow of such an air-water mixture, it becomes a bit difficult to define the flow depth; the water surface is often covered by *white water.*

3° The schematic distribution of the concentration of air :

$$C(z) = \frac{\text{volume (air)}}{\text{volume (air + water)}}$$

is given with Fig. 3.20. Two regions are to be distinguished : bubbles in the water and droplets in the air.

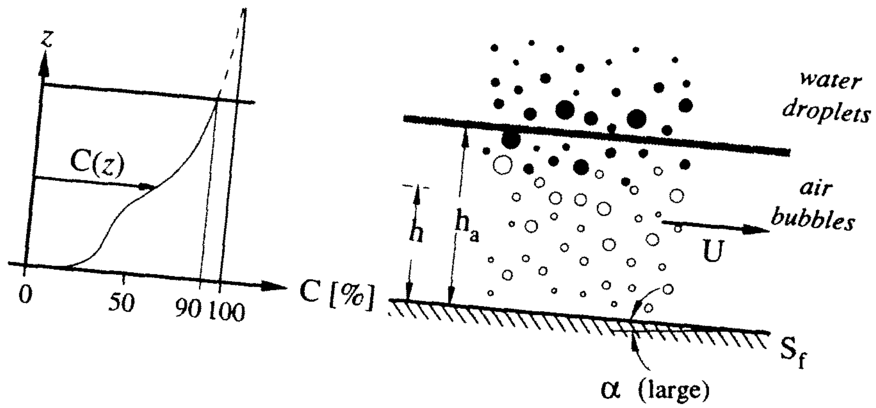


Fig. 3.20 Flow with air entrainment.

4° The equivalent flow depth of water (without the air) is defined by :

$$h = \int_0^{\infty} (1-C)\, dz \tag{3.50}$$

and the average velocity of water by :

$$U = q/h$$

where q is the unit discharge of the water.

The depth of the mixture, which is the depth where the concentration is equal to C = 90%, is given (see *Wood*, 1985, p.21) by :

$$h_a = \frac{h}{(1 - \bar{C})} \tag{3.51}$$

where $\bar{C}$ is the average concentration in the cross section, given (see *Henderson*, 1966, p. 185) by the relation :

$$\bar{C} = 0.7 \log (S_f / q^{1/5}) + 0.9 \tag{3.52}$$

which was obtained from experimental studies performed by *Straub* et *Anderson* for $0.14 < q[m^2/s] < 0.93$. Usually (see *Wood*, 1985, p.21) one takes :

$\bar{C} = 0.14$ for slopes of $S_f = 7.5°$
$\bar{C} = 0.71$ for slopes of $S_f = 75°$

5° The air entrainment, or the aspiration of air on the downstream face of a weir, begins at a point where the boundary layer is completely developed, $\delta \equiv h$ (see *Wood*, 1985, p. 18).

3.7 EXERCISES

3.7.1 Problems, solved

Ex. 3.A

A channel is to be built of medium-quality concrete to convey a discharge of $Q = 80$ [m^3/s]. The channel should have a trapezoidal cross section with a bottom width of $b = 5$ [m] and side slopes of $m = 3$. The channel slope will follow the geomorphology of the terrain which is determined to be $S_f = 0.1$ %. It is admitted that the flow is uniform and the water has a temperature of $T = 10$ [°C].

i) Calculate the flow depth using both the coefficient of Manning and the coefficient of Weisbach-Darcy.

ii) Verify whether the flow is laminar or turbulent and subcritical or supercritical.

SOLUTION :

The geometrical characteristics of a trapezoidal cross section are (see Table 1.1) :

wetted surface : $A = (b + mh)\,h = (5 + 3h)\,h$

hydraulic radius : $R_h = \dfrac{(b + mh)\,h}{b + 2h\sqrt{1+m^2}} = \dfrac{(5 + 3h)\,h}{5 + 2h\sqrt{1+3^2}}$

i) a) *Calculation of normal flow depth using the coefficient of Manning:*

If one uses the Manning-Strickler formula, eq. 3.16, to express the average flow velocity, the discharge in the channel is given by :

$$Q = U\,A = \frac{1}{n}\,R_h^{2/3}\,S_f^{1/2}\,A$$

The coefficient of Manning is obtained from the Table 3.2 :

For a medium-quality concrete, one can take : $n = \dfrac{1}{K_s} \cong \dfrac{1}{70} = 0.0143$ [$m^{-1/3}s$]

By introducing the expressions for A and R_h, as well as the values of n and S_f into the equation for the discharge, Q , one obtains :

$$Q = 80 = 70\left(\frac{(5 + 3h)\,h}{5 + 2h\sqrt{10}}\right)^{2/3}\sqrt{0.001}\;(5 + 3h)\,h$$

This equation can be solved by trial-and-error :

h [m]	Q [m^3/s]
2.20	69
2.50	91
2.36	*80*

The *normal depth* is therefore : $h = h_n = 2.36$ [m]

b) *Calculation of normal flow depth using the coefficient of Weisbach-Darcy* :

The average flow velocity in a channel is given by :

$$U = \sqrt{8g/f}\ \sqrt{R_h\,S_f} \tag{3.10}$$

The discharge is therefore : $Q = U\,A = \sqrt{\frac{8g}{f}}\ \sqrt{R_h\,S_f}\ A$

The coefficient of Weisbach-Darcy, f, can be obtained (either by using the Moody-Stanton diagram or) by using the Colebrook-White formula :

$$\sqrt{\frac{1}{f}} = -2\log\left(\frac{k_s/R_h}{a_f} + \frac{b_f}{Re\sqrt{f}}\right) \tag{3.13}$$

For a trapezoidal channel one generally uses : $a_f = 12$ and $b_f = 2.5$. However this equation can only be solved by trail-and-error.

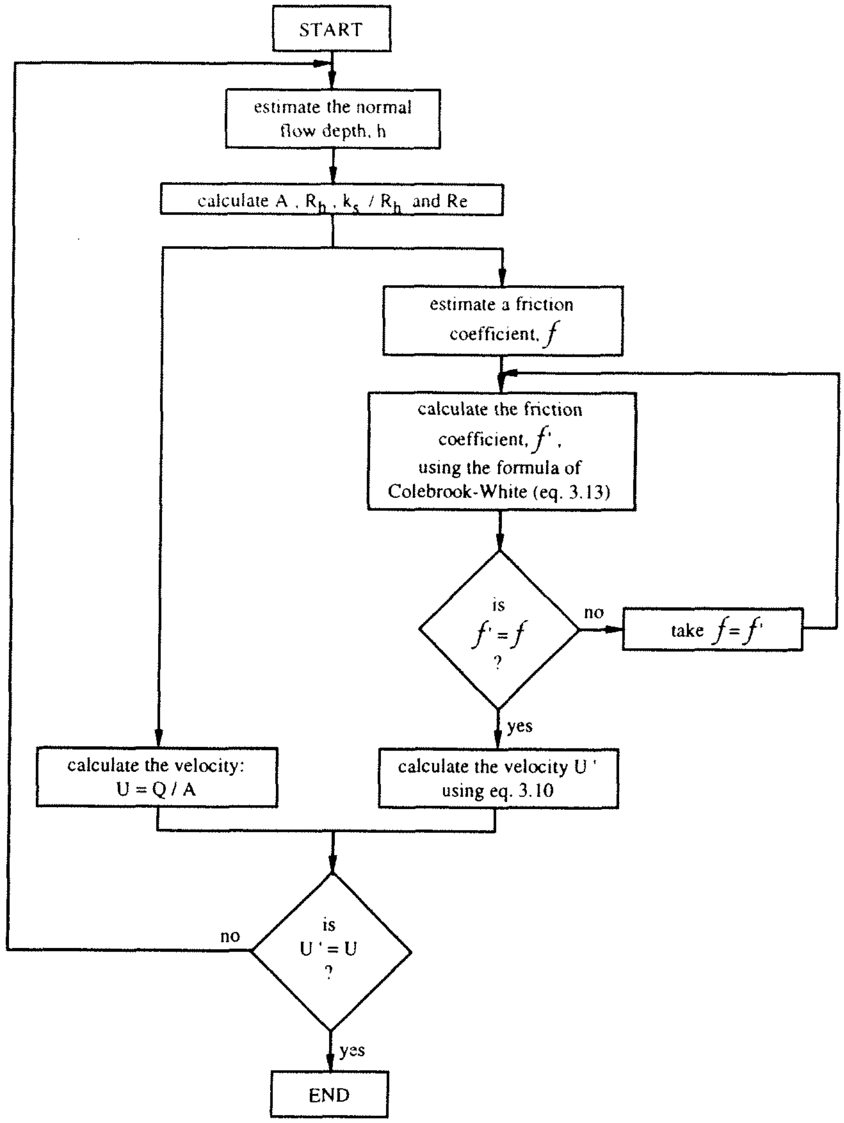
The uniform equivalent roughness, k_s, is obtained from the Table 3.1 :
For a medium-quality concrete, one can take : $k_s = 0.001$ [m] .

It is evident that the velocity, U', calculated with eq. 3.10 for an *estimated* normal depth, h_n , should also satisfy the relationship $U = Q/A$, where A is the wetted surface corresponding to h_n. Using this fact, a trial-and-error type calculation can be devised for calculating the normal depth. The flowchart of this trial-and-error calculation is given hereafter. As it can be seen the algorithm is based on two nested calculation loops. The internal loop calculates the friction coefficient, f, whereas the external loop calculates the normal depth, h_n .

This procedure can be programmed on a microcomputer using a spreadsheet program. The calculation sheet presented below simulates the sequence of calculations depicted in the flowchart. The order of execution of the nested loops is shown in the leftmost column. The detailed explanations of the other columns, numbered from 1 to 10, are given in the table.

The iteration starts with an estimation of $h_n = 2.20$ [m] in the external loop. The internal loop is here executed three times for this same value of h_n. Although the estimated and the calculated values of the Weisbach-Darcy friction coefficient are equal: $f = f' = 0.01419$, at the end of the third iteration, the velocities, U and U', are still different : $\Delta U \neq 0$. Consequently, the calculations must continue by next taking $h_n = 2.50$ [m] and then $h_n = 2.35$ [m]. After the first execution of the internal loop with $h_n = 2.35$ [m] and $f = 0.02$, one finds already that $f' = 0.01402$ and $\Delta U = 0.00$. The iterations can be stopped here without waiting for the convergence of the value of f. The following two calculation lines only help to refine the value of f.

The *normal depth* is therefore : $h = h_n = 2.35$ [m]



ii) For $h_n = 2.36$ [m] , $R_h = 1.43$ [m] and $U = 2.81$ [m/s] , one has :

Reynolds number : $$Re' = \frac{R_h\, U}{\nu} = \frac{(1.43)\,(2.81)}{1.31 \times 10^{-6}} = 3.1 \times 10^{6} > 2000$$

The flow is therefore *turbulent.*

Froude number : $$Fr = \frac{U}{\sqrt{g h_n}} = \frac{2.81}{\sqrt{(9.81)\,(2.36)}} = 0.58 < 1$$

The flow is therefore *fluvial.*

Calculation sheet for determining the normal flow depth
by using the coefficient of Weisbach-Darcy,
calculated using the formula of Colebrook-White (eq. 3.13)

$Q = 80$ [m³/s] | $S_f = 0.001$ [-] | | $a_f = 12$ [-]
$b = 5.0$ [m] | $k_s = 0.001$ [m] | $m = 3.0$ [-] | $b_f = 2.5$ [-]
For $T = 10$ [°C] $\nu = 1.31 \times 10^{-6}$ [m²/s] (*Graf & Altinakar*, 1991, Table I.3)

iteration number for h_n and f	1 h_n [m]	2 A [m²]	3 R_h [m]	4 U [m/s]	5 k_s / R_h [-]	6 Re [-]	7 f [-]	8 f' [-]	9 U' [m/s]	10 ΔU [m/s]
1 1	2.20	25.52	1.35	3.13	7.41×10^{-4}	1.29×10^{7}	0.02000	0.01417		
2							0.01417	0.01419		
3							0.01419	0.01419	2.73	0.40
2 1	2.50	31.25	1.50	2.56	6.66×10^{-4}	1.18×10^{7}	0.02000	0.01388		
2							0.01388	0.01389		
3							0.01389	0.01389	2.91	-0.35
3 1	2.35	28.32	1.43	2.83	7.01×10^{-4}	1.23×10^{7}	0.02000	0.01402	2.82	0.00
2							0.01402	0.01403	2.82	0.00
3							0.01403	0.01403	2.82	0.00

col.	symbol	explanations	expression
1	h_n	*estimated* normal depth	h
2	A	wetted surface	$(b + mh)\,h$
3	R_h	hydraulic radius	$\frac{(b + mh)\,h}{b + 2h\sqrt{1+m^2}}$
4	U	average velocity :	Q / A
5	k_s / R_h	relative roughness	k_s / R_h
6	Re	Reynolds number	$4\,U\,R_h / \nu$
7	f	*estimated* friction coefficient	
8	f	friction coefficient calculated using eq. 3.13 :	

$$\left[-2\log\left(\frac{k_s/R_h}{a_f} + \frac{b_f}{Re\sqrt{f}}\right)\right]^{-2}$$

9	U'	average velocity obtained using eq. 3.10 :	$\sqrt{8g/f'}\;\sqrt{R_h S_f}$
10	ΔU	difference between the velocities	U - U'

Ex. 3.B

The river *Happy* has a variable discharge in the range of $10 < Q\ [m^3/s] < 1000$. In the city of Ste-Justice, the width of the bed is $b = 90$ [m] and the non erodible banks have a slope of 1:1. A bridge crosses the river (without causing any obstruction to the flow) and it is planned to install a measuring gauge at the mid-span of the bridge. A grain-size analysis of the bed material yielded : $s_s = 2.65$ [-] and $d_{50} = 0.32$ [mm], $d_{35} = 0.29$ [mm] and $d_{90} = 0.48$ [mm]. The water temperature is $T = 14$ [°C]. A survey of the river bed showed that the bed slope is $S_f = 0.0005$ [-].

i) Determine the stage-discharge curve, $Q = f(h_n)$, by assuming that the flow is turbulent rough.

ii) At what depth will erosion and deposition begin to occur ?

SOLUTION :

For water at $T = 14$ [°C] (interpolating the values in Table I.3, in *Graf & Altinakar*, 1991), one has : $\rho = 999.1$ [kg/m^3] and $\nu = 1.186 \times 10^{-6}$ [m^2/s] .

Using the definition of the specific density (see *Graf & Altinakar*, 1991, p.9), one writes:

$s_s = \rho_s / \rho_{eau} \quad \Rightarrow \quad \rho_s = \rho_{eau}\, s_s = 1000 \times 2.65 = 2650$ [kg/m^3]

where $\rho_{eau} = 1000$ [kg/m^3] is the density of water at $p_a = 1$ [atm] and $T = 3.98$[°C].

i) The bed (and banks) of a river are generally mobile, composed of erodible granular material. If it is desired to obtain the hydraulic radius, R_h, or the bed shear stress, τ_0, it becomes necessary to make a distinction (see eq. 3.24) between the contribution of the grain roughness, R_h' or τ', and the one of the bed forms, R_h'' or τ''.

The calculations can readily be programmed on a microcomputer using a spreadsheet program. The tabular computation sheet prepared in this way is presented below. Each line of this table represents the computation of the discharge, Q, and some other useful parameters for a given water depth, h. The detailed explanations concerning all the columns are given on the bottom of the table. A brief description of the computation sequence is presented hereafter.

The calculations on a line start by *assuming* a value for the hydraulic radius due to the grain roughness, R_h'. The values of R_h' should be chosen in a way to cover the whole range of the possible discharges, $10 < Q\ [m^3/s] < 1000$. The calculations in the columns 2 and 3 are straightforward. It is interesting to note that, according to the method of Einstein-Barbarossa, the average velocity, U, is calculated using only R_h'. The hydraulic radius due to bed forms, R_h'', calculated in the columns 4 to 7, influences only the flow depth. It is to be noted that the value of U/u_*'' is obtained from Fig. 3.6 with the value of ψ'.

Computation sheet for determining the stage-discharge curve . . .

$b = 90$ [m] $m = 1$ $S_f = 0.0005$ [-]	$T = 14$ [°C] $\rho = 999.1$ [kg / m^3] $\nu = 1.186 \times 10^{-6}$ [m^2/s]	$\rho_s = 2650$ [kg/m^3]

1 R_h' [m]	2 u_*' [m/s]	3 U [m/s]	4 ψ' [-]	5 $\frac{U}{u_*''}$ [-]	6 u_*'' [m/s]	7 R_h'' [m]	8 R_h [m]	9 u_* [m/s]	10 h [m]
0.02	0.01	0.16	47.92	4.5	0.04	0.26	0.28	0.04	0.29
0.05	0.02	0.29	19.17	6.6	0.04	0.39	0.44	0.05	0.44
0.10	0.02	0.45	9.58	8.7	0.05	0.55	0.65	0.06	0.65
0.15	0.03	0.58	6.39	10.4	0.06	0.63	0.78	0.06	0.79
0.20	0.03	0.69	4.79	11.9	0.06	0.68	0.88	0.07	0.89
0.40	0.04	1.05	2.40	18.3	0.06	0.67	1.07	0.07	1.09
0.60	0.05	1.33	1.60	23.4	0.06	0.66	1.26	0.08	1.29
0.80	0.06	1.58	1.20	32.4	0.05	0.49	1.29	0.08	1.32
1.00	0.07	1.81	0.96	42.6	0.04	0.37	1.37	0.08	1.41
1.25	0.08	2.06	0.77	56.2	0.04	0.27	1.52	0.09	1.57
1.50	0.09	2.30	0.64	73.1	0.03	0.20	1.70	0.09	1.76
2.00	0.10	2.72	0.48	107.2	0.03	0.13	2.13	0.10	2.23
2.50	0.11	3.11	0.38	163.0	0.02	0.07	2.57	0.11	2.71
3.00	0.12	3.46	0.32	6844.0	0.00	0.00	3.00	0.12	3.19

<u>col.</u>	<u>symbol</u>	<u>explanations</u>	<u>expression</u>
1	R_h'	hydraulic radius due to grain roughness (*assumed* value)	
2	u_*'	friction velocity due to grain roughness, eq. 3.24,	$\sqrt{g\ R_h'\ S_f}$
3	U	average velocity in the cross section	$u_*' \sqrt{8/f'}$
		with (see eq. 3.13b) : $\sqrt{8/f'} = 5.6 \log (R_h'/k_s) + 6.25$	
4	ψ'	parameter of Einstein-Barbarossa, eq. 3.31,	$\frac{\rho_s - \rho}{\rho} \frac{d_{35}}{R_h'\ S_f}$
5	$\frac{U}{u_*''}$	ratio of velocities corresponding to ψ' (see eq. 3.31 and Fig. 3.6)	
6	u_*''	friction velocity due to bed forms	$U / (U/u_*'')$
7	R_h''	hydraulic radius due to bed forms	$(u_*'')^2 / (g\, S_f)$
8	R_h	total hydraulic radius, eq. 3.24,	$R_h' + R_h''$
9	u_*	friction velocity , eq. 3.7,	$\sqrt{g\ R_h\ S_f}$

. . . using the method of *Einstein-Barbarossa*

$d_{35} = 0.00029$ [m]

$U_{cr} = 0.2$ [m/s] $\Leftarrow$ $k_s = d_{50} = 0.00032$ [m] $\Rightarrow$ $d_* = 7.21$ [-] $\Rightarrow$ $\tau_{*cr} = 0.04$ [-]

(see Fig. 3.12) $d_{90} = 0.00048$ [m] (see Fig. 3.13)

11	12	13	14	15	16	17	18	19	20
A	P	Q	$\frac{U}{\sqrt{g\ h}}$	$\frac{R_h}{d_{50}}$	$\frac{U}{\sqrt{g\ R_h}}$	$\frac{U}{\sqrt{g\ d_{50}}}$	τ_o	τ_*	Notes
[m²]	[m]	[m³/s]	[-]	[-]	[-]	[-]	[N/m²]	[-]	
25.83	90.81	4.2	0.10	889	0.10	2.88	1.39	0.27	¥
40.20	91.26	11.7	0.14	1377	0.14	5.18	2.16	0.42	† ¥
59.25	91.85	26.5	0.18	2016	0.18	7.99	3.16	0.61	† ¥
71.92	92.24	41.4	0.21	2437	0.21	10.27	3.82	0.74	† ¥
81.34	92.53	55.8	0.23	2747	0.23	12.25	4.31	0.83	† ¥
99.28	93.08	103.7	0.32	3333	0.32	18.65	5.23	1.01	† ¥
118.03	93.66	157.4	0.37	3938	0.38	23.80	6.18	1.19	† ¥
120.78	93.74	191.3	0.44	4026	0.45	28.26	6.31	1.22	† ¥
128.53	93.98	232.4	0.49	4274	0.49	32.28	6.70	1.29	† ¥
144.00	94.45	297.3	0.53	4765	0.53	36.84	7.47	1.44	† ¥
161.62	94.98	371.6	0.55	5317	0.56	41.04	8.34	1.61	† ¥
205.28	96.30	559.3	0.58	6662	0.60	48.63	10.45	2.02	† ¥
251.41	97.67	780.9	0.60	8044	0.62	55.44	12.61	2.43	† ¥
297.05	99.02	1026.7	0.62	9375	0.64	61.69	14.70	2.84	† ¥

col.	symbol	explanations	expression
11	A	wetted surface (see Table 1.1)	$(b + mh)\,h$
12	P	wetted perimeter (see Table 1.1)	$b + 2h\sqrt{1 + m^2}$
13	Q	discharge (see eq. 3.2a)	U A
14	$U/\sqrt{g\,h}$	Froude number using the flow depth	Fr
15	R_h/d_{50}	relative depth	
16	$U/\sqrt{g\,R_h}$	Froude number using the hydraulic radius	
17	$U/\sqrt{g\,d_{50}}$	parameter proposed by Alam-Kennedy (see eq. 3.32 and Fig. 3.7)	
18	τ_o	total bed shear stress, eq. 3.6,	$\rho\, u_*^2$
19	τ_*	dimensionless bed shear stress, eq. 3.38,	$\frac{\tau_o}{(\gamma_s - \gamma)\, d_{50}}$
20	†	$U > U_{cr} \Rightarrow$ erosion according to Hjulstrom's criteria (see Fig. 3.12)	
	¥	$\tau_* > \tau_{*cr} \Rightarrow$ motion according to Shields' criteria (see Fig. 3.13)	

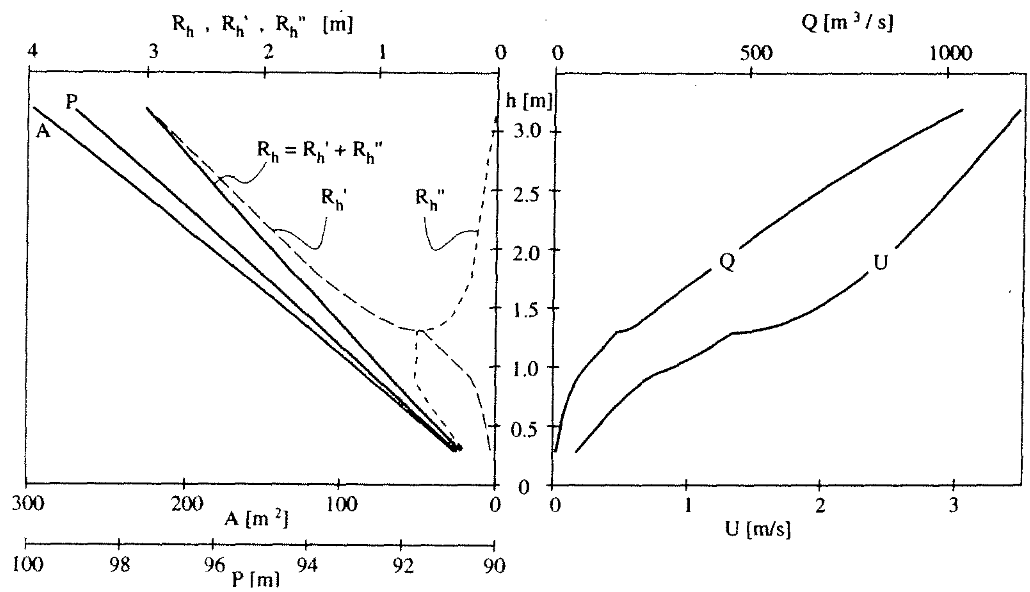
The friction coefficient due to bed forms, f'', can be evaluated using either the method of Einstein-Barbarossa, or the method of Alam-Kennedy (see sect. 3.2.6). The second method is more general but, it necessitates an iterative solution; the calculations are also more elaborate. The method of Einstein-Barbarossa, on the other hand, relies on a straightforward and simple calculation and it will be used for solving the present problem. However, as was already mentioned (see sect. 3.2.6), according to Alam-Kennedy (see Fig. 3.7), the relationship of Einstein-Barbarossa is valid in the region where the friction coefficient due to the bed forms, f'', does not depend on the relative depth, R_h / d_{50}. At the end of the calculation, a verification must be made to check that the calculated f''-value lies in that region.

After calculating the total hydraulic radius, $R_h = R_h' + R_h''$, the flow depth, h, can be obtained using the geometrical relationships. For a trapezoidal cross section (see Table 1.1), the problem is reduced to finding the positive square root of the following quadratic equation :

$$m\,h^2 + (b - 2\,R_h \sqrt{1 + m^2}\,)\,h - b\,R_h = 0$$

The calculations for the remaining columns are explained in the computation sheet. The Froude numbers in column 14 show that the flow is subcritical for all flow depths. Using the values in columns 15 to 17 it can now be checked on Fig 3.7 that all the points fall into the region where the value of f'' is independent from R_h/d_{50}.

The stage-discharge curve, $Q = f(h_n)$, as well as the variation of other useful parameters, U, P, A, R_h', R_h'', R_h, are plotted on the following figure. On this figure, it is interesting to observe the evolution of the curves corresponding to R_h' and R_h''.



ii) a) According to the *criteria of Hjulstrom*, for a given grain size, the critical velocities for the erosion and the sedimentation can be obtained from Fig. 3.12 :

for $d_{50} = 0.00032$ [m] $\Rightarrow$ $U_E = U_{cr} \cong 0.2$ [m/s] and $U_D \cong 0.03$ [m/s]

As it can be seen in the column 20 of the computation sheet, for the average velocities calculated in column 3. one has :

always $U > U_D$: sediment transport takes place,
always(except for h = 0.29 [m]) $U > U_E$: erosion of the bed should be expected.

It is to be noted that the erosion of the bed *does not mean* the formation of a scour hole in the river bed. Since the sediment transported from the upstream compensates globally the erosion, one should rather talk about a *transport of sediments*.

b) According to the *criteria of Shields*, the initiation of erosion should be verified using the total bed-shear stress, τ_* (see eq. 3.40), whose critical value, τ_{*cr} , for a given dimensionless grain diameter, d_*, is obtained from Fig. 3.13; namely:

$$d_{50} = 0.00032 \text{ [m]} \Rightarrow d_* = d_{50}\left(\frac{\rho_s-\rho}{\rho}\,\frac{g}{\nu^2}\right)^{1/3} = 7.21 \text{ [-]} \Rightarrow \tau_{*cr} = 0.04 \text{ [-]}$$

As it can be seen in the column 20 of the computation sheet, the calculated dimensionless total bed-shear stress values (column 19) are :

always $\tau_* > \tau_{*cr}$: erosion of the bed should be expected.

In order to compute the flow conditions corresponding to $\tau_* = \tau_{*cr}$, one has to consider the values of $R_h < 0.28$ [m]. The flow depth at which the erosion starts can be calculated as being $h_{cr} = 0.05$ [m], which is too shallow for a river of this importance.

Ex. 3.C

A channel excavated in earth should convey a water discharge of $Q = 57$ [m^3/s] at an average temperature of $T = 14$ [°C]. The bed slope, $S_f = 0.001$ [-] , is given; it will be assumed that the banks will have side slopes of 1.5 horizontal for 1 vertical. A grain-size analysis yielded : $d_{50} = 37$ [mm], $\varphi = 37°$, $s_s = 2.65$ [-] and $n = 0.02$ [m$^{-1/3}$s]. What should be the dimensions of this channel, if no erosion is allowed either at the bottom or on the banks ?

SOLUTION :

For water at $T = 14$ [°C] (interpolating the values in Table I.3, in *Graf & Altinakar*, 1991), one has : $\rho = 999.1$ [kg/m^3] and $\nu = 1.186 \times 10^{-6}$ [m^2/s] .

Using the definition of the specific density (see *Graf & Altinakar*, 1991, p.9), one writes:

$s_s = \rho_s / \rho_{eau} \quad \Rightarrow \quad \rho_s = \rho_{eau}\, s_s = 1000 \times 2.65 = 2650\ [kg/m^3]$

where $\rho_{eau} = 1000\ [kg/m^3]$ is the density of water at $p_a = 1$ [atm] and $T = 3.98[°C]$.

The stability of the banks requires that the side slopes, θ, should be *smaller* than the angle of repose, $\varphi = 37°$ (see sect. 3.4.3) :

$tg\theta = 1 / 1.5 = 0.667 \quad \Rightarrow \quad \theta = 33.7° < 37° \quad \Rightarrow$ the banks are thus stable.

The critical bed-shear stress on the banks can be calculated using eq. 3.41 :

$$(\tau_{o_{cr}})^w = \tau_{o_{cr}}\left[\cos\theta \left(1 - \frac{tg^2\theta}{tg^2\varphi}\right)^{1/2}\right] = \tau_{o_{cr}}\left[\cos(33.7)\left(1 - \frac{tg^2(33.7)}{tg^2(37)}\right)^{1/2}\right] = 0.39\ \tau_{o_{cr}}$$

To determine the critical shear stress, the Shields criteria will now be used (see sect. 3.4.2).

The dimensionless grain diameter corresponding to $d_{50} = 0.037$ [m] is :

$$d_* = d_{50}\left(\frac{\rho_s - \rho}{\rho}\ \frac{g}{\nu^2}\right)^{1/3} = 0.037\left(\frac{2650 - 999.1}{999.1}\ \frac{9.81}{(1.186 \times 10^{-6})^2}\right)^{1/3} = 835\ [-]$$

According to Shields' criteria, the critical value of the shear stress at the bed, τ_{*cr}, is obtained from Fig. 3.13. In the present case, this value falls in the region where $d_* > 2 \times 10^2$ [-] , having a constant value of $\tau_{*cr} \cong 0.055$ [-] $\cong$ Cte.

The critical shear stress for the *bed* is (see eq. 3.38) :

$$\tau_{o_{cr}} = \tau_{*cr}\, g\, (\rho_s - \rho)\, d_{50} = 0.055 \times 9.81 \times (2650 - 999.1) \times 0.037 \cong 33\ [N/m^2]$$

On the *banks*, however, the critical value is already reached for :

$$(\tau_{o_{cr}})^w = 0.39\ \tau_{o_{cr}} = 0.39 \times 33 = 12.9\ [N/m^2]$$

The flow depth should be chosen not to exceed these critical values. By taking the critical value of shear stress at the bed as the design criteria and by assuming a wide channel the flow depth can readily be calculated using eq. 3.5a :

$$h = \frac{\tau_{o_{cr}}}{\rho g\, S_f} = \frac{33}{999.1 \times 9.81 \times 0.001} = 3.37\ [m]$$

According to Fig. 3.14, for a wide channel with a trapezoidal cross section having side slopes of m = 1.5, the maximum value of the shear stress on the banks is : $\tau_o = 0.75\, \rho g\, h\, S_f$. By taking this value as the critical value, one concludes that the flow depth should not exceed :

$$h = \frac{(\tau_{o_{cr}})^w}{0.75\, \rho g\, S_f} = \frac{12.9}{0.75 \times 999.1 \times 9.81 \times 0.001} = 1.75\ [\mathrm{m}]$$

One should therefore choose h = 1.75 [m] as the *maximum flow depth.*

Now the channel width, b , should be determined such that for a discharge of Q = 57 [m^3/s] the uniform flow can be established at a flow depth of h = 1.75 [m]. The discharge in a channel is calculated using eq. 3.33 :

$$Q = U\, A = \frac{1}{n}\, R_h^{2/3}\, S_f^{1/2}\, A$$

where U represents the average flow velocity. Given that both the hydraulic radius, R_h , and the wetted surface, A , depend on the channel width, b, which is what we are trying to calculate, the above relationship can not be solved directly for b. Therefore, a trial-and-error calculation should be carried out by varying b until the desired discharge is obtained. The computation sheet for the trial-and-error calculation is presented below.

Computation sheet for determining the channel width, b , by trial-and-error.

m = 1.5 [-] $S_f = 0.001$ h = 1.75 [m] n = 0.02 [m$^{-1/3}$s]

b [m]	A = h (b + mh) [m^2]	$P = b + 2h\sqrt{1 + m^2}$ [m]	$R_h = A / P$ [m]	U (eq. 3.16) [m/s]	Q = UA [m^3/s]
5.00	13.34	11.31	1.18	1.77	23.56
10.00	22.09	16.31	1.35	1.94	42.77
14.00	29.09	20.31	1.43	2.01	58.46
13.63	28.45	19.94	1.43	2.00	57.00

The channel width should therefore be : b = 13.63 [m]. In calculating the shear stress at the bed the channel was assumed to be a wide one. Given that b > h, this hypothesis is now justified.

The uniform flow velocity is U = 2.0 [m/s] (see the above computation sheet). This velocity can be compared with the critical velocity for the erosion according to Hjulstrom. As it can be read on Fig. 3.12, the erosion velocity for $d_{50} = 0.037$ [m] is $U_{cr} \cong 2$ [m/s] approximately. Given that the uniform flow depth was selected considering the critical shear stress on the banks, the flow is at the limit of the beginning of erosion in accordance with the previous calculations. To have a better security against the erosion it is necessary to choose a flow depth smaller than the one corresponding to $(\tau_{o_{cr}})^w$.

Ex. 3.D

An artificial channel is going to be constructed in a mountainous region. The slope of the channel imposed by the terrain is $S_f = 0.01$ [-]. This channel should convey a discharge of $Q = 30$ [m^3/s] at a temperature of $T = 14$ [°C], *without causing any erosion.* The grain-size analysis has shown that the granular material is non cohesive with $d_{50} = 50$ [mm] and $s_s = 2.65$ [-]. The angle of repose of this material is $\varphi = 37°$. The Manning coefficient is estimated to be $n = 0.025$ [$m^{-1/3}s^1$].

i) What should be the dimensions of this channel which should have a rectangular section with the sides made of wooden boards ? The use of two different approaches is suggested : the critical velocity, U_{cr} , and the critical shear stress, $\tau_{o_{cr}}$.

ii) What will be the dimensions of the channel, if it were to be constructed entirely in its bed material having an ideal stable cross section ?

iii) Compare the dimensions of the channel, obtained using the different methods.

SOLUTION :

For water at $T = 14$ [°C] (interpolating the values in Table I.3, in *Graf & Altinakar*, 1991), one has : $\rho = 999.1$ [kg/m^3] and $\nu = 1.186 \times 10^{-6}$ [m^2/s] .

Using the definition of the specific density (see *Graf & Altinakar*, 1991, p.9), one writes:
$s_s = \rho_s / \rho_{eau} \quad \Rightarrow \quad \rho_s = \rho_{eau}\, s_s = 1000 \times 2.65 = 2650$ [kg/m^3]
where $\rho_{eau} = 1000$ [kg/m^3] is the density of water at $p_a = 1$ [atm] and $T = 3.98$[°C].

i) a) *Design of the channel using the critical velocity criteria :*

The critical velocity, U_{cr} , according to Hjulstrom (see Fig. 3.12) is :

$$d_{50} = 0.05 \text{ [m]} \quad \Rightarrow \quad U_{cr} = 2.5 \text{ [m/s]}$$

The hydraulic radius corresponding to this velocity can be calculated using the formula of Manning-Strickler :

$$U \equiv U_{cr} = \frac{1}{n} R_h^{2/3} S_f^{1/2} \tag{3.16}$$

With $n = 0.025$ [$m^{-1/3}s^1$] and $S_f = 0.01$ [-], one finds :

$$R_h = \left(\frac{U\ n}{S_f^{1/2}}\right)^{3/2} = \left(\frac{2.5 \times 0.025}{0.01^{1/2}}\right)^{3/2} = 0.494 \text{ [m]}$$

The wetted surface is (see eq. 1.3) : $A = Q / U = 30 / 2.5 = 12\ [m^2]$

The wetted perimeter is (see eq. 1.1) : $P = A / R_h = 12 / 0.494 = 24.3\ [m]$

Given that (see Table 1.1) : $A = b\,h$ and $P = b + 2h$,

the *dimensions* of the rectangular channel are : $b = \mathit{23.1}$ [m] and $h = \mathit{0.52}$ [m]

b) *Design of the channel using the critical shear-stress criteria:*

The dimensionless grain diameter corresponding to $d_{50} = 0.05$ [m] is (see sect. 3.4.2) :

$$d_* = d_{50}\left(\frac{\rho_s - \rho}{\rho}\ \frac{g}{\nu^2}\right)^{1/3} = 0.05\left(\frac{2650 - 999.1}{999.1}\ \frac{9.81}{\left(1.186 \times 10^{-6}\right)^2}\right)^{1/3} = 1129\ [-]$$

Since d_* is known, the critical shear stress at the bed, τ_{*cr} , according to the Shields criteria, can be obtained from Fig. 3.13. In the present case, the calculated value of d_* falls outside of the limits of this figure. Nevertheless, it can be assumed that for $d_* > 2 \times 10^2$ [-] the critical shear stress has a constant value, being $\tau_{*cr} \cong 0.055\ [-] \cong$ Cte.

Given the definition of τ_* (see eq. 3.40) : $\tau_{o_{cr}} = \tau_{*_{cr}} (\gamma_s - \gamma)\, d = \tau_{*_{cr}}\, g\, (\rho_s - \rho)\, d_{50}$

The bed-shear stress is given by eq. 3.5 : $\tau_{o_{cr}} = \gamma\, R_h\, S_f = g\, \rho\, R_h\, S_f$

Combining these two expressions, the hydraulic radius can be calculated :

$$R_h = \frac{\tau_{*cr}\, d_{50}}{S_f}\ \frac{(\rho_s - \rho)}{\rho} = \frac{0.055 \times 0.05}{0.01}\ \frac{(2650 - 999.1)}{999.1} = 0.45\ [m]$$

The velocity corresponding to this hydraulic radius can now be calculated using the formula of Manning-Strickler, eq. 3.16. With $n = 0.025\ [m^{-1/3}s^{1}]$ and $S_f = 0.01$ [-], one has :

$$U \equiv U_{cr} = \frac{1}{n}\ R_h^{2/3}\, S_f^{1/2} = \frac{1}{0.025}\ (0.45)^{2/3}\, (0.01)^{1/2} = 2.35\ [m/s]$$

The remaining part of the calculations are the same as in the first case (see above) :

$A = Q / U = 30 / 2.35 = 12.8\ [m^2]$ and $P = A / R_h = 12.8 / 0.45 = 28.4\ [m]$

The *dimensions* of the rectangular channel are : $b = \mathit{27.5}$ [m] and $h = \mathit{0.47}$ [m]

ii) The critical dimensionless shear stress at the bed, τ_{*cr} , according to the Shields criteria has already been obtained above :

$$d_{50} = 0.05 \text{ [m]} \quad \Rightarrow \quad d_* = 1129 \text{ [-]} \quad \Rightarrow \quad \tau_{*cr} \cong 0.055 \text{ [-]}$$

By using the definition of τ_* (see eq. 3.40), one finds :

$$\tau_{o_{cr}} = \tau_{*cr}\, g\, (\rho_s - \rho)\, d_{50} = 0.055 \times 9.81 \times (2650 - 999.1) \times 0.05 = 44.5 \text{ [N/m}^2\text{]}$$

The maximum depth in the middle of ideal section will be (see sect. 3.4.4) :

$$h = \frac{\tau_{o_{cr}}}{\gamma\, S_f} = \frac{\tau_{o_{cr}}}{g\, \rho\, S_f} = \frac{44.5}{9.81 \times 999.1 \times 0.01} = 0.45 \text{ [m]}$$

The expression for the ideal section is given by eq. 3.42 :

$$h' = h \cos\left(\frac{tg\varphi}{h}\, y\right) = 0.45 \cos\left(\frac{tg(37)}{0.45}\, y\right) \quad \Rightarrow \quad h' = 0.45 \cos(1.675\, y)$$

Other characteristics of the section are calculated using eqs. 3.42a :

$$A = 2\, h^2 / tg\varphi = 2 \times (0.45)^2 / tg(37) = 0.54 \text{ [m}^2\text{]}$$

$$B = \pi\, h / tg\varphi = \pi \times 0.45 / tg(37) = 1.88 \text{ [m]}$$

Approximating the elliptic integral by :

$$E \approx (\pi/2)\,(1 - \tfrac{1}{4} \sin^2\varphi) = (\pi/2)\left(1 - \frac{\sin^2(37)}{4}\right) = 1.429 \text{ [-]}$$

one obtains :

$$U = \frac{1}{n}\, S_f^{1/2} \left(\frac{h \cos\varphi}{E}\right)^{2/3} = \frac{1}{0.025}\, (0.01)^{1/2} \left(\frac{0.45 \times \cos(37)}{1.429}\right)^{2/3} = 1.59 \text{ [m/s]}$$

The discharge, Q_i , which *can* be conveyed through this ideally stable section is :

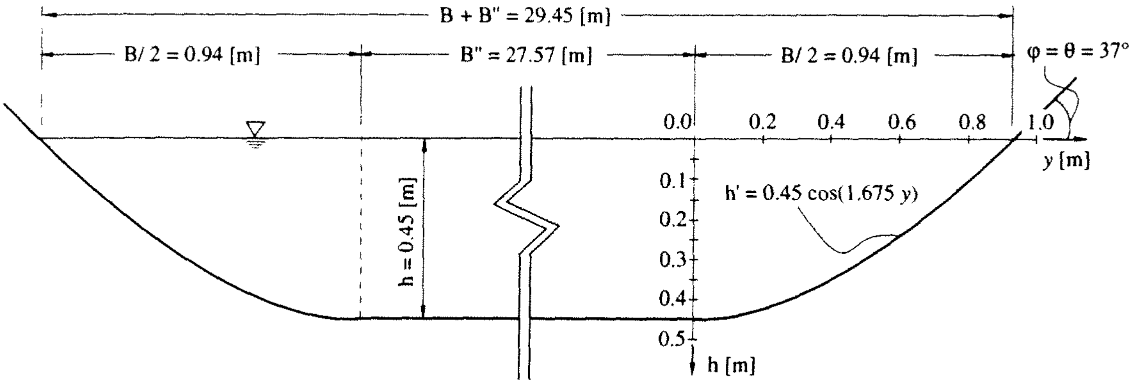
$$Q_i = UA = 1.59 \times 0.54 = 0.86 \text{ [m}^3\text{/s]}$$

This discharge is considerably less than the *required* discharge of $Q = 30$ [m^3/s] . Therefore the wetted surface should be increased by adding a rectangular central part which has a flow depth of $h = 0.45$ [m] and a width of :

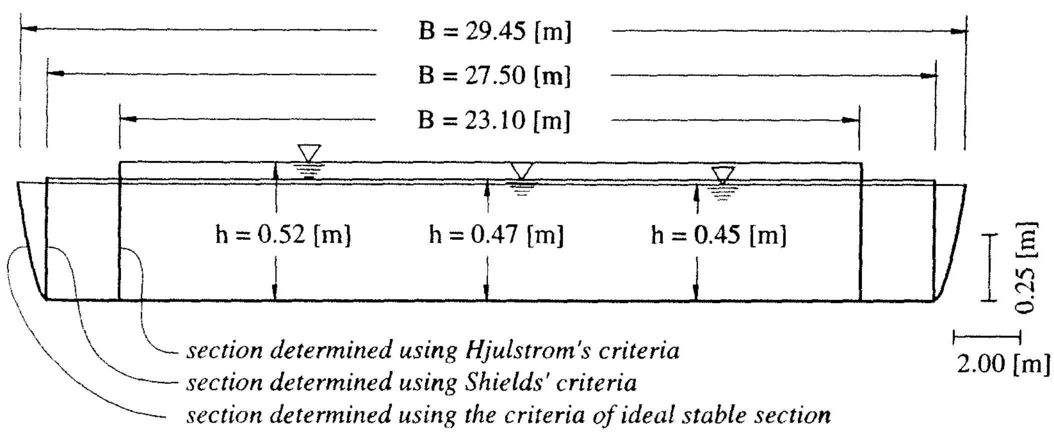
$$B'' = n \frac{(Q - Q_i)}{h^{5/3} J_f^{1/2}} = 0.025 \frac{(30 - 0.86)}{(0.45)^{5/3} (0.01)^{1/2}} = 27.57 \text{ [m]}$$

The total width of the channel will thus be : $B + B'' = 1.88 + 27.57 = 29.45$ [m]

This section is drawn below :



iii) The figure below shows the superposition of the sections, which were determined using the three different methods. It is interesting to note the differences in the dimensions of channels designed according to the erosion criteria and the one designed according to the criteria of an ideal stable section. Each of the three methods rely on different assumptions and a perfect agreement of the results should not be expected.



Ex. 3.E

In a riveted steel channel, the uniform flow is established at a depth equal to 70% of the critical depth. This channel has a rectangular cross section with a width of $b = 9$ [m]. An average velocity of $U = 12$ [m/s] has been assumed for its design.

i) What bed slope should the channel have for this flow ?

ii) What flow regime should one expect ?

iii) What is the shear stress at the bottom of the canal ?

iv) Verify if there is air entrainment into the flow and if so determine its influence on the flow depth.

v) After a long straight reach, the channel makes a $\alpha = 60°$ curve with a radius of curvature of $r_0 = 100$ [m] . How much super-elevation should one expect ? Will cross waves develop in the curved reach ?

SOLUTION :

i) Assuming a turbulent rough, uniform flow regime the average velocity in the channel can be calculated using the formula of Manning-Strickler :

$$U = K_s R_h^{2/3} S_f^{1/2} \tag{3.16}$$

from which one can obtain the bed slope : $S_f = \left[U / (K_s R_h^{2/3})\right]^2$

The average velocity, U, of the flow is specified. From Table 3.2 the friction coefficient is estimated as being $K_s \cong 65$ [$m^{1/3}s^{-1}$] .

For a rectangular channel (see also Table 1.1), one has : $R_h = \dfrac{b\, h_n}{b + 2h_n}$

The width of the channel, b, is given, but the normal depth, $h = h_n$, has to be calculated using the following relationship: $h_n = 0.7\, h_c$. The critical flow depth, h_c, for a rectangular channel is given by :

$$\frac{h_c}{2} = \frac{q^2}{2gh_c^2} \qquad \text{or} \qquad h_c = \sqrt[3]{\frac{q^2}{g}} \tag{2.23}$$

where the unit discharge, q, for the uniform flow is :

$$q = Q / b = (U\, A) / b = (U\, bh_n) / b = Uh_n = U\,(0.7\, h_c)$$

One can therefore write :

$$h_c = \sqrt[3]{\frac{q^2}{g}} = \sqrt[3]{\frac{(0.7\, Uh_c)^2}{9.81}} = \sqrt[3]{\frac{[(0.7)\,(12)\,h_c]^2}{9.81}} = 1.93\, h_c^{2/3}$$

which yields : $h_c = 7.10$ [m] and $h_n = 0.7\,(7.10) = 5.03$ [m]

The discharge for uniform flow, $h = h_n$, is :

$Q = (h\ b)\ U = (5.03 \times 9.0)\ 12 = 543\ [m^3/s]$

One can now compute : $R_h = \dfrac{b\ h_n}{b + 2h_n} = \dfrac{9\ (5.03)}{9 + 2\ (5.03)} = 2.38\ [m]$

and the bed slope : $S_f = \left[\dfrac{U}{K_s\ R_h^{2/3}}\right]^2 = \left[\dfrac{12}{(65)\ (2.38)^{2/3}}\right]^2 = 0.0107\ [-].$

For such a uniform flow, the bed slope should be : $S_f = \mathit{0.0107}\ [-]$.

ii) Given that : $h_c > h_n \quad \Rightarrow$ the flow is supercritical, namely : $Fr > 1$.

For a water temperature of $T = 20$ [°C], the viscosity is $\nu = 1.004 \times 10^{-6}$ [m²/s] (see *Graf & Altinakar*, 1991, Table I.3). The Reynolds number is then (see eq. 1.7) :

$$Re' = \frac{R_h\ U}{\nu} = \frac{(2.38)\ (12)}{1.004 \times 10^{-6}} = 2.8 \times 10^7 > 2000 \quad \Rightarrow \qquad \text{the flow is turbulent.}$$

Moreover, on the Moody-Stanton diagram (see *Graf & Altinakar*, 1991, p.438), it can be verified that the flow is in the region of turbulent rough flow, where the friction coefficient is independent of the Reynolds number.

The flow regime in this channel is therefore *supercritical and turbulent rough.*

iii) The bed-shear stress can be calculated using eq. 3.5. By taking $\rho = 998.2\ [kg/m^3]$ for water at 20 [°C] (see *Graf & Altinakar*, 1991, Table I.3) one has :

$$\tau_o = \rho\ u_*^2 = \gamma\ R_h\ S_f = \rho g\ R_h\ S_f = 998.2 \times 9.81 \times 2.38 \times 0.0107 = \mathit{249.4}\ [N/m^2]$$

iv) The Froude number for the flow is :

$$Fr = \frac{U}{\sqrt{g\ h_n}} = \frac{12}{\sqrt{9.81 \times 5.03}} = 1.71\ [-]$$

For a turbulent rough flow the surface instabilities are expected only for $Fr > 2$. Given that in the present case $Fr < 2$, there will be no roll-wave forming on the free-water surface.

Since the channel slope is weak, it can also be surmised that there will not be any air entrainment. This can be verified using the following equation to compute the mean air concentration in the cross section :

$$\overline{C} = 0.7 \log (\sin\alpha / q^{1/5}) + 0.9 \qquad (3.52)$$

The unit discharge is : $q = Q / b = U\ h_n = 12 \times 5.03 \cong 60.4\ [m^2/s]$

The calculated value of the mean air concentration is negative, which is an impossibility. It can therefore be concluded that no air entrainment takes place across the water surface.

v) The Froude number, Fr = 1.71, calculated above indicates that the flow is supercritical. The cross waves will form in the curved reach. The angle formed between the upstream tangent and the positive and negative waves is given by eq. 3.45 :

$$\sin\beta = \frac{1}{Fr} = \frac{1}{1.71} \qquad \Rightarrow \qquad \beta = 35.8°$$

By using now data for the curve, $r_o = 100$ [m] and B = b = 9 [m], the central angle between a successive maximum and minimum can be calculated using eq. 3.46a :

$$tg\,\theta = \frac{B}{(r_o + B/2)\,tg\,\beta} = \frac{9}{(100 + 9/2)\,tg(35.8)} = 0.119 \qquad \Rightarrow \qquad \theta = 6.8°$$

Knowing these values, β and θ, one can then calculate the maximum and minimum water depths on the concave and convex side walls of the channel, respectively by using eq. 3.46 :

$$h_{max} = h_n\,Fr^2 \sin^2(\beta + \theta/2) = 5.03 \times (1.71)^2 \times \sin^2(35.8 + 6.8/2) = 5.88 \text{ [m]}$$

$$h_{min} = h_n\,Fr^2 \sin^2(\beta - \theta/2) = 5.03 \times (1.71)^2 \times \sin^2(35.8 - 6.8/2) = 4.22 \text{ [m]}$$

The super-elevation is: $h_{max} - h_{min} = 5.88 - 4.22 = 1.66$ [m]

It is interesting to compare this value with the one obtained using eqs. 3.43a and 3.47 (see Fig. 3.18) :

$$\Delta z + \Delta z' = 2\,\frac{B}{r_o}\,\frac{U^2}{2g} = 2\,\frac{9}{100}\,\frac{12^2}{2 \times 9.81} = 1.32 \text{ [m]} < 1.66 \text{ [m]}$$

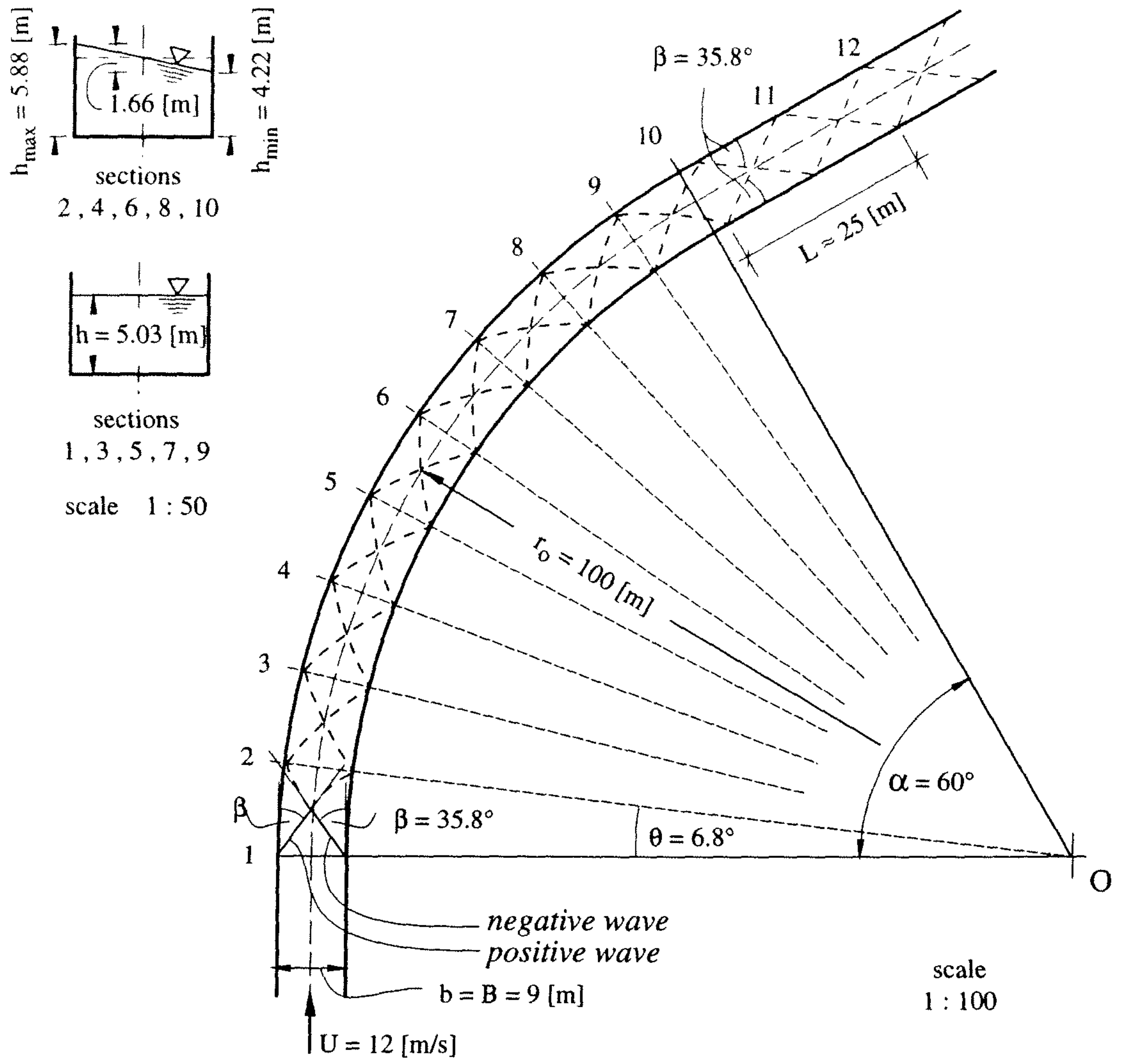
One can see that the first relationship yields a value larger than the second one. For a better safety a super-elevation of 1.66 [m] can be assumed.

The super-elevation with respect to the normal depth, $h_n = 5.03$ [m] , is :

$h_{max} - h_n = 5.88 - 5.03 = 0.85$ [m]

This value will be used in dimensioning the channel cross section in the curved reach.

The formation of cross waves in the curved reach and the downstream straight reach as well as the channel cross sections at different central angles are represented in the following figure. It is important to emphasize that this figure represents a rather simplified image of the reality. A more realistic representation of the water surface can be calculated using the method of characteristics which will be studied in sect. 5.2.2.

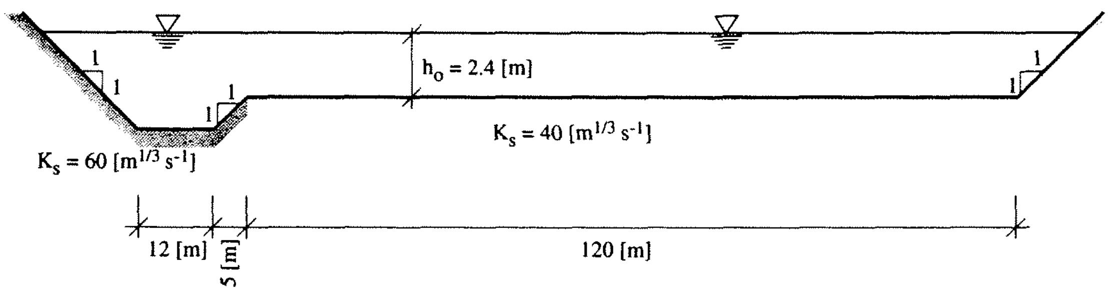


The situation downstream of the curved reach is difficult to predict by a simple method. At the end of the curved reach, at the point of passage from a curved channel to a straight channel, new positive and negative waves are created. According to the ratio between the total angle of the curve, α , and the central angle between a successive maxima and minima, θ, these new waves can be *in* phase or *out* of phase with the existing waves in the curved reach. In some cases a cross-wave pattern of considerable amplitude continues to exist in the downstream channel foı some distance before being gradually attenuated by the friction. In the present case, given that the water surface is inclined at the end of the curved reach, it can be expected that the cross waves do continue in the downstream reach. The distance between two maximum (or two minimum) will be in the order of : $L = 2B / tg\beta = (2 \times 9) / tg(35.8) \cong 25$ [m]. This situation is schematically represented on the figure.

3.7.2 Problems, unsolved

Ex. 3.1
The width at the bed of a trapezoidal channel is b = 2.30 [m] and the side slopes are at an angle of 50° with the horizontal plane. The bed drops by 150 [cm] over a distance of 1.2 [km]. Determine the average velocity and the discharge for a flow depth of h = 1.60 [m] and a coefficient of friction after Bazin of $m_B = 0.46$ [$m^{1/2}$]. Calculate also the shear stress on the bed.

Ex. 3.2
Calculate the discharge for the channel as given below. The bed slope was measured, being $S_f = 0.09\%$. Consider the flow as steady and uniform.



Ex. 3.3
The circular pipe of a sewage system has a coefficient of Manning of n = 0.015 [$m^{-1/3}$s] and is put at an inclination of S_f = 0.0002. When the flow depth attains 0.9 of its diameter, the pipe must still transport a discharge of Q = 2.5 [m^3/s]. What should be the diameter ? Try to make the same calculations using the coefficient of Weisbach-Darcy.

Ex. 3.4
Water at a temperature of 20 [°C] flows in a large rectangular channel, whose coefficient of Manning was determined previously as being n = 0.011 [$m^{-1/3}$s]. The channel slope is $S_f = 0.0004$ and the flow depth is put at h_n = 1.20 [m]. Calculate the value of the corresponding coefficient of Chezy, C, using the formula of Strickler and the logarithmic expression of $U = 2.5 u_* \ln (41.2 R_h/\delta)$; compare these two resulting values. The viscous sublayer is given by $\delta = 11.6\, \nu/u_*$.

Ex. 3.5
A semi-circular canal is made of smooth metal, with n_1 = 0.012 [$m^{-1/3}$s]; the diameter is D = 2 [m] and the bed slope is S_f = 0.005. What diameter should the canal have if it will be reconstructed of corrugated metal, with n_2 = 0.022 [$m^{-1/3}$s] ?

Ex. 3.6
A trapezoidal channel must convey a discharge of $Q = 5.25$ [m^3/s] at a flow velocity of $U = 1$ [m/s]. This channel is made of brickwork (dry, rough stones); the lateral walls are inclined at 45° and the channel slope is $S_f = 0.0005$. Determine the flow depth and the width of the channel.

Ex. 3.7
A stream should be channelised between the sections 15.2 [km] and 17.5 [km] with a circular pipe having a diameter of 1.5 [m]; the flow drops 4.6 [m] between these two sections. This pipe should transport a maximum discharge of $Q = 0.8$ [m^3/s] at a flow depth of 1/3 of its diameter. What will be the coefficient of Manning, n , for this flow and what pipe material should be selected ?

Ex. 3.8
A rectangular channel being $B = 3.6$ [m] wide conveys a discharge of $Q = 5.50$ [m^3/s]. What will be the critical flow depth and the corresponding velocity ? For what slope, S_f, will the velocity be critical, if a coefficient of Manning of $n = 0.02$ [$m^{-1/3}s$] is assumed ?

Ex. 3.9
Find the critical flow depth in a trapezoidal canal, whose width at the bottom is $b = 4.0$ [m] and whose side slopes are inclined by 1 : 2. The discharge is given as $Q = 90$ [m^3/s].

Ex. 3.10
In a trapezoidal channel the bottom width is $b = 6.10$ [m] and the side slopes are inclined at $m = 2$. The elevation of the bed at km 35.0 is 681.30 [m] and the one at km 48.7 is 659.38 [m]. Determine the flow depth for a discharge of $Q = 11.33$ [m^3/s]. A coefficient of Manning of $n = 0.025$ [$m^{-1/3}s$] was previously determined. Subsequently determine the flow regime, using the average depth as the characteristic length.

Ex. 3.11
A trapezoidal canal must convey a discharge of 16.70 [m^3/s] at a flow depth of 1.05 [m] over a distance of 5 [km]. The side slopes are 2 horizontal to 1 vertical; the total drop is given as 8.5 [m]. What will be the bottom width, b, if the flow velocity is supposed to be critical (for the characteristic length use the ratio of the transversal section to the surface width). Give the corresponding coefficient of Manning.

Ex. 3.12
A rectangular channel having a width of $B = 6.5$ [m] conveys a water discharge of $Q = 18$ [m^3/s]. Establish the specific-energy curve, $H_s = f(h)$, in the range of $0 < h$ [m] < 8. What is the critical depth ? What would be the specific energy, H_s , if the flow depth is $h = 2h_{cr}$. What will be the flow depth of uniform, supercritical flow having the *same* specific energy, H_s ?

Ex. 3.13
A rectangular channel, made of smooth concrete and having a width of $B = 20$ [m], conveys a discharge of $Q = 200$ [m^3/s] at a specific energy of $H_s = 3.75$ [m]. Determine the flow depth, h_n, and the bed slope, S_f, for uniform, steady flow. Is this flow a supercritical one ?

Ex. 3.14
A trapezoidal channel, made of rather rough concrete and having a side slope of $m = 2$, is envisioned to transport a discharge of $Q = 17$ [m^3/s] at an average velocity of $U = 1.2$ [m/s]. Determine the width at the bed, the flow depth, and the bed slope for the hydraulically optimal cross section.

Ex. 3.15
A trapezoidal channel – the coefficient of Strickler is estimated to be $K_s = 40$ [$m^{1/3}s^{-1}$] – should be designed having a cross section of maximum discharge. The bottom width is $b = 2$ [m] and the side slopes are of $m = 3$. The average velocity is fixed at $U = 1.98$ [m/s]. What will be the geometry of this section, the discharge, Q, and the channel slope, S_f ?

Ex. 3.16
A drainage channel on a highway, running on a slope of $S_f = 0.0001$, has a triangular section whose side slopes are $m = 4$ and $m = 2$. The coefficient of Manning is $n = 0.02$ [$m^{-1/3}s$]. If flow in the channel is uniform, what will be the flow depth for a discharge of $Q = 0.1$ [m^3/s] ? By how much can the wetted surface be reduced if the channel is made semi-circular ?

Ex. 3.17
A channel of a trapezoidal cross section is built in an alluvium whose granulate is $d_{50} = 1$ [mm]. The flow depth is $h_n = 3$ [m] and the width at the bed is $b = 4$ [m]. The bed slope is $S_f = 0.001$ and the side slopes of worked stone are 45° inclined. What will be the corresponding velocity, U, and the discharge, Q. Check if the bed will be subject of erosion. May one expect the formation of dunes ?

Ex. 3.18
A very large canal in an alluvium, whose granulate of quartz is $d_{50} = 1$ [mm], has a bed slope of $S_f = 10^{-4}$. At what flow depth, h, will erosion commence ? What is the velocity which corresponds to this critical condition ?

Ex. 3.19
One envisions the construction of a non-erodible canal having an ideal, stable cross section. The bed slope is $S_f = 10^{-3}$. The analysis of the granulometry gave : $\rho_s/\rho = 2.65$, $d_{50} = 6.5$ [mm], and an angle of repose of $\varphi = 30°$. Establish the design dimensions for the following discharges : $Q_1 = 1.5$ [m^3/s] and $Q_2 = 4$ [m^3/s].

Ex. 3.20

Calculate the profile of a trapezoidal channel for a discharge of $Q = 12$ [m^3/s] on a bed slope of $S_f = 0.0016$. This channel should be excavated in an alluvium of large gravel. For the side slopes it is recommended to take $m = 2$.

Ex. 3.21

In the town of Ste-Justice, the construction of a road along the river Happy is envisioned. This project can be realised if the river's width is reduced to 67.5 [m]. All other parameters are the same as the ones in Ex. 3.B.

i) Establish the rating curve, $Q = f(h_n)$, assuming the flow is rough and turbulent.

ii) At which flow depth should one expect the commencement of erosion and deposition ?

iii) Compare these results with the ones of Ex. 3.B. Remark on the consequence of such a reduction in the river width.

Ex. 3.22

The downstream slope of the back of a large weir is 30°; it is long enough such that its flow can be considered uniform. The coefficient of Manning is assumed to be $n = 0.0149$ [$m^{-1/3}s$]. The width of the weir is $B = 7$ [m] and the flood discharge is set at $Q = 0.3$ [m^3/s]. Determine if one may expect air-entrainment and calculate the flow depth of the mixture as well as the pressure on the back of the weir.

Ex. 3.23

On a slope of 20°, a canal in concrete was projected to evacuate a unit discharge of $q = 35$ [m^2/s]. Calculate the flow depth and the pressure on the floor. Will air-entrainment take place ?

Ex. 3.24

A rectangular canal made of wood has a width of $B = 8$ [m] and should evacuate the flood discharge. The flow depth is $h = 1.0$ [m] and the flow velocity should be $U = 10$ [m/s]. Determine the radius of the curve which should be foreseen, under the condition that the maximum flow depth does not exceed a height of $h_{max} = 2.0$ [m].

Ex. 3.25

A rectangular channel, having a bed of sand and side walls of concrete, should be designed to have a cross section of maximum discharge :

i) Knowing that the flow in the channel has a depth of $h = 1.0$ [m] and runs at a Froude number of $Fr = 0.7$, what will be the discharge ?

ii) A change in the flow direction of $\alpha = 30°$ should be envisioned. Determine the radius of the curve for the case that the maximum super-elevation is not larger than 15% of the normal depth. What is the head loss in this curve ?

RANSPORT OF
EDIMENTS

flow in a watercourse is a particularly difficult task, since the channel bed is
orm which varies in space and time. The movement of the sediments, which
ed, represents a rather complex phenomenon.

ter, the hydrodynamic equations of the flow over a mobile bed are
some solutions are given. The different modes of the transport of (non-
iments as bed load and as suspended load are presented. The formulae for
ns of the transport of the total load will be exposed, as well as their domain
.

OF CONTENTS

6.1 GENERALITIES

6.1.1 Notions

1° The flow of water over a mobile bed has the ability to entrain the sediments (solid particles) ; a water-sediment mixture will consequently displace itself in the watercourse. The movement of the sediments — *erosion, transport, deposition* — will modify the flow, but also the channel bed, thus its elevation, its slope and its roughness. The interaction between the water and the sediments makes the problem a coupled one.

2° When the bed is a *mobile* one, the fluvial hydraulics must concern itself with both the flow of the liquid phase, namely the mixture, and the movement of the solid phase, namely the sediments in the mixture.

3° A characterisation of the liquid and the solid phase of a water-sediment mixture is a difficult task.

The *liquid* phase is rather well described by :

i) its density, ρ ,
ii) its viscosity, μ ,
iii) the average velocity of the flow, U , and
iv) the friction velocity, u_*.

The *solid* phase is more difficult to characterise ; considered should be :

i) the size of the solid particles, given by its granulometric curve, which includes different types of diameters such as d_{50} , d_{90} , d_{35} , etc.,
ii) the form of these particles,
iii) the density of the particles, ρ_s ,
iv) together, these parameters can be defined by the settling velocity of the particles, v_{ss} , and
v) possibly, the cohesion between the particles.

All these parameters could vary along the watercourse. Furthermore, they will depend on the way the bed samples (in the nature) are taken and are analysed.

4° The dimensions of the sediments are relatively small compared to the ones of the flow ; thus the *turbulence* will play an essential role in all flows of a water-sediment mixture.

5° The transport of these sediments plays an important, if not the most important role in all problems of fluvial hydraulics. This phenomenon is very complex and consequently a theoretical study can only be performed in simple or simplified cases. The formulae, developed for the quantitative determination of the transport of sediments, are based on experimental results, being often limited, and thus should be used with much caution. Such formulae are of great value for the hydraulic engineer, but must be applied within hydraulic conditions under which they have been established.

6.1.2 Flow of a Mixture

1° For gravitational flow of a water-sediment mixture, one may distinguish three types of movement (see Table 6.1). :

i) The mixture may be considered Newtonian, if the volumic concentration of the particles is very small, $C_s << 1$ %. The difference between the density of the mixture and of the water, $\Delta\rho = (\rho_m - \rho) = (\rho_s - \rho)\ C_s$ (see eq. 7.2), remains also small, $\Delta\rho << 16$ [kg/m^3].

The *transport of sediments* (see chap. 6), as bed load and as suspended load, falls into this category. It is this type of transport of solid particles, which is most often encountered in watercourses.

ii) The mixture behaves quasi-Newtonian, if the volumic concentration remains small, $C_s < 8\%$. The difference between the density of the mixture and of the water becomes important, $\Delta\rho < 130$ [kg/m^3].

The transport of sediments as *concentrated suspension* (see *Graf*, 1971, p. 182-186) notably close to the bed, as well as the *turbidity currents* (see chap. 7) fall into this category.

Table 6.1 Classification of flows of a mixture.

Gravitational flow of a water-sediment mixture

non-Newtonian mixture $C_s > 8$ %	quasi-Newtonian mixture $C_s < 8$ %	Newtonian mixture $C_s << 1$ %
* transport of sediments as hyperconcentrated suspension	* transport of sediments as concentrated suspension	* transport of sediments as bed load and suspended load
* debris flow	* turbidity currents	

iii) The mixture behaves non-Newtonian, if the volumic concentration becomes of importance, $C_s > 8\%$. The difference between the density of the mixture and of the water is also very large, $\Delta\rho > 130$ [kg/m^3].

The flow of a non-Newtonian fluid modifies all concepts of Newtonian hydraulics, such as the resistance to the flow, as well as the distribution of velocity and of concentration ; the settling velocity is also influenced and the solid particles stay longer in suspension.

The transport of sediments as *hyperconcentrated suspension* (see *Wan* et *Wang*, 1994), the *debris flow* (see *Takahashi*, 1991), as well as *hyperconcentrated turbidity currents* (see *Wan* et *Wang*, 1994) fall into this category.

- The transport of sediments as an hyperconcentrated suspension is encountered in watercourses of small slopes. Usually enormous quantities of sediments — being of small sizes — enter the channel due to surface erosion caused by extensive rainfalls in the catchment basin. These solid particles stay usually for long time periods in suspension, as wash load.

- Torrential flows of debris may establish themselves at rather steep slopes, $\alpha > 15°$. All kinds of particles, from the finest (having cohesion) to the largest (blocks of 1 [m^3]), participate in the movement, which is rather rare in occurrence and of short duration, and is usually caused by severe rainfalls.

2° It should be stressed, that the above schematic classification (see Table 6.1) is a simplification of the reality, where limits can often not readily be defined and where the different cases can coexist.

6.1.3 Modes of Transport (see Fig. 6.1)

1° The (total) transport of sediments by flow of water is the entire solid transport (of the particles) which passes through a cross section of a watercourse.

Traditionally (but a bit artificially) the transport of sediments is classified in different modes of transport (see Table 6.2) which correspond to distinctly different physical mechanisms.

Table 6.2 Modes of transport of sediments (T.S.).

- total T.S.
  - T.S. of bed material, q_s
    - T.S. as bed load, q_{sb}
    - T.S. as suspended load, q_{ss}
  - T.S. as wash load, q_{sw}

2° In a watercourse the sediments, namely the solid phase, are transported :

i) as *bed load*, q_{sb}, — volumic solid discharge per unit width [m^3/sm] — when the particles stay in close contact with the bed ; the particles displace themselves by gliding, rolling or (shortly) jumping ; this type of transport concerns the relatively larger particles ;

ii) as *suspended load*, q_{ss}, when the particles stay occasionally in contact with the bed ; the particles displace themselves by making more or less large jumps and remain often surrounded by water ; this type of transport concerns the relatively smaller particles ;

iii) as bed load + suspended load, being the (total) *bed-material load*, $q_s = q_{sb} + q_{ss}$, when the particles stay more or less in continuous contact with the bed.

iv) as *wash load*, q_{sw}, when the particles are almost never in contact with the bed ; the particles are washed through the cross section by the flow ; this type of transport concerns the relatively finest particles.

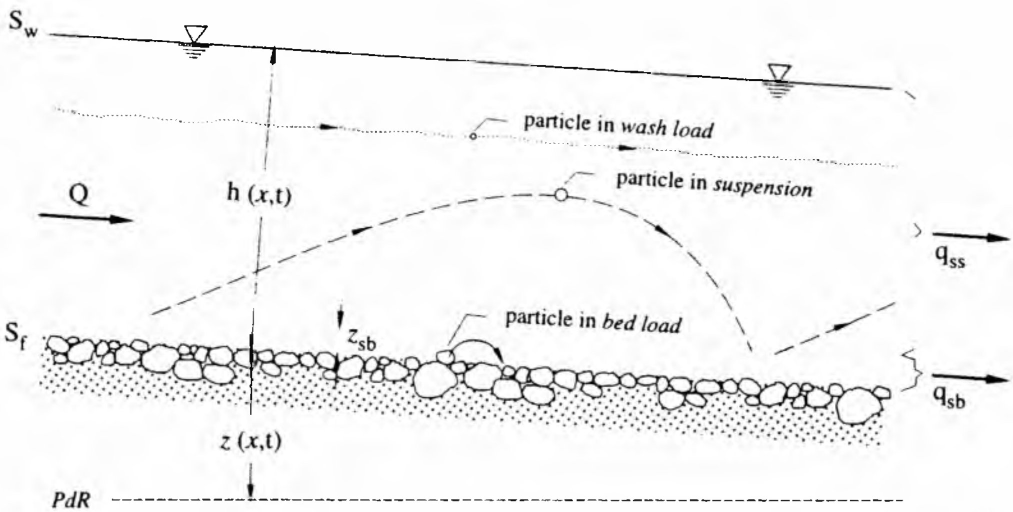


Fig. 6.1 Scheme of the modes of transport.

3° The transport of sediments , namely the erosion of the bed (see sect. 3.4.2), commences upon attainment of a certain critical value, which can be parametrised, for example, by the critical shear stress, τ_{ocr}.

4° It will be useful, but it is also rather imprecise, to give limiting values for the separation of the different modes of transport. Given here are purely indicative values, which use the ratio of the shear velocity of the flow, u_* , and the settling velocity of the particles, v_{ss} (see *Graf*, 1971) :

- $\dfrac{u_*}{v_{ss}} > 0.10$ beginning of bed-load transport,
- $\dfrac{u_*}{v_{ss}} > 0.40$ beginning of suspended load transport.

5° To determine quantitatively the transport of sediments, there are three possibilities available, namely :

- using existing formulae (see sects. 6.3, 6.4 and 6.5),
- obtaining field measurements with adequate instruments (see *Graf*, 1971, chap. 13),
- performing physical models (see *Graf*, 1971, chap. 14).

6.1.4 Types of Problems

1° Many of the hydraulic problems, which require a knowledge of the transport of sediments, can readily be put into one of the following categories :

- determination of a sedimentological rating curve, $q_s = f(q)$, for a given cross section of the channel (see Fig. 6.2);
- determination of the stability of the bed in a given cross section (see sect. 3.4.4);
- determination of the stability of the channel slope (aggradation and degradation) in a given reach of the channel (see sect. 6.2.4).

2° The different modes of transport of sediments, quantified in form of solid discharge, q_{sb} , q_{ss} and q_s, should be related to the liquid discharge, q. This will give the relation of the "sedimentological" rating curve (see Fig. 6.2) for a given cross section of the channel. This curve together with the "liquid" rating curve (see Fig. 3.8) give a rather complete hydraulic description for a given cross section of a channel having a mobile bed.

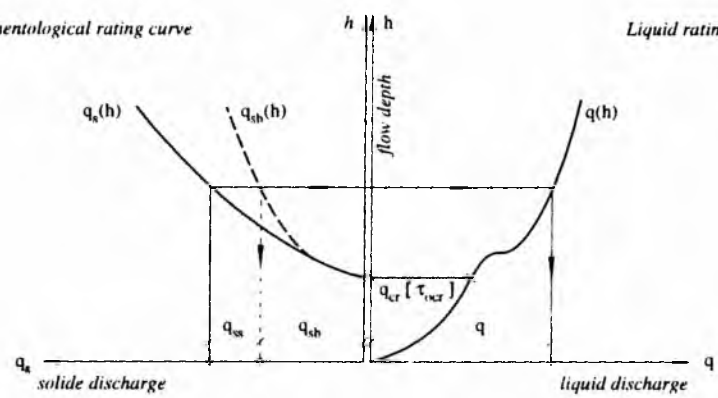


Fig. 6.2 Rating curves for the liquid discharge and the solid discharge.

3° The formulae, which are used to calculate the solid discharge, q_s , allow to know the *capacity* of the transport of sediments for a given flow. Under such conditions, the transport of sediments is said to be in equilibrium.

However it could happen, that the supply of solid discharge is not equal to the capacity of the transport. The transport of sediments is then not in equilibrium :

- if the capacity is larger than the supply, erosion and transport occurs,
- if the supply is larger than the capacity, deposition and transport occurs,
- if the supply is equal to the capacity, transport without erosion or deposition occurs,
- if the bed is armoured, the capacity may not be satisfied (see sect. 6.3.4).

One sees here the complexity of the problem, where along a watercourse the different scenarios can coexist or overlap.

6.2 HYDRODYNAMIC EQUATIONS

Presented will be the hydrodynamic equations, and some solutions, for flow in an open channel over a mobile bed, when entrainment of sediments is possible.

6.2.1 Equations of Saint-Venant - Exner

1° The equations of Saint-Venant (see sect. 5.11) for unsteady and non-uniform flow over a *fixed bed* in a prismatic open channel with a small bed slope (see Fig. 5.1), have been given before (see eq. 5.2 and eq. 5.3) ; for flow over a *mobile bed* they can be written as :

$$\frac{\partial h}{\partial t} + h\frac{\partial U}{\partial x} + U\frac{\partial h}{\partial x} = 0 \qquad B = Cte \tag{6.1}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial h}{\partial x} + g\frac{\partial z}{\partial x} = -\,g\,S_e \tag{6.2}$$

The energy slope, S_e, shall be expressed with a relationship established for uniform flow, by using a friction coefficient, f, for a mobile bed (see sect. 3.2.6), or :

$$S_e = f(f, U, h) \tag{6.3}$$

where h is the flow depth, U is the average velocity of the flow and $z(x,t)$ gives the elevation of the channel bed.

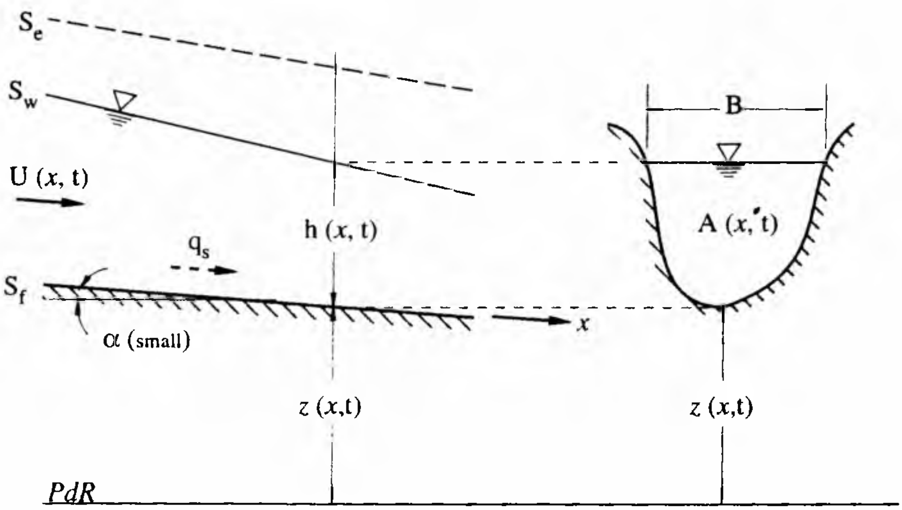


Fig. 6.3 Scheme of unsteady and non-uniform flow over a mobile bed, $z(x,t)$.

2° For flow over a *mobile bed* (see Fig. 6.3), the elevation (level) of the channel bed, $z(x,t)$, may vary. According to the relation of *Exner* (see *Exner*, 1925, and *Graf*, 1971, p. 288), such a variation can be expressed by :

$$\frac{\partial z}{\partial t} = -a_E \frac{\partial U}{\partial x} \tag{6.4}$$

where a_E is an erosion coefficient. This relationship, eq. 6.4, can be written (see *Graf*, 1971, p. 152 and *Krishnappan*, 1981, p. 91) in form of a continuity equation for the solid phase, namely :

$$\frac{\partial z}{\partial t} + \left(\frac{1}{1-p}\right)\left[\frac{\partial}{\partial t}(\tilde{C}_s h) + \frac{\partial}{\partial x}(C_s U h)\right] \equiv \frac{\partial z}{\partial t} + \left(\frac{1}{1-p}\right)\frac{\partial q_s}{\partial x} = 0 \tag{6.4a}$$

where p is the porosity of the sediments of the bed, being defined as the ratio of the volume of empty space (occupied by water) and of the total volume. $q_s = C_s Uh$ is the volumic solid discharge per unit width and C_s ($\tilde{C}_s$) is the volumic concentration (in the cross section) of the solid phase, being defined by the ratio of the volume of the sediments and of the volume of the mixture. In general one admits that the solid discharge, q_s , is a function — still to be determined (see sect. 6.3 to sect. 6.5) — of the liquid discharge, $q = Uh$, or :

$$q_s = f(U, h \,;\, \text{sediment}) \tag{6.5}$$

3° The three basic (differential) equations, eqs. 6.1, 6.2 and 6.4a, contain three unknowns, $U(x, t)$, $h(x, t)$ and $z(x,t)$, with their independent variables, x and t . U and h are the average velocity and the flow depth of the water-sediment mixture (the liquid phase), or of the water only, if the concentration of the sediments, C_s, is negligible.

The two other unknowns, S_e (see eq. 6.3) and q_s (see eq. 6.5), have to be expressed with semi-empirical relationships.

4° The five relations, eqs. 6.1, 6.2 and 6.4a together with eq. 6.3 and eq. 6.5, are the *equations of Saint-Venant - Exner.*

The three relations, eqs. 6.1, 6.2 and 6.3, describe the flow of the liquid phase over a mobile bed ; the two other relations, eqs. 6.4a and 6.5, describe the transport (erosion and deposition) of the solid phase.

5° The liquid and the solid phase are *implicitly* coupled by the semi-empirical relations, eqs. 6.3 and 6.5. After the solution for the liquid phase, eqs. 6.1 and 6.2, a solution for the solid phase, eq. 6.4a, can be obtained, giving the variation of the bed elevation, $z(x,t)$.

The equations of Saint-Venant - Exner can be *explicitly* coupled, if the equation of continuity for the liquid phase, eq. 6.1, is expressed (see *Krishnappan*, 1981, p. 93) as follows :

$$\frac{\partial h}{\partial t} + \frac{\partial z}{\partial t} + \frac{\partial}{\partial x}(Uh) = 0 \tag{6.1a}$$

A direct coupling is thus achieved (see *Correia* et *al.*, 1992), since the term, $\partial z/\partial t$ — which is however often rather small — exists now in both eq. 6.1a and eq. 6.4a. One looks now for a solution by solving simultaneously the equations for the liquid and for the solid phase.

6° To obtain solutions to the equations of Saint-Venant - Exner, use can be made of :

i) analytical methods (see sect. 6.2.3) for simple problems, and

ii) numerical methods (see sect. 6.2.5) for complex problems.

6.2.2 Propagation of Perturbations

1° The propagation of a perturbation, being a wave of small amplitude on the mobile bed, can now be investigated by using the equations of Saint-Venant - Exner, eq. 6.1 to eq. 6.5.

2° For a rectangular channel, these equations — see also the system of equations, eq. 5.2 to eq. 5.10 — are written as six equations of partial derivatives :

$$\frac{\partial h}{\partial t} + h\frac{\partial U}{\partial x} + U\frac{\partial h}{\partial x} = 0 \tag{6.1}$$

$$\frac{1}{g}\frac{\partial U}{\partial t} + \frac{U}{g}\frac{\partial U}{\partial x} + \frac{\partial h}{\partial x} + \frac{\partial z}{\partial x} = -S_e \tag{6.2}$$

$$(1-p)\frac{\partial z}{\partial t} + \frac{\partial q_s}{\partial U}\frac{\partial U}{\partial x} = 0 \tag{6.4b}$$

with :

$$\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial t}dt = dh$$

$$\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial t}dt = dU$$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial t}dt = dz$$

The three last equations are expressions for the total derivatives of the three dependent variables, $h(x,t)$, $U(x,t)$ and $z(x,t)$.

In writing eq. 6.4b, it was assumed that the solid discharge is only a function of the flow velocity, $q_s = f(U)$, thus :

$$\frac{\partial q_s}{\partial x} = \frac{\partial q_s}{\partial U} \frac{\partial U}{\partial x}$$

3° Upon mathematical manipulations of these six equations — the determinant of the matrix of the coefficients must become zero — the following cubic relationship (see *de Vries*, 1965 and 1973, p. 2) is obtained :

$$-c_w^{\ 3} + 2\,U\,c_w^{\ 2} + \left(gh - U^2 + g\frac{\partial q_s}{\partial U} \right) c_w - gU\frac{\partial q_s}{\partial U} = 0 \tag{6.6}$$

where the absolute celerity (the characteristic) is defined by :

$$c_w = \frac{dx}{dt} \tag{2.34}$$

4° This equation, eq. 6.6, has evidently three real roots, thus three characteristics :

i) Two roots, c_{w_1} and c_{w_2} , are an expression of the celerity of the perturbation (wave) on the water surface ; the third root, c_{w_3} , gives the celerity of a perturbation (undulation) on the mobile bed.

ii) c_{w_3} is positive for subcritical flow, $U < \sqrt{gh}$; the form (undulation) of the bed, usually called dunes (see sect. 3.2.5), displaces itself in the same direction as the flow (see *de Vries*, 1973, p. 3).

c_{w_3} is negative for supercritical flow, $U > \sqrt{gh}$; the form (undulation) of the bed, usually called antidunes (see sect. 3.2.5), displaces itself in the opposite direction of the flow.

iii) For a fixed bed without solid discharge, $q_s = 0$, one gets :

$$c_{w_3} = 0$$

$$c_{w_1} = U + \sqrt{gh} \qquad \text{and} \qquad c_{w_2} = U - \sqrt{gh} \tag{5.14a}$$

This solution has already been presented in sect. 5.2.1 and sect. 2.4.3 (see Fig. 2.9).

5° It seems reasonable (see *de Vries*, 1973, p. 4, and *Jansen* et *al.*, 1979, p. 96) to assume that for $Fr = U/\sqrt{gh} \neq 1$ the celerities, c_{w_1} and c_{w_2}, of the waves on the surface are much larger than the celerity, c_{w_3}, of the undulations on the bed, or :

$$c_{w_1} \text{ and } c_{w_2} >> c_{w_3}$$

When studying the perturbations on the bed having a weak celerity, c_{w_3}, it is now possible to consider the flow of the liquid phase as quasi-steady ; thus :

$$\partial U/\partial t = 0 \qquad \text{and} \qquad \partial h/\partial t = 0$$

Consequently, combining eq. 6.1 with eq. 6.2, one can write a single differential equation, which is the equation of the free-surface flow (see eq. 4.7), or :

$$\frac{\partial U}{\partial x}\left(U - \frac{h}{U} g \right) + g\frac{\partial z}{\partial x} = - g S_e \tag{6.7}$$

By eliminating $\partial U/\partial x$ between eq. 6.7 and eq. 6.4b, one obtains :

$$\frac{\partial z}{\partial t} + c_{w_3}\frac{\partial z}{\partial x} = - c_{w_3} S_e = \mathbf{F}(U) \tag{6.8}$$

where $\mathbf{F}(U)$ is a friction (roughness) term — being responsible for the decay of the perturbation on the bed — and

$$c_{w_3} = \frac{g}{(1-p)}\,\frac{(\partial q_s/\partial U)}{(gh/U-U)} = \frac{1}{(1-p)}\,\frac{U(\partial q_s/\partial U)}{h(1-Fr^2)} \tag{6.9}$$

where $Fr^2 = U^2/gh$. For subcritical flow, when $Fr^2 << 1$, the following approximation is possible :

$$c_{w_3} \cong \frac{1}{(1-p)}\,\frac{U}{h}\,\frac{\partial q_s}{\partial U} \tag{6.9a}$$

If the solid discharge is expressed by a power law of the form :

$$q_s = a_s U^{b_s} \qquad \text{and} \qquad \frac{dq_s}{dU} = b_s\left(\frac{q_s}{U}\right) \tag{6.5a}$$

one may write :

$$c_{w_3} = \frac{1}{(1-p)}\, b_s\, \frac{q_s}{h} \tag{6.9b}$$

It is to be noted (see eq. 6.9a), that the celerity of propagation of the undulations c_{w_3}, on the bed is usually rather small compared to the average velocity, U, of the flow itself.

6.2.3 Analytical Solutions

1° To obtain analytical solutions to the equations of Saint-Venant - Exner, which are non-linear and hyperbolic, is a very difficult and often impossible task. However simplifications are nevertheless possible, if one assumes that for flow at small Froude numbers, Fr < 0.6, a *quasi-steadiness* is maintained. This hypothesis of a steadiness of flow can be justified : in general a variation of liquid discharge, $\partial(Uh)/\partial t$, is a short-term phenomenon, while a variation of the bed elevation, $\partial z/\partial t$, is a long-term phenomenon, which produces itself when the variation of the discharge has already disappeared ; thus the flow may be considered reasonably constant, $q = Uh = Cte$. Under such conditions solutions are of great interest, notably if one studies the variation of the bed, $z(x,t)$, as a long-term phenomenon.

2° Using the hypothesis of quasi-steadiness of the flow, a system of two differential equations can be written as :

$$\frac{\partial U}{\partial x}\left(U - g\frac{h}{U}\right) + g\frac{\partial z}{\partial x} = -\,g\,S_e \tag{6.7}$$

$$(1-p)\frac{\partial z}{\partial t} + \frac{\partial q_s}{\partial U}\frac{\partial U}{\partial x} = 0 \tag{6.4b}$$

These two equations are non-linear ones ; only numerical solutions are possible. For certain special cases, analytical solutions (after linearisation) can be of help to understand the problem, notably the relative importance of the different parameters.

3° If one further assumes (see *Vreugdenhil* et *de Vries*, 1973, p. 8) that the quasi-steady flow is also quasi-uniform, $\partial U/\partial x = 0$, the above equation, eq. 6.7, becomes :

$$0 + g\frac{\partial z}{\partial x} = -\,g\,S_e = -\,g\frac{U^2}{C^2 h} = -\,g\frac{U^3}{C^2 q} \tag{6.10}$$

where C is the coefficient of Chézy and $q = Uh$ is the unit discharge.

By eliminating $\partial U/\partial x$ after differentiation of eq. 6.10 with respect to x, one obtains for the above equation, eq. 6.4b :

$$\frac{\partial z}{\partial t} - K(t)\frac{\partial^2 z}{\partial x^2} = 0 \tag{6.11}$$

where the coefficient (of diffusion) , K(t), being a function of time, is given by :

$$K = \frac{1}{3}\,\frac{\partial q_s}{\partial U}\,\frac{1}{(1-p)}\,\frac{C^2 h}{U} \tag{6.12}$$

This model, eq. 6.11, is a *parabolic* one and is limited to large values of x and o namely for $x > 3h/S_e$ (see *de Vries*, 1973, p. 9). The expression for the coefficie eq. 6.12, can also be written (see *de Vries*, 1973, p. 6) in the following way :

$$K = \frac{1}{3} \frac{\partial q_s}{\partial U} \frac{1}{(1-p)} \frac{U}{S_{e_o}} \left(\frac{U_o}{U}\right)^2 \qquad (6.12$$

and upon linearisation (possible for $U \simeq U_o$), one obtains :

$$K \equiv K_o \approx \frac{1}{3} \frac{\partial q_s}{\partial U} \frac{1}{(1-p)} \frac{U_o}{S_{e_o}} \qquad (6.12$$

where the index, o , refers to the uniform (initial) condition. Using the power-l expression for the solid discharge, namely :

$$q_s = a_s U^{b_s} \qquad (6.5$$

one obtains :

$$K \cong \frac{1}{3} b_s q_s \frac{1}{(1-p)} \frac{1}{S_{e_o}} \qquad (6.12$$

The parabolic model — obtained by using the different important assumptions is of interest, since it allows to obtain analytical solutions in certain well-defin cases.

Depending on the applied mathematical techniques and on the hypothesis us some solutions — which often are rather similar — have been communicated *Vreugdenhil* et *de Vries* (1973, p. 9), *Ashida* et *Michiue* (1971), *Jaramillo* et *J* (1984), *Ribberink* et *Sande* (1985) and *Gill* (1987).

4° A *hyperbolic* model for quasi-steady flow, but being non-uniform, has be proposed (see *Vreugdenhil* et *de Vries*, 1973, p. 5 and *Exner*, 1925) :

$$\frac{\partial z}{\partial t} - K \frac{\partial^2 z}{\partial x^2} - \frac{K}{c_{w_3}} \frac{\partial^2 z}{\partial x \, \partial t} = 0 \qquad (6.1$$

where K and c_{w_3} are respectively given by eq. 6.12 and eq. 6.9. Solving t equation, one fixes K and c_{w_3} at their initial values, K_o and c_{w_3o}. However, sin an analytical solution to eq. 6.13 is rarely possible (see *de Vries*, 1985, *Ribberi* et *Sande*, 1985 and *Lenau* et *Hjelmfeld*, 1992), this model turns out to be not that useful.

5° A model of a *simple wave* (see *Vreugdenhil* et *de Vries* , 1973, and *Exner*, 1925) is obtained by reduction of eq. 6.8, or :

$$\frac{\partial z}{\partial t} + c_{w_3} \frac{\partial z}{\partial x} = 0 \tag{6.14}$$

where c_{w_3} is given by eq. 6.9. Since the friction term, $\mathbf{F}(U)$, is now neglected, the application of this model remains limited (see *Ribberink* et *Sande*, 1985) to small values of x and of t, namely for $x << 3h/S_{e_0}$.

6.2.4 Degradation and Aggradation

1° A *degradation* (or *aggradation*) in a reach of a watercourse is encountered if the entering solid discharge is smaller (or larger) than the capacity of the transport of sediments. The sediments of the bed will be eroded (or deposited) and as a consequence the elevation of the channel bed decreases (or increases).

Degradation (erosion) and aggradation (deposition) are long-term processes of the evolution of the channel bed, $z(x,t)$.

The flow, being steady and uniform at the beginning, will also be steady and uniform at the end of the process ; in between the flow becomes non-uniform and quasi-steady. If one assumes, that during this transition the flow can be considered as being quasi-uniform, $\partial U/\partial x = 0$, one may make use of the *parabolic model.*

2° The equation for this parabolic model was given as :

$$\frac{\partial z}{\partial t} - K \frac{\partial^2 z}{\partial x^2} = 0 \tag{6.11}$$

where K is a function of time ; the other variables (see eq. 6.12) must be kept constant to facilitate a resolution of eq. 6.11.

Note that this model is limited to large values of x and of t, namely for $x > 3h/S_e$, and for Froude numbers of $Fr < 0.6$.

3° Analytical solutions to the parabolic model, eq. 6.11, can be obtained for cases, where the initial and boundary conditions are well specified. Such solutions will not only clarify a physical problem, but can often be considered as a first tentative for an understanding of the problem. Caution is however necessary since this model was established using different assumptions.

4° The analytical solutions will certainly help to explain the long-term evolution of th bed of the channel, when the variation of the liquid discharge can be readil neglected. The following examples can be cited (see Fig. 6.4) :

Degradation :
- the supply of solid discharge is reduced (interrupted) at the upstream ;
- the liquid discharge is increased ;
- a lowering of a fixed point on the channel bed at the downstream.

Aggradation :
- the supply of solid discharge is increased at the upstream ;
- the liquid discharge is decreased ;
- a mounting of a fixed point on the channel bed at the downstream.

Some applications of the parabolic model will be presented in the following pages.

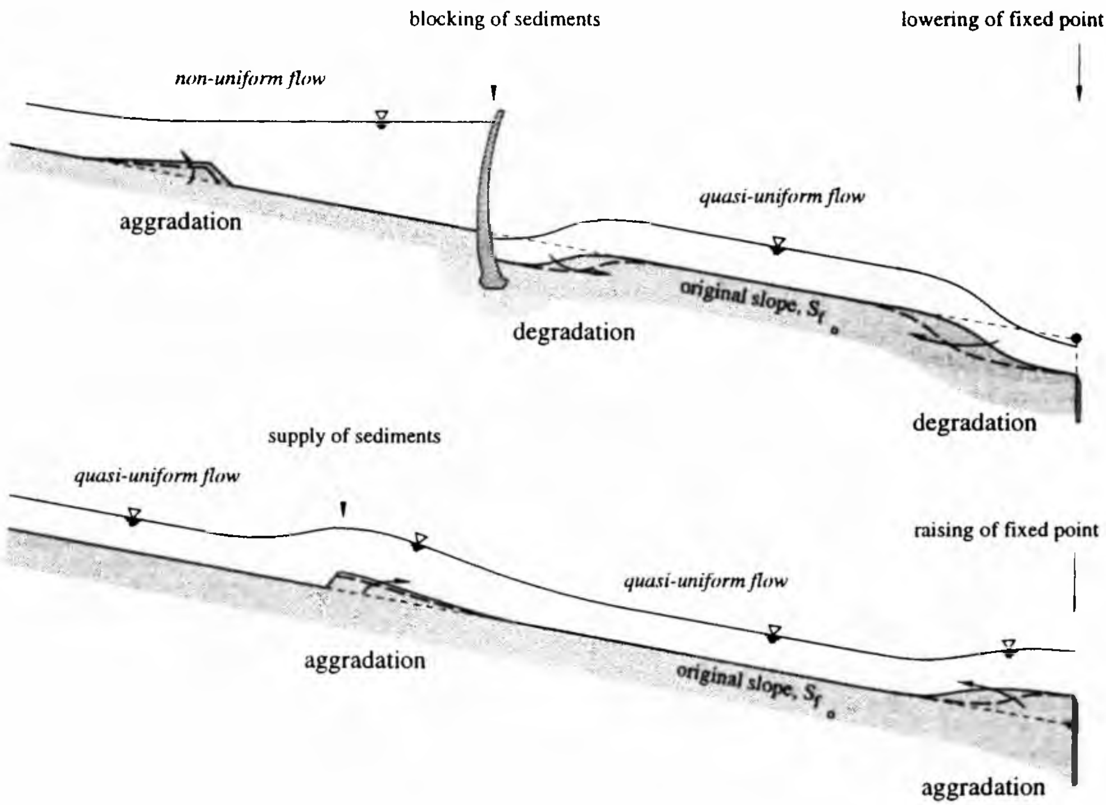


Fig. 6.4 Scheme of a degradation or an aggradation.

5° *Degrading channel* (see Fig. 6.5) :

i) Consider a channel with a mobile bed, having a uniform flow of constant unit discharge, q , at an initial time, t = 0 , and a flow depth of $h = h^o$. This discharge enters into a reservoir, whose water level is lowered by Δh_w, causing a local lowering of the fixed point of the bed of Δh. Consequently, a degradation of the bed is initiated and a long time after, $t = \infty$, one will notice throughout the reach of the channel a lowering of the bed and of the water surface ; the flow depth will then be again the initial depth, $h^o \equiv h^\infty$. During the period of degradation, t = t , the discharge, q , as well as the flow depth, h , remain quasi-constant.

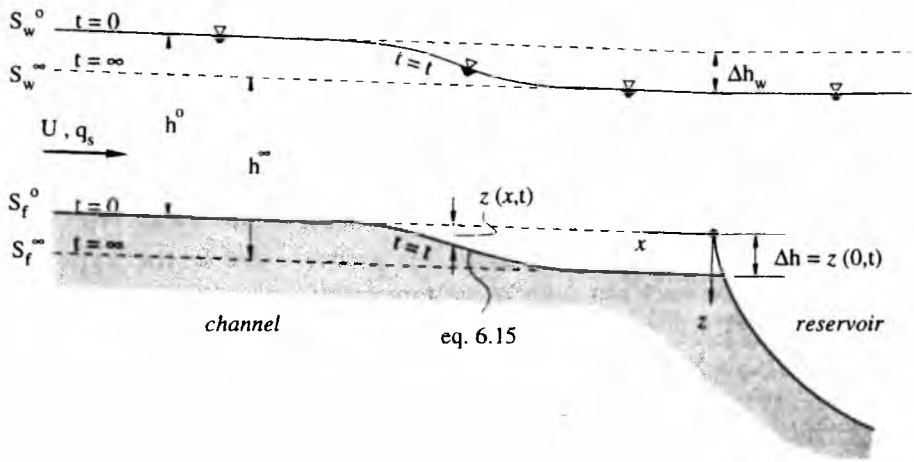


Fig. 6.5 Degradation due to lowering, Δh, of the fixed point on the bed.

ii) The flow is considered as being steady and quasi-uniform ; the use of the parabolic model, eq. 6.11, seems justified. Since the discharge stays constant, the coefficient K (see eq. 6.12) remains also constant.

iii) The problem has to be mathematically specified. The x-axis is put into the initial bed pointing upstream ; z stands now for the variation of the bed elevation with respect to the initial bed slope, S_f^o.

The initial and boundary conditions are :

$$z\,(x,0) = 0 \qquad ; \qquad \lim_{x \to \infty} z\,(x,t) = 0$$

$$z\,(0,t) = \Delta h$$

iv) The solution to eq. 6.11 — making use of Laplace transformations (see *Vreugdenhil* et *de Vries*, 1973, p. 9 and p. 11) — is :

$$z(x,t) = \Delta h \; erfc \left(\frac{x}{2\sqrt{Kt}} \right) \qquad (6.15)$$

where the complementary error function is given as :

$$erfc\,(\mathrm{Y}) = \frac{2}{\sqrt{\pi}} \int_{\mathrm{Y}}^{\infty} \mathrm{e}^{-\xi^2} \;\; d\xi \tag{6.16}$$

which can be found in mathematical tables (see Ex.6.A).

v) One may now ask (see *de Vries*, 1973), after what time period, $t_{50\%}$, at a certain section, $x_{50\%}$, the bed elevation will be lowered by 50 % with respect to the final bed elevation, namely $z/\Delta h = 1/2$. Using eq. 6.15, one writes :

$$\frac{z(x,t)}{\Delta h} = \frac{1}{2} = erfc\left(\frac{x_{50\%}}{2\sqrt{Kt_{50\%}}}\right) = erfc\;(\mathrm{Y})$$

and, upon consulting the tables of the error function, one finds $\mathrm{Y} \cong 0.48$, thus obtaining :

$$x_{50\%} = 0.48\,(2\sqrt{Kt_{50\%}}) \quad \text{where} \quad t_{50\%} \approx x^2_{50\%} / (0.96^2\,K) \tag{6.17}$$

vi) It has been shown (see *Vreugdenhil* et *de Vries*, 1973, p. 11) that the parabolic model — which approaches itself to the hyperbolic one, being more correct but also more complex — is rather valuable, if the value of $(x^2/(2Kt)) < 1/2$ is small, or $x > 3h/S_{e_0}$, namely if the time, t , and the distance, x , are large and/or if the bed slope, S_{e_0} , is relatively large.

6° *Aggrading Channel* (see Fig. 6.6)

i) Consider a channel with a mobile bed, having a uniform flow. A particular cross section, till now in equilibrium, is overloaded, namely the supply of solid discharge, Δq_s, is increased (caused by an earth slide or by a mining operation). Aggradation on the channel bed will take place. Subsequently, after a lapse of time, Δt , the elevation of the bed as well as the water surface will increase by Δh. During the period of aggradation, t = t , the discharge, q, will remain essentially constant.

ii) The flow is considered as being steady and quasi-uniform ; the use of the parabolic model, eq. 6.11, seems justified. Since the discharge remains constant, the coefficient K (see eq. 6.12) stays also constant.

iii) The problem has to be mathematically specified. The x-axis is put into the initial bed, being positive towards the downstream ; z stands now for the variation of the bed elevation with respect to the initial bed slope, S_f^o.

The initial and boundary conditions are :

$$z\,(x,0) = 0 \quad ; \quad \lim_{x \to \infty} z\,(x,t) = 0$$

$$z\,(0,t) = \Delta h(t)$$

iv) The solution to eq. 6.11 is :

$$z(x,t) = \Delta h(t)\ erfc\left(\frac{x}{2\sqrt{Kt}}\right) \tag{6.15a}$$

Evidently this solution is of the same type as the one in the preceding problem ; however now $\Delta h(t)$ is a function of time, to be determined. The coefficient $K \equiv K_0$, eq. 6.12, must be evaluated for the initial situation, thus not taking into account the overloading, Δq_s .

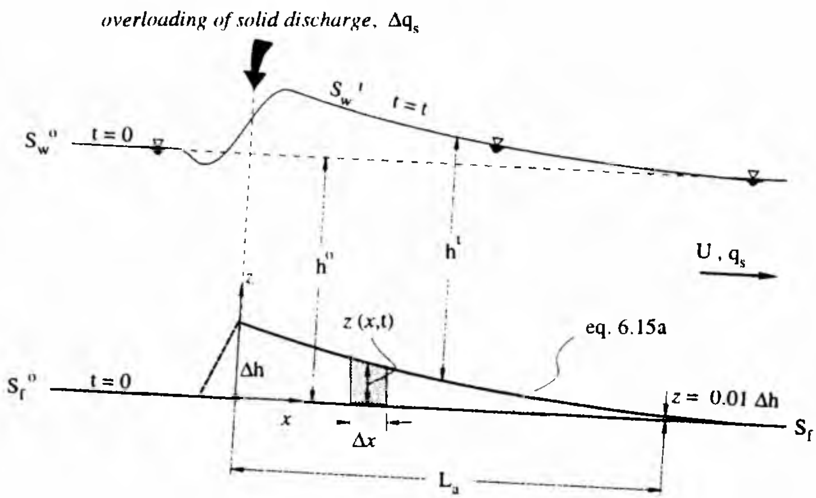


Fig. 6.6 Aggradation by overloading the supply of solid discharge, Δq_s .

v) It is necessary to define the length of the zone of aggradation, L_a , taken as being the one corresponding to a deposition of $z/\Delta h = 0.01$ ($Y \approx 1.80$); according to eq. 6.15a, one writes :

$$L_a = x_{1\%} \cong 3.65\sqrt{Kt_{1\%}} \tag{6.19}$$

The volume of the supply of solid discharge, Δq_s , during a certain time, Δt , is given by $\Delta q_s \cdot \Delta t$; this quantity is distributed over the bed of the channel (see Fig. 6.6) as follows :

$$\Delta q_s \cdot \Delta t = (1 - p) \int_0^{L_a} z\ dx \tag{6.18}$$

Subsequently, eq. 6.15a can be used to calculate (see *Soni* et *al.*, 1980, p. 122) the thickness of the layer, Δh , due to the aggradation :

$$\Delta h(t) = \frac{\Delta q_s \cdot \Delta t}{1.13\ (1-p)\ \sqrt{K \Delta t}} \tag{6.20}$$

where it becomes evident that Δh increases with time, t.

vi) Agreement of experimental work (in the laboratory) with eq. 6.15a has been communicated (see *Soni* et *al.*, 1980); but the coefficient of aggradation, K, had to be slightly adjusted. Also, it was remarked, that the parabolic model, eq. 6.11, remains valid over the entire region ; thus not limited to $x > 3h/S_f$.

7° Computation of a degradation and of an aggradation, such as was discussed in the present section, is only possible if the conditions assumed for a parabolic model are well fulfilled, namely :

i) quasi-steadiness of the flow (long-term variation of the bed) ;
ii) quasi-uniformity of the flow at $Fr < 0.6$;
iii) validity for $x > 3h/S_e$.

If these conditions (hypothesis) are not fulfilled, it is obviously necessary to solve the equations of Saint-Venant - Exner using numerical methods.

6.2.5 Numerical Solutions

1° Analytical solutions of the equations of Saint-Venant - Exner are only possible when the hypothesis of quasi-steadiness of the flow is justified. Furthermore, it is often necessary to assume also quasi-uniformity of the flow. However these assumptions are *no* more possible, if the temporal variation of the discharge, $\partial(Uh)/\partial t$, and the one of the elevation of the bed, $\partial z/\partial t$, are of the same order of magnitude, namely relatively rapid.

2° If the flow is unsteady and non-uniform (see chap. 5) or steady and non-uniform (see chap. 4), no analytical solutions, which are reasonably simple, are available.

The system of the equations of Saint-Venant - Exner, eq. 6.1b, eq. 6.2 and eq. 6.4a, together with eq. 6.3 and eq. 6.5, can be resolved — without making too severe assumptions — by numerical methods ; this may be well achieved with the use of computers.

3° The numerical methods are essentially the same which are used to solve the equations of Saint-Venant, namely for flow over a *fixed* bed (see sect. 5.2). They become however rather complicated, if they are applied for the modelisation of flow over *mobile* bed.

The *implicit* methods (see sect. 5.2.4) using finite differences are the ones which are at the present frequently used to solve the equations of Saint-Venant - Exner.

4° Here we shall only give reference to a selection of the existing literature, which employ numerical methods for the solutions of the equations of Saint-Venant - Exner for steady and unsteady flow over mobile bed : *Chen* et *al.* (1975) and *Cunge* et *al.* (1980, p. 271); *Yucel* et *Graf* (1971), *Krishnappan* (1981), *Holly* et *Rahuel* (1990) and *Correia* et *al.* (1992).

6.3 BED-LOAD TRANSPORT

6.3.1 Notions

1° Transport as bed-load is the mode of transport of sediments (see Fig. 6.1) where the solid particles glide, roll or (briefly) jump, but stay very close to the bed, $0 < z < z_{sb}$, which they may leave only temporarily. The displacement of the particles is intermittent ; the random concept of the turbulence plays an important role.

2° There exist a number of formulae, which can be used for the prediction of the bed-load transport (see *Graf*, 1971, chap. 7, *Yalin*, 1972, chap. 5, and *Raudkivi*,1976, chap. 7).

3° Many of theses formulae are of empirical nature, but often have incorporated dimensionless numbers. This allows to make experiments in the laboratory, where the hydraulic conditions can be well controlled ; subsequently it is possible to use such formulae for field conditions.

6.3.2 Theoretical Considerations

1° Considered will be that the bed of a channel (see Fig. 6.1) is plane but mobile, composed of solid particles of uniform size and being non-cohesive. These particles displace themselves under the action of the flow, which be uniform and steady.

For such simplified conditions — bed forms (see sect. 3.2.5) may form, the granulometric distribution may be non-uniform (see sect. 6.3.4) and cohesion may exist —, one tries to obtain functional relations, such as the ones given by eq. 3.40 and eq. 6.29. The form of such functions, being often rather complex, will be established by experiments, which more or less will take care of the reality of the problem.

2° The forces, which enter (see *Graf*, 19' 1, chap. 6) into the description of the uniform and steady motion of a single pa: cle, isolated and without cohesion, are :

- the hydrodynamic force : $F_H \propto f(\frac{u_* d}{\nu}) \rho d^2 u_*^2$

- the submerged weight of the particle : $W_p \propto g(\rho_s - \rho) d^3$

where u_* is the friction velocity, considered as being proportional to the velocity the particle.

3° The components of this two-phase flow are :

- the *fluid*, by its density, ρ , and its viscosity, ν ;
- The *solid material* , by its density, ρ_s , and a characteristic diameter, d ;
- the *flow*, by its flow depth, h or R_h, the slope, S_f, and the gravity, g ; th by the friction velocity, $u_* = \sqrt{\rho g R_h S_f}$, which characterises the turbulen (see sect. 2.6.4).

In all, there are thus 7 parameters.

4° A dimensionless analysis, using the Π-theorem (see *Yalin*, 1972, p. 61), shov that the arguments which quantify the two-phase flow, such as the bed-loa transport, can now be expressed by 4 dimensionless quantities, namely :

- a Reynolds number of the particle :

$$Re_* = \frac{u_* d}{\nu} \quad (6.21$$

- a dimensionless shear stress (see eq. 3.38) :

$$\tau_* = \frac{\rho\, u_*^2}{(\gamma_s - \gamma) d} = \frac{\tau_0}{(\gamma_s - \gamma) d} = \frac{\gamma\, R_h\, S_f}{(\gamma_s - \gamma) d} \quad (6.22$$

or a densimetric Froude number of the particle :

$$Fr_{*D} = \frac{u_*}{\sqrt{(s_s - 1)\, gd}} = \frac{\sqrt{\tau_0}}{\sqrt{(\gamma_s - \gamma) d}} = \sqrt{\tau_*} \quad (6.23$$

- a relative depth :

$$\frac{h}{d} \text{ ou } \frac{R_h}{d} \quad (6.24$$

- a relative density :

$$s_s = \frac{\rho_s}{\rho} \quad (6.25$$

- In addition, a dimensionless particle diameter can be obtained by combining eq. 6.21 with eq. 6.22 (see point 3.4.2.7°), or :

$$d_* = d \left((s_s - 1) \frac{g}{\nu^2} \right)^{1/3} \qquad (6.26)$$

5° Combining eq. 6.22 and eq. 6.21, a relation was proposed by *Shields* (see sect. 3.4.2 and Fig. 3.13), such as :

$$\tau_* = f(Re_*) \qquad \text{or} \qquad \tau_* = f(d_*) \qquad (3.40)$$

for the study of the commencement of erosion, expressed by the dimensionless shear stress, τ_{*cr} . Furthermore, a relation of the form :

$$\tau_{*cr} = f(Re_*) \qquad (6.27)$$

gives a delimitation of the zone of "motion" from the zone of "no motion" of the particles ; this was developed experimentally from laboratory date, showing a rather large spread. The function of Shields, eq. 6.27, is generally agreed upon as being valuable and useful, notably for the hydraulic engineers, if the granulometry is uniform or almost so.

6° The transport of sediments can be expressed as a function of these 4 dimensionless quantities, namely :

$$\Phi = f(d_*, \tau_*, R_h/d, \rho_s/\rho)$$

Utilising the Π-theorem, one obtains (see *Yalin*, 1972, p. 67) an expression for a dimensionless *intensity of the solid discharge* as the bed load, or :

$$\Phi \equiv q_{sb*} = \frac{q_{sb}}{\sqrt{(s_s-1)gd^3}} \qquad (6.28)$$

with q_{sb} [m^2/s] as the volumic solid discharge per unit width.

Expressions, which are similar to the one of eq. 6.28, can be written (see *Yalin*, 1972, p. 65) as :

$$\Phi' = \frac{q_{sb}}{u_* d} \qquad \text{or} \qquad \Phi'' = \frac{q_{sb}}{Ud} \qquad (6.28a)$$

Since some terms, R_h/d and ρ_s/ρ, are included in the term of τ_*, and taking $\tau_* = f(Re_*)$, one can formulate now a rather simple relationship :

$$\Phi = f(\tau_*) \qquad \text{or} \qquad \frac{q_{sb}}{\sqrt{(s_s-1)gd^3}} = f\left(\frac{\tau_o}{(\gamma_s-\gamma)d}\right) \qquad (6.2$$

which is often written as :

$$\Phi = f(\Psi) \qquad (6.29$$

where $\tau_* \equiv \Psi^{-1}$ and Ψ is called the dimensionless *intensity of shear stre:* applied upon the solid particles.

This expression, eq. 6.29, links the solid transport, q_{sb} (see eq. 6.28), to the she stress, τ_* (see eq. 6.22). Thus an increase in τ_* — passing by $\tau_{*_{cr}}$, where erosi begins — is responsible for an increase in q_{sb}.

The form of this function, eq. 6.29, must still be established ; it is given by t formulae of bed-load transport, which are established by experiments performed the laboratory and in the field.

7° One often assumes that this relation, eq. 6.29, can be expressed in form of a pow law, or :

$$\Phi = \alpha(\tau_*)^\beta \qquad (6.3$$

Making use of the ratio, which defines the coefficient of friction :

$$\frac{U}{\sqrt{\tau_o/\rho}} = \sqrt{\frac{8}{f}} \qquad (2.5$$

one can formulate the following proportionalities :

$$U^2 \propto \tau_o \propto \tau_*$$

Thus it is possible to express the above equatioin, eq. 6.29, by an approxim: relation (see *de Vries*, 1973) in the form of :

$$q_{sb} = a_s U^{b_s} \qquad (6.5$$

where a_s , α and $b_s = 2\beta$, β are the coefficients which depend essentially on t granulometry. This simple, but often useful relation, shows that the avera velocity, U , of the flow is the predominant parameter for the determination of t solid discharge, q_{sb}.

6.3.3 Bed-load Relations

1° At the present, the formulae for a determination of the solid discharge as bed load give only reasonably satisfying results within a domain of the parameters for which the chosen formula has been established. Consequently, the application and use of such formulae has to be done with great care.

Here will be given a selection (in chronological order) of some of the many available formulae ; their most characteristic hydraulic aspects will be pointed out.

2° From the different empirical formulae, proposed by *Schoklitsch* in 1934 and 1950 (see *Graf*, 1971, p. 133), the last one is presented, namely :

$$q_{sb} = \frac{2.5}{s_s} S_e^{3/2} (q - q_{cr}) \tag{6.31}$$

The critical liquid discharge, q_{cr}, characterises the commencement of erosion — usually expressed by τ_{cr} — ; it is given with the use of the Manning-Strickler formula, eq. 3.16 and eq. 3.18, such as :

$$q_{cr} = 0.26 \, (s_s - 1)^{5/3} \, \frac{d^{3/2}}{S_e^{7/6}} \tag{6.31a}$$

valid for $d \geq 0.006$ [m] (see *Bathurst* et *al.*, 1987); for a non-uniform granulometric mixture one takes $d = d_{40}$, as the equivalent diameter.

This relation, eq. 6.31, is applicable for larger grain sizes, $d \geq 6$ [mm], being rather uniform and for bed slopes being moderate to strong (see Table 6.3).

3° From the different empirical formulae — using the condition of similitude of Froude — which *Meyer-Peter* et *al.* have developed in 1934 and 1948 (see *Graf*, 1971, p. 136 or *Yalin*, 1972, p. 112), the last one is presented, namely :

$$0.25 \, \rho^{1/3} \frac{(g_{sb}')^{2/3}}{(\gamma_s - \gamma) d} = \frac{\gamma \, R_{hb} \, \xi_M \, S_e}{(\gamma_s - \gamma) d} - 0.047 \tag{6.32}$$

where $g_{sb}' = g_{sb} \, (\gamma_s - \gamma)/\gamma_s$ is the solid discharge in weight under water and $g_{sb}/\gamma_s = q_{sb}$; R_{hb} is the hydraulic radius of the bed. For a non-uniform granulometry, the mean diameter, $d = d_{50}$, is taken as the equivalent diameter.

This relation can be written in the dimensionless form (see eq. 6.29) such as :

$$\Phi = 8 \, (\xi_M \, \tau_* - \tau_{*cr})^{3/2} \tag{6.32a}$$

where $\tau_{*_{cr}}$ is the dimensionless critical shear stress (see eq. 6.27 and Fig. 3.13) ξ_M is a roughness parameter, given by :

$$\xi_M = \left(\frac{K_S}{K_S'}\right)^{3/2}$$

where K_S' is the roughness of the granulates, to be evaluated with the formula of Strickler, eq. 3.18, and K_S is the total roughness of the bed, evaluated with the formula of Manning-Strickler, eq. 3.16, or :

$$K_S = \frac{U}{R_{hb}^{2/3}\, S_e^{1/2}} \qquad \text{and} \qquad K_S' = \frac{26}{d_{90}^{1/6}}$$

In the absence of bed forms it is recommended to take $\xi_M = 1$; but $1 > \xi_M > 0.35$ if bed forms are present.

This relation, eq. 6.32, is applicable for rather large grain sizes, $d > 2$ [mm], being uniform as well as non-uniform, and for bed slopes, being moderate to strong (see Table 6.3).

4° Using extensively the concepts of hydrodynamics, *Einstein* has developed in 1942 and 1950 (see *Graf*, 1971, pp. 139-150) a probabilistic model for the transport of sediments as bed load.

i) Determination of the probability of erosion

The probability, p_e, of erosion of a particle at any time instant depends (see *Einstein*, 1950, p. 35) on the hydrodynamic force, F_H, here the lift force and the submerged weight of the particle, W_P :

$$p_e = f\left[\frac{W_P}{F_H}\right] = f\left[\frac{k_2\, g(\rho_s-\rho)\, d^3}{1/2\; k_1 C_L(1+\eta)\, \rho\, d^2\, (5.75^2 u_*'^2 \beta_x^{\,2})}\right] =$$

$$= f\left[\frac{1}{(1+\eta)}\,(B\beta_x^{-2}\Psi')\right] \qquad (6.33)$$

k_1 and k_2 are form factors of the particle ; $C_L = 0.178$ is a lift coefficient and η is a random variable of lift, where $\eta_o = 0.5$ is the most probable value According to eq. 6.29a and eq. 6.22, the intensity of shear stress, Ψ', is defined by :

$$\Psi' = \frac{(\gamma_s-\gamma)\,d}{\rho u_*'^2} = \frac{(\gamma_s-\gamma)}{\gamma}\,\frac{d}{R_{hb}'S_f} = \frac{1}{\tau_*'} \tag{6.34}$$

where R_{hb}' is the hydraulic radius of the bed due to the granulate (see sect. 3.2.5); for a non-uniform granulometry, $d = d_{35}$ is taken. $B = 2k_2/(5.75^2 C_L k_1)$ is a numerical constant and β_x is a relation which takes in account the logarithmic velocity distribution as well as the roughness, $k_s = d_{65}$. The following expression (see *Einstein*, 1950, p. 36), where d = X is a characteristic diameter of the granulometry, was given :

$$\beta_x = \log\,(10.6\,X/\Delta) \tag{6.35a}$$

$$\text{with} \qquad X \begin{cases} = 0.77\Delta & \text{if} \quad \Delta/\delta > 1.80 \\ = 1.39\delta & \text{if} \quad \Delta/\delta < 1.80 \end{cases}$$

$$\text{where} \quad \Delta = f(k_s/\delta) \qquad \text{according to Fig. 6.7a}$$

$$\delta = 11.5\,\nu/u'_*$$

The above functional relation, eq. 6.33 , is valid for a uniform granulometry, but can be generalised for a non-uniform one, in the following way :

$$p_e = f\left[\zeta_H\,\zeta_P\,\frac{1}{(1+\eta)}\,(B\beta_x^{-2}\,\Psi')\right] \tag{6.35}$$

ζ_H is a hiding coefficient — the smaller particles hide behind the larger ones — ; it was obtained experimentally (see Fig. 6.7b). ζ_P is a lift-force correction coefficient, also obtained experimentally (see Fig. 6.7c). This expression, eq. 6.35, can also be written (see *Einstein*, 1950, p. 37) as :

$$p_e = f\,(B_*\Psi_*)$$

where Ψ_* is the intensity of shear stress after Einstein :

$$\Psi_* = \zeta_H\,\zeta_P\,(\beta^2/\beta_x^2)\,\Psi' \tag{6.36}$$

and B_* is a constant to be determined experimentally (see eq. 6.42b) :

$$B_* = \frac{B}{\beta^2\eta_o} \qquad \text{with} \qquad \beta = \log\,(10.6) \tag{6.36a}$$

Note, that for a uniform granulometry, one takes (see *Einstein*, 195(p. 36) $\zeta_H = 1$, $\zeta_P = 1$ and $(\beta^2/\beta_x^2) = 1$; thus one writes :

$$\Psi_* = \Psi' \tag{6.36b}$$

In order to express the probability of motion, *Einstein* (1950, p. 37 postulated the following function, being rather similar to the normal function or :

$$p_e = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_*\Psi_*-1/\eta_0}^{+B_*\Psi_*-1/\eta_0} e^{-\xi^2} d\xi \tag{6.37}$$

where ξ is a variable of integration.

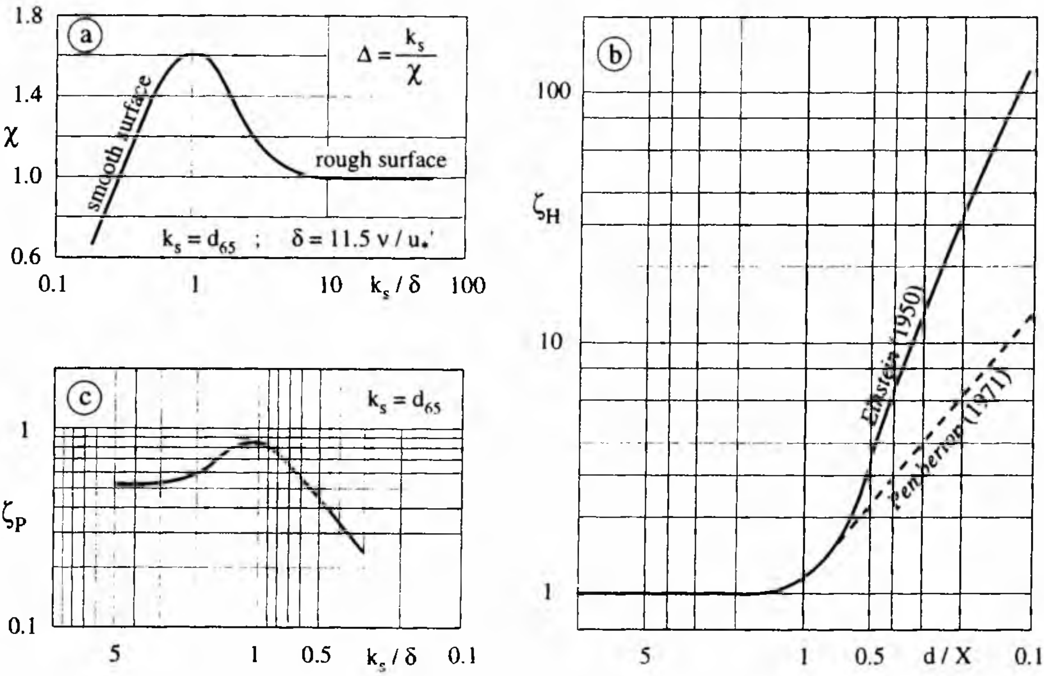


Fig. 6.7 Correction coefficients : (a) of velocity distribution, (b) of hiding and (c) of lift force (see *Graf*, 1971, p. 146).

ii) Equation of bed load

The number of particles, which are *deposited* per unity of time and of bed surface, $A_L d \cdot 1$, is given by :

$$N_D = \frac{g_{sb}\, i_{sb}}{(A_L d)(\gamma_s k_2 d^3)} \tag{6.38}$$

where $(\gamma_s k_2 d^3)$ is the weight of a particle and A_L is a constant. i_{sb} is a fraction (see Fig. 6.9) of the granulometric curve of the unit solid discharge, g_{sb} , in weight.

The number of particles, which are *eroded* per unity of time and of bed surface, is given by :

$$N_E = \frac{i_b}{k_1 d^2}\,(p_e/t_e) \tag{6.39}$$

where $(i_b/k_1 d^2)$ is the number of particles in a unit bed surface ; i_b is a fraction (see Fig. 6.9) of the granulometric curve of the bed material. p_e is the probability of erosion of a particle ; the exchange time, t_e , necessary for the replacement of a particle of the bed by another particle, is expressed by :

$$t_e \propto \frac{d}{v_{ss}} = k_3 \sqrt{\frac{\rho d}{g(\rho_s - \rho)}}$$

where v_{ss} is the settling velocity of the particle.

The *equation of bed load* after *Einstein* (1950) postulates that the rate of erosion, eq. 6.39, is equal to the rate of deposition, eq. 6.38 ; thus one takes $N_D = N_E$ and consequently :

$$\frac{g_{sb} i_{sb}}{(A_L d)(\gamma_s k_2 d^3)} = \frac{i_b p_e}{k_1 k_3 d^2} \sqrt{\frac{g(\rho_s - \rho)}{\rho d}} \tag{6.40}$$

Furthermore, one admits that a solid particle displaces itself by making jumps of a length of $A_L d$ (see eq. 6.38), which are linked to the exchange probability (see *Einstein*, 1950, p. 34) in the following way :

$$A_L d = \lambda d \left(\frac{1}{1 - p_e}\right)$$

where λ is a constant of the jump of the particles. Introducing this expression into the above relation, eq. 6.40, yields :

$$\left(\frac{p_e}{1 - p_e}\right) = A_* \left(\frac{i_{sb}}{i_b}\right) \Phi = A_* \Phi_* \tag{6.41}$$

Φ is the intensity of transport aft r Einstein, given by :

$$\Phi = \frac{q_{sb}}{\sqrt{(s_s - 1)\, g\, d^3}} \qquad (6.28$$

where $q_{sb} = g_{sb}/\gamma_s$ is the volumic solid discharge per unit width and A_* is a empirical constant to be determined experimentally (see eq. 6.42b). For uniform granulometry, one takes (see *Einstein*, 1950, p. 36) simply :

$$\Phi_* = \Phi$$

The above relations, eq. 6.37 and eq.6.41 put together, give now the fin: form of the *equation of bed load of Einstein* (1950):

$$p_e = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_*\Psi_* - 1/\eta_o}^{+B_*\Psi_* - 1/\eta_o} e^{-\xi^2} d\xi = \frac{A_* \Phi_*}{1 + A_* \Phi_*} \qquad (6.42$$

namely a functional relation (see eq. 6.29a), such as :

$$\Phi_* = f(\Psi_*) \qquad (6.42a$$

The (universal) constants have now to be determined experimentally both fc uniform and non-uniform granulometries (see *Einstein*, 1950, pp. 3 and 43) ; they are given (see *Graf*, 1971, p. 149) as being :

$$A_* = 43.6 \quad ; \quad B_* = 0.143 \quad ; \quad \eta_o = 0.5 \qquad (6.42b$$

This relation, eq. 6.42, is plotted in Fig. 6.8 — using the tables of the erroı function — together with the data of *Meyer-Peter* et *al.* and *Gilbert*. Th graphical representation facilitates the use of the above relation, eq. 6.42.

Since a non-uniform granulometry can be broken down into its fraction: i_{sb}/i_b , this relation is rather flexible. For a quasi-uniform granulometry, a equivalent diameter of $d = d_{35}$ can be taken.

It is interesting to remark, that the relation of *Einstein*, eq. 6.42, and the on of *Meyer-Peter* et *al.*, eq. 6.32, give rather similar results (see *Graf*, 1971, ╒ 150), and this notably for $\Phi < 10$. Note also, that in the relation of *Einstei* the notion of a critical value (for erosion), $\Psi_{cr} = \tau_{*cr}^{-1}$, has nowhere bee used explicitly. Nevertheless, one may ask now, what numerical value for Ч one would get, if the value of Φ becomes very small ; for example :

$$\Phi \cong 0.0004 \quad \Rightarrow \quad \Psi \cong 25 \quad \Rightarrow \quad \tau_{*cr} \cong 0.04$$

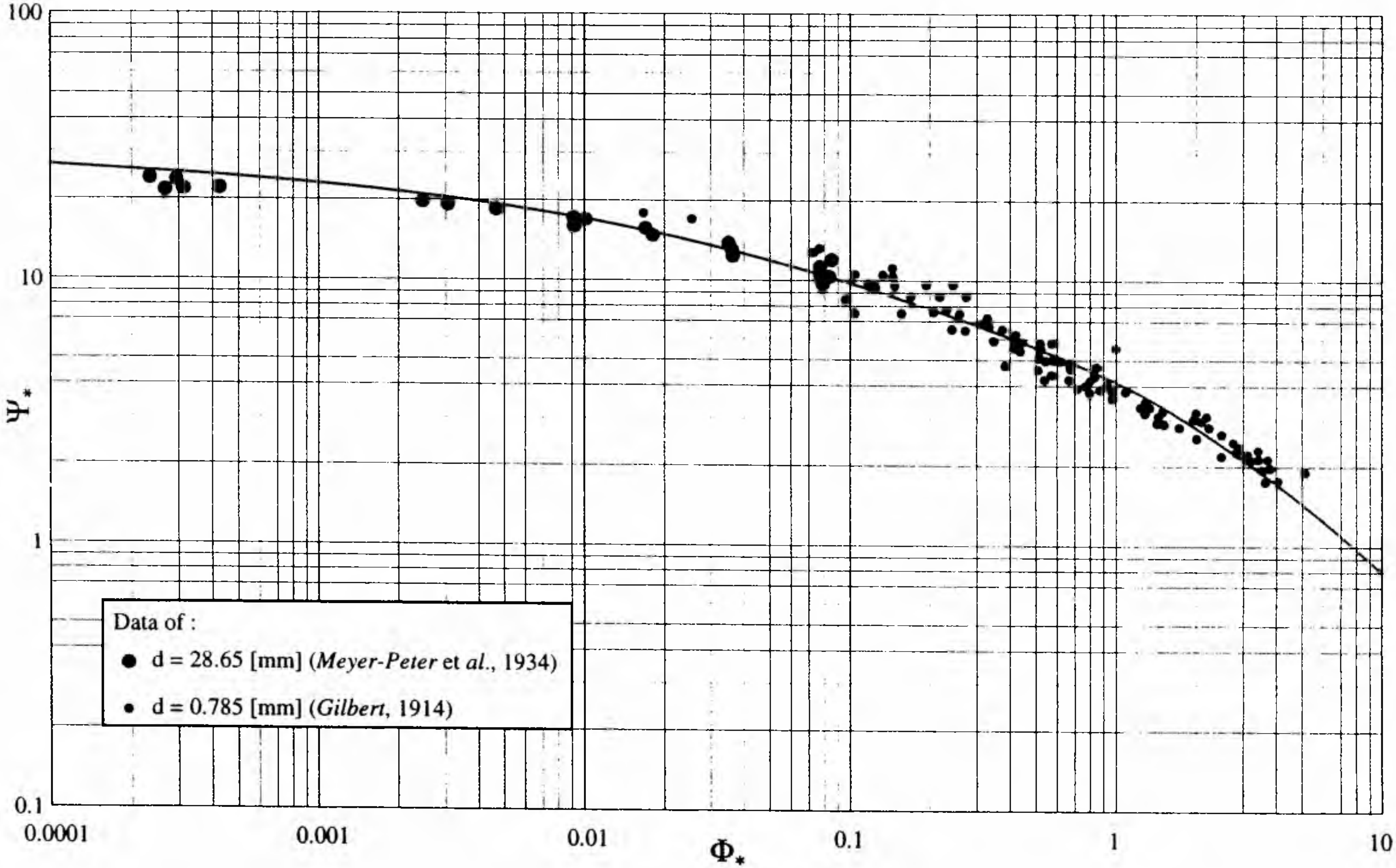


Fig. 6.8 Equation of bed load, $\Phi_* = f(\Psi_*)$, of Einstein (see *Graf,* 1971, p. 148).

Here one sees that there exists a rather good agreement with the critical valı τ_{*cr} , taken from the diagram of Shields (see Fig. 3.13).

The equation of Einstein, eq. 6.42 and Fig. 6.8, is well suitable for unifoı and non-uniform granulates over a large range of diameters, d > 0.7 [mr and of bed slopes (see Table 6.3). It is world-wide used with great succes:

6.3.4 Granulometry, Armouring

1° The non-cohesive sediments (solid particles), which make up the bed of watercourse, are in general of different sizes, being given by the granulomet: curve of the bed material (see Fig. 6.9a).

This curve, which is in general half-logarithmic, can be divided into fractio (percentages), i_{b_i} , whose sum is :

$$1(100\ \%) = \Sigma\, i_{b_i} = i_{b_1} + i_{b_2} + \dots + (5\ \% + 5\ \%)$$

Usually this curve is partitioned into 4 or 5 (unequal) fractions, after havi eliminated small fractions, namely ≈ 5 % , of the finest particles — they are part the wash load — and of the coarsest particles.

For each fraction the average diameter, d_i, is determined and the correspondi solid discharge, $i_{sb_i} q_{sb_i}$, is calculated, using one of the formulae for bed-lo transport. For the entire granulometric mixture, the solid discharge is now obtain as :

$$q_{sb} = \Sigma\, i_{sb_i}\, q_{sb_i} = i_{sb_1}\, q_{sb_1} + i_{sb_2}\, q_{sb_2} + \dots$$

2° The granulometric curve of the bed material is in general different from the one the material moving as bed load or as suspended load (see Fig. 6.9l Consequently, for an average diameter of the granulate, d_i, the given fraction of t granulometric curve of the bed material, i_{b_i}, will be different from t corresponding fraction of the granulometric curve of the solid discharge, i_{sb_i}.

This subtlety was elaborated by *Einstein* (1950, p. 32) by introducing the ratio i_{sb}/i_b and the hiding factor, ζ_H , into the equation of bed-load transport, eq. 6.42.

For a very intensive sediment transport, all sizes (fractions) of particles will readi participate ; consequently $i_{b_i} \equiv i_{sb_i}$, since the curves L and C become identical (s Fig. 6.9b).

3° For cohesive material, the determination of the solid discharge represents a ve difficult task ; literature specialised on this topic should be consulted (see *Grı* 1971, chap. 12 and *Raudkivi*, 1976, chap. 9).

4° The granulometric curve of the bed material is obtained by taking samples from the bed of the channel. Recommended is (see *Einstein*, 1950, p. 48) to take many samples at different sections of the watercourse under study, and obtain an average granulometric curve. Each sample should be taken at (up to) a, to-be expected, maximal erosion depth, namely at a depth of 0.70 [m] below the bed surface.

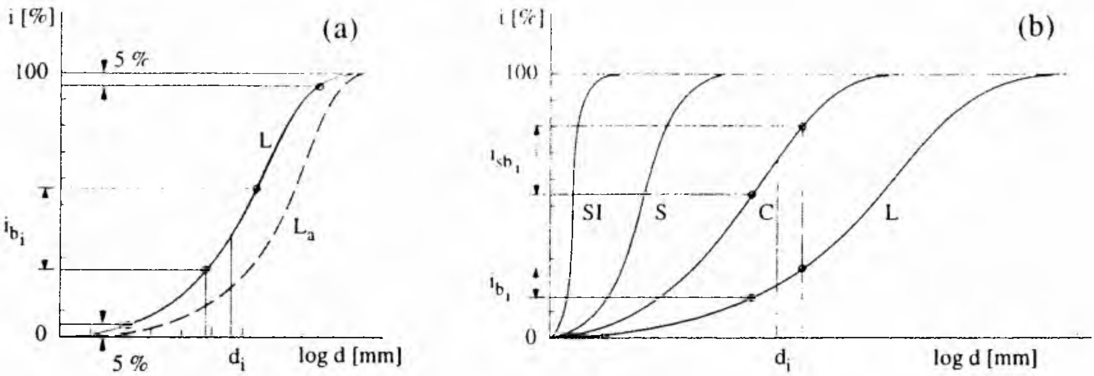


Fig. 6.9 Scheme of granulometric curves for the bed material, L , and the armoured bed material, L_a , the bed-load material, C , and the suspended (wash) load material, S (SI).

5° On a channel bed of non-uniform granulometry, the smaller particles are more easily eroded than the larger ones. a grain-size sorting takes place. An accumulation of the remaining larger particles results in an *armouring* of the bed, which subsequently protects the underlying "original" granulate (see *Graf*, 1971, p. 102).

It can thus happen, that erosion does not take place, if the bed becomes (naturally) *armoured* with the larger particles which remain at the bed surface after an important erosion process during a previous flood event. In such a case the flow cannot take its (full) capacity of sediments transport (see point 6.1.4.3°), and this until another exceptional flood will destroy the armour layer and the original granulate reappears to form once more another new armour layer.

The formulae developed for the capacity of transport are thus only valid for such watercourses, which pass through their own alluvium, namely in a bed being made up of material, which was also transported and can again be transported.

From above it becomes evident that granulometric samples taken in situ — if armouring takes place — have to be interpreted with great caution.

The development of an armour layer is an asymptotic process. When the friction velocity, u_* , increases, the smaller particles are eroded and the larger ones stay in place. The corresponding friction velocity is used to define the critical friction velocity for armouring, $u_{*a,cr}$. The (*maximum* possible) armoured bed will now be formed by the largest particles, d_{90} or larger, which are found in the granulometry of the original bed. For high discharges, when $u_* > u_{*a,cr}$, the armoured bed becomes unstable and will be destroyed. A bed, being composed of the sizes of the granulometric curve of the original composition, arrives at the surface and a extremely active erosion will take place.

There exists only limited conclusive information about the ratio of the origin. granulometry, d , and the granulometry of the armour layer, d_a. For some Swi: rivers, having large bed slopes, $S_f > 0.03$ [-], and large grain sizes, $d_{50} > 6$ [mm an indicative relationship of :

$$d_{50_a}/d_{50} \approx 1.4 \qquad ; \qquad d_{50_a}/d_{90} \leq 0.6$$

was developed by *Correia* et *Graf* (1988).

An empirical relationship for a prediction of the stability of the armour layer w: given by *Raudkivi* (1990, p. 113) as :

$$\tau_{*a,cr} = \frac{(u_{*a,cr})^2}{(s_s-1)g\, d_{50_{a,max}}} = \tau_{*cr}\left[0.4\,(\frac{d_{50}}{d_{50_{a,max}}})^{1/2} + 0.6\right]^2$$

$$\text{with} \qquad d_{50_{a,max}} \lesssim 0.55\, d_{100}$$

where $d_{50_{a,max}}$ is the median diameter of the (maximal possible) armour and τ_{*cr} the dimensionless critical shear stress, taken as $\tau_{*cr} \approx 0.05$ (see Fig. 3.13 Evidently the armouring process is controlled by the largest fraction, d_{90} or large of the granulometric curve of the channel bed. No armouring takes place, if tl granulometry is uniform.

6.4 SUSPENDED-LOAD TRANSPORT

6.4.1 Notions

1° Transport of sediments in suspension is the mode of transport where the sol particles displace themselves by making large jumps, but remain (occasionally) contact with the bed load and also with the bed. The zone of suspension is delimit by : $z_{sb} < z < h$ (see Fig. 6.1).

2° Transport as suspended load could be considered as an advanced stage of transpc as bed load ; however the analytical methods do not allow a description of the two modes of transport with the same (or single) relationship.

6.4.2 Theoretical Considerations

1° The transport of sediments in suspension can be explained with the concept diffusion-convection, which gives the vertical distribution of the (loca concentration, $c_s(z)$, of the suspended particles.

2° For steady uniform flow, the vertical distribution of the concentration of the suspended particles, $c_s(z)$, in the fluid, can be obtained by using the equation of one-dimensional diffusion-convection (see sect. 8.4 or *Graf*, 1971, p. 166) :

$$0 = v_{ss} \frac{\partial c_s}{\partial z} + \frac{\partial}{\partial z} (\varepsilon_s \frac{\partial c_s}{\partial z}) \qquad (6.43)$$

where $c_s(z)$ is the local volumic concentration, ε_s is the diffusivity of the suspended particles, whose units are $[L^2/T]$, and v_{ss} is the settling velocity of the particles.

This equation, eq. 6.43, relates the vertical exchange of solid particles due to the turbulence (upwards) with the gravitational motion (downwards), expressed with the settling velocity, v_{ss} ; it is valid only for weak concentrations, namely for $(1 - c_s) \cong 1$ or $c_s < 0.1$ [%].

3° Integration of the above equation, eq. 6.43 , yields:

$$v_{ss}\, c_s + \varepsilon_s \frac{dc_s}{dz} = Cte = 0 \qquad (6.44)$$

where the constant of integration is taken to be Cte = 0 , implying that $c_s = 0$ at the water surface for $\varepsilon_s = 0$.

The above equation expresses that, at all levels, $z_{sb} < z < h$, there is a (vertical) equilibrium between the movement in the direction of gravity and the one due to the concentration gradient in the direction against gravity. In other words, the rate of sedimentation of particles per unit volume is equal to the rate of turbulent diffusion per unit volume.

For not so weak concentrations (see *Graf*, 1971, p. 185) the above equation, eq. 6.44, should be written as :

$$v_{ss}\, c_s\, (1 - c_s) + \varepsilon_s \frac{dc_s}{dz} = 0 \qquad (6.44a)$$

4° The following remarks, concerning the diffusivity, should be made :

A relation between the diffusivity of suspended particles in the fluid, ε_s , and the turbulent diffusivity of a (soluble) substance in the fluid, ε_t , is in general admitted (see *Graf*, 1971, pp. 167 and 177) :

$$\varepsilon_s = \beta_s \varepsilon_t \qquad (6.45)$$

where β_s is a factor of proportionality. For fine particles, which follow readily th fluid motion, one takes $\beta_s = 1$; for larger particles, one takes $\beta_s \leq 1$. Som researchers (see *Graf*, 1971, p. 178 and *Raudkivi*, 1990, p. 172) advance arguments to show that $\beta_s \geq 1$.

For weak concentrations it is usually assumed that :

$$\varepsilon_s \approx \varepsilon_t \tag{6.46}$$

thus one takes $\beta_s = 1$.

Furthermore, one may also postulate (see sect. 8.1.3) that :

- diffusion (per unity of surface) of matter, namely of a substance in the fluid, i given by :

$$\rho(\varepsilon_m + \varepsilon_t)\frac{\partial c}{\partial z} \approx \rho\varepsilon_t\frac{\partial c}{\partial z} = q_m$$

- diffusion (per unity of surface) of momentum is given (see eq. 2.49) by :

$$\rho(\nu + \nu_t)\frac{\partial u}{\partial z} \approx \rho\nu_t\frac{\partial u}{\partial z} = \tau_{zx}$$

Here is assumed that the turbulent diffusivity, ε_t , and the turbulent viscosity, ν_t are far more important than the molecular diffusivity, ε_m , and the viscosity, ν respectively (see *Graf*, 1971, p. 166).

According to the analogy of Reynolds (see *Taylor*, 1954, p. 451), the transfer o matter (as well as the one of heat) and the transfer of momentum by the turbulenc are analogous ; this is strictly correct close to a solid surface (the bed) Consequently, one may also take :

$$\varepsilon_t \equiv \nu_t \equiv \varepsilon_s \tag{6.47}$$

5° For the case, where the diffusivity is independent of the level, ε_s = Cte, the abov equation, eq. 6.44, can be integrated and yields :

$$\frac{c_s}{c_{sa}} = \exp\left[-\frac{v_{ss}}{\varepsilon_s}(z - a)\right] \tag{6.48}$$

where c_{sa} is the concentration at a reference level, a. This relation has bee experimentally verified (see *Graf*, 1971, p. 167).

6° In the open-channel flow, the turbulence and thus the diffusivity are vertically distributed, $\varepsilon_s(z)$, (see sect. 2.6). The distribution of the diffusivity, $\varepsilon_s \approx \nu_t$, is given (see *Graf*, 1971, p. 173) by :

$$\varepsilon_s = \kappa u_*' \frac{z}{h} (h - z) \tag{6.49}$$

This parabolic relation, established for unidirectional flow, has been obtained by assuming :

- the vertical distribution of the tangential turbulent shear stress :

$$\tau_{zx} = \tau_o \left(\frac{h-z}{h}\right) \tag{2.47b}$$

- the vertical distribution of the velocity (see sect. 2.5.2) :

$$\frac{du}{dz} = \frac{\sqrt{\tau_o/\rho}}{\kappa} \frac{1}{z}$$

  where $\kappa = 0.4$ is the Karman constant, which is independent of the concentration (see *Colemann*, 1981) ;

- the expression of the Reynolds stress :

$$\tau_{zx} = \rho \nu_t \frac{du}{dz} \tag{2.49}$$

- the analogy of Reynolds :

$$\varepsilon_s \cong \varepsilon_t \cong \nu_t \tag{6.47}$$

This distribution of the diffusivity, eq. 6.49, has been experimentally verified (see *Raudkivi*, 1990, p. 170).

Substitution of eq. 6.49 into eq. 6.44, and separation of the variables, yields :

$$\frac{dc_s}{c_s} = -\frac{v_{ss}}{\kappa u_*'} \left(\frac{h}{h-z}\right) \frac{dz}{z} \tag{6.50}$$

where one defines the *Rouse* exponent as :

$$\mathfrak{z} = \frac{v_{ss}}{\kappa u_*'} \qquad \left(\text{or } \mathfrak{z}' = \frac{v_{ss}}{\beta_s \kappa u_*'} \right) \tag{6.50a}$$

This expression, eq. 6.50, can now be integrated by parts, within the limits $a < z < h$ (see *Rouse*, 1938, p. 341, and *Graf*, 1971, p. 173) and renders :

$$\frac{c_s}{c_{sa}} = \left(\frac{h - z}{z} \cdot \frac{a}{h - a} \right)^{\mathfrak{z}} \qquad (6.5$$

where c_{sa} is the concentration at a reference level, a. This equation, eq. 6.51, giv the distribution of the relative concentration, c_s/c_{sa}, for one single particle size, and $\mathfrak{z}$. Note that in the definition of the Rouse exponent, $\mathfrak{z}$, the friction velocit u_*', due to the granulate must be used.

In the Rouse exponent, eq. 6.50a, one should take the settling velocity, v_{ss}, of t particle in clear and quiescent water, thus being not influenced by turbulence or concentration. For *natural* particles of quartz, $s_s = 2.65$ [-], falling in quiesce water at $T = 20$ [C°], the settling velocity can be determined using the Fig. 6.10 (s *Graf*, 1971, p. 45).

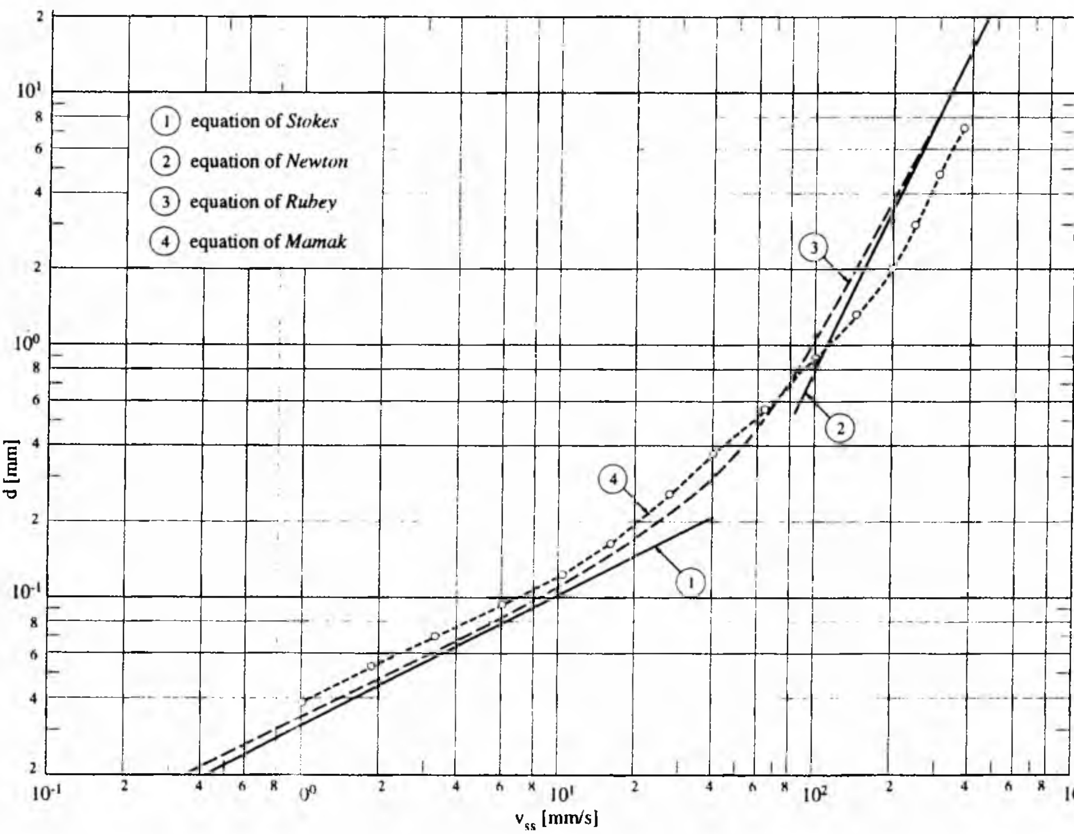


Fig. 6.10 Sett ng velocity , v_{ss} , as function of particle diameter, d.

The equation, giving the distribution of the relative concentration, eq. 6.51, for different values of the Rouse exponent, ζ , is shown in Fig. 6.11. The following is to be observed :

- For small ζ-values, the relative concentration is large and tends to become uniform over the entire flow depth, h.
- For large ζ-values, the relative concentration is small at the water surface and is large close to the bed.
- The size of the particles, expressed with the settling velocity, v_{ss} , is directly responsible for these distributions.
- Close to the bed, $z \cong 0$, the concentration goes towards infinity, $c_s = \infty$, thus to an impossible value. Thus one delimits this level usually by $a \equiv z_{sb} \cong 0.05h$ or by $z_{sb} = 2d$, below which there exists the bed load (see Fig. 6.1).
- The reference concentration, c_{sa} , is usually taken at a level of $a \equiv z_{sb}$; it will be calculated later (see eq. 6.57) with one of the bed-load formulae, q_{sb} .

Numerous are the investigations, both in laboratory and in situ, which give evidence of the validity of the above equation, eq. 6.51 (see *Graf*, 1971, p. 175).

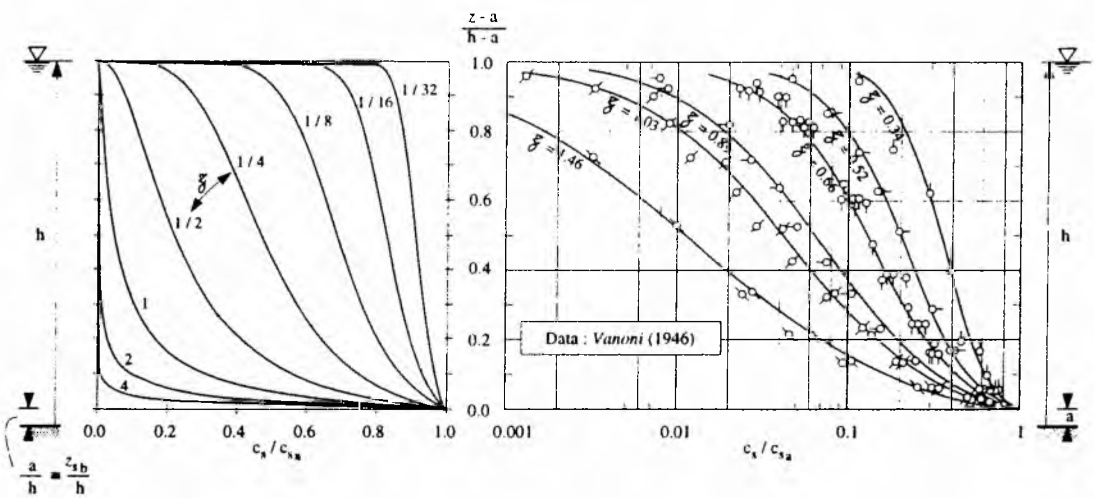


Fig. 6.11 Vertical distribution of the relative concentration, c_s/c_{sa} , in a suspension.

6.4.3 Suspended-Load Relation

1° The volumic solid discharge in suspension per unit width, in a region delimited by $z_{sb} < z < h$, is obtained by :

$$q_{ss} = \int_{z_{sb}}^{h} c_s \, u \, dz \qquad (6.52)$$

where $c_s(z)$ is the local concentration, eq. 6.51, and $u(z)$ is the local velocity. Thi relation is valid for a single particle size, d or v_{ss}.

There exist different methods for the calculation of the suspended-load transpoı (see *Graf*, 1971, p. 189), but only the one of *Einstein* (1950) will be presentec being presently the most popular one.

2° The distribution of the velocity shall be given by a logarithmic relation (se *Einstein*, 1950, p. 17), of the form :

$$u(z) = u_*' \, 5.75 \log \left(30.2 \frac{z}{\Delta}\right) \tag{6.53}$$

where Δ is a correction term, given in Fig. 6.7a, and u_*' is the friction velocity duı to the granulate.

3° Upon substitution of eq. 6.51 and of eq. 6.53 into the above equation, eq. 6.52 one obtains :

$$q_{ss} = \int_{z_{sb}}^{h} c_{sa} \left(\frac{h-z}{z} \cdot \frac{a}{h-a} \right)^{\mathfrak{z}} u_*' \, 5.75 \log \left(30.2 \frac{z}{\Delta}\right) dz \tag{6.54}$$

Replacing $a \equiv z_{sb}$ by a dimensionless expression, $z_{sb}/h = A_E$, and using h as thı unity of z (see *Einstein*, 1950, p. 18), yields :

$$q_{ss} = \int_{z_{sb}}^{h} c_s \, u \, dz = \int_{A_E}^{1} c_s \, u \, h \, dz \tag{6.52a}$$

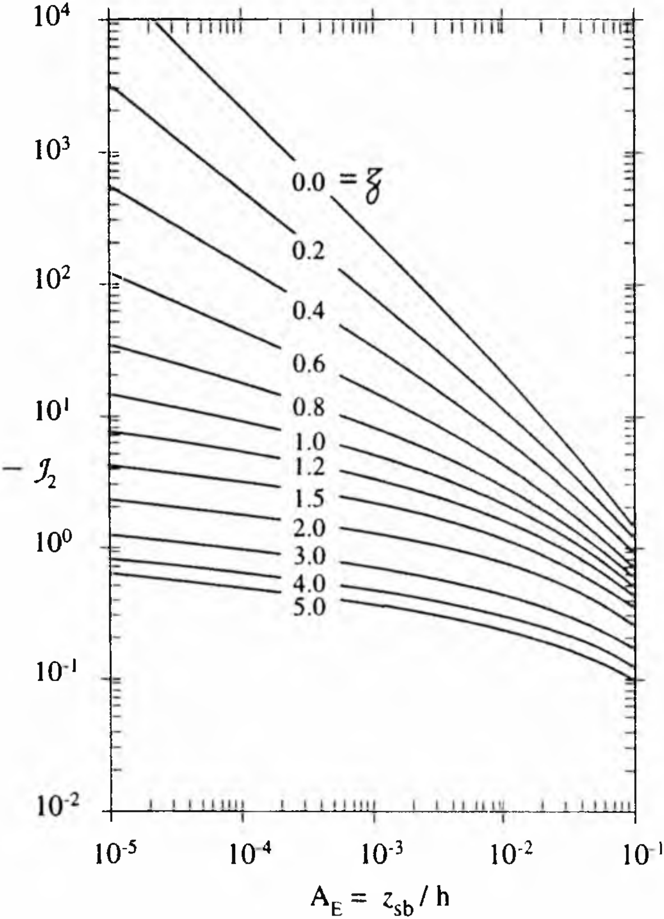
After some mathematical manipulations, one gets :

$$q_{ss} = c_{sa} \, u_*' \; 5.75 \, h \left(\frac{A_E}{1-A_E} \right)^{\mathfrak{z}} \cdot$$

$$\cdot \left\{ \log \left(30.2 \frac{h}{\Delta}\right) \int_{A_E}^{1} \left(\frac{1-z}{z} \right)^{\mathfrak{z}} dz + 0.434 \int_{A_E}^{1} \left(\frac{1-z}{z} \right)^{\mathfrak{z}} \ln z \, dz \right\} \tag{6.55}$$

The values of the following integrals :

$$\mathcal{J}_1 = 0.216 \frac{A_E^{\mathfrak{z}-1}}{(1-A_E)^{\mathfrak{z}}} \int_{A_E}^{1} \left(\frac{1-z}{z} \right)^{\mathfrak{z}} dz$$



, used in the method of *Einstein* (1950).

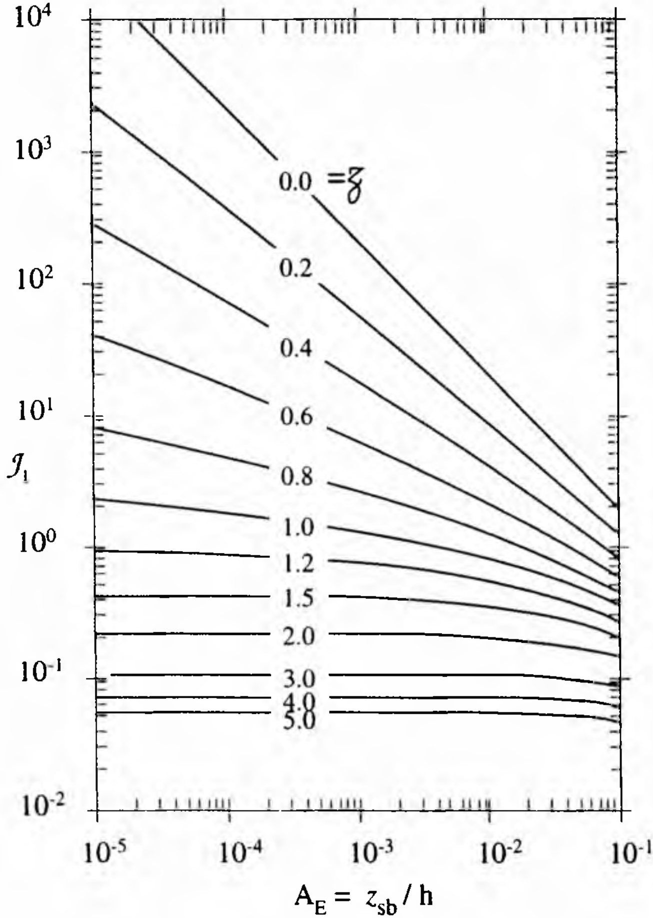


Fig. 6.12 The integrals, $\mathcal{J}_1(A_E, \zeta)$ and $\mathcal{J}_2(A_E, \zeta)$

$$\mathcal{J}_2 = 0.216 \frac{A_E^{\zeta-1}}{(1-A_E)^{\zeta}} \int_{A_E}^{1} \left(\frac{1-z}{z} \right)^{\zeta} \ln z \, dz$$

are numerically evaluated (see *Einstein*, 1950, p. 19-24) and this for diffe values of A_E and ζ ; they are given in Fig. 6.12.

Finally, the above equation, eq. 6.55, can be put into the following form :

$$q_{ss} = 11.6 \, c_{sa} \, u_*' \, z_{sb} \left[2.303 \log \left(30.2 \frac{h}{\Delta} \right) \mathcal{J}_1 + \mathcal{J}_2 \right] \quad (6$$

where q_{ss} is the volumic solid discharge per unit width of the suspended load.

4° The *reference concentration*, c_{sa} , shall be taken there, where the concentra distribution, eq. 6.51, lacks any physical sense, namely very close to the be will thus be positioned within the layer where the bed load moves (see Fig. 6.1

One usually assumes (see *Graf*, 1972, p. 191), that the thickness of this la called *bed layer*, is twice the grain diameter, $z_{sb} \approx 2d$; for a granulometric mixt the bed layer takes different values for each granulometric fraction.

It is now of the foremost interest, to establish a relation between the bed-load suspended-load transport ; the reference concentration, c_{sa} , will make this link.

The formula of Einstein for bed-load transport, eq. 6.42, for one sir granulometric fraction, $q_{sb} \, i_{sb}$, shall be used for the determination of the (aver concentration in this bed layer ; one writes :

$$c_{sa} = \frac{q_{sb} \, i_{sb}}{u_b \, z_{sb}} \quad (6.$$

Exploiting experiments (see *Einstein*, 1950, p. 40) which rendered the velocit bed load as being $u_b = 11.6 \, u_*'$, one obtains an expression for the refere concentration, such as :

$$c_{sa} = \frac{q_{sb} \, i_{sb}}{11.6 \, u_*' \, z_{sb}} \quad (6.5$$

5° Consequently, the solid discharge as suspended load per unit width — using expression, eq. 6.57a — is given by :

$$q_{ss} \, i_{ss} = q_{sb} \, i_{sb} \left[2.303 \log \left(30.2 \frac{h}{\Delta} \right) \mathcal{J}_1 + \mathcal{J}_2 \right] \quad (6$$

$q_{ss}i_{ss}$ being the volumic solid discharge per unit width of the suspended load for one single granulometric fraction.

This relation, eq. 6.58, establishes the link between bed-load and suspended load transport for all particle sizes, which are found in the granulometric fraction of the bed-load.

6.5 TOTAL-LOAD TRANSPORT

6.5.1 Notions

1° Total-load transport of sediments — or better called *total bed-material load transport* — is made up of transport as bed load (see sect. 6.3) and of transport as suspended load :

$$q_s = q_{sb} + q_{ss} (+ q_{sw}) \tag{6.59}$$

Added should (possibly) be the transport as wash load, q_{sw} .

2° Different formulae (see *Graf*, 1971, chap. 9 or *White* et *al.*, 1973) exist, which can be used for the prediction of the bed-material load in a watercourse.

The formulae for determination of the total load — just as the ones of the bed load — give only reasonable results in the domain of their established parameters. Thus an application of any formula must be done with great care.

Here will be given a selection (in chronological order) of some existing formulae.

6.5.2 Total-Load Relations

1° The *indirect* methods determine the bed-material load by addition of the calculated bed load and the calculated suspended load. Thus these methods take into account that the hydromechanics of each mode of transport is not the same. However, a clear distinction between the two modes is not easily possible.

The *direct* methods determine the bed-material load directly, without making a distinction between the two modes of transport.

2° *Einstein* (1950, p. 40) proposed a formula for bed-load transport, eq. 6.42, and one for suspended-load transport, eq. 6.58 ; by combining up these two relations, it is possible to get a formula for the bed-material load transport :

$$q_s\, i_s = q_{sb}\, i_{sb} + q_{ss}\, i_{ss} = q_{sb}\, i_{sb} \left[1 + 2.303 \log \left(30.2 \frac{h}{\Delta}\right) \mathcal{J}_1 + \mathcal{J}_2 \right] \tag{6.60}$$

This relation gives the sediment-transport capacity, but does not, of course, include the wash-load transport.

This formula, eq. 6.60, can be used if the hydraulic and sedimentolog parameters are known in advance. If in addition a measurement of the suspe load is also available, there exists a modified version (see *Graf*, 1971, p. 20 the above relation.

In many ways, the indirect method of *Einstein*, eq. 6.60, is hydraulically ra complete, but its application might seem laborious. Notably, the non-uniformit the granulometry is accounted for by using the ratio of i_s/i_{sb}. Furthermore, considers also the influence of the water temperature (see *Graf*, 1971, p. 238 the velocity distribution, eq. 6.35, and of the concentration distribution, eq. 6 using the exponent of Rouse, eq. 6.50a.

3° A relation for the direct prediction of the bed-material transport, valid for o channel flow [but also for flow in pipes], was developed by *Graf* et *Acar* (1968).

A *parameter of shear intensity* was elaborated as a criteria of solid transport *Graf*, 1971, pp. 218 et 443), such as :

$$\Psi_A = \frac{(s_s - 1)d}{S_e R_h} \qquad (6$$

which is the inverse of the dimensionless shear stress, given by eq. 6.22.

Applying the concept of power (work) of a flow system, a *parameter of trans* was proposed (see *Graf*, 1971, pp. 219 and 446), such as :

$$\Phi_A = \frac{C_s\, UR_h}{\sqrt{(s_s - 1)gd^3}} = \frac{(q_s/q)UR_h}{\sqrt{(s_s - 1)gd^3}} \qquad (6$$

which is similar to the dimensionless intensity of solid discharge, give eq. 6.28.

Note, that the hydraulic radius, R_h , is here taken as the total one ; for a na channel, the hydraulic radius of the bed, R_{hb} , should be taken. C_s is the vol concentration in the section and $d = d_{50}$ is the equivalent diameter.

It could be shown, that a functional relation between these parameters, Ψ_A and (see eq. 6.29a) is possible :

$$\Phi_A = f(\Psi_A)$$

whose form was experimentally determined. Using close to 800 experiments the laboratory and close to 80 experiments in the field (see Table 6.3), all for surface flow [and close to 300 experiments for pipe-line flow], the follo relationship (see *Graf*, 1971, pp. 220 and 448) was established :

$$\Phi_A = 10.39\,(\Psi_A)^{-2.52} \qquad (6$$

This relationship is found valid for $10^{-2} < \Phi_A < 10^3$ or for $\Psi_A \leq 14.6$. An extension of this work by *Graf* et *Acaroglu* (1968) has been done by *Graf* et *Suszka* (1987) ; it provided the following relationship :

$$\Phi_A = 10.4\, K\, (\Psi_A)^{-1.5} \tag{6.63a}$$

with
$$K = \Psi_A^{-1} \qquad \text{if} \qquad \Psi_A \leq 14.6$$
$$K = (1 - 0.045\, \Psi_A)^{2.5} \qquad \text{if} \qquad 22.2 > \Psi_A > 14.6$$
$$K = 0 \qquad \text{if} \qquad \Psi_A > 22.2$$

The trend, for very weak solid transport, $10^{-5} < \Phi_A < 10^{-2}$, with $\Psi_A > 14.6$, is also evident in other experiments (see *Pazis* et *Graf*, 1977).

If one takes in the above relation, eq. 6.61, the energy slope, S_e, defined by eq. 6.2, the functional relation between Ψ_A and Φ_A, eq. 6.63, can also be used for the calculation of the sediment transport during unsteady flow (see *Graf* et *Song*, 1995).

The relations, eq. 6.63 and eq. 6.63a, are also valid when taking an equivalent diameter, $d \cong d_{50}$, if the granulometry is a non-uniform one.

4° For a direct determination of the total-load transport, q_s, *Ackers* et *White* (1973) proposed the use of some sedimentological parameters ; employed were hydraulic considerations and dimensional analysis.

A *parameter of mobility* of sediments was defined as :

$$F_{gr} = \frac{u_*^{\,n_w}}{\sqrt{(s_s - 1)gd}} \left[\frac{U}{\sqrt{32}\,\log(10 h_m/d)}\right]^{(1-n_w)} \tag{6.64}$$

which becomes $F_{gr} = \sqrt{\tau_*}$ (see eq. 6.23) for very fine particles, where $n_w = 1$.

A *parameter of transport* of sediments was postulated as :

$$G_{gr} = C_w \left(\frac{F_{gr}}{A_w} - 1\right)^{m_w} \tag{6.65}$$

The total-load transport is calculated according to :

$$C_s = \frac{q_s}{q} = G_{gr} \frac{d}{h_m} \left(\frac{U}{u_*}\right)^{n_w} \tag{6.66}$$

where C_s is the volumic average concentration in a section and $h_m = A/B$ is the average flow depth.

The coefficients in the above relations were determined by regression an using close to 1000 experiments in the laboratory and close to 250 experim the field, with sediments having a uniform and a non-uniform granulo $0.04 < d_{50}$ [mm] < 4.0 and for flow at $Fr < 0.8$ (see Table 6.3). The res values of these coefficients are the following :

coefficient	$d_* > 60$ $d > 2.5$ [mm]	$1.0 < d_* \leq 60$	$d_* < 1$ $d < 0.04$ [
n_w	0.0	$(1.0 - 0.56 \log d_*)$	1.0
m_w	1.50	$(9.66/d_*) + 1.34$	
A_w	0.17	$(0.23/\sqrt{d_*}) + 0.14$	
C_w	0.025	$\log C_w = 2.86 \log d_* - (\log d_*)^2 - 3.53$	

Above, the dimensionless particle diameter, d_*, is used, defined as :

$$d_* = d \left[(s_s - 1) \frac{g}{v^2} \right]^{1/3} \qquad ($$

For a non-uniform granulometry, one takes $d = d_{35}$ as the equivalent diamete

6.5.3 Applications of Relations

1° Different formulae for the determination of the solid transport have been pres However, none of these relations can pretend to translate the intrinsic comple the transport of sediments.

2° Most of these formulae should not be used beyond the conditions within whic were established. Table 6.3 contains a summary of the range of the param d and S_f, investigated for the establishment of each formula by their auth other author(s) may have extended this range. Also listed are the recommen by the author(s) for the choice of the equivalent diameter, d_x, if the granulom quasi or non-uniform.

3° The formulae for the transport of sediments are often established, using labo data and less often using field data.

A verification of these formulae in watercourses is a very delicate task, sinc difficult to measure correctly the solid discharge in the field. Furthermore, it is a rather subjective evaluation, since the zones of the modes of transport c easily be separated.

4° Numerous studies have been reported, comparing measurements in waterco with the different existing formulae.

For a better appreciation of the validity of the above presented formulae, it wil be of interest to compare the computed results with the direct measurements solid discharge in the field.

Table 6.3 Parameters used for establishing the different formulae.

Formula	d [mm]	S_f [-]	d_τ [mm], equivalent diameter for a non-uniform granulate
Schoklitsch (eq. 6.31)	0.3 – 7.0 (44.0)	0.003 – 0.1	d_{40}
Meyer-Peter et *al.* (eq. 6.32)	3.1 – 28.6	0.0004 – 0.020	$d_m(d_{50})$
Einstein (eq. 6.42)	0.8 – 28.6	–	d_{35}
Graf et *Acaroglu* (eq. 6.63)	0.3 – 1.7 (23.5)	–	d_{50}
Ackers et *White* (eq. 6.66)	0.04 – 4.0	Fr < 0.8	d_{35}

Many (nineteen) of the existing formulae for the calculation of the total transport have been studied by *White* et *al.* (1973) and compared with experimental results. They evaluated almost 1000 laboratory experiments with uniform and non-uniform sediments of $0.04 < d_{50}$ [mm] < 4.9, at flow depth of $h < 0.4$ [m], and almost 270 experiments in watercourses with sediments of $0.1 < d$ [mm] < 68.0 and a width/depth ratio of $9 < B/h < 160$.

Each formula was applied to all the data of the solid-discharge measurements. Subsequently was established a ratio of the values calculated, C_{calc} , and the values observed, C_{obs}, where $C \equiv C_s$ is the total-load transport, expressed in concentration.

Some results of this investigation are given in Fig. 6.13, where one may see the success of a prediction (in percentage) for different ranges of the ratio, C_{calc}/C_{obs}. For the formulae, which are presented in this book — considering only the range of $1/2 < C_{calc}/C_{obs} < 2$ — it can be seen that the percentage is for the formula of

Einstein (1950), eq. 6.60 : 44 % of success

Graf et *Acaroglu* (1968), eq. 6.63 : 40 % of success

Ackers et *White* (1973), eq. 6.66 : 64 % of success

This implies that with the formula of *Ackers* et *White*, 64 % of the experimen data can be predicted in the above-mentioned range. This is usually considered a good (or a not-so-bad) result ; more than half of the studied (nineteen) formu give results which are less good, namely < 40 %. Also noticed is that with formula of *Einstein* there is a slight under-estimation of the solid discharge ; wh the one of *Graf* et *Acaroglu* gives a slight over-estimation.

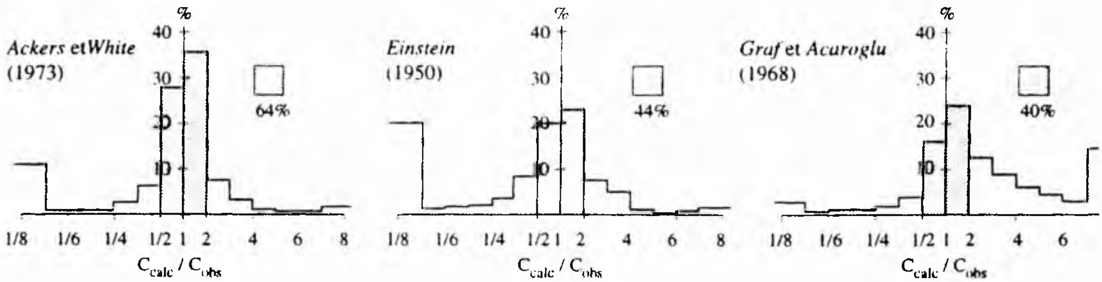


Fig. 6.13 Comparison, with respect to C_{calc}/C_{obs}, of the success of prediction for the presented formulae.

The comparative study of *White* et *al.* (1973) is reasonably objective, but certair not conclusive. Other studies exist (see *Raudkivi*, 1976, p. 227) which shc clearly that an objective validation is nearly impossible.

5° Amongst the different existing formulae for the determination of the total-lo transport, but equally for the ones of the bed-load and suspended-load transpo each one will give an answer, but none will be very precise nor very true.

Finally, it must be said, that the results obtained with these formulae give on valuable guide-lines for the engineer. For practical purposes, it is advised to consı more than one formula ; the obtained result may however render different valu (see *Graf*, 1971, p. 156).

6.5.4 Wash Load

1° The wash load, q_{sw}, contains all these particles which are never in contact with tl bed and displace themselves by being carried (washed) through the channel by tl flow (see Fig. 6.1).

This mode of the transport of sediments (see Table 6.2) is limited to the very fine particles which are rare in the granulometry of the bed material. The distribution these particles is rather uniform over the entire flow depth (see Fig. 6.11).

Einstein (1950, p. 7) has proposed that the granulometry of the wash load is the fraction of granulometry of the bed which is smaller than 10 %. It was also proposed (see *Raudkivi*, 1976, p. 220) that the wash load is composed of the fine particles having a diameter of $d < 0.06$ [mm].

2° Since there exists no physical relationship to the flow, it has been difficult to advance an analytical method for the determination of the wash load.

The wash load depends more on the hydrological, geomorphological and meteorological conditions within the drainage basin (see *Graf*, 1971, p. 232), namely on the overland surface erosion and less on the erosion in the stream bed.

3° Thus it is to be remarked, that at the present no methods exist for the prediction of the wash load.

In order to obtain a quantitative information on the wash load, measurements in the field must be performed. One measures thus the total suspended load, $q_{ss} + q_{sw}$. Subsequently is calculated the suspended load, q_{ss}, (see sect. 6.4) and consequently the suspended wash load, q_{sw}, can be obtained.

4° In some watercourses, the transport as wash load can be much more important than the bed-material load, $q_{sw} > q_s$. Obviously, this makes the problem of sediment transport hopelessly complicated.

If the total suspended-load transport, $q_{ss} + q_{sw}$, becomes very large, one may well imagine that this influences on the flow behaviour ; such a mixture of water-sediments is probably not anymore a Newtonian one (see Table 6.1). The flow of such a non-Newtonian mixture will modify the hydraulics, thus the distribution of the velocity and of the concentration, but also the flow resistance as well as the bed forms.

An early version of section 6.2 was published as :
Graf W.H. (1994) : *Les équations de Saint-Venant-Exner*.
Österr. Ing. und Arch. Zeitschrift, Jgg. 139, N°9, Wien, A

6.6 EXERCISES

6.6.1 Problems, solved

Ex. 6.A

A rectangular channel has a width of B = 5 [m]. At some point, the bed of the chann changes from a fixed bed to a mobile bed with a uniform sediment of d_{50} = 1 [mm] ar s_s = 2.6 [-]. The discharge of Q = 15 [m^3/s] remains constant and the water depth h = 2.2 [m]. In the fixed-bed reach of the channel there is no sediment transport. Th flow initiates however erosion in the mobile-bed reach of the channel, where the porosi of the bed material is p = 0.3 [-].

A *degradation* of the channel starts at the junction between the fixed bed and the mobi bed. Determine the time it will take to lower the bed level down to $z = 0.4\Delta h$ at a static located at $L = 6R_h/S_f$ downstream from the junction; subsequently draw the bed profi for this particular moment. Furthermore, show the temporal variation of the degradatic at this station. Calculate also the resulting bed profile if the mobile bed is limited to length of x_f = 90 [km].

SOLUTION :

i) The steady flow will be considered to be quasi-uniform during the phase (degradation (see Fig. Ex. 6.A.1); therefore the *parabolic model* can be used :

$$\frac{\partial z}{\partial t} - K\frac{\partial^2 z}{\partial x^2} = 0 \qquad (6.1$$

where x is positive towards the downstream and follows the initial bed profil z represents the bed-level variation with respect to the initial bed, S_f^o. Note that tl use of the parabolic model is limited to : Fr < 0.6 and $x > 3R_h/S_e$.

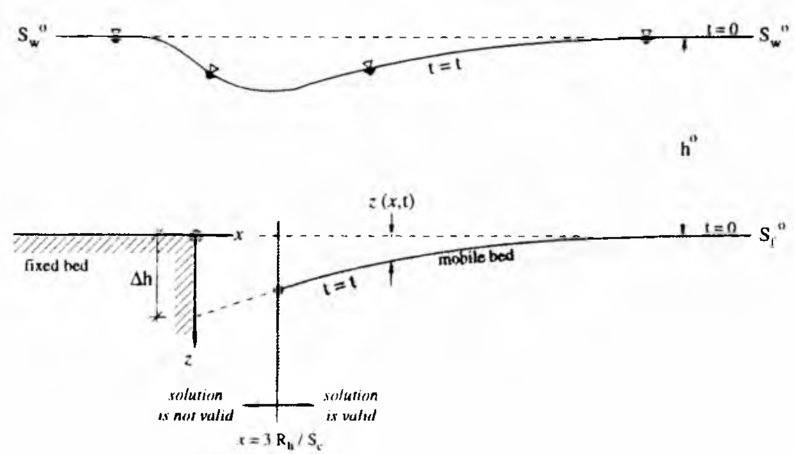


Fig. Ex.6.A.1 Scheme of the degradation.

The initial and boundary conditions are given as :

$$z(x,0) = 0 \quad ; \quad \lim_{x \to \infty} z(x,t) = 0$$

$$z(0,t) = \Delta h(t)$$

The solution to eq. 6.11 is given by :

$$z(x,t) = \Delta h \, erfc \left(\frac{x}{2\sqrt{Kt}} \right) \tag{6.15}$$

ii) Calculation of the quasi-uniform *flow* in the mobile-bed channel.

The slope of the energy line, S_e, is calculated using the Manning-Strickler formula:

$$U = \frac{Q}{Bh} = K_s \, R_h^{2/3} \, S_e^{1/2} \tag{3.16}$$

with
$$\begin{aligned} K_s &= 21.1/d_{50}^{1/6} = 66.7 \, [m^{1/3}/s] \\ h &= 2.2 \, [m] \quad , \quad B = 5.0 \, [m] \quad , \quad R_h = 1.17 \, [m] \\ Q &= 15.0 \, [m^3/s] \quad , \quad q = Q/B = 3 \, [m^2/s] \\ U &= q/h = 1.36 \, [m/s] \end{aligned} \tag{3.18}$$

The slope of the energy line : $S_e = \mathit{0.00034}$ [-]

The Froud number is : $Fr = \dfrac{U}{\sqrt{gh}} = 0.29$ [-]

It should be emphasized that the Froude number has to be small, $Fr < 0.6$, being one of the conditions (see sect. 6.2.3) for the validity of the parabolic model, eq. 6.11.

iii) Calculation of the *solid discharge* in the mobile-bed channel.

The solid discharge, $q_s = C_s U h$, is calculated using the *Graf* et *al.* (1968) formula :

$$\frac{C_s \, U R_h}{\sqrt{[(\rho_s - \rho)/\rho] \, g \, d_{50}^3}} = 10.39 \left\{ \frac{[(\rho_s - \rho)/\rho] \, d_{50}}{S_f \, R_h} \right\}^{-2.52} \tag{6.63}$$

with
$$\begin{aligned} (\rho_s - \rho)/\rho &= 1.6 \, [-] \\ d_{50} &= 1 \, [mm] \\ S_f \equiv S_e &= 0.00034 \, [-] \end{aligned}$$

$$C_s \, U R_h = 3.9 \cdot 10^{-5} \, [m^2/s]$$

The solid discharge is : $q_s = C_s \, U \, h \, \dfrac{R_h}{R_h} = 3.9 \cdot 10^{-5} \, \dfrac{2.2}{1.17} = \mathit{7.3 \cdot 10^{-5}} \, [m^2/s]$

iv) The *coefficient*, K , in the parabolic model, eq. 6.11, is approximately given by :

$$K_o \equiv K \approx \frac{1}{3}\, b_s q_s \, \frac{1}{(1-p)} \, \frac{1}{S_e^{\,o}} \qquad (6.12$$

with $S_e^{\,o}$ = 0.00034 [-]

$(1-p)$ = 0.7 [-]

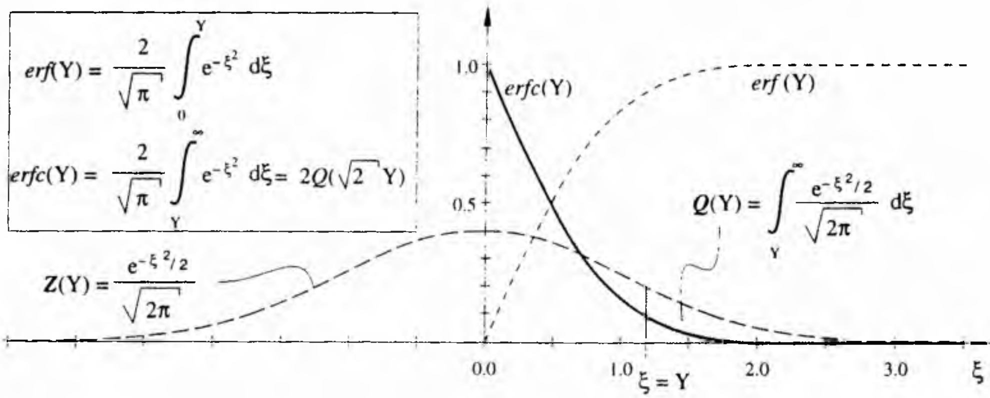
b_s = 2 (2.52) ≅ 5 (where β = 2.52 is the exponent in eq. 6.6 according to eq. 6.5a and eq. 6.30)

The coefficient is : K = *0.511* [m²/s]

Table for the complementary error function

(see *Handbook of Mathematical Functions*, 1964, National Bureau of Standards, pp. 310-311, formula 7.1.2

Y	*erfc*(Y)	Y	*erfc*(Y)	Y	*erfc*(Y)
0.00	*1.00000*	1.20	*0.08969*	2.30	*0.00114*
0.10	*0.88754*	1.30	*0.06599*	2.40	*0.00069*
0.20	*0.77730*	1.40	*0.04772*	2.50	*0.00041*
0.30	*0.67137*	1.50	*0.03390*	2.60	*0.00024*
0.40	*0.57161*	1.60	*0.02365*	2.70	*0.00013*
0.50	*0.47950*	1.70	*0.01621*	2.80	*0.00008*
0.60	*0.39614*	1.80	*0.01091*	2.90	*0.00004*
0.70	*0.32220*	1.90	*0.00721*	3.00	*0.00002*
0.80	*0.25790*	2.00	*0.00468*	3.10	*0.00001*
0.90	*0.20309*	2.10	*0.00298*	3.20	*0.00001*
1.00	*0.15730*	2.20	*0.00186*	3.30	*0.00000*
1.10	*0.11979*				



The complementary error function can be calculated approximately using the following expression :

$erfc(Y) = 1 / (1 + a_1 Y + a_2 Y^2 + a_3 Y^3 + a_4 Y^4 + a_5 Y^5 + a_6 Y^6)^{16} + \varepsilon(Y)$ where $|\varepsilon(Y)| \le 3 \cdot 10^{-7}$

$a_1 = 0.0705230784$; $a_2 = 0.0422820123$; $a_3 = 0.0092705272$

$a_4 = 0.0001520143$; $a_5 = 0.0002765672$; $a_6 = 0.0000430638$

v) In the present problem, it is asked to determine the time it takes to lower the bed level down to $z = 0.4\Delta h$, thus :

$$\frac{z(x,t)}{\Delta h} = \frac{0.4\Delta h}{\Delta h} = 0.4$$

The eq. 6.15 is now written as :

$$0.4 = erfc\left(\frac{x}{2\sqrt{Kt}}\right) = erfc\,(Y)$$

Using the table of the complementary error function yields :

$$Y \cong 0.6 = \left(\frac{x}{2\sqrt{Kt}}\right) \qquad \Rightarrow \qquad t \cong \frac{x^2}{4\ Y^2\ K} \cong \frac{x^2}{1.44\ K}$$

At the station $x \equiv L = 6R_h/S_e = 20.73$ [km], the lowering of the bed down to a level of $z = 0.4\Delta h$ occurs at the time :

$$t = \frac{(20.73 \cdot 10^3)^2}{(1.44)\ (0.511)} = 5.84 \cdot 10^8\ [s] = \mathit{1.62 \cdot 10^5}\ [h] \cong \mathit{18.52}\ [years]$$

To draw the bed profile for the entire channel at this particular moment, $t = 5.84 \cdot 10^8$ [s], the calculations are repeated for different values for the distance x (see following table).

Calculation of the bed profile				
$R_h = 1.17$ [m] ; $S_f = 0.00034$ [-] ; $K = 0.511$ [m^2/s] ; $\Delta h =$ **3.11** [m] ; $t = 5.84 \cdot 10^8$ [s]				
x [m]	$x\ (S_e / R_h)$ [-]	$Y = x/(2\sqrt{Kt})$ [-]	$z/\Delta h = erfc(Y)$ [-]	z [m]
10500	3.04	0.30	0.66735	2.073
11000	3.18	0.32	0.65253	2.027
13000	3.76	0.38	0.59465	1.847
15000	4.34	0.43	0.53923	1.675
20000	5.79	0.58	0.41299	1.283
20730	*6.00*	*0.60*	*0.39615*	*1.231*
30000	8.68	0.87	0.21946	0.682
40000	11.58	1.16	0.10157	0.316
50000	14.47	1.45	0.04070	0.126
60000	17.37	1.74	0.01405	0.044
70000	20.26	2.03	0.00417	0.013
80000	23.15	2.32	0.00106	0.003
90000	26.05	2.60	0.00023	0.001
100000	28.94	2.89	0.00004	0.000

The depth of degradation of the channel bed due to a solid discharge $q_s = 7.3 \cdot 10^{-5}$ [m^2/s] , during a time period of t = $5.84 \cdot 10^8$ [s] is given by :

$$\Delta h = \frac{q_s \cdot \Delta t}{1.13\,(1-p)\sqrt{K\,\Delta t}} = \frac{(7.3 \cdot 10^{-5})\,\sqrt{5.84 \cdot 10^8}}{(1.13)\,(0.7)\,\sqrt{0.511}} = \mathbf{3.11}\ [\mathrm{m}] \qquad (6.2$$

and $z = 0.4\Delta h = 1.23$ [m] .

The bed profile, $z(x)$, for t = $5.84 \cdot 10^8$ [s] = 18.52 [years], is plotted Fig. Ex. 6.A.2. This solution is valid only if $x > 3R_h/S_e$. For distances $x < 3R_h/S_e$, the solution is only an indicative one.

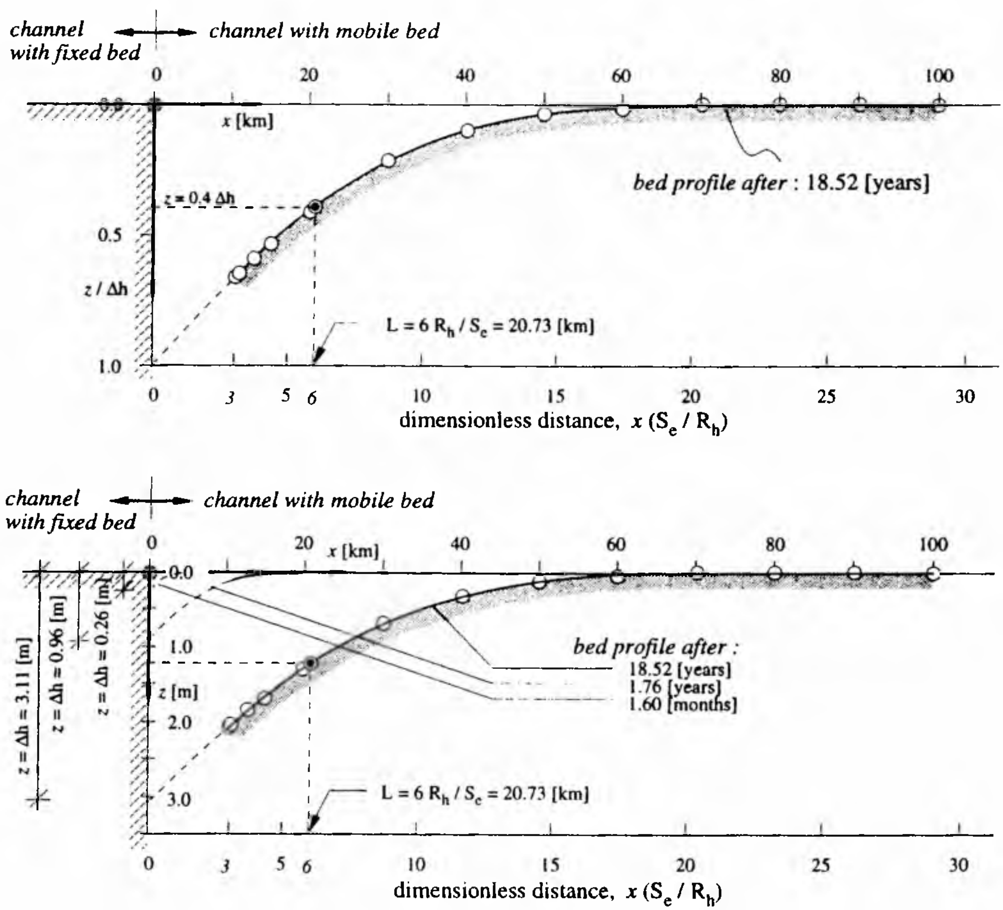


Fig. Ex.6.A.2 Bed profile after *18.52* [years] of degradation.

For sake of comparison, the bed profiles, $z(x)$, for t = 1.76 [year] and f t = 1.6 [month] are also plotted (without giving the calculations) in Fig. Ex. 6.A.

vi) The *temporal evolution of the degradation* at the station located at $x \equiv L = 6R_h/S_e = 20.73$ [km] is given by :

$$z(t) = \Delta h \; erfc\left(\frac{x}{2\sqrt{K\;\Delta t}}\right) = \Delta h \; erfc\left(\frac{20730}{2\sqrt{0.511\;\Delta t}}\right) \tag{6.15}$$

where, $\Delta h(t)$ can be evaluated by :

$$\Delta h = \frac{q_s \cdot \Delta t}{1.13\;(1-p)\sqrt{K\;\Delta t}} \tag{6.20}$$

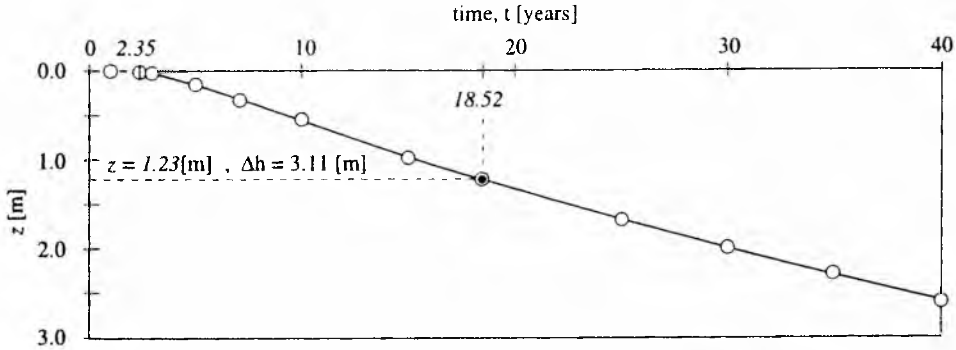


Fig. Ex.6.A.3 Evolution of the degradation at the station $x \equiv L = 6R_h/S_e = 20.73$ [km].

Calculation of the evolution of the degradation					
$R_h = 1.17$ [m] ; $S_f = 0.00034$ [-] ; $K = 0.511$ [m^2/s] $x \equiv L = 6R_h/S_{f_0} = 20730$ [m]					
t [years]	t [s]	$Y = x/(2\sqrt{Kt})$ [-]	$z/\Delta h = erfc(Y)$ [-]	Δh [m]	z [m]
1	3.15E+07	2.58	0.00026	0.72	0.0002
3	9.46E+07	1.49	0.03502	1.25	0.0438
5	1.58E+08	1.15	0.10248	1.61	0.1654
7	2.21E+08	0.98	0.16756	1.91	0.3201
10	3.15E+08	0.82	0.24823	2.28	0.5667
15	4.73E+08	0.67	0.34579	2.80	0.9669
18.52	*5.84E+08*	*0.60*	*0.39618*	*3.11*	*1.2309*
25	7.88E+08	0.52	0.46522	3.61	1.6794
30	9.46E+08	0.47	0.50500	3.95	1.9970
35	1.10E+09	0.44	0.53711	4.27	2.2941
40	1.26E+09	0.41	0.56371	4.57	2.5740
45	1.42E+09	0.38	0.58622	4.84	2.8391
50	1.58E+09	0.37	0.60559	5.11	3.0916

The evolution of the bed degradation can now be calculated by assuming diff values for $\Delta t \equiv t$. By using the approximate formula for the complementary function (see before), the calculation can easily be programmed on a spreads The table above summarizes these calculations; Fig. Ex. 6.A.3 shows the evol of the erosion, $z(t)$, at the station, $x \equiv L$.

This solution is however only valid (see *Ribberink* et *Sande*, 1984, p. 30) for :

$$t > \frac{40}{30}\frac{R_h^2}{S_f}\frac{1}{q_s} = \frac{40}{30}\frac{1.17^2}{0.00034}\frac{1}{7.3\cdot 10^{-5}} = 7.42\cdot 10^7 \text{ [s]} \cong 2.35 \text{ [years]}$$

vii) Calculation of the final bed profile if the channel reach with the mobile b *limited* to a length of $x_f = 90$ [km].

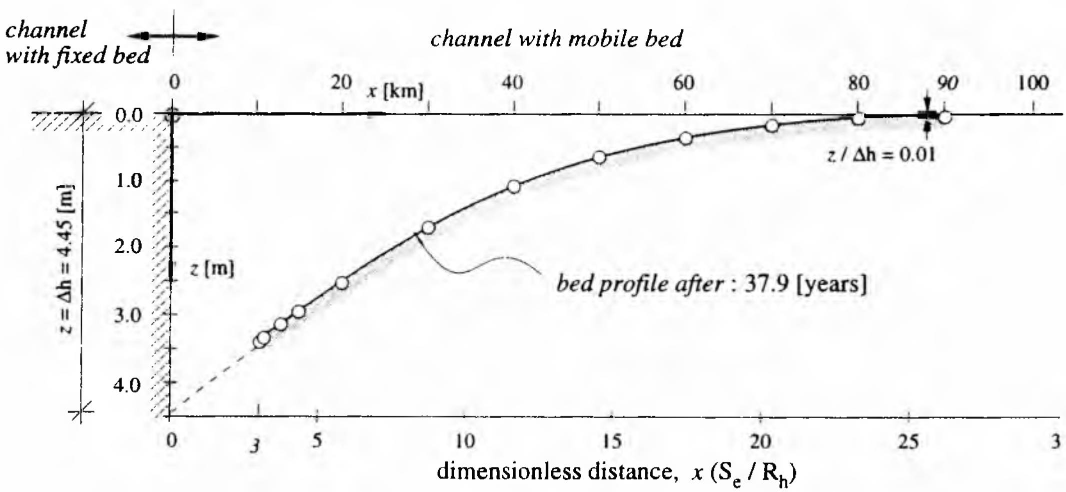


Fig. Ex.6.A.4 The channel-bed profile after *37.9* [years] of degradation

By assuming a very small amount of erosion, such as $z = 0.01\Delta h$, at the st $x_f = 90$ [km] , one can write :

$$\frac{z(x,t)}{\Delta h} = 0.01 = erfc\left(\frac{x_f}{2\sqrt{Kt}}\right) = erfc(Y)$$

Using the table of the complementary error function yields :

$$Y = 1.82 = \left(\frac{x}{2\sqrt{Kt}}\right) \qquad \Rightarrow \qquad t = \frac{x^2}{4\,Y^2\,K} \cong \frac{x^2}{13.25\,K}$$

and with $K = 0.511$ [m²/s], one calculates :

$$t = \frac{(90 \cdot 10^3)^2}{(13.25)\,(0.511)} = 1.2 \cdot 10^9 \text{ [s]} = 3.3 \cdot 10^5 \text{ [h]} \cong 37.93 \text{ [years]}$$

To obtain the bed profile for the entire channel at this moment, $t = 1.2 \cdot 10^9$ [s], the calculations for the degradation are repeated using different values for x (see the following table). The final bed profile, calculated in this way, is plotted in Fig. Ex. 6.A.4.

This solution is valid only if $x > 3R_h / S_e$.

The depth of the bed degradation due to a solid discharge, $q_s = 7.3 \cdot 10^{-5}$ [m²/s] , during a time period of $t = 1.2 \cdot 10^9$ [s] , is given by the eq. 6.20 :

$$\Delta h = \frac{q_s \cdot \Delta t}{1.13\,(1-p)\sqrt{K\ \Delta t}} = \frac{(7.3 \cdot 10^{-5})\sqrt{1.2 \cdot 10^9}}{(1.13)\,(0.7)\,\sqrt{0.511}} = \mathbf{4.45} \text{ [m]}$$

Calculation of the final bed profile				
$R_h = 1.17$ [m] ; $S_f = 0.00034$ [-] ; $K = 0.511$ [m²/s] $\Delta h = \mathbf{4.45}$ [m] ; $t = 1.2 \cdot 10^9$ [s]				
x [m]	$x\ (S_e / R_h)$ [-]	$Y = x/(2\sqrt{Kt})$ [-]	$z/\Delta h = erfc(Y)$ [-]	z [m]
10500	3.04	0.21	0.76396	3.397
11000	3.18	0.22	0.75307	3.349
13000	3.76	0.26	0.71005	3.157
15000	4.34	0.30	0.66793	2.970
20000	5.79	0.40	0.56734	2.523
30000	8.68	0.61	0.39091	1.738
40000	11.58	0.81	0.25264	1.123
50000	14.47	1.01	0.15273	0.679
60000	17.37	1.21	0.08617	0.383
70000	20.26	1.42	0.04529	0.201
80000	23.15	1.62	0.02214	0.098
90000	*26.05*	*1.82*	*0.01006*	*0.045*

Ex. 6.B

A river on a bed slope of $S_f = 0.0005$ [-] conveys a unit discharge of $q = 1.5$ [m^2/s]. T river bed is made of granular material of uniform size of $d_{50} = 0.00032$ [m] witl specific gravity of $s_s = 2.6$ [-]; the porosity of the bed material is $p = 0.4$ [-]. There exi a weak transport of sediments.

At a certain station on this river, the solid discharge is locally increased $\Delta q_s = 0.0001$ [m^2/s] for a time period of $\Delta t = 50$ [h]. Determine the *aggradation* of t bed to be expected.

SOLUTION :

i) The flow is steady and is considered to be quasi-uniform during the period aggradation (see Fig. Ex. 6.B.1); thus the *parabolic model* can be used :

$$\frac{\partial z}{\partial t} - K\frac{\partial^2 z}{\partial x^2} = 0 \qquad (6.1$$

where x is positive towards the downstream and follows the initial bed profi z represents the bed-level variation with respect to the initial bed, S_{f_0}. Note that t use of the parabolic model is limited to : Fr < 0.6 and $x > 3R_h/S_e$.

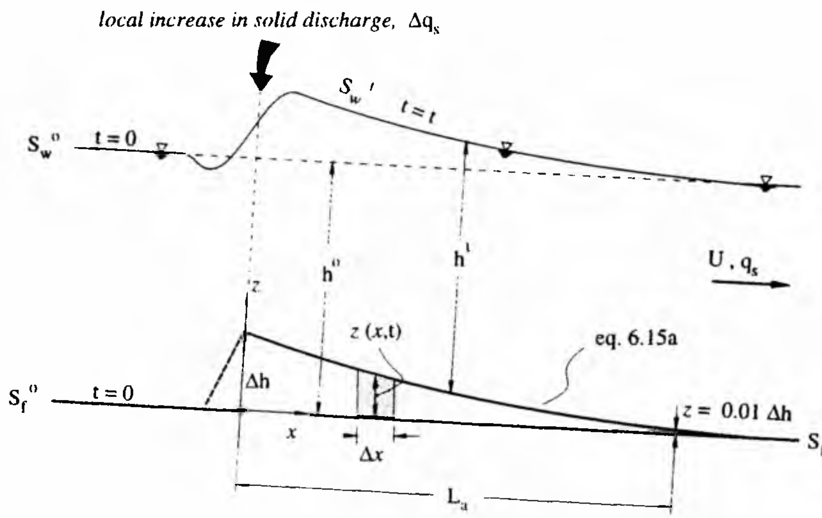


Fig. Ex.6.B.1 Sketch of the aggradation.

The initial and boundary conditions are given as :

$$z(x,0) = 0 \qquad ; \qquad \lim_{x \to \infty} z(x,t) = 0$$

$$z(0,t) = \Delta h(t)$$

The solution to eq. 6.11 is given by :

$$z\,(x,t) = \Delta h\ erfc\ \left(\frac{x}{2\sqrt{Kt}}\right) \tag{6.15}$$

ii) Calculation of the quasi-uniform *flow* in the river having a mobile bed.

The normal depth is calculated using the Manning-Strickler formula :

$$U = \frac{q}{h} = K_s\ h^{2/3}\ S_f^{\ 1/2} \tag{3.16}$$

with $K_s = 21.1/d_{50}^{\ 1/6} = 80.7\ [m^{1/3}/s]$ (3.18)
$q = 1.5\ [m^2/s]$
$S_f = 0.0005\ [-]$

The flow depth is : $h = \mathit{0.895}$ [m]
The average velocity is : $U = 1.676$ [m/s]
The Froude number is : $Fr = \dfrac{U}{\sqrt{gh}} = 0.566$

It should be remembered that the Froude number has to be small, namely $Fr < 0.6$.

iii) Calculation of the *solid discharge* in the river having a mobile bed.

The solid discharge, $q_s = C_s\,Uh$, is calculated using the relationship given by *Graf* et *al.* (1968) :

$$\frac{C_s\ UR_h}{\sqrt{[(\rho_s-\rho)/\rho]g\ d_{50}^3}} = 10.39\ \left\{\frac{[(\rho_s-\rho)/\rho]\ d_{50}}{S_f\ R_h}\right\}^{-2.52} \tag{6.63}$$

with $(\rho_s-\rho)/\rho = 1.6\ [-]$
$d_{50} = 0.32$ [mm]
$R_h \cong h = 0.895$ [m]

The solid discharge is : $q_s = \mathit{1.678 \cdot 10^{-4}}\ [m^2/s]$

iv) The *coefficient*, K , in the parabolic model, eq. 6.11, is approximately given by :

$$K_o \equiv K \approx \frac{1}{3}\ b_s q_s\ \frac{1}{(1-p)}\ \frac{1}{S_e^{\ o}} \tag{6.12c}$$

with $S_f^{\ o} \equiv S_e^{\ o} = 0.0005\ [-]$
$(1-p) = 0.6\ [-]$
$b_s = 2\,(2.52) \cong 5$ (where $\beta = 2.52$ is the exponent in eq. 6.63, according to eq. 6.5a and eq. 6.30)

The coefficient is : $K = \mathit{0.932}\ [m^2/s]$

v) The thickness of the aggradation of the bed (see Fig. Ex. 6.B.1) due to a loc increase in solid discharge, $\Delta q_s = 0.0001$ [m^2/s], during a time period ($\Delta t = 50$ [h] $= 1.8 \cdot 10^5$ [s] , is given by eq. 6.20, or :

$$\Delta h(t) = \frac{\Delta q_s \cdot \Delta t}{1.13\,(1-p)\sqrt{K\ \Delta t}} = \frac{(0.0001)\sqrt{1.8 \cdot 10^5}}{(1.13)\,(0.6)\,\sqrt{0.932}} = \mathbf{0.065}\ [\mathrm{m}]$$

The length of the zone of aggradation, L_a, can be calculated with eq. 6.15 b assuming, for example, a precision of $z/\Delta h = 0.01$:

$$\frac{z(x,t)}{\Delta h} = \frac{0.01\Delta h}{\Delta h} = 0.01 = erfc\left(\frac{x}{2\sqrt{K\ \Delta t}}\right) = erfc\,(Y)$$

Using the table of the complementary error function (see Ex. 6.A), yields :

$$Y = 1.821 = \left(\frac{x}{2\sqrt{K\ \Delta t}}\right)$$

The length of the zone of aggradation (see eq. 6.19) can now be calculated a follows :

$$L_a \equiv x_{1\%} = 2Y\sqrt{K\ \Delta t} = (2)\,(1.821)\,\sqrt{(0.932)\,(1.8 \cdot 10^5)} = \mathit{1492.3}\ [\mathrm{m}]$$

vi) To plot the bed profile after a time period of $\Delta t = 50$ [h] $= 1.8 \cdot 10^5$ [s], calculation are made using eq. 6.15 for different distances, x. (see the following table).

The resulting bed profile, $z(x)$, is plotted in Fig. Ex. 6.B.2.

The calculations, summarized in the following table, are valid only if $x > 3h/S_e$ In the present case, it can be shown that :

$$x = 3h/S_e = (3)\,(0.895)\,/\,(5 \cdot 10^{-4}) = 5370\ [\mathrm{m}] \gg L_a = 1492.3\ [\mathrm{m}]$$

However, experimental data (see *Soni* et *al.*, 1980), have shown that the calculate value is only indicative, but nevertheless acceptable.

Calculation of the bed profile due to aggradation				
$R_h = h = 0.895$ [m] ; $S_f = 0.0005$ [-] ; $K = 0.932$ [m^2/s]				
$\Delta h = \mathbf{0.065}$ [m] ; $\Delta t = 1.8 \cdot 10^5$ [s]				
x [m]	$x\ (S_e / R_h)$ [-]	$Y = x/(2\sqrt{Kt})$ [-]	$z/\Delta h = erfc(Y)$ [-]	z [m]
10.0	0.01	0.01	0.98623	0.064
50.0	0.03	0.06	0.93123	0.060
100.0	0.06	0.12	0.86296	0.056
300.0	0.17	0.37	0.60459	0.039
500.0	0.28	0.61	0.38813	0.025
700.0	0.39	0.85	0.22696	0.015
900.0	0.50	1.10	0.12032	0.008
1000.0	0.56	1.22	0.08434	0.005
1100.0	0.61	1.34	0.05761	0.004
1300.0	0.73	1.59	0.02484	0.002
1492.3	*0.83*	*1.82*	*0.01000*	*0.001*
1500.0	0.84	1.83	0.00962	0.001
1600.0	0.89	1.95	0.00575	0.000

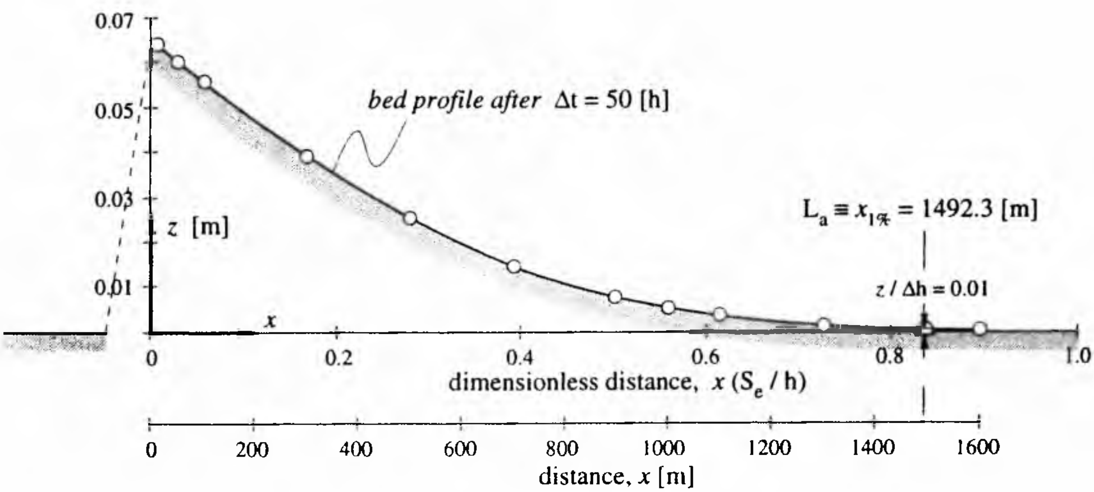


Fig. Ex.6.B.2 Bed profile after 50 [hours] of aggradation.

Ex. 6.C

The unit discharge of a river is kept constant at $q = 2.5$ [m^2/s]. The bed slope $S_f = 5.4 \cdot 10^{-4}$ [-]. The river bed is composed of quasi-uniform sediments ($s_s = 2.65$ [- with an average grain size of $d_{50} = 6$ [mm] and a porosity of $p = 0.3$ [-]. The Mannin coefficient of the bed was determined as being $n = 0.032$ [m$^{-1/3}$s].

This river enters into a reservoir, created by a dam which keeps the water at a height ($H = 23.5$ [m] at the immediate vicinity of the dam.

Determine the deposition pattern of bed-load material, which is carried by the river int the reservoir, after 20 [years] and 100 [years], respectively.

SOLUTION :

a) *General comments on the method of solution* :

Fig. Ex.6.C.1 shows the longitudinal profile of this river-reservoir system. The dar creates a backwater curve extending to a certain upstream distance. This curve can b calculated by using one of the methods presented in chap. 4. The backwater calculatic enables one to know the hydraulic parameters (average velocity, water depth, slope (energy grade line, etc.) for the entire length of the system.

Let there be two stations, (i) and (i+1), separated by a distance of Δx. If th characteristics of the sediments at the bed are known, one can calculate the bed-loa discharge for these two stations, $q_{sb}(i)$ and $q_{sb}(i+1)$, by using one of the bed-loa formulae presented in sect. 6.3.3. It will then be seen that the bed-load transport at th upstream station, (i+1) , is larger than the one at the downstream station, (i). In fact th closer a station is to the dam, the larger is the water depth, resulting in a smaller averag velocity and as a consequence in a decrease in the bed-load transport capacity. Th difference of the transport capacities between the two consecutive stations causes deposition (or erosion) of the sediments, which in turn modifies the bed level. Th modification of the bed level causes a change in the water-surface profile and therefo modifies all hydraulic parameters. This cycle repeats itself.

To calculate the deposition of the sediments, namely the *delta* formation, one has t simulate the process described above. Such a simulation involves a large number (calculations and therefore is particularly well suited to treatment on a computer.

In this exercise, a computer program in FORTRAN IV language has been written to carr out this simulation. The program is written in standard FORTRAN and can be run c most of the personal computers. Although a basic knowledge of computers and (programming in languages like FORTRAN, BASIC or PASCAL will certainly be helpfi in understanding this exercise, it is not essential. Special care is taken to make the gener programming techniques understandable to everybody, even to those who do not hav any experience in programming.

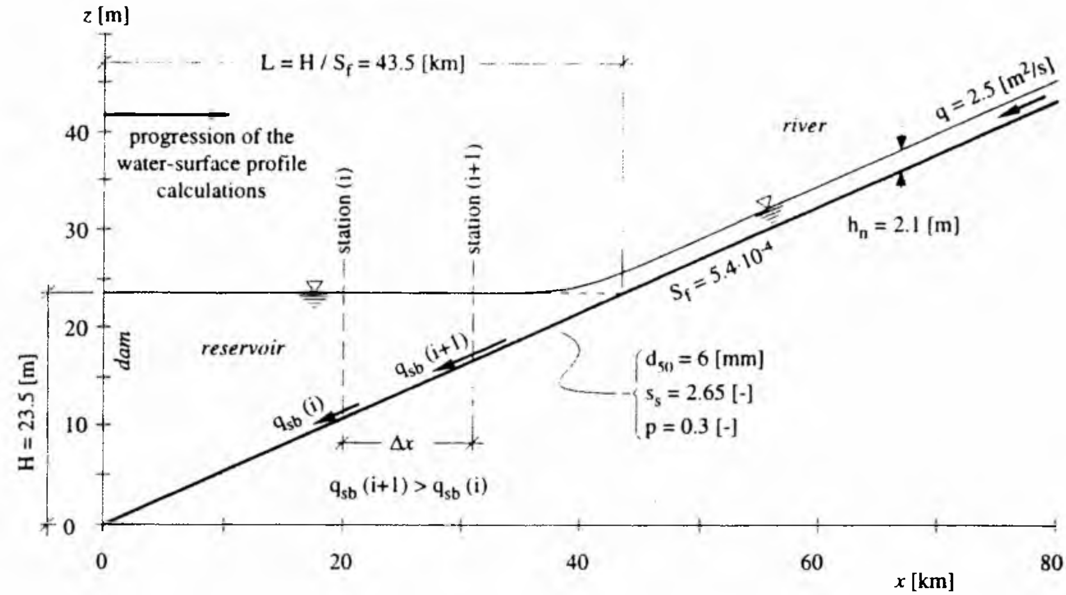


Fig. Ex.6.C.1 Modeling of the river-reservoir system.

b) *Definition of solution domain and boundary conditions* :

In reality, the cross sections of a natural river have complex forms. Nevertheless, by considering that the length of the river-reservoir system is much larger than the water depth, a simplified one-dimensional approach, where the hydrodynamic equations of the water flow and of the bed-load transport are expressed for a unit width, will be adopted.

The modeled river-reservoir system is presented in Fig. Ex.6.C.1. The origin of the coordinate system coincides with the dam location. The dam constitutes a control section and gives the boundary condition necessary for the water-surface profile calculations. Since the flow regime is subcritical, the calculations start at the dam, where the flow depth is known, and proceed towards the upstream.

The length of the reach to be modeled upstream of the dam can be decided by considering the boundary condition for the sediment transport at the upstream end. In fact it is necessary to extend the calculations up to a point where the river atteins its normal depth, h_n. It is even better to include in the calculations a certain length of the river with the normal depth. This insures a sufficiently long river reach at the upstream end, where the bed-load transport is in equilibrium, namely where the bed-load transport capacity at two consecutive sections will be the same.

For the calculation of the unit discharge, q, one can take $R_h = h$; the Manning-Strickler equation, eq. 3.16, can be written as :

$$q = \frac{Q}{B} = \frac{h}{n} R_h^{2/3} S_f^{1/2} = \frac{h^{5/3}}{n} S_f^{1/2}$$

The normal depth can now be calculated with the following expression :

$$h_n = \left(\frac{q\ n}{S_f^{1/2}} \right)^{3/5} = \left(\frac{(2.5)\ (0.032)}{\sqrt{S_f}} \right)^{3/5} = 2.1\ [m]$$

The Froude number for the uniform flow is :

$$Fr = \frac{U}{\sqrt{gh_n}} = \frac{q}{h_n \sqrt{gh_n}} = \frac{q}{\sqrt{gh_n^3}} = \frac{2.5}{\sqrt{(9.81)\ (2.1)^3}} = 0.26\ [-]$$

The flow is therefore subcritical.

It is not necessary to compute an initial water-surface profile to guess the point where t river reaches its normal depth. By considering the known values of the water depth at t dam, H = 23.5 [m], and of the bed slope, $S_f = 5.4 \cdot 10^{-4}$ [-], the approximate length the reservoir can be calculated : $L = H / S_f \cong 43.5$ [km]. To be able to guarantee sufficiently long river reach at the upstream, where the flow remains uniform throughc the whole simulation period, namely 100 [years], a computational reach length of, 1 example, TL = 120 [km] shall be adopted. This total length of the system (TL) is divid in ND reaches having a length of DX; this yields NS = ND +1 stations. Starting from t dam location, the stations are numbered from the downstream to the upstream end.

c) *Structure of the program DELTA*:

A *decoupled* algorithm has been used in writing the program DELTA. The adjecti "decoupled" means that the calculations for the liquid and solid phases are carried c separately and successively (see Fig. Ex.6.C.2).

The calculations start at time t = 0, when the bed-level elevations are known. The wat surface profile is calculated without considering the sediment transport. Once the wat surface profile is calculated, the hydraulic parameters are known at all the stations. T bed-load transport rate is now calculated for all the stations. The balance of the sedimen entering and leaving is subsequently calculated for all reaches to find the volume deposition (or erosion). These volumes are then translated into a deposition heig Finally the bed levels are modified by using these deposition heights. This concludes t computational cycle for the time t = 0. The time is then advanced by Δt, and a new wat surface calculation is carried out with the new bed profile; and so on. It should be notic that during the calculations for one phase, the characteristics of the other phase are ke constant.

The program DELTA is written in a didactic style and does not have the pretention being optimized. The complete program code is presented Fig. Ex.6.C.11. Numerous comments inserted in the code explain the flow of t program almost step by step. As far as possible, the names of the variables are chosen recall the notation used in the text. An exhaustive list of variables together with the tyr of variables and explanations, are provided at the beginning of the main program and t related subprograms.

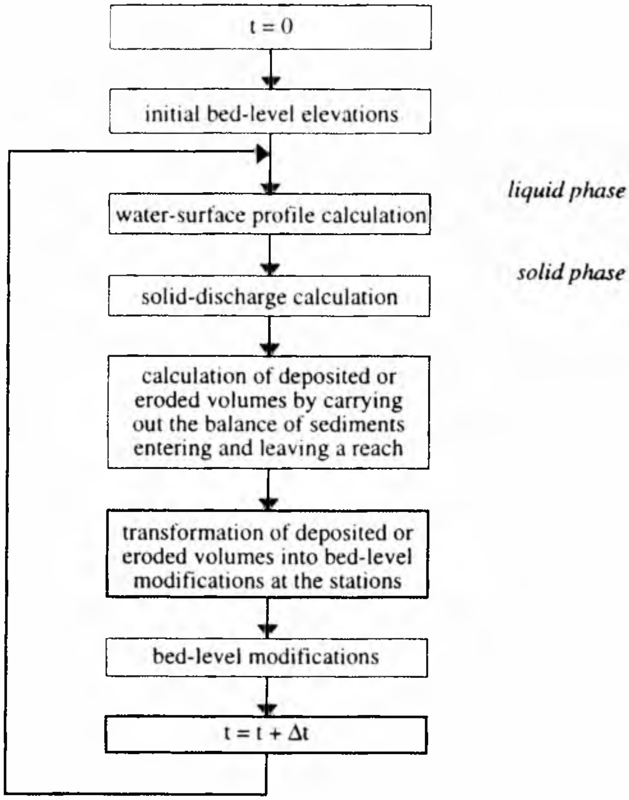


Fig. Ex.6.C.2 Decoupled simulation algorithm.

The program has a modular structure. It is composed of a main program DELTA and nine subprograms, each accomplishing a specific pre-defined task : DREAD; TITLES; RK4; DERIVE; SCHOKL; MEYPET; EINS42; FORMUL and DWRITE. The flowchart, given in Fig. Ex.6.C.3, shows not only the relations between different program units but also the calculation loops inside the principal program. However, it is important to note that the flowchart is somewhat simplified; it does not show all the details of the code. The specific tasks carried out by the main program and each subprogram are described below in detail (see Figs. Ex.6.C.3 and 11).

d) *Working principles of the program DELTA* :

The main program DELTA controls the flow of the entire program. The working principles of the program and the algorithms are described below step by step. The reader is advised to follow these explanations in parallel with the flowchart, given in Fig. Ex.6.C.3, and the program code presented in Fig. Ex.6.C.11.

- The main program first calls the subroutine subprogram DREAD to read the program data by questioning the user. The interactive dialog between the program and the user is presented in Fig. Ex.6.C.5. This dialog will be explained later in detail. The program data are read into the computer in 6 groups :

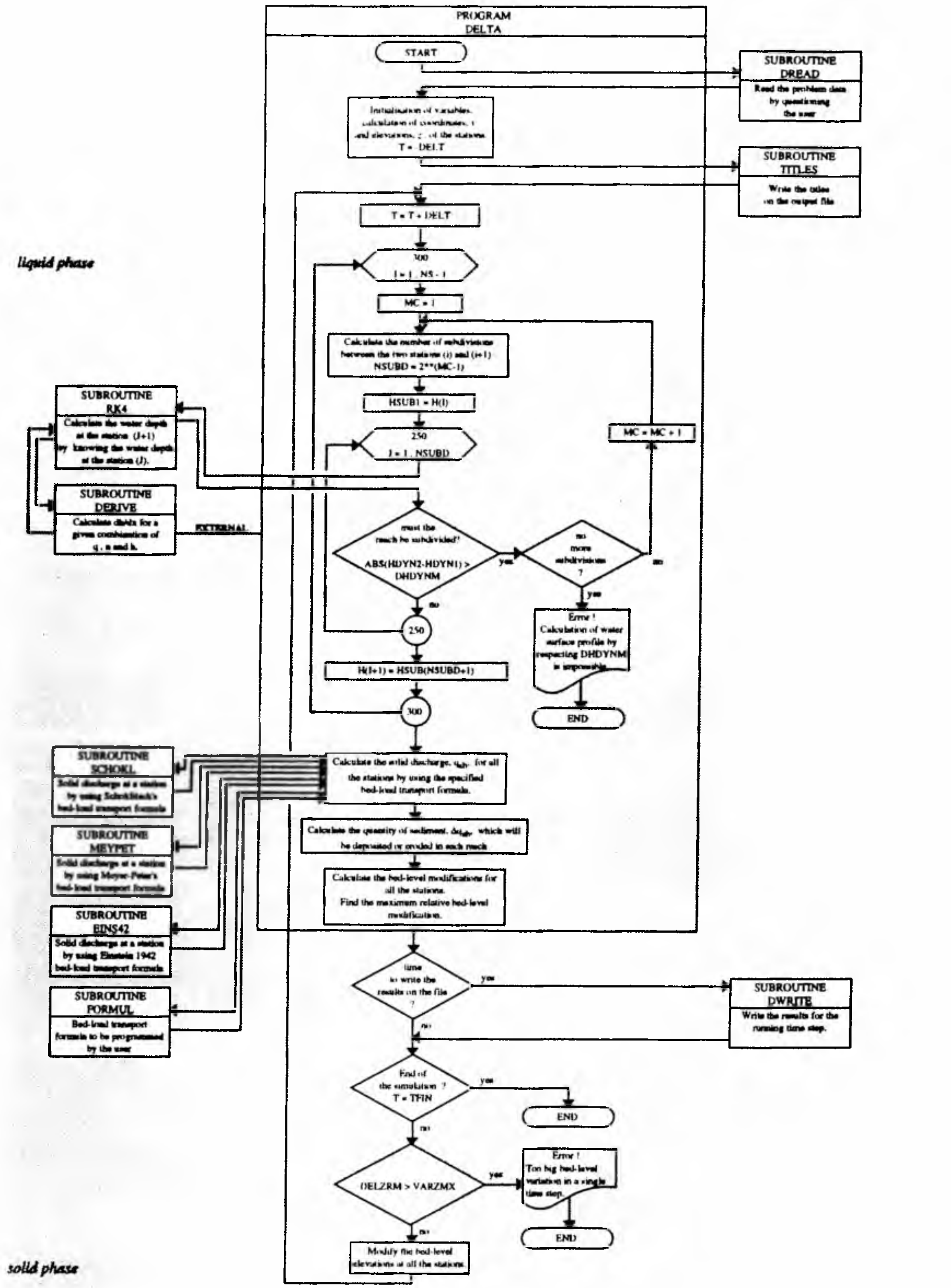


Fig. Ex.6.C.3 Flowchart of the program DELTA.

- *Physical characteristics* (initial bed slope, average sediment diameter, Manning-Strickler coefficient, densities of the water and of the sediments, discharge per unit width),
- *Choice of the bed-load transport formula* (number of the formula to be used),
- *Data concerning the modification of the bed profile* (maximum relative bed-level change tolerated, porosity of the sediments, the ratio of the upstream/downstream heights of the sediment deposition or erosion),
- *Information concerning the computational domain* (coordinates of the first and the last station, space-step length, maximum tolerated dynamic-head variation, maximum number of subdivisions which can be automatically created),
- *Boundary conditions* (water depth at the downstream end),
- *Simulation time and the printing of results* (time step, duration of the simulation, frequency of the printing of the results and the name of the output file).

- According to the data supplied by the user, the program calculates the coordinates of the stations and the initial bed level at these stations. It also initializes certain variables, such as the calculation-steps counter, eroded or deposited cumulative volumes, etc.

- The subprogram TITLES is called to echo-print of the program data on the output file.

- The time is advanced one time step. The calculation loop starts in fact at this point. It can be seen that before entering the calculation loop the time is initialized as T = – DELT; in this way the time for the first calculation step is correctly obtained as being T = 0.

- The calculation of the water-surface profile, using the running bed profile, is carried out using the 4th-order Runge-Kutta method. The differential equation for the free-surface flow in a rectangular channel is given by:

$$\frac{d}{dx}\left(\frac{Q^2}{2g(Bh)^2}\right) + \frac{dh}{dx} - S_f = -S_e \tag{4.5}$$

For a very wide river, B >> h , with a constant unit discharge of q = Q/B, this equation becomes :

$$\frac{q^2}{2g}\frac{d}{dx}\left(\frac{1}{h^2}\right) + \frac{dh}{dx} - S_f = -S_e$$

The slope of the energy-grade line can be expressed with the Manning-Strickler formula for uniform flow, eq. 3.16. Recalling that for a very wide river one can take $R_h = h$, the following is written :

$$S_e = \frac{q^2 n^2}{h^{10/3}} \tag{6.α}$$

Substituting this expression into the differential equation for the free-surface flow one obtains (see eq. 4.8a) :

$$\frac{dh}{dx} = -\frac{S_f - \dfrac{q^2\,n^2}{h^{10/3}}}{1 - \dfrac{q^2}{gh^3}}$$

It is to be noted that a negative sign is added before the term on the right-hand side of the equation to take into account that the calculation progresses from downstream to upstream. This differential equation is solved using the 4th-order Runge-Kutta method (presented in detail in Ex.7.A.b) by calling the subroutine subprogram RK4. The differential equation is programmed in the subroutine subprogram DERIVE, whose name is passed to the subprogram RK4 in the argument list. For this reason, according to the rules of FORTRAN language, the subprogram DERIVE is declared as EXTERNAL in the main program.

- The calculations for the liquid phase are finished for this time step. The program now proceeds with the calculations for the solid phase. During these calculations it is assumed that the water surface does not vary. The bed-load transport at the stations is calculated by calling the subprogram corresponding to the method specified by the user :

- The subprogram SCHOKL calculates the bed-load discharge with the method of *Schoklitsch* (1950), whose formula is given by eq. 6.31 :

$$q_{sb} = \frac{2.5}{s_s}\, S_e^{\;3/2}\, (q - q_{cr}) \quad \Rightarrow \quad QSU = \frac{2.5*SEFF^{3/2}*(QU - QCRIT)}{SS}$$

  The critical liquid discharge is calculated using eq. 6.31a :

$$q_{cr} = 0.26\, (s_s - 1)^{5/3}\; d_{40}^{\;3/2}\; S_e^{\;-7/6} \quad \Rightarrow \quad QCRIT = 0.26*(SS - 1)^{5/3}*D50^{3/2}*SEFF^{-7/6}$$

  Since the grain-size distribution is uniform, one may take $d_{40} = d_{50}$. If at a station $q_{cr} > q$, the program will assume $q_{sb} = 0$. The slope of the energy-grade line, S_e (SEFF), is calculated by the main program using the Manning-Strickler formula (see the explanations for the water-surface profile calculation) and sent to the subprogram in the argument list.

- The subprogram MEYPET calculates the bed-load discharge with the method of *Meyer-Peter* et *al.* (1948), whose formula is given by eq. 6.32 :

$$q_{sb} = \frac{1}{g\,(\rho_s - \rho)} \left(\frac{g\,\rho\,R_{hb}\,\xi_M\,S_e - 0.047\,g\,(\rho_s - \rho)\,d_{50}}{0.25\,\rho^{1/3}} \right)^{3/2}$$

  In the program this equation is written as:

$$QSU = \frac{1}{G*(ROS-ROE)} \left(\frac{G*ROE*RH*FCOR*SEFF - 0.047*G*(ROS-ROE)*D50}{0.25*ROE^{1/3}} \right)^{3/2}$$

Since the calculations are done for a unit width, the program uses R_{hb} = h (RH=H). It is assumed that $q_{sb} = 0$, if $(g\,\rho\,R_{hb}\,\xi_M\,S_e) < (0.047\,g\,(\rho_s - \rho)\,d_{50})$. The user may or may not use the roughness parameter, ξ_M (FCOR). This parameter is calculated in the subprogram DREAD, during data input, according to the expression :

$$\xi_M = \left(\frac{n'}{n}\right)^{3/2} \qquad \Rightarrow \qquad FCOR = \left(\frac{CN50}{CN}\right)^{3/2}$$

where n' (CN50) is the grain roughness calculated using the formula of Strickler, eq. 3.18 :

$$n' = \frac{d_{50}^{\;1/6}}{21.1} \qquad \Rightarrow \qquad CN50 = \frac{D50^{1/6}}{21.1}$$

and n (CN) represents the total bed roughness, which is introduced by the user during the data input. If the user chooses not to make any corrections, the program takes FCOR = 1.

- The subprogram EINS42 calculates the bed-load discharge with the method of *Einstein* (1942), described by *Graf* (1971, p. 145) :

$$q_{sb} = \frac{\sqrt{(s_s - 1)\,g\,d_{50}^{\;3}}}{0.465} \exp\left(\frac{-0.391\,(s_s - 1)\,d_{50}}{R_{hb}'\,S_e}\right)$$

In the program this equation is written as:

$$QSU = \frac{\sqrt{(SS-1)*G*D50^3}}{0.465} \exp\left(\frac{-0.391*(SS-1)*D50}{RH*SEFF}\right)$$

Since the calculations are done for a unit width, the program uses RH = H ($R_{hb}' \equiv R_{hb} \equiv h$).

- The subprogram FORMUL does not do any calculations, but simply displays a message, inviting the user to program a bed-load transport formula of his choice in this subprogram.

- Let us now go back to the main program DELTA. The quantity of the sediments to be deposited (or eroded) in a reach, Δq_{sb}, depends on the difference between the bed-load transport capacities at the upstream, $q_{sb}(i+1)$, and at the downstream, $q_{sb}(i)$, stations :

$$\Delta q_{sb}(i) = q_{sb}(i+1) - q_{sb}(i) \qquad \Rightarrow \qquad DELQS(I) = QSU(I+1) - QSU(I)$$

In the program the reaches between two consecutive stations are numbered frc downstream to upstream. The number of a reach is therefore the same as t number of the station at its downstream end (i.e.: the **reach** (i) is limited by t **station** (i) at the downstream and by the **station** (i+1) at the upstream). This fa is used in the program as a *programming trick*. The same variable I is used denote both a reach and the station. In this way, the solid discharges, QSU(I+ and QSU(I), refer to the **stations** (I+1) and (I), respectively, whereas DELQS(I) the difference in transport capacities for the **reach** (I). While reading the progra this double role of the variable I must be kept in mind.

\- The procedure used by the program to translate the transport-capacity differenc between the upstream and the downstream ends of a reach into a deposition heig — this is Exner's relationship — is presented in Fig. Ex.6.C.4. The virtual volur of the sediments to be deposited at the **reach** (i), by taking into account the volur increase due to the porosity, p. is the following :

Deposition volume for the **reach** (i) = $\Delta q_{sb}(i)\ \Delta t\ \dfrac{1}{(1-p)}$

The program admits that this volume creates a trapezoidal deposit whose upstrea and downstream heights are $\delta z_{am}(i)$ and $\delta z_{av}(i)$, respectively. For the **reach** (one can therefore write :

$$\Delta q_{sb}(i)\ \Delta t\ \frac{1}{(1-p)} = \frac{\delta z_{am}(i) + \delta z_{av}(i)}{2}\ \Delta x \qquad (6.4$$

During data input the user specifies the ratio between the upstream and tl downstream heights of this trapezoidal deposit, $\lambda = \delta z_{am}(i)/\delta z_{av}(i)$ (in the progra $\lambda \equiv$ HAMHAV). By using this information one obtains :

$$\delta z_{av}(i) = \frac{2}{(1+\lambda)}\left(\frac{\Delta q_{sb}(i)\ \Delta t}{\Delta x\ (1-p)}\right) = \text{CRAV}\left(\frac{\Delta q_{sb}(i)\ \Delta t}{\Delta x\ (1-p)}\right)$$

$$\delta z_{am}(i) = \frac{2\lambda}{(1+\lambda)}\left(\frac{\Delta q_{sb}(i)\ \Delta t}{\Delta x\ (1-p)}\right) = \text{CRAM}\left(\frac{\Delta q_{sb}(i)\ \Delta t}{\Delta x\ (1-p)}\right)$$

For an internal **station** (i), the upstream (i–1) and the downstream (i) **reach** give two different deposition heights : $\delta z_{am}(i-1)$ and $\delta z_{av}(i)$, respectively. The fin value of the deposition height at such a **station** (i) corresponds therefore to tl average of these two values :

$$\Delta z(i) = \frac{1}{2}\left[\frac{2\lambda}{(1+\lambda)}\left(\frac{\Delta q_{sb}(i-1)\ \Delta t}{\Delta x\ (1-p)}\right) + \frac{2}{(1+\lambda)}\left(\frac{\Delta q_{sb}(i)\ \Delta t}{\Delta x\ (1-p)}\right)\right]$$

At the end of each time step the program searches for the maximum relati variation of the bed level, $\Delta z(i)/h_i$ (= DELZRM), and compares it with tl maximum tolerated value (VARZMX), as specified by the user during data input. DELZRM > VARZMX , the program displays an error message and stops.

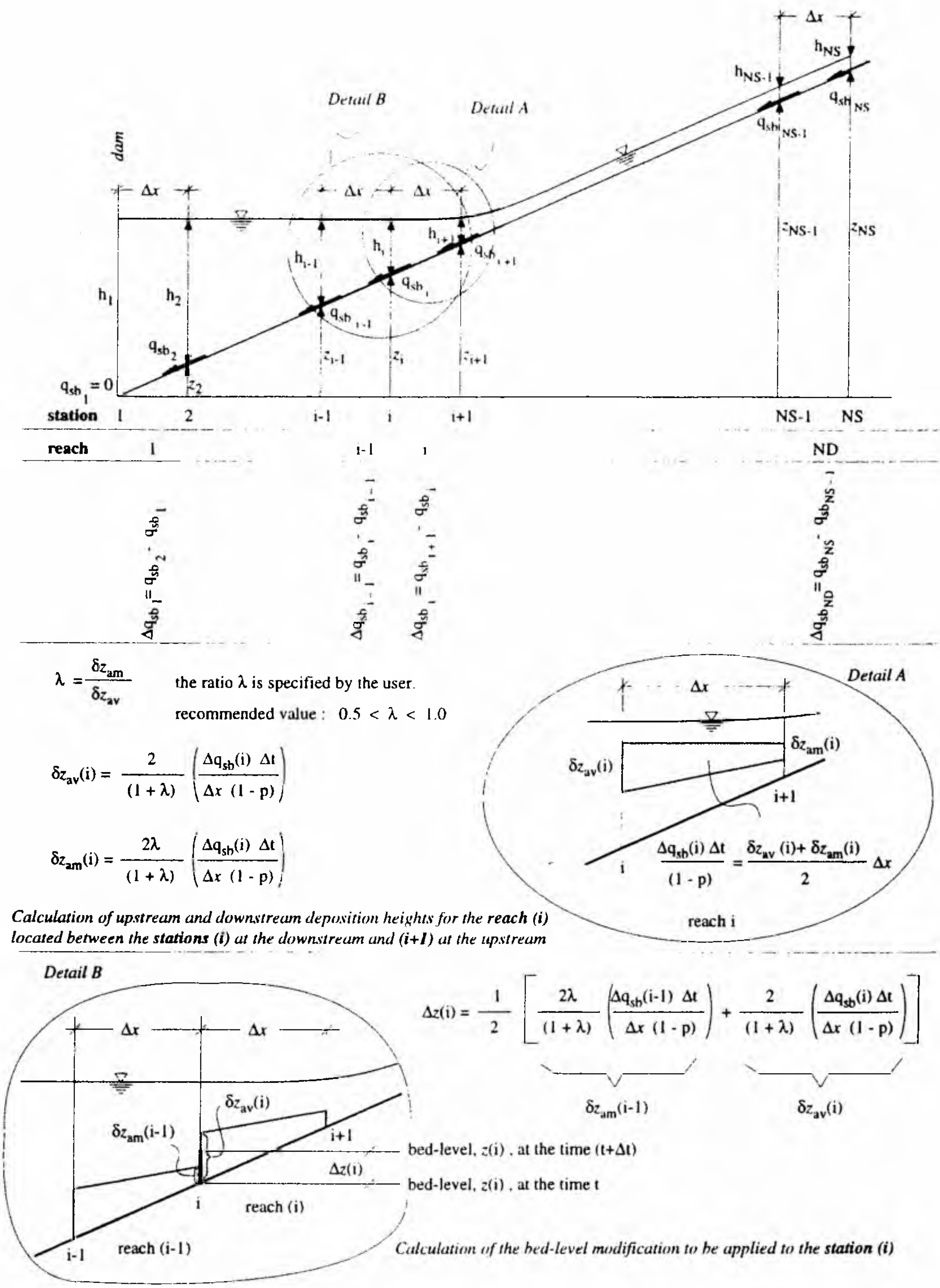


Fig. Ex.6.C.4 Procedure for calculating the volume of the sediments to be deposited at each reach and the modification of the bed level.

- At the end of a time step — if the printing time has come — the main program cal the subprogram DWRITE for printing the results for the current time step on th users specified output file as well as on a second file called GRAPH.DAT.
- If the simulation period specified by the user is not yet reached, the progra modifies the bed-level heights at the stations :

  $$z_i^{t+\Delta t} = z_i^{t} + \Delta z_i$$

  and then goes to the beginning of the calculation loop to start a new time step with water-surface profile calculation, using this new bed profile.

The program DELTA is a simplified way of solving numerically the equations of Sain Venant - Exner (see sect. 6.2.1). The equations of Saint-Venant, eq. 6.1 and eq. 6.2, a written for a steady flow in the form of the equation for the free-surface flow, eq. 4.5; th slope of the energy-grade line, eq. 6.3, is expressed here by the Manning-Strickle formula, eq. 6.α; the relationship of Exner, eq. 6.4a, gives the volume of the deposit; th solid discharge, eq. 6.5, is expressed using the formula of *Schoklitsch* (1950), eq. 6.3 and the formula of *Meyer-Peter* et *al.* (1948), eq. 6.32.

e) *Use of the program DELTA for solving the problem* :

The source code of the program DELTA presented in the Fig. Ex.6.C.11 is first *compile* and *linked* to obtain an executable code. A FORTRAN compiler is of course necessary t do these operations on your computer. The reader should consult the manuals of hi computer to learn the exact procedure to follow.

The user feeds the program data into the computer interactively by answering th questions asked by the program. Numerous comments has been introduced into th conversation to remind the user of certain important points treated in chap. 6 and to guid the user in making his choices. The program also checks some of the likely errors in th data introduction and warns the user. In case of an error, the question is repeated until th user answers correctly.

The dialog between the user and the computer for solving the present problem i presented in Fig. Ex.6.C.5. The values typed in by the user are highlighted by a whit background. They are followed by a sign representing the RETURN (CR or ENTER o some computers) key on the keyboard.

The user begins the work by introducing the initial bed slope (SF) and the average grai diameter (D50) of the sediments. Using the average grain diameter the program calculate the Manning-Strickler coefficient due to grain roughness and displays it on the screen CN50 = 0.0202 ($m^{-1/3}s$). The program then asks the user to enter the total Manning Strickler coefficient (CN). It is the value of CN that is used later in all calculation (specifying a value of CN = CN50 will mean that the grain roughness is the only cause o the regular head loss). If the user has good reason to think that the head loss is larger the the one given by CN50 — since bed forms (such as dunes) or other irregularities ar present — the estimated total value should be entered. For the present problem the valu given in the problem will be entered, namely CN = 0.032 ($m^{-1/3}s$).

Supplying the density of the sediments (ROS) and of the water (ROE) as well as th discharge per unit width (QU) is sufficiently clear.

The program proposes now 4 formulae for solid discharge as bed-load transport and asks the user to select one of them. To select the formula of Meyer-Peter et al., one enters "2". Up to this point the dialog with the computer is the same for all cases. Now comes a short conversation with the computer which depends on the selected method. Some methods do not need supplementary data. The program only displays the method used. In case of the selection of the formula of Meyer-Peter et al., if CN > CN50, the program asks if the user wishes to use the roughness parameter, FCOR.

From here on the text of the dialog with the computer is again the same, regardless of the choice of the bed-load transport formula. The conversation continues with the introduction of the data concerning the modification of the bed profile. First, the desired value of the maximum relative variation of the bed level during a single time step, VARZMX = 0.1, must be entered. The program sees to it that at none of the stations, during a single time step, the bed-level modification, DZ, is higher than 10% of the water depth. The user enters also the porosity of the deposited sediments, p (POROS).

The procedure used by the program to transform the volume of the sediments deposited in reaches into upstream and downstream bed-level modification heights was described above in detail. For the ratio of the upstream/downstream heights a value of λ = HAMHAV = 0.75 is entered; this means that the deposition at the downstream end of the reach is higher than the one at the upstream end. It is important to note that this ratio must be between 0.5 and 1.0. In case of high bed-load transport rates, a uniform distribution of the sediments over the reach (HAMHAV = 1.0) may cause instabilities in the calculation of the bed levels at the stations.

The data input continues with the information on the calculation domain. The first station is located on the dam; by entering X1 = 0 [m], the origin of the coordinate system is placed at the dam. The calculations will be carried out up to an upstream distance of XF = 120000 [m].

The choice of the step length in the longitudinal direction, DX, is important. A very small step length necessitates a very small time step; this increases of course the calculation time. For the calculation of the deposition one can choose relatively large time steps. In the present case a step length of DX = 600 [m] is used. A too big step length may cause errors in the calculation of the water-surface profile (especially around the region where the river meets the reservoir) and may cause the program to stop the execution. To overcome this difficulty without loosing the advantages of working with long steps the program uses a clever programming trick.

The water-surface profile is calculated using the 4th-order Runge-Kutta method. The calculations start at the station located at the dam where the water depth is known. By sending this value to the subprogram RK4, the main program obtains in return the water depth at a station immediately upstream. To be able to guarantee a sufficient precision in the water-surface profile calculations, the program checks if the difference of the dynamic heads, $U^2/2g$, between these two successive stations is less than a value specified by the user, in the present case DHDYNM = 0.01 [m], or not. If in a reach this value is not respected, the program divides this reach into $2^1 = 2$ *sub-reaches*, and redoes the calculation in two steps. If the criteria is still not respected the program tries this time $2^2 = 4$ sub-reaches and so on. It is the user who specifies up to which power of 2 the

```
PROGRAM FOR THE BED-LOAD TRANSPORT CALCULATION
BY TAKING INTO ACCOUNT A BED-LEVEL MODIFICATION.

NOTES :
- UNIT SYSTEM = SI
- NUM. METHOD FOR WATER-SURFACE PROFILE CALCULATION = 4th ORDER RUNGE-KUTTA
- THE FLOW IS SUBCRITICAL (Fr < 1 ). THE WATER-SURFACE PROFILE CALCULATION
  STARTS AT THE DOWNSTREAM END AND PROGRESSES TOWARDS UPSTREAM
- SEDIMENT TRANSPORT CALCULATIONS ARE CARRIED OUT FROM UPSTREAM TO DOWNSTREAM
- THE CALCULATIONS ARE MADE FOR A UNIT WIDTH


PHYSICAL CHARACTERISTICS DATA :
Initial bed slope,                 SF (-)        ? = 5.4e-4     RETURN

Average diameter of sediments,     D50 (mm)      ? = 6          RETURN


According to eq. 3.18, the Manning coefficient
due to the grain roughness is :

CN50 = d50^(1/6) / 21.1 =   .0202(s/m^1/3)

Total Manning-Strickler coeff.,    CN (s/m^1/3) ? = 0.032      RETURN

Density of sediments,              ROS (kg/m3)  ? = 2650       RETURN
Density of water,                  ROE (kg/m3)  ? = 1000       RETURN

Unit discharge,                    QU (m2/s)    ? = 2.5        RETURN


CHOICE OF THE BED-LOAD FORMULA :
The program allows the use of one of the
following 4 bed-load formulae :
1- Schocklitsch          (1950)
2- Meyer-Peter et al.    (1948)
3- Einstein              (1942)
4- Your formula (to be programmed !)
Which formula do you choose ?     (1 a 4)      = 2            RETURN

MEYER-PETER (1934) bed-load formula is chosen.

Do you want to use the roughness parameter, FCOR ?
Answer Y(es) or N(o)/CR ? = Y

The roughness parameter is then,    FCOR = (CN50 / CN)**1.5 =    .502


BED-LEVEL MODIFICATION DATA :
The bed-level variation during a single time step
must not be too big; otherwise instabilities of
the bed profile may appear. At the end of each time
step, the program calculates the maximum relative bed-
level variation, DELZ/H , (bed-level variation divided
by the water depth) and checks that it is not bigger
than a value specified by the user.

Max. rel. bed-level variation,    VARZMX (-)    ? = 0.1        RETURN

Porosity of deposited sediments,  POROS (-)     ? = 0.3        RETURN

The volume of sediments deposited in a reach is
transformed into a bed-level variation height at the
downstream and upstream, by defining a trapezoid.
The user controls the sediment distribution by choosing
the ratio of upstream/downstream heights of the trapezoid.
We recommend to use the values : 0.5 < HAMHAV < 1.

Ratio of upstr./downstr. heights, HAMHAV (-)   ? = 0.75       RETURN
```

Fig. Ex.6.C.5 Interactive dialog with the computer for the data input (continuec

```
INFORMATION ON COMPUTATIONAL DOMAIN :
x-coordinate of first station,       X1  (m)      ? = 0            RETURN
x-coordinate of last station,        XF  (m)      ? = 120000       RETURN

Total reach length is therefore,     TL  (m)        =  120000.00

Now you have to specify the step length in x-direction.
If the step length is too long to guarantee a correct
prediction of the water-surface profile, the program
will automatically add some intermediate stations. The
results, however, are only printed at the stations
specified by the user; others remain invisible. The step
length for the space must therefore be specified to
guarantee a correct representation of physical processes
involved in the bed-load transport. In case of doubt, you
can repeat the simulation with different step lengths and
compare the results.

Step length in x-direction,          DX  (m)      ? = 600          RETURN

Number of reaches,                   ND             =  200
Number of stations,                  NS             =  201

Max. tolerated var. of dyn. head,   DHDYNM (m) ? = 0.01            RETURN

In case this value is exceeded the reach will be subdivided
in order to refine the calculations.
The number of divisions is specified in powers of 2.
The maximum value is      MCMAX =    7
Which corresponds to  2.0^MCMAX = 128 subdivisions.

Maximum number of subdiv. in powers of 2, NMC  ? = 7               RETURN


INFORMATION ON BOUNDARY CONDITIONS :
Note that the sediment transport at downstream end
is automatically taken as zero,     QSU(1)        = 0.0
Water depth at the downstream end,  H1  (m)     ? = 23.5           RETURN


PARAMETERS RELATED TO TIME AND PRINTING OF RESULTS :
Time step,                    DELT  (days)      ? = 10             RETURN

Duration of simulation,       TFIN  (days)      ? = 36500          RETURN

Results will be printed every NPP step          ? = 730            RETURN

Name of output file (max. 40 char.)             ? = MEYPET.OUT     RETURN

TIME STEP     =              1.
TIME (days)   =         .000

TIME STEP     =              2.
TIME (days)   =       10.000
....
....
TIME STEP     =           3651.
TIME (days)   =    36500.000

  2 subdivisions between stations      57 and     58

  4 subdivisions between stations      57 and     58

  8 subdivisions between stations      57 and     58

 16 subdivisions between stations      57 and     58

NORMAL END OF PROGRAM

PRESS ON RETURN KEY TO EXIT                                        RETURN
```

Fig. Ex.6.C.5 Interactive dialog with the computer for the data input (end).

program should continue to subdivide the reach. The maximum number of subdivis provided in the program is $2^7 = 128$, but this can be modified. In the present cas maximum number of subdivisions is chosen as NMC ≤ 7. While running the prograı case of subdivision of a reach, a message is displayed on the screen indicating the nuı of the reach and the number subdivisions applied (as a power of 2). In this way the can follow the calculations.

The program needs two boundary conditions to solve the problem. The boun condition for the calculation of sediment deposition is implicit. The program automati considers a zero solid discharge at the dam section (station). For the water-suı calculations the user enters the water depth at the dam, H1 = 23.5 [m].

The last group of data concerns the time and the printing of the results. The choice o time step depends of course on the step length in the longitudinal direction. In the pr case a time step of DT = 10 [days] is chosen. The calculations are done for a simul period of hundred years (TFIN = 36500 [days]). The results are written in the outpu MEYPET.OUT every 730 steps; this corresponds to a period of 20 [years].

The program creates two other files in the current directory (the directory in whicl program is run) :

DIALOG.DAT : contains the text of the dialog with the computer to enter the data.
GRAPH.DAT : contains the station numbers, the bed level and the water-suı elevation separated by commas. This file can be easily read by commercial spreadsheet program to draw the longitudinal profile.

Before running the program on a micro-computer, the user should make sure tha current directory does not contain files with these two names; in such a case the pro will refuse to start. If you do not want to erase these files you must either rename the move them to another directory. You may also try to run the program in anc directory.

f) *Results of the calculations with the program DELTA* :

The formation and the advance of the delta in the river-reservoir system were simul for a period of 100 [years] with the program DELTA, using the methods of *Meyer-I* et *al.* (1948) and of *Schoklitsch* (1950).

The output file MEYPET.OUT containing the results of the simulation of the delta u the method of *Meyer-Peter* et *al.* (1948) is partially, namely for T = 0, 20 100 [years], presented in the Fig. Ex.6.C.6. Due to lack of space the results for T = 60 and 80 [years] are omitted. For the same reason the output file for the simulation u the method of *Schoklitsch* (1950) is not presented either. The results for these simulations are presented in a graphical form in Figs. Ex.6.C.7 and 8, respectively.

At the beginning of the output file one finds the problem title and a summary of al data fed in by the user as well as some useful parameters calculated by the program results at different time steps follow. Since a time step of 10 [days] was used, and it decided that the results are to be printed at every 730 steps (see Fig. Ex.6.C.5) results are printed with a time interval of 20 [years], starting with the initial t T = 0 [years].

The results for each time step are preceded by a header, indicating the time step-number as well as the time itself in seconds, hours, days and years. The total volumes of the deposited (DQSDET) and eroded (DQSERT) sediments refer to the cumulative sums of the positive and negative values of the variable DELQS, since the beginning of the simulation (T = 0). The absolute (DELZMX) and relative (DELZRM) maximum bed-level variations refer to the bed-level variations (DELZ) for the runnung time step; being the one printed at the top right. The explanations for the different columns of the output file are given below :

STATION NO		:	Station number. There are 201 numbered stations, starting with station 1 (the dam at the downstream end) up to the station 201 (at the upstream end).
X	(m)	:	Distance between the station and the downstream end (station no 1).
ZF	(m)	:	Bed-level elevation with respect to the first station. The bed-level elevations, ZF, at time T = 0 are also stored as initial bed-level elevations in the variable ZFI.
ZF – ZFI	(m)	:	Difference between the running bed-level elevation, ZF (at the time indicated on the top left) and the initial bed-level elevation, ZFI (at time T = 0). It is this column which shows the *height of the deposition* (or the erosion). On Fig. Ex.6.C.6, the region of the delta formation is highlighted by a thin frame.
H	(m)	:	Running actual water depth.
ZF + H	(m)	:	Water-surface elevation with respect to the bed-level elevation at the downstream end (at the dam).
U	(m/s)	:	Average velocity.
Fr	(-)	:	Froude number.
QSU	(m^3/s/m)	:	Solid discharge at a station.
DELQS	(m^3/s/m)	:	Difference of solid discharge between the extremities of a reach. There are only 200 reaches for NS = 201 stations. This is the reason for having DELQS = 0 on the line 201; however, this value is not used in the calculations. To calculate the deposition height at the upstream end of the computational domain, DELZ(NS), the program assumes implicitly that DELQS(NS) = DELQS(NS–1).
DELZ	(m)	:	Modifications to be applied to the bed, before starting the calculations for the next time step.

The different stages of the formation and the progression of the delta and its influence on the water-surface profile can readily be observed in Figs. Ex.6.C.7 and 8. These figures have been prepared by plotting the information in columns ZF, bed-level elevation, and (ZF+H), water-surface elevation, as a function of the distance given in column X. By studying the contents of the output file, presented in Fig. Ex.6.C.6, and the profiles plotted in Figs. Ex.6.C.7 and 8, one can make several interesting observations.

First the column QSU for the time **T = 0 [years]** in Fig. Ex.6.C.6 will be conside From the dam up to the station 69 (x = 40.8 [km]), there is no solid discha The velocity in this reach is equal to or smaller than 1 [m/s]. From station 71 (x = 42.0 [km]) onward the solid discharge increases towards the upstrean attain a constant value of QSU $\cong 0.16 \cdot 10^{-4}$ [m^3/s/m], which is the solid discharge for river cross-section. The column DELQS shows that the deposition of the sedime occurs between the station 69 (x = 40.8 [km]) and station 95 (x = 56.4 [kn Downstream of this reach there is no sediment transport whereas at the upstre constitutes the river reach where the bed-load transport is in equilibrium. The format of the delta starts therefore at the point where the river enters the reservoir. The colu DELZ gives the modifications to be applied to the bed elevation between the station and the station 95, for the next time step.

At time **T = 20 [years]**, as the column ZF–ZFI shows, the downstream end of the d is at the station 62 (x = 36.6 [km]). The maximum height of the delta is h_d = 1.23 [m the station 64 (x = 37.8 [km]). By integrating the data in this column, using trapezoidal rule, the total volume of deposited sediments is found as being 14396 [m^3/ This volume takes into account the porosity. In the header of the results T = 20 [years], the cumulative volumes of deposition and erosion given as : DQSDET = 12650.7 [m^3/m] and DQSERT = –2559.5 [m^3/ The net volume of the deposited sediments is now calculated as be (DQSDET + DQSERT) = 10091.2 [m^3/m]. By multiplying this volume with coefficient of swelling CFOI = 1 / (1–p) = 1.4286, one gets the swelled volume deposited sediment of 14416 [m^3/m]; this value is very close to 14396 [m^3/m] calcul before. In Fig. Ex.6.C.6, the extent of the delta is highlighted by enclosing a portio the column ZF–ZFI in a rectangle. The bed-level modifications near the upstream end in fact very small (see Fig. Ex.6.C.7).

```
              PROGRAM FOR THE BED-LOAD TRANSPORT CALCULATION
           BY TAKING INTO ACCOUNT THE BED-LEVEL MODIFICATION.

NOTES :
- UNIT SYSTEM = SI
- NUM. METHOD FOR WATER-SURFACE PROFILE CALCULATION = 4th ORDER RUNGE-KUTTA
- THE FLOW IS SUBCRITICAL (Fr < 1 ). THE WATER-SURFACE PROFILE CALCULATION
  STARTS AT THE DOWNSTREAM END AND PROGRESSES TOWARDS THE UPSTREAM END
- SEDIMENT TRANSPORT CALCULATIONS ARE CARRIED OUT FROM UPSTREAM TO DOWNSTREAM
- CALCULATIONS ARE MADE FOR A UNIT WIDTH

PHYSICAL CHARACTERISTICS DATA :                              CHOICE OF BED-LOAD FORMULA:
Initial bed slope,              SF (-)        = .0005400     Bed-load transport formula by Meyer-Peter et al. (1934) is used.

Average diameter of sediments,  D50 (mm)      = 6.00         Roughness coefficient,  FCOR = (CN50 / CN)^(3/2) =   .502
Manning coeff. for sed. grains, CN50 (s/m^1/3) = .0202
Manning-Strickler coefficient,  CN (s/m1/3)   = .0320
Density of sediments,           ROS (kg/m3)   = 2650.00
Density of water,               ROE (kg/m3)   = 1000.00
Unit discharge,                 QU (m2/s)     =    2.50

INFORMATION RELATED TO WATER-SURFACE CALCULATIONS            INFORMATION RELATED TO CALCULATION OF DEPOSITION VOLUME :
Max. tolerated var. of dyn. head,  DHDYNM (m) = .010         Max. rel. bed-level variation, VARZMX (-)       =  .10
Maximum number of subdiv. in powers of 2, NMC = 7            Coefficient of swelling,         CFOI (-)       = 1.4286
                                                             Ratio of upst./dwnst. heights, HAMHAV (-)       =  .750

INFORMATION ON CALCULATION DOMAIN                            BOUNDARY CONDITIONS :
x-coordinate of first station,  X1 (m)        =       .00    Bed-load transport at the downstream end is zero.
x-coordinate of last station,   XF (m)        = 120000.00    Water depth at downstream end,   H1 (m)         =  23.500
Total reach length is therefore, TL (m)       = 120000.00
Step length in x-direction,     DX (m)        =    600.00    PARAMETERS RELATED TO TIME :
Number of reaches,              ND (m)        =   200        Time step,                DELT (days)           =    10.000
Number of stations,             NS            =   201        Duration of simulation,   TFIN (jours)          = 36500.000

PRINTING OF RESULTS :
Printing of results every NPP step            = 730
Name of output file (max. 40 char.)           = MEYPET.OUT
```

Fig. Ex.6.C.6 Output file of the program DELTA, resulting from a simulation using method of *Meyer-Peter* et *al.* (1948) (continued): file header.

Time step = [illegible] Total volume of deposited sediments DVDET (m3/m) = 0.138047D+02
Time (s) = 0.0000000D+00 Total volume of eroded sediments DVERT (m3/m) = 0.000000D+00
Time (hours) = 000
Time (days) = 000 Max. absolute bed level variation DELZMX (m) = 0.759922D-02 at the station, IMAXZ = 71
Time (years) = 000 Max. relative bed level variation DELZRM = [illegible] at the station, IMAXR = 71

STATION NO	X (m)	ZF (m)	ZF-ZFI (m)	H (m)	ZF+H (m)	V (m/s)	Fr	QS (m3/s/m)	DELQS (m3/s/m)	DELZ (m)
1	.00	0.000000D+00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
3	1200.00	0.648000D+00	0.000000D+00	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
5	2400.00	0.129600D+01	0.000000D+00	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
7	3600.00	0.194400D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
9	4800.00	0.259200D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
11	6000.00	0.324000D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
13	7200.00	0.388800D+01	[illegible]	[illegible]	[illegible]	[illegible]	.004	0.000000D+00	0.000000D+00	0.000000D+00
15	8400.00	0.453600D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
17	9600.00	0.518400D+01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
19	10800.00	0.583200D+01	[illegible]	[illegible]	[illegible]	[illegible]	.011	0.000000D+00	0.000000D+00	0.000000D+00
21	12000.00	0.648000D+01	[illegible]	[illegible]	[illegible]	.147	.011	0.000000D+00	0.000000D+00	0.000000D+00
23	13200.00	0.712800D+01	[illegible]	[illegible]	[illegible]	[illegible]	.012	0.000000D+00	0.000000D+00	0.000000D+00
25	14400.00	0.777600D+01	0.000000D+00	[illegible]	[illegible]	.154	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
27	15600.00	0.842400D+01	0.000000D+00	[illegible]	[illegible]	.166	.014	0.000000D+00	0.000000D+00	0.000000D+00
29	16800.00	0.907200D+01	0.000000D+00	14.414	[illegible]	.173	.015	0.000000D+00	0.000000D+00	0.000000D+00
31	18000.00	0.972000D+01	0.000000D+00	[illegible]	[illegible]	.181	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
33	19200.00	0.103680D+02	0.000000D+00	[illegible]	[illegible]	.190	.017	0.000000D+00	0.000000D+00	0.000000D+00
35	20400.00	0.110160D+02	0.000000D+00	12.493	[illegible]	.200	.018	0.000000D+00	0.000000D+00	0.000000D+00
37	21600.00	0.116640D+02	0.000000D+00	[illegible]	[illegible]	.211	.020	0.000000D+00	0.000000D+00	0.000000D+00
39	22800.00	0.123120D+02	0.000000D+00	[illegible]	[illegible]	.223	.021	0.000000D+00	0.000000D+00	0.000000D+00
41	24000.00	0.129600D+02	0.000000D+00	[illegible]	[illegible]	.237	.023	0.000000D+00	0.000000D+00	0.000000D+00
43	25200.00	0.136080D+02	0.000000D+00	[illegible]	[illegible]	.252	.026	0.000000D+00	0.000000D+00	0.000000D+00
45	26400.00	0.142560D+02	0.000000D+00	[illegible]	0.235219D+02	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
47	27600.00	0.149040D+02	0.000000D+00	[illegible]	[illegible]	[illegible]	.032	0.000000D+00	0.000000D+00	0.000000D+00
49	28800.00	0.155520D+02	0.000000D+00	[illegible]	[illegible]	[illegible]	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
51	30000.00	0.162000D+02	0.000000D+00	[illegible]	0.235402D+02	[illegible]	.040	0.000000D+00	0.000000D+00	0.000000D+00
53	31200.00	0.168480D+02	0.000000D+00	[illegible]	[illegible]	[illegible]	.049	0.000000D+00	0.000000D+00	0.000000D+00
55	32400.00	0.174960D+02	0.000000D+00	[illegible]	0.235651D+02	.412	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
57	33600.00	0.181440D+02	0.000000D+00	[illegible]	0.235857D+02	.459	[illegible]	0.000000D+00	0.000000D+00	0.000000D+00
59	34800.00	0.187920D+02	0.000000D+00	[illegible]	0.236160D+02	[illegible]	.075	0.000000D+00	0.000000D+00	0.000000D+00
61	36000.00	0.194400D+02	0.000000D+00	[illegible]	0.236626D+02	.592	.092	0.000000D+00	0.000000D+00	0.000000D+00
63	37200.00	0.200880D+02	0.000000D+00	[illegible]	0.237376D+02	.685	.114	0.000000D+00	0.000000D+00	0.000000D+00
65	38400.00	0.207360D+02	0.000000D+00	3.127	0.238630D+02	.799	.144	0.000000D+00	0.000000D+00	0.000000D+00
67	39600.00	0.213840D+02	0.000000D+00	[illegible]	0.240754D+02	.929	.181	0.000000D+00	0.000000D+00	0.000000D+00
69	40800.00	0.220320D+02	0.000000D+00	2.364	0.244164D+02	1.048	.217	0.000000D+00	0.411567D-07	0.483801D-04
71	42000.00	0.226800D+02	0.000000D+00	[illegible]	[illegible]	1.124	.242	0.338982D-05	0.395311D-05	0.759922D-02
73	43200.00	0.233280D+02	0.000000D+00	[illegible]	[illegible]	1.164	.255	0.104630D-04	0.213861D-05	0.526467D-02
75	44400.00	0.239760D+02	0.000000D+00	2.114	[illegible]	[illegible]	.260	0.139624D-04	0.830167D-06	0.217562D-02
77	45600.00	0.246240D+02	0.000000D+00	[illegible]	0.267284D+02	1.188	.261	0.152869D-04	0.290155D-06	0.776868D-03
79	46800.00	0.252720D+02	0.000000D+00	[illegible]	[illegible]	1.190	.262	0.157460D-04	0.979595D-07	0.264109D-03
81	48000.00	0.259200D+02	0.000000D+00	[illegible]	0.280201D+02	[illegible]	.262	0.159005D-04	0.326765D-07	0.883289D-04
83	49200.00	0.265680D+02	0.000000D+00	[illegible]	[illegible]	1.191	.262	0.159521D-04	0.108409D-07	0.293921D-04
85	50400.00	0.272160D+02	0.000000D+00	[illegible]	[illegible]	1.191	.262	0.159692D-04	0.359736D-08	0.973265D-05
87	51600.00	0.278640D+02	0.000000D+00	[illegible]	[illegible]	1.191	.262	0.159749D-04	0.117617D-08	0.323533D-05
89	52800.00	0.285120D+02	0.000000D+00	2.100	0.306116D+02	1.191	.262	0.159768D-04	0.397835D-09	0.107764D-05
91	54000.00	0.291600D+02	0.000000D+00	2.100	0.312596D+02	1.191	.262	0.159774D-04	0.147028D-09	0.393955D-06
93	55200.00	0.298080D+02	0.000000D+00	2.100	0.319076D+02	1.191	.262	0.159776D-04	0.345948D-10	0.864164D-07
95	56400.00	0.304560D+02	0.000000D+00	2.100	0.325556D+02	1.191	.262	0.159777D-04	0.000000D+00	0.381249D-07
97	57600.00	0.311040D+02	0.000000D+00	2.100	0.332036D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
99	58800.00	0.317520D+02	0.000000D+00	2.100	0.338516D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
101	60000.00	0.324000D+02	0.000000D+00	2.100	0.344996D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
103	61200.00	0.330480D+02	0.000000D+00	2.100	0.351476D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
105	62400.00	0.336960D+02	0.000000D+00	2.100	0.357956D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
107	63600.00	0.343440D+02	0.000000D+00	2.100	0.364436D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
109	64800.00	0.349920D+02	0.000000D+00	2.100	0.370916D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
111	66000.00	0.356400D+02	0.000000D+00	2.100	0.377396D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
113	67200.00	0.362880D+02	0.000000D+00	2.100	0.383876D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
115	68400.00	0.369360D+02	0.000000D+00	2.100	0.390356D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
117	69600.00	0.375840D+02	0.000000D+00	2.100	0.396836D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
119	70800.00	0.382320D+02	0.000000D+00	2.100	0.403316D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
121	72000.00	0.388800D+02	0.000000D+00	2.100	0.409796D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
123	73200.00	0.395280D+02	0.000000D+00	2.100	0.416276D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
125	74400.00	0.401760D+02	0.000000D+00	2.100	0.422756D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
127	75600.00	0.408240D+02	0.000000D+00	2.100	0.429236D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
129	76800.00	0.414720D+02	0.000000D+00	2.100	0.435716D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
131	78000.00	0.421200D+02	0.000000D+00	2.100	0.442196D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
133	79200.00	0.427680D+02	0.000000D+00	2.100	0.448676D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
135	80400.00	0.434160D+02	0.000000D+00	2.100	0.455156D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
137	81600.00	0.440640D+02	0.000000D+00	2.100	0.461636D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
139	82800.00	0.447120D+02	0.000000D+00	2.100	0.468116D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
141	84000.00	0.453600D+02	0.000000D+00	2.100	0.474596D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
143	85200.00	0.460080D+02	0.000000D+00	2.100	0.481076D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
145	86400.00	0.466560D+02	0.000000D+00	2.100	0.487556D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
147	87600.00	0.473040D+02	0.000000D+00	2.100	0.494036D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
149	88800.00	0.479520D+02	0.000000D+00	2.100	0.500516D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
151	90000.00	0.486000D+02	0.000000D+00	2.100	0.506996D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
153	91200.00	0.492480D+02	0.000000D+00	2.100	0.511476D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
155	92400.00	0.498960D+02	0.000000D+00	2.100	0.519956D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
157	93600.00	0.505440D+02	0.000000D+00	2.100	0.526436D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
159	94800.00	0.511920D+02	0.000000D+00	2.100	0.532916D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
161	96000.00	0.518400D+02	0.000000D+00	2.100	0.539396D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
163	97200.00	0.524880D+02	0.000000D+00	2.100	0.545876D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
165	98400.00	0.531360D+02	0.000000D+00	2.100	0.552356D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
167	99600.00	0.537840D+02	0.000000D+00	2.100	0.558836D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
169	100800.00	0.544320D+02	0.000000D+00	2.100	0.565316D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
171	102000.00	0.550800D+02	0.000000D+00	2.100	0.571796D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
173	103200.00	0.557280D+02	0.000000D+00	2.100	0.578276D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
175	104400.00	0.563760D+02	0.000000D+00	2.100	0.584756D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
177	105600.00	0.570240D+02	0.000000D+00	2.100	0.591236D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
179	106800.00	0.576720D+02	0.000000D+00	2.100	0.597716D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
181	108000.00	0.583200D+02	0.000000D+00	2.100	0.604196D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
183	109200.00	0.589680D+02	0.000000D+00	2.100	0.610676D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
185	110400.00	0.596160D+02	0.000000D+00	2.100	0.617156D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
187	111600.00	0.602640D+02	0.000000D+00	2.100	0.623636D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
189	112800.00	0.609120D+02	0.000000D+00	2.100	0.630116D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
191	114000.00	0.615600D+02	0.000000D+00	2.100	0.636596D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
193	115200.00	0.622080D+02	0.000000D+00	2.100	0.643076D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
195	116400.00	0.628560D+02	0.000000D+00	2.100	0.649556D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
197	117600.00	0.635040D+02	0.000000D+00	2.100	0.656036D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
199	118800.00	0.641520D+02	0.000000D+00	2.100	0.662516D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00
201	120000.00	0.648000D+02	0.000000D+00	2.100	0.668996D+02	1.191	.262	0.159777D-04	0.000000D+00	0.000000D+00

Fig. Ex.6.C.6 Output file of the program DELTA, resulting from a simulation using the method of *Meyer-Peter* et *al.* (1948) (continued) : **T = 0 [years].**

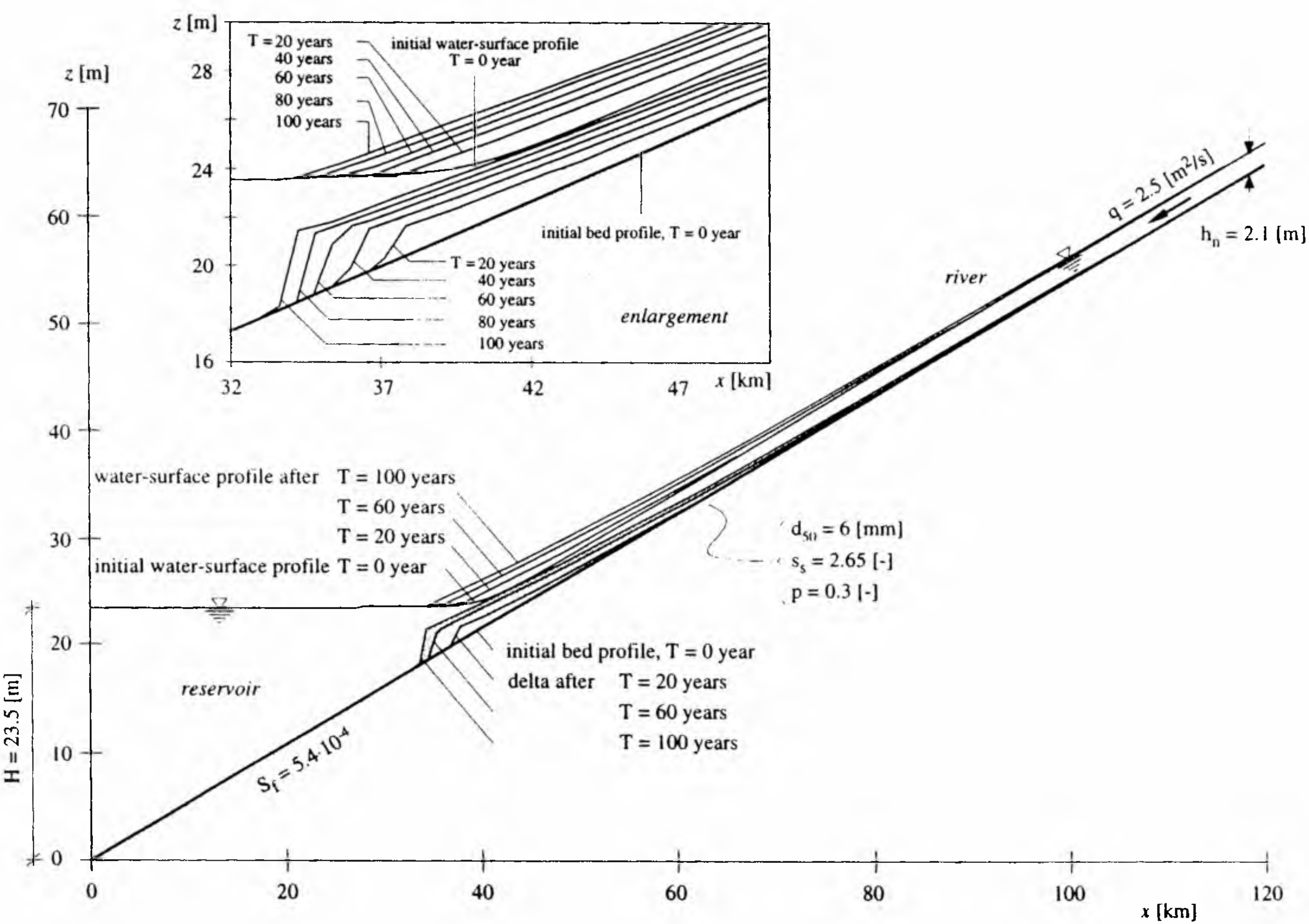


Fig. Ex.6.C.7 Time history of formation and advancement of the delta in a river-reservoir system, simulated for a period of 100 years with the program DELTA by using the bed load formula of ...

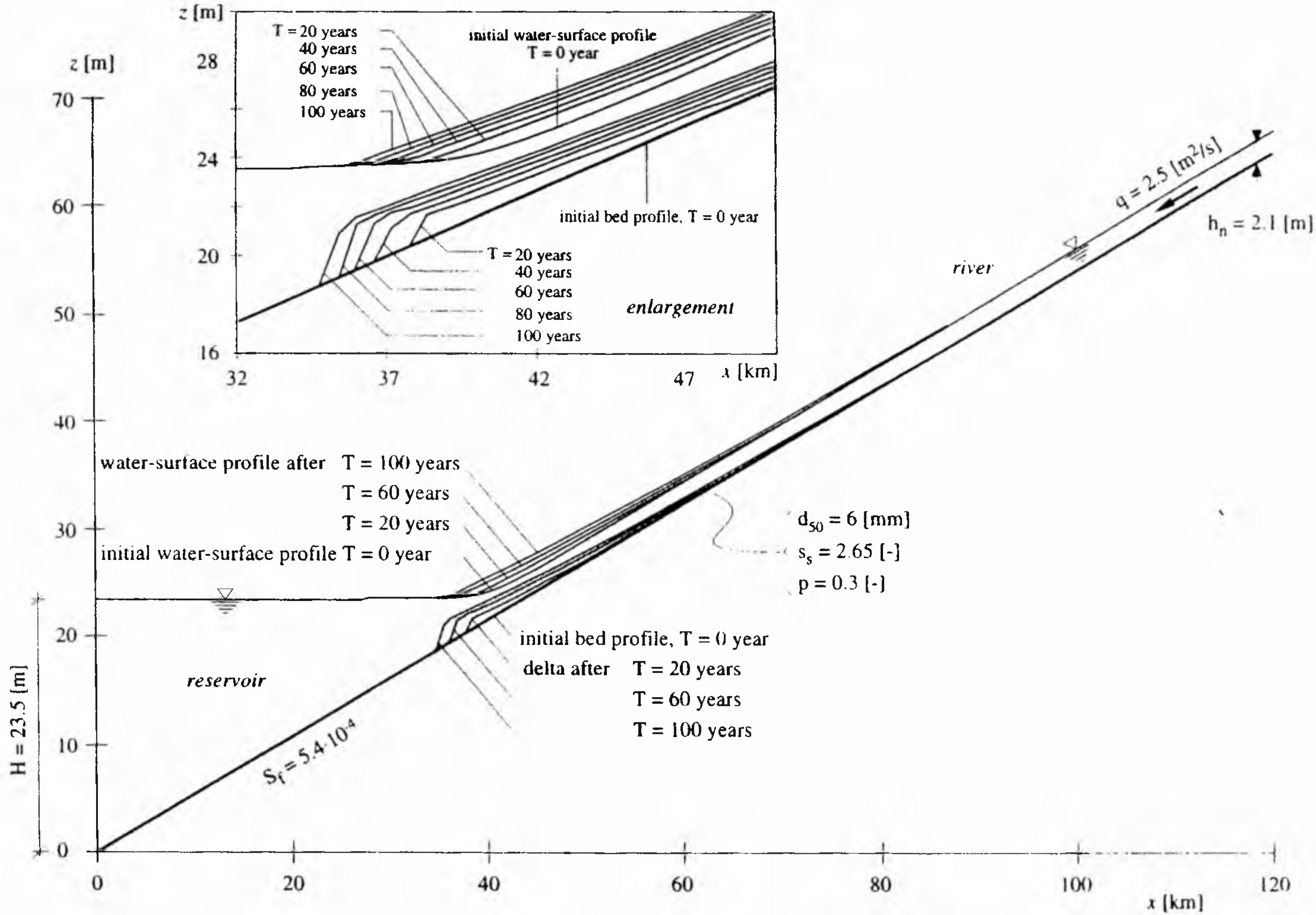


Fig. Ex.6.C.8 Time history of formation and advancement of the delta in a river-reservoir system, simulated for a period of 100 years with the program DELTA by using the bed-load formula of *Schoklitsch* (1950).

At time **T = 100 [years]**, the downstream end of the delta has reached the statio (x = 33.3 [km]) (see column ZF–ZFI). The maximum height of the delta is 2.91 (station 58, x = 34.2 [km]). The total volume of deposited sediments is 71887 [m^3] unit width.

On Figs. Ex.6.C.7 and 8, it is interesting to note that the influence of the delta can be up to a quite long distance upstream.

The table below summarizes the most important characteristics of the deltas, calcul using the two methods.

Results of the calculations, using the bed-load formula of *Meyer-Peter* et *al.* (1948						
	Nose of delta		Maximum height of delta			Vol. o
time [years]	station no	x [m]	h_d (m)	station no	x [m]	delta [m^3.
20	62	36600	1.23	64	37800	14396
40	60	35400	1.78	62	36600	28789
60	59	34800	2.17	61	36000	43173
80	57	33600	2.56	59	34800	57539
100	56	33000	2.91	58	34200	71887

Results of the calculations, using the bed-load formula of *Schoklitsch* (1950)						
	Nose of delta		Maximum height of delta			Vol. o
time [years]	station no	x [m]	h_d (m)	station no	x [m]	delta [m^3/
20	63	37200	0.91	65	38400	7395
40	62	36600	1.31	64	37800	14790
60	61	36000	1.62	63	37200	22185
80	60	35400	1.90	62	36600	29580
100	59	34800	2.14	61	36000	36973

The above table shows that the bed-load formula of *Meyer-Peter* et *al.* (1948) predic delta which is *larger* than the one predicted by the bed-load formula of *Schokli* (1950). After a period of 100 [years], although their height is almost the same, the d obtained by using the method of *Meyer-Peter* et *al.* (1948) has a volume 1.94 times volume of the delta obtained by using the method of *Schoklitsch* (1950). A graph comparison of the results obtained using these two methods is presented Fig. Ex.6.C.9. The difference between the two methods comes from the differe between the predicted bed-load transport rates. For the same hydraulic conditions, formula of *Meyer-Peter* et *al.* predicts a solid discharge which is larger than the predicted by the formula of *Schoklitsch* (the same observation is also valid for Ex.6 In Fig. Ex.6.C.6., the bed-load formula of *Meyer-Peter* et *al.* predicts a solid disch of $q_{sb} = 0.16 \cdot 10^{-4}$ [m^3/s/m] at time T = 0 for the initial river cross-section. The ou file for the simulation using the method of *Schoklitsch* is not given here; the can easily run the program himself and will obtain a solid discharge $q_{sb} = 0.82 \cdot 10^{-5}$ [m^3/s/m]. The ratio between the two solid discharges is 1.95; this is close to the ratio between the different values for the volume of the deltas.

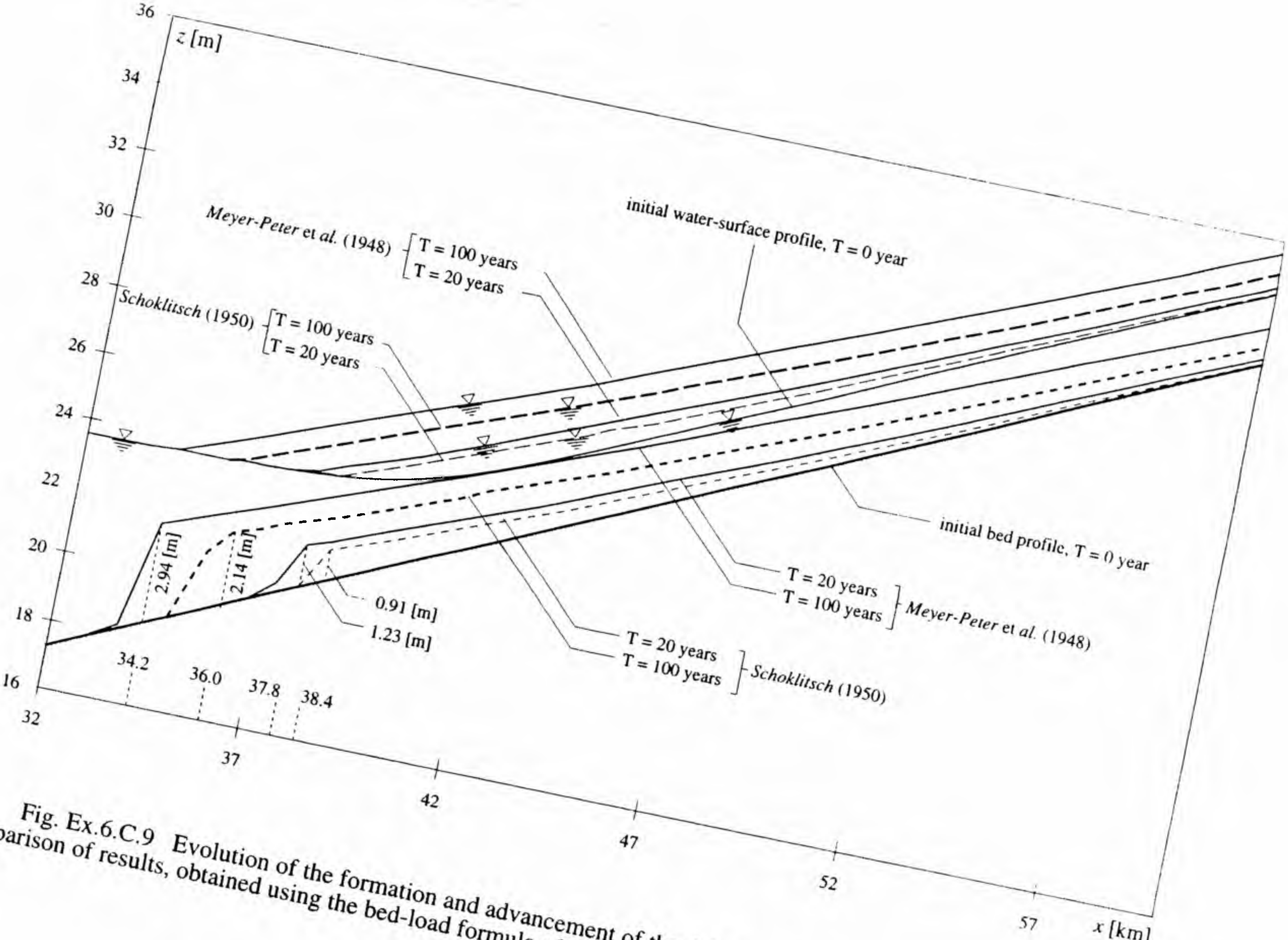


Fig. Ex.6.C.9 Evolution of the formation and advancement of the delta in the river-reservoir system; comparison of results, obtained using the bed-load formula of *Meyer-Peter* et *al.* (1948) and of *Schoklitsch* (1950).

Differences of this order of magnitude between the results obtained using different load formulae are not rare. In a real study of the sediment transport in a river or in a ı reservoir system, it is necessary to do the calculations using different methods aı compare the results against in-situ measurements, in order to determine the formula v is best suited to the particular case. Sometimes it may even be necessary to calibrat sediment-transport formula and/or the simulation program to adapt it to the parti problem in hand.

It is left to the user to run the program using the method of *Einstein* (1942) (see ; 6.3.3, 4 °) and to compare the results with those presented above. It is also importa recall that the option number 4 in choosing the bed-load formula is provided ı programmed by the reader. The user should select a bed-load formula of his choic program it into this subroutine.

g) *Remarks* :

Before finishing this exercise, it is in place to call attention of the reader to a few cr points :

- A computer cannot represent a number internally with an absolute precision.] computers on the market use a set of 32 bits (= 4 bytes), called *word*, for storing ε floating-point number of single precision (R*4). In such a computer a real numt stored according to a standard format as a combination of an integer number (23 l called *mantissa*, and an *exponent* of 2 (8 bits). The last remaining bit is reserve the *sign*. The relative precision that can be obtained with such a storage meth approximately $3 \cdot 10^{-8}$. In many cases this precision may appear to be suffic However, the arithmetic operations with real floating-point numbers reserve a surprises. When a very small number is added on a very large number or whei difference between two numbers being nearly equal is calculated, the round-off e may become very important. In the present program, very often the difference bet two nearly equal numbers are calculated (for example for the calculation of DEL We also add very small values contained in the variable DELZ on the variabl containing much larger values. To avoid the errors caused by a round off, some o variables of the program are declared as "DOUBLE PRECISION" Fig. Ex.6.C.11). The computer uses then two words (= 8 octets = 64 bits representing a real floating-point number (R*8). In this way a relative precision iı order of 10^{-15} is obtained.

- Since the program uses a decoupled algorithm, one cannot talk about a stab condition of the Courant type, for example (see sect. 5.2.3). The choice of the sp and time-step lengths remains nevertheless important from the point of view of al mentioned round-off errors. The step length in the longitudinal direction must be s enough to allow a correct representation of the form of the delta. A too short space and/or a too small time step may lead to deposition volumes and/or bed-l modification values close to the precision of the internal representation of the floaı point numbers in the computer. The reader is advised to try the program with diffe space and time step combinations.

- One of the most important variables controlling the delta formation is the HAMHAV-variable, which defines the way the deposited sediments will be distributed over a reach. It is recalled that a value of HAMHAV = 1, which means a uniform distribution of the sediments depositing in the reach, generates instabilities at the upstream and brings the program to a halt either because of an impossibility to calculate the water-surface profile or because of the detection of a deposition height being larger than the one specified by the user as the maximum allowable value. The reader is encouraged to try the program with different values of HAMHAV ≤ 1.

- A sufficiently long river reach, where the flow remains uniform during the whole simulation period, must be provided at the upstream end of the river-reservoir system to insure a correct functioning of the program DELTA. The upstream limit of the delta should not, therefore, touch the upstream end of the computational reach. To illustrate the influence of the computational domain on the results of the delta formation and progression, simulations were done by modeling three different lengths of the river-reservoir system, namely : TL = 60 [km], 80 [km] and 160 [km]. The form of the data for these three simulations are presented in Fig. Ex.6.C.10 (to be also compared with Fig. Ex.6.C.9). The results are summarized in the table below.

Delta after 100 [years]
Simulations using the bed-load formula of *Meyer-Peter* et *al.* (1948)
with different computational domain lengths, TL.

Modeled length	Space step	Nose of delta	Maximum height of delta		Apparent volume of delta after 100 [years] from the downstream end (dam) up to a distance of			
TL	DX	x	h_d	x				
[km]	[m]	[m]	[m]	[m]	60 [km]	80 [km]	120 [km]	160 [km]
60	400	32800	3.30	33600	69617	—	—	—
80	400	33600	2.85	34400	47576	57524	—	—
120	600	33000	2.91	34200	49205	63830	71887	—
160	400	34000	2.87	34400	49131	63789	71302	71977

The solution with TL = 60 [km] predicts a longer and higher delta (see also Fig. Ex.6.C.10). The other three simulations yield similar values. The difference between the simulation with TL = 120 [km] and the simulations with TL = 80 and 160 [km] is probably due to the difference in the space-step length.

Each simulation is done with a different computational domain length. In the first three simulations, the delta touches the most upstream station; it is therefore not simulated in its total length. Thus it is necessary to calculate the volume of the delta at intermediate intervals, as shown in the table above, for a comparison of deposition volumes obtained from different runs. The simulation with TL = 60 [km] overestimates considerably the volume of the delta up to that distance. It is also interesting to note that the last two simulations predict almost identical delta formations, despite the fact that only the solution with TL = 160 [km] has still a river reach with a uniform flow after 100 years of simulation. The solution with TL = 80 [km] gives a good prediction of the maximum height of the delta; the volume of deposition is, however, smaller.

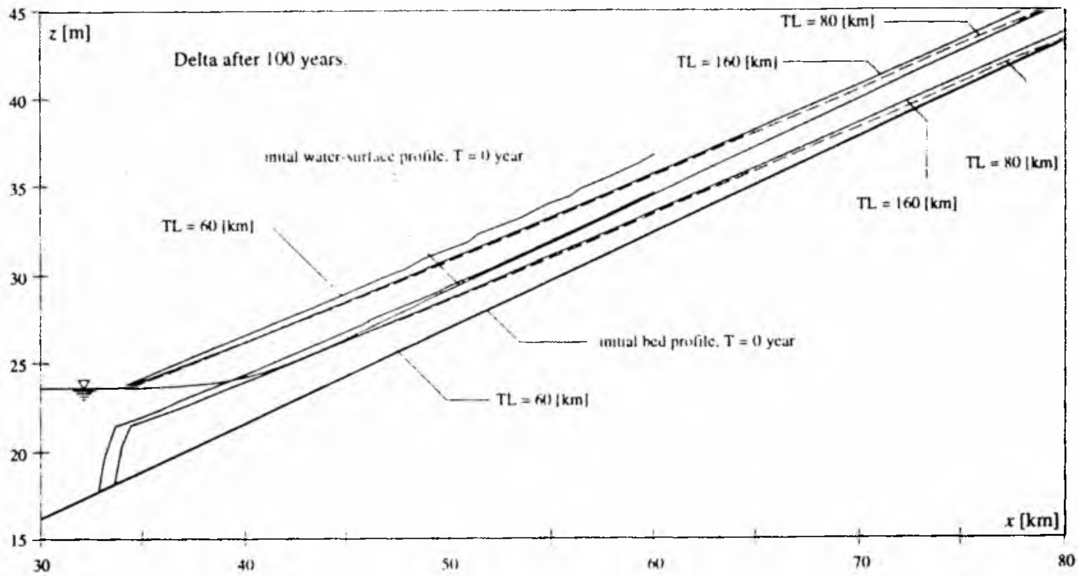


Fig.Ex.6.C.10 Comparison of deltas, simulated with differents lengths of the river reservoir system by using the bed-load transport formula of *Meyer-Peter* et *al.*

```
      PROGRAM DELTA
C************************************************************************
C                                                                       *
C MAIN PROGRAM FOR CALCULATING THE BED-LOAD TRANSPORT IN A              *
C RIVER-RESERVOIR SYSTEM BY TAKING INTO ACCOUNT THE MODIFICATION        *
C OF THE BED LEVEL.                                                     *
C                                                                       *
C THE PROGRAM CONSIDERS A SUBCRITICAL FLOW WITH A CONTROL SECTION       *
C LOCATED AT THE DOWNSTREAM.                                            *
C                                                                       *
C************************************************************************
C
C LIST OF VARIABLES DEFINED GLOBALLY FOR THE MAIN PROGRAM AND THE
C SUBPROGRAMS. VARIABLES DEFINED LOCALLY IN SUBPROGRAMS ARE LISTED AT
C THE BEGINNING OF EACH SUBPROGRAM.
C
C TYPE NAME   DIMEN       EXPLANATIONS
C ==== ====== =======     ================================================
C R*4  CFOI               = COEFFICIENT OF VOLUME INCREASE DUE TO POROSITY
C R*4  CN                 = TOTAL MANNING COEFFICIENT
C R*4  CN50               = MANNING COEFFICIENT DUE TO GRAIN ROUGHNESS
C R*4  CPTT               = TIME-STEP COUNTER
C R*4  CPAM               = WEIGHTING COEFFICIENT FOR UPSTREAM STATION
C                           USED IN CALCULATION OF DEPOSITION
C R*4  CPAV               = WEIGHTING COEFFICIENT FOR DOWNSTREAM STATION
C                           USED IN CALCULATION OF DEPOSITION
C R*4  D50                = AVERAGE SEDIMENT DIAMETER
C R*4  DELQS (NSMAX-1)    = SEDIMENTS DEPOSITED OR ERODED IN A REACH
C                           BOUNDED BY TWO PRINCIPAL SECTIONS
C R*8  DELT               = TIME STEP
C R*8  DELZ  (NSMAX)      = ARRAY CONTAINING THE BED LEVEL CHANGES AT ALL
C                           STATIONS
C R*8  DELZMX             = MAXIMUM BED-LEVEL CHANGE
C R*8  DELZRM             = DIMENSIONLESS MAXIMUM BED-LEVEL CHANGE
C R*4  DHDYNM             = MAXIMUM DIFFERENCE IN DYNAMIC HEAD (U2/2g)
C                           BETWEEN TWO CONSECUTIVE STATIONS
C R*8  DQSDET             = TOTAL VOLUME DEPOSITED SINCE T = 0
C R*8  DQSERT             = TOTAL VOLUME ERODED SINCE T = 0
C R*4  DX                 = DISTANCE BETWEEN TWO PRINCIPAL STATIONS
C R*4  DXSUB              = DISTANCE BETWEEN INTERPOLATED STATIONS
C R*4  FCOR               = ROUGHNESS PARAMETER IN BED-LOAD FORMULA OF
C                           MEYER-PETER (1948) (K/K'=n'/n)
C C*40 FICHS              = NAME OF OUTPUT FILE
C R*4  FRNAM              = FROUDE NUMBER AT UPSTREAM STATION OF A REACH
C R*4  FRNAV              = FROUDE NUMBER AT DOWNSTREAM STATION OF A REACH
C R*4  G                  = GRAVITATIONAL ACCELERATION
C R*4  H     (NSMAX)      = WATER DEPTH AT PRINCIPAL STATIONS
C R*4  HAMHAV             = RATIO OF UPSTREAM/DOWNSTREAM HEIGHTS OF TRAPE-
C                           ZOID FORMED BY SEDIMENTS DEPOSITED IN A REACH
C R*4  HDYN1              = DYNAMIC HEAD (U^2/2G) AT STATION 1
C R*4  HDYN2              = DYNAMIC HEAD (U^2/2G) AT STATION 2
C R*4  HSUB  (2**MCMAX+1) = WATER DEPTH AT INTERPOLATED STATIONS
C I*4  I                  = DO-LOOP COUNTER VARIABLE
C I*4  IAM                = UPSTREAM STATION NUMBER FOR A GIVEN REACH
C I*4  IAV                = DOWNSTREAM STATION NUMBER FOR A GIVEN REACH
C I*4  II                 = DO-LOOP COUNTER VARIABLE
C I*4  IMAXR              = NUMBER OF THE STATION WHERE WE HAVE 'DELZRM'
C I*4  IMAXZ              = NUMBER OF THE STATION WHERE WE HAVE 'DELZMX'
C J*4  J                  = DO-LOOP COUNTER VARIABLE
C I*4  MC                 = NUMBER OF SUBDIVISIONS (IN POWERS OF 2)
C I*4  MCMAX              = MAXIMUM NUMBER OF SUBDIVISIONS (IN POWERS OF 2)
C I*4  ND                 = NUMBER OF PRINCIPAL REACHES
C I*4  NFTS               = NUMBER OF BED-LOAD FORMULA TO BE USED
C I*4  NMC                = NUMBER OF SUBDIVISIONS (IN POWERS OF 2)
C                           SPECIFIED BY USER
C I*4  NOUT               = UNIT NUMBER OF OUTPUT FILE
```

```
C I*4  NPP              = PRINTING FREQUENCY
C I*4  NPT              = PRINTING TIME COUNTER
C I*4  NS               = NUMBER OF PRINCIPAL STATIONS
C I*4  NSMAX            = NUMBER OF MAXIMUM STATIONS ALLOWED BY THE PROGRAM
C I*4  NSUBD            = NUMBER OF SUBDIVISIONS AT A GIVEN REACH
C R*4  PI               = PI NUMBER
C R*8  QCRIT            = CRITICAL DISCHARGE IN FORMULA OF SCHOKLITSCH
C R*8  QSU    (NSMAX)   = SEDIMENTS TRANSPORTED THROUGH A STATION
C R*4  QU               = UNIT DISCHARGE (WATER)
C C*1  QUIT             = READ FOR TERMINATING THE PROGRAM
C R*4  ROE              = DENSITY OF WATER
C R*4  ROS              = DENSITY OF SEDIMENTS
C R*4  SEFF             = SLOPE OF ENERGY GRADE LINE AT A STATION
C R*4  SF               = INITIAL BED SLOPE
C R*4  SFTR             = LOCAL BED SLOPE AT A GIVEN REACH
C R*4  SS               = SPECIFIC DENSITY OF THE SEDIMENTS
C R*8  T                = TIME
C R*8  TFIN             = FINAL TIME
C R*4  TJ               = TIME (NUMBER OF DAYS)
C R*4  TL               = TOTAL LENGTH OF RIVER REACH STUDIED
C R*4  VARZMX           = MAXIMUM RELATIVE BED-LEVEL CHANGE ALLOWED BY
C                         THE PROGRAM IN A SINGLE TIME STEP
C R*4  X      (NSMAX)   = X COORDINATES OF PRINCIPAL STATIONS
C R*4  X1               = X COORDINATE OF MOST DOWNSTREAM STATION
C R*4  XF               = X COORDINATE OF MOST UPSTREAM STATION
C R*4  XSUB (2**MCMAX+1) = X-COORDINATES OF INTERPOLATED STATIONS
C R*8  ZF     (NSMAX)   = BED-LEVEL ELEVATIONS AT ANY TIME
C R*8  ZFI    (NSMAX)   = INITIAL BED-LEVEL ELEVATIONS
C
C
C*********************************************************************
C
C PARAMETERS
      PARAMETER ( NOUT = 10 , NSMAX = 1000 , MCMAX = 7 )
C
C LABELLED COMMON BLOCKS (SHARED VARIABLES)
      COMMON /DONNE1/ SFTR , QU , CN
C
C THE MAIN PROGRAM "DELTA" SENDS THE NAME OF THE SUBROUTINE SUBPROGRAM
C "DERIVE" TO THE SUBROUTINE SUBPROGRAM "RK4" IN THE LIST OF ARGUMENTS.
C ACCORDING TO THE FORTRAN PROGRAMMING RULES, THE SUBROUTINE SUBPROGRAM
C "DERIVE" SHOULD BE DECLARED AS "EXTERNAL".
      EXTERNAL DERIVE
C
C DECLARATION OF VARIABLES
      CHARACTER*1    QUIT
      CHARACTER*40   FICHS
      DOUBLE PRECISION  ZF (NSMAX) , ZFI (NSMAX)
      DOUBLE PRECISION  QSU(NSMAX) , DELZ(NSMAX) , DELQS(NSMAX-1)
      DOUBLE PRECISION  DQSDET , DQSERT , DELZMX , DELZRM
      DOUBLE PRECISION  DELT , TFIN , T
      DIMENSION         X(NSMAX) , H(NSMAX)
      DIMENSION         XSUB(2**MCMAX+1) , HSUB(2**MCMAX+1)
C
C
      OPEN( UNIT = 8 , FILE = 'GRAPH.DAT' , STATUS = 'NEW' )
C
C READ THE PROBLEM DATA
      CALL DREAD( NOUT , NSMAX , MCMAX ,
     1            QU , SF , CN  , D50 , ROS , ROE , SS , CN50 ,
     2            X(1) , XF , TL  , DX , ND  , NS , NMC ,
     3            H(1) , DHDYNM  , VARZMX ,
     4            DELT , TFIN ,
     5            NPTS , CFOI , FCOR , HAMHAV , CRAM , CRAV ,
     6            FICHS , NPP )
C
C INITIALISATIONS
      PI      = 3.1415927
      G       = 9.81
      T       = -DELT
      CPTT    = 0.0
      NPT     = -1
      DQSDET  = 0.0
      DQSERT  = 0.0
C
C CALCULATION OF X COORDINATES AND INITIAL BED LEVELS OF COMPUTATIONAL
C SECTIONS
      X(1)    = 0.0
      ZF(1)   = 0.0
      DO 10 I = 2 , NS
      X(I)       = X(I-1) + DX
      ZF(I)      = ZF(I-1) + SF * DX
      ZFI(I)     = ZF(I)
10    CONTINUE
C
C WRITE THE TITLES ON OUTPUT FILE
      CALL TITLES (NOUT , SF , D50 , CN50 , CN , ROS , ROE , QU ,
     1            DHDYNM , NMC , VARZMX , CFOI , HAMHAV , FCOR ,
     2            NPTS , X1 , XF , TL , DX , ND , NS ,  H(1) ,
     3            DELT , TFIN , NPP , FICHS)
C
C CALCULATION LOOP
100   T     = T + DELT
      CPTT = CPTT + 1.0
      NPT  = NPT + 1
      TJ   = T / 86400
      WRITE(*,110) CPTT , TJ
110   FORMAT(/' TIME STEP       = ',F15.0/
     1        ' TIME (days)     = ',F10.3)
C
C*************************************************************************
C**************      BEGINNING OF LIQUID PHASE CALCULATIONS     **********
C*************************************************************************
C
C BACKWATER CALCULATION
      DO 300 I = 1 , NS-1
      SFTR     = (ZF(I+1) - ZF(I)) / DX
      MC       = 0
200   NSUBD    = 2**MC
      DXSUB    = (X(I+1) - X(I)) / NSUBD
      HSUB(1) = H(I)
      XSUB(1) = X(I)
C
      DO 250 J = 1 , NSUBD
      XSUB(J+1)   = X(I) + DXSUB * NSUBD
      CALL RK4( G , HSUB(J) , DXSUB , HSUB(J+1) , DERIVE )
      HDYN1 = QU**2 / (2*G*HSUB(J)**2)
      HDYN2 = QU**2 / (2*G*HSUB(J+1)**2)
      IF(ABS(HDYN2-HDYN1).GT.DHDYNM)THEN
C
C INTERMEDIATE SECTIONS MUST BE GENERATED BY INTERPOLATION
        MC = MC+1
        WRITE(*,223) 2**MC , I , I+1
223     FORMAT(/1X,I2,' subdivisions between sections ',I5,' and ',I5)
        IF(MC.GT.NMC)THEN
          FRNAM = QU / (H(I) * SQRT(G * H(I)))
```

Fig.Ex.6.C.11   Program DELTA (continued).

Ex. 6.D

An artificial channel has been constructed to divert a certain discharge from a river. Thi channel has an approximately rectangular cross section with a width of $B = 46.5$ [m] an a bed slope of $S_f = 6.5 \cdot 10^{-4}$ [-]. The uniform flow is established when the flow dept is $h_n = 5.6$ [m]. The velocity-profile measurements carried out in this channel allowe to obtain the average velocity of $U = 1.8$ [m/s] for a friction coefficient c $n' = 0.0212$ [$m^{-1/3}s$]. The granulometry of the bed material has not been analyzed.

Estimate the bed-load transport in this channel. Subsequently, express the solid discharg as a concentration. Is suspended-load transport to be expected ?

SOLUTION :

i) First, preliminary calculations concerning the hydraulics of the channel and th sedimentology of the bed material should be carried out.

Unit discharge : $q = Uh = 1.8\,(5.6) = 10.08$ [m^3/sm]

Discharge : $Q = qB = 10.08\,(46.5) = 468.72$ [m^3/s]

Hydraulic radius : $R_h = \dfrac{Bh}{B + 2h} = \dfrac{(46.5)\,(5.6)}{46.5 + 2\,(5.6)} = 4.51$ [m]

Hydraulic radius of the channel bed (the channel banks are assumed to be smooth)

$$R_{hb} \cong h_n = 5.6 \text{ [m]}$$

The granulometry can be estimated using the calculated friction coefficient, n which, being obtained from a measured velocity profile, corresponds to the frictio coefficient due to grain roughness. By using the Strickler formula :

$$\frac{1}{n'} = K_s' = \frac{1}{0.0212} = \frac{26}{d_{90}^{1/6}} \quad \text{and} \quad K_s' = \frac{21.1}{d_{50}^{1/6}} \qquad (3.18$$

one obtains :

$$d_{90} = 0.0280 \text{ [m]} \quad ; \quad d_{50} = 0.0080 \text{ [m]}$$

By assuming a granulometric distribution to be logarithmic, one finds :

$d_{35} = 0.0055$ [m] ; $d_{40} = 0.0062$ [m] ; $d_{65} = 0.0117$ [m]

The total friction coefficient, n , which is due to the combined effect of grai roughness and bed forms, can now be obtained using the Manning-Strickle formula :

$$K_s = \frac{1}{n} = \frac{U}{R_{hb}^{2/3} S_f^{1/2}} = \frac{1.8}{(5.6)^{2/3}(0.00065)^{1/2}} = 22.41 \ [m^{1/3}/s] \qquad (3.16)$$

whereas :

$$K_s' = \frac{1}{n'} = \frac{1}{0.0212} = 47.17 \ [m^{1/3}/s].$$

The roughness parameter is therefore :

$$\xi_M = \left(\frac{K_s}{K_s'}\right)^{3/2} = \left(\frac{22.41}{47.17}\right)^{3/2} = 0.327$$

The settling velocity for d_{50} is (see Fig. 6.10) : $v_{ss} \cong 0.4$ [m/s]

ii) Three different bed-load equations will be used to estimate the solid discharge; namely the bed-load equation of *Schoklitsch*, eq. 6.31, of *Meyer-Peter* et *al.*, eq. 6.32, and of *Einstein*, eq. 6.42.

a) The bed-load equation of *Schoklitsch* is given by :

$$q_{sb} = \frac{2.5}{s_s} S_e^{3/2} (q - q_{cr}) \qquad (6.31)$$

The critical liquid discharge is calculated using :

$$q_{cr} = 0.26 (s_s - 1)^{5/3} \ d^{3/2} \ S_e^{-7/6} \qquad (6.31a)$$

where $d = d_{40} = 0.0062$ [m] for a non-uniform granulometry and assuming that the specific density of the bed material is $s_s = 2.65$ [-] :

$$q_{cr} = 0.26 \ (1.65)^{5/3} \ (0.0062)^{3/2} \ (0.00065)^{-7/6} = 1.53 \ [m^2/s]$$

The volumic solid discharge for a unit width is then :

$$q_{sb} = \frac{2.5}{2.65} (0.00065)^{3/2} (10.08 - 1.53) = \mathit{1.33 \cdot 10^{-4}} \ [m^2/s].$$

b) The bed-load equation of *Meyer-Peter* et *al.* is given by :

$$\frac{\gamma R_{hb} \xi_M S_e}{(\gamma_s - \gamma) d} - 0.047 = 0.25 \ \rho^{1/3} \frac{g'_{sb}{}^{2/3}}{(\gamma_s - \gamma) d} \qquad (6.32)$$

where $d = d_{50} = 0.008$ [m] for a non-uniform granulometry .

The solid discharge by submerged weight for a unit width can then calculated as :

$$g'_{sb}{}^{2/3} = \left(\frac{9.81\ (1650)\ (0.008)}{0.25\ (1000)^{1/3}}\right)\ \left(\frac{5.6\ (0.327)\ (0.00065)}{(2.65-1.0)\ 0.008} - 0.047\right)$$

$$g'_{sb} = [(51.91)\ (0.090 - 0.047)]^{3/2} = (2.23)^{3/2} = 3.35\ [\mathrm{N/ms}]$$

The volumic solid discharge for a unit width is :

$$q_{sb} = \frac{g_{sb}}{\gamma_s} = \frac{g'_{sb}}{(\gamma_s-\gamma)} = \frac{3.35}{9.81\ (1650)} = 2.07 \cdot 10^{-4}\ [\mathrm{m^2/s}]$$

c) The bed-load equation of *Einstein* is given by :

$$\Phi_* = f(\Psi'_*) \tag{6.42}$$

Considering the non-uniform granulometry with $d = d_{35} = 0.0055$ [m], o shall take :

$$\Phi = f(\Psi') \qquad \text{or} \qquad \frac{q_{sb}}{\sqrt{(s_s-1)gd^3_{35}}} = f\left(\frac{(\gamma_s-\gamma)\ d_{35}}{\tau_o'}\right)$$

First the shear-stress intensity parameter should be calculated :

$$\Psi' = \frac{(\gamma_s-\gamma)d_{35}}{\rho u_*'^2} = (s_s-1)\ \frac{d_{35}}{R_{hb}'S_f} \tag{6.3}$$

The friction velocity due to the grain roughness will be calculated using tl logarithmic velocity distribution :

$$\frac{U}{u_*'} = 5.75 \log\left(\frac{h}{k_s}\right) + 6.25 = 21.66 \tag{3.131}$$

with $k_s = d_{65} = 0.0117$ [m] and $h = 5.6$ [m]. It is found that :

$$u_*' = \frac{U}{21.66} = \frac{1.8}{21.66} = 0.083\ [\mathrm{m/s}].$$

The value of Ψ' can then be calculated as follows :

$$\Psi' = \frac{g(\rho_s-\rho)d_{35}}{\rho\ u_*'^2} = \frac{9.81(2.65-1)0.0055}{(0.083)^2} = 12.92\ [-]$$

The hydraulic radius due to grain groughness is :

$$u_*' = \sqrt{g\ R_{hb}'\ S_f} \qquad \Rightarrow \qquad R_{hb}' = \frac{u_*'^2}{g\ S_f} = 1.08\ [m]$$

whereas the hydraulic radius due to the bed forms is :

$$R_{hb}'' = R_{hb} - R_{hb}' = 5.6 - 1.08 = 4.52\ [m]$$

This shows the importance of bed forms in this cross section of the channel. (If these values, $\Psi' = 12.92$ and $U/u_*'' = 10.60$, are compared with the relationship of Einstein-Barbarossa, represented in Fig. 3.6, a slight difference will be observed).

One can now either evaluate the function given by eq. 6.42, or read it directly on Fig. 6.8, to find :

$$\text{for} \quad \Psi' = 12.92 \qquad \Rightarrow \qquad \Phi \cong 0.033$$

The volumic solid discharge per unit width can then be calculated as :

$$q_{sb} = \Phi \sqrt{(s_s-1)\ g d_{35}^{\ 3}} = 0.033\ [(1.65)\ (9.81)\ (0.0055^3)\ ]^{1/2} \qquad (6.28)$$

$$= 5.41 \cdot 10^{-5}\ [m^2/s]$$

iii) The transport of sediments as bed load, obtained using these three bed-load equations, is presented in the table below, both by volume and by mass.

The difference between the values obtained using the different formulae is considerable but this is not surprising; one should in fact never expect to find exactly the same values using different formulae.

Formula	Solid discharge as bed load			
	q_{sb} [m²/s]	g_{sb} [kg/ms]	Q_{sb} [m³/s]	G_{sb} [kg/s]
Schoklitsch eq. 6.31	$1.33 \cdot 10^{-4}$	0.35	$6.18 \cdot 10^{-3}$	16.39
Meyer-Peter et *al.* eq. 6.32	$2.07 \cdot 10^{-4}$	0.55	$9.62 \cdot 10^{-3}$	25.50
Einstein eq. 6.42	$0.54 \cdot 10^{-4}$	0.14	$2.51 \cdot 10^{-3}$	6.65

iv) The liquid discharge of

$$Q = 468.72 \ [m^3/s] \qquad \text{or} \qquad G = 468.72 \cdot 10^3 \ [kg/s]$$

is responsible for the solid discharge — taking the average of the values listed in above table — of :

$$Q_{sb} = 6.1 \cdot 10^{-3} \ [m^3/s] \qquad \text{or} \qquad G_{sb} = 16.2 \ [kg/s]$$

The average sediment *concentration*, C_s , can be expressed in different manr Here are the possible definitions :

concentration by volume : $C_s = \dfrac{\text{volume of sediments}}{\text{total volume}} \ \dfrac{[m^3/s]}{[m^3/s]}$

concentration by mass : $C_s' = \dfrac{\text{mass of sediments}}{\text{total mass}} \ \dfrac{[kg/s]}{[kg/s]}$

concentration by unit mass : $C_s'' = \dfrac{\text{mass of sediments}}{\text{total volume}} \ \dfrac{[kg/s]}{[m^3/s]}$

These definitions are related to one another in the following way :

$$C_s'' = \rho_s C_s \quad ; \qquad C_s' = \frac{\rho_s C_s}{\rho + (\rho_s - \rho) C_s} = \frac{\rho_s}{\rho_m} C_s$$

where ρ_m is the average density of the water-sediment mixture, defined by :

$$\rho_m = C_s \rho_s + (1 - C_s) \rho = \rho + (\rho_s - \rho) C_s$$

With above definitions, the concentrations can be obtained.

The density of the solid particles is : $\rho_s = 2650 \ [kg/m^3]$ or $2.65 \ [g/cm^3]$.

The concentrations can now be calculated :

$$C_s = \frac{6.1 \cdot 10^{-3}}{468.72} = 0.000013 \ [-]$$

$$C_s' = \frac{16.2}{468.72 \cdot 10^3} = 0.000035 \ [-]$$

$$C_s'' = \frac{16.2}{468.72} = 0.0345 \left[\frac{kg}{m^3}\right] \text{ or } \left[\frac{g}{l}\right] \text{ or } \frac{1}{1000} \left[\frac{g}{cm^3}\right]$$

The average density of the mixture is :

$$\rho_m = 1000.00 + 0.02 = \mathit{1000.02}\ [\mathrm{kg/m^3}]$$

v) It is already shown that there is a strong bed-load transport in this channel. One can now ask, if there will be also suspended-load transport.

According to an indicative criteria, given in sect. 6.1.3, the suspended-load transport starts when :

$$\frac{u_*}{v_{ss}} > 0.40$$

For the present problem one obtains :

$$\frac{u_*}{v_{ss}} = \frac{0.189}{0.400} = 0.47$$

Therefore, a weak transport of sediments as suspended load is to be expected.

This expectation can still be controlled by determining the Rouse exponent, eq. 6.50a :

$$\mathfrak{z} = \frac{v_{ss}}{\kappa u_*'} = \frac{0.4}{0.4\ (0.083)} = 12.05$$

On Fig. 6.11, it can be seen that, for this $\mathfrak{z}$-value, the relative concentration distribution, consequently the suspended-load transport, will be indeed weak.

Ex. 6.E

A mountain river with a bottom slope of $S_f = 0.0062$ [-] has an approximately rectangula cross section, being $B = 23.5$ [m] wide. Analysis of the sediment samples taken from well below the armour layer show that $d_{50} = 60$ [mm] and $d_{90} = 200$ [mm] and the density of sediments is $s_s = 2.65$ [-].

Determine the diameter, d_{50_a}, of the maximum possible armouring. At which flow depth does the armour layer become unstable ?

SOLUTION :

i) The grain diameter of the armour layer is calculated (see point 6.3.4.5°) using the relationship :

$$d_{50_a} \cong 0.6\ d_{90}$$

$$d_{50_a} \cong 0.6\ (200) = \mathit{120}\ [\mathrm{mm}]$$

ii) The stability of the armour layer (see point 6.3.4.5°) can be estimated using the expression :

$$\tau_{*a,cr} = \frac{u_{*\,a,cr}^{2}}{(s_s-1)\ g\ d_{50_a}} = \tau_{*cr}\left[0.4\ \left(\frac{d_{50}}{d_{50_a}}\right)^{1/2} + 0.6\right]^2$$

$$\tau_{*a,cr} = \tau_{*cr}\left[0.4\ \left(\frac{60}{120}\right)^{1/2} + 0.6\right]^2 = 0.05[0.88]^2 = \mathit{0.04}\ [-]$$

from which one obtains :

$$u_{*a,cr} = \sqrt{0.04\ [(s_s-1)\ g\ d_{50_a}]} = \sqrt{0.04\ [1.94]} = 0.28\ [\mathrm{m/s}].$$

According to the definition of the friction velocity :

$$u_* = \sqrt{g\ R_h\ S_f} \qquad \Rightarrow \qquad R_h = u_*^2/g\ S_f$$

$$R_h = \frac{0.28^2}{9.81\ (0.0062)} = 1.25\ [\mathrm{m}]$$

Now the limiting depth for the stability of the armour layer can be calculated as :

$$R_h = \frac{h\ B}{2h+B} \qquad \text{or} \qquad 1.25 = \frac{h\ (23.5)}{2h + 23.5} \qquad \Rightarrow \qquad h = \mathit{1.40}\ [\mathrm{m}]$$

If the flow depth becomes larger than $h = 1.40$ [m], the armour layer is no longer stable and an important erosion of the bed material may be expected.

Ex. 6.F

The river *Happy* — whose *stage—water-discharge curve* was established in Ex. 3.B — has a variable discharge in the range of $10 < Q\ [m^3/s] < 1000$. The width of the river bed is b = 90 [m] and its non-erodible banks have a slope of 1:1. The topographical survey of the river showed that the bed slope is $S_f = 0.0005$ [-]. The sediment forming the bed has a specific density of $s_s = 2.652$ [-] and the grain-size analysis yielded: $d_{50} = 0.32$ [mm], $d_{35} = 0.29$ [mm] and $d_{90} = 0.48$ [mm]. The water temperature in the river is $T = 14$ [°C].

Determine the *stage—sediment-discharge curve*, $Q_s = f(h)$, for this river.

SOLUTION :

First the hydraulic calculations should be done to determine the *stage—water*-discharge curve, $Q = f(h)$, for the river. Once this is done, one can carry out the sediment-transport calculations to determine the *stage—solid*-discharge curve, $Q_s = f(h)$.

i) *Hydraulic calculations* :
The hydraulic calculations were presented and commented in Ex. 3.B. The calculation table, where each line represents the calculation of a discharge, Q , and other useful hydraulic parameters, R_h' , R_h'' , R_h , etc., is partially reprinted in Table Ex.6.F.1, together with the explanations for the columns. On every line, the calculations start by *assuming* a hydraulic radius due to grain roughness, R_h'. The corresponding discharge is obtained by following the procedure described in Ex. 3.B. The values for R_h' are selected such that the calculations cover the entire range of the desired water discharges in the river, $10 < Q\ [m^3/s] < 1000$.

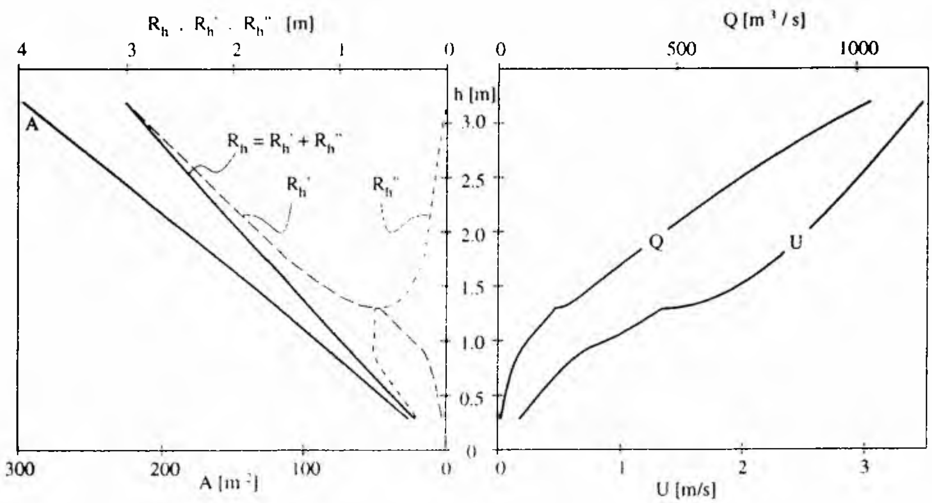


Fig. Ex. 6.F.1 *Stage—liquide*-discharge curve.

The *stage—water*-discharge curve, $Q = f(h)$, and the variation of other parameters, U, A, R_h', R_h'' et R_h , as a function of the flow depth, h, are presented in Fig. Ex. 6.F.1.

Table Ex.6.F.1

Computation sheet for determining the *stage*—**water**-discharge curve, using the method of *Einstein-Barbarossa* (1952)

b = 90 [m]	T = 14 [°C]	ρ_s = 2650 [kg/m³]
m = 1	ρ = 999.1 [kg / m³]	d_{35} = 0.00029 [m]
S_f = 0.0005 [-]	$\nu = 1.186 \times 10^{-6}$ [m²/s]	$k_s = d_{50}$ = 0.00032 [m]

1	2	3	4	5	6	7	8	9	10	1
R_h'	u_*'	U	Ψ'	U/u_*''	u_*''	R_h''	R_h	u_*	h	(
[m]	[m/s]	[m/s]	[-]	[-]	[m/s]	[m]	[m]	[m/s]	[m]	[m
0.02	0.01	0.16	47.92	4.5	0.04	0.26	0.28	0.04	*0.29*	[illegible]
0.05	0.02	0.29	19.17	6.6	0.04	0.39	0.44	0.05	*0.44*	*1*
0.10	0.02	0.45	9.58	8.7	0.05	0.55	0.65	0.06	*0.65*	*2*
0.15	0.03	0.58	6.39	10.4	0.06	0.63	0.78	0.06	*0.79*	*4*
0.20	0.03	0.69	4.79	11.9	0.06	0.68	0.88	0.07	*0.89*	*5*
0.40	0.04	1.05	2.40	18.3	0.06	0.67	1.07	0.07	*1.09*	*10*
0.60	0.05	1.33	1.60	23.4	0.06	0.66	1.26	0.08	*1.29*	*15*
0.80	0.06	1.58	1.20	32.4	0.05	0.49	1.29	0.08	*1.32*	*19*
1.00	0.07	1.81	0.96	42.6	0.04	0.37	1.37	0.08	*1.41*	*23*
1.25	0.08	2.06	0.77	56.2	0.04	0.27	1.52	0.09	*1.57*	*29*
1.50	0.09	2.30	0.64	73.1	0.03	0.20	1.70	0.09	*1.76*	*37*
2.00	0.10	2.72	0.48	107.2	0.03	0.13	2.13	0.10	*2.23*	*55*
2.50	0.11	3.11	0.38	163.0	0.02	0.07	2.57	0.11	*2.71*	*78*
3.00	0.12	3.46	0.32	6844.0	0.00	0.00	3.00	0.12	*3.19*	*102*

<u>col.</u>	<u>symbol</u>	<u>explanations</u>	<u>expression</u>
1	R_h'	hydraulic radius due to grain roughness (*assumed* value)	
2	u_*'	friction velocity due to grain roughness, eq. 3.24,	$\sqrt{g\ R_h'}$
3	U	average velocity in the cross section	$u_*'\sqrt{8\ /}$
		with (see eq. 3.13b) : $\sqrt{8/f'}$ = 5.6 log (R_h' / k_s) + (	
4	Ψ'	parameter of Einstein-Barbarossa, eq. 3.31,	$\frac{(s_s - 1)}{R_h'\ S_f}$
5	U/u_*''	ratio of velocities corresponding to Ψ' (see eq. 3.31 and Fig. 3.6)	
6	u_*''	friction velocity due to bed forms	U / (U/u
7	R_h''	hydraulic radius due to bed forms	$(u_*'')^2$ / (g
8	R_h	total hydraulic radius, eq. 3.24,	R_h' + I
9	u_*	total friction velocity, eq. 3.7,	$\sqrt{g\ R_h}$
10	h	flow depth (see Tableau 1.1)	
11	Q	water discharge, eq. 3.2a,	Uh (b + n

ii) *Sediment-transport calculations* :
The sediment-transport calculations will be made for the representative grain diameter, using three different total-load relations, namely : (1) *Einstein*, (2) *Graf* and *Acaroglu* and also (3) *Ackers* et *White*.

1° The formula of *Einstein* (1950), which allows the calculation of the total load transported by the flow, is given by :

$$q_s = q_{sb} + q_{ss} = q_{sb}\left[1 + 2.303 \log(30.2\, h / \Delta)\, \mathcal{I}_1 + \mathcal{I}_2\right] \tag{6.60}$$

Since the grain-size distribution is *quasi uniform*, the calculations can be done using an equivalent grain diameter (see Table 6.3) of :

$d = d_{35} = 0.00029$ [m]

The intensity of transport is :

$$\Phi_* \equiv \Phi = \frac{q_{sb}}{\sqrt{(s_s-1)g d_{35}^3}} \tag{6.28}$$

where $q_{sb} = g_{sb}/\gamma_s$ is the volumic solid discharge for a unit width and g_{sb} the solid discharge by weight; both transported as bed load.

The intensity of shear is :

$$\Psi_* \equiv \Psi' = (s_s-1)\, \frac{d_{35}}{R_{hb}' S_f} \tag{6.34}$$

where $R_{hb}' \equiv R_h'$ is the hydraulic radius of the bed due to the granulats.

With the functional relationship of :

$$\Phi_* = f(\Psi_*) \tag{6.42a}$$

given by eq. 6.42 and presented in Fig. 6.8, one can obtain the solid discharge, q_{sb}, transported as bed load.

Next, the integrals, $\mathcal{I}_1$ and $\mathcal{I}_2$, which appear in the suspended-load formula, eq. 6.56 (see also eq. 6.60), should be determined to calculate the solid discharge, q_{ss}, transported as suspended load.

The total load, $q_s = q_{sb} + q_{ss}$, transported by the river can then be calculated using eq. 6.60.

The calculations can be programmed on a microcomputer using a spreadsheet program. The table of calculations prepared in this way is presented in Table Ex. 6.F.2. Each line of this table gives the calculations of the solid discharge (by volume, mass and weight), as well as other useful parameters calculated for each flow depth, h. The detailed explanations on the contents of the columns are given below the table of calculations.

Table Ex.6.F.2

Computation sheet for determining the *stage*—**solid**-discharge curve,
using the method of *Einstein* (1950)

$b = 90$ [m] $\rho = 999.1$ [kg/m³]

$S_f = 0.0005$ [-] $\nu = 1.186 \times 10^{-6}$ [m²/s]

1	2	3	4	5	6	7	8	9	10	11	12
h	R_h'	u_*'	δ	k_s/δ	χ	Δ	P_e	Ψ'	Φ	q_{sb}	Q_{s}
[m]	[m]	[m/s]	[m]	[-]	[-]	[m]	[-]	[-]	[-]	[m³/s/m]	[m³/
0.29	0.02	0.01	1.39E-03	0.256	0.91	3.92E-04	10.01	47.92	0.00	0.00E+00	*0.00E-*
0.44	0.05	0.02	8.81E-04	0.405	1.25	2.86E-04	10.76	19.17	0.00	7.96E-08	*7.16E-*
0.65	0.10	0.02	6.23E-04	0.573	1.48	2.42E-04	11.32	9.58	0.10	2.00E-06	*1.80E-*
0.79	0.15	0.03	5.09E-04	0.702	1.56	2.29E-04	11.56	6.39	0.35	6.91E-06	*6.22E-*
0.89	0.20	0.03	4.41E-04	0.811	1.60	2.24E-04	11.71	4.79	0.71	1.41E-05	*1.27E-*
1.09	0.40	0.04	3.12E-04	1.147	1.60	2.23E-04	11.91	2.40	2.51	5.00E-05	*4.50E-*
1.29	0.60	0.05	2.54E-04	1.404	1.57	2.28E-04	12.06	1.60	4.32	8.59E-05	*7.73E-*
1.32	0.80	0.06	2.20E-04	1.621	1.49	2.40E-04	12.03	1.20	6.06	1.21E-04	*1.08E-*
1.41	1.00	0.07	1.97E-04	1.813	1.42	2.51E-04	12.05	0.96	7.77	1.54E-04	*1.39E-*
1.57	1.25	0.08	1.76E-04	2.027	1.37	2.61E-04	12.12	0.77	9.86	1.96E-04	*1.76E-*
1.76	1.50	0.09	1.61E-04	2.220	1.31	2.72E-04	12.19	0.64	11.94	2.37E-04	*2.14E-*
2.23	2.00	0.10	1.39E-04	2.564	1.24	2.87E-04	12.37	0.48	16.06	3.19E-04	*2.87E-*
2.71	2.50	0.11	1.25E-04	2.866	1.19	2.99E-04	12.53	0.38	20.18	4.01E-04	*3.61E-*
3.19	3.00	0.12	1.14E-04	3.140	1.16	3.07E-04	12.66	0.32	24.28	4.83E-04	*4.34E-*

col.	symbol	explanations	expression
1	h	flow depth (see Table Ex.6.F.1)	
2	R_h'	hydraulic radius due to grain roughness (see Table Ex.6.F.1)	
3	u_*'	friction velocity due to grain roughness (see Table Ex.6.F.1)	
4	δ	thickness of viscous sublayer	$\delta = 11.5\ \nu/$
5	k_s/δ	relative roughness (see Fig. 6.7a)	$k_s/\delta = d_{65}/$
6	χ	correction term for logarithmic velocity distribution (see Fig. 6.7a)	
7	Δ	apparent roughness diameter (see Fig. 6.7a)	$\Delta = d_{65}/$
8	P_e	transport parameter (see eq. 6.58)	$P_e = 2.303 \log(30.2h/$
9	Ψ'	intensity of shear, eq. 6.34,	$\Psi' = (s_s - 1)\ \frac{d_{35}}{R_h'}$
10	Φ	intensity of transport, eq. 6.42,	$\Phi = f(\Psi$
11	q_{sb}	solid discharge, as bed load, by volume and by unit width, eq. 6.28,	$q_{sb} = \Phi \sqrt{(s_s - 1)\, g\, d_3}$
12	Q_{sb}	solid discharge, as bed load, by volume	$Q_{sb} = q_{sb}$

Table Ex.6.F.2 (suite)

Computation sheet for determining the *stage*—**solid**-discharge curve, using the method of *Einstein* (1950)

$d_{35} = 0.00029$ [m] $v_{ss}(d_{35}) = 0.0365$ [m/s]
$d_{65} = 0.00036$ [m] $\rho_s = 2650$ [kg/m³] $\kappa = 0.4$ [-]

13	14	15	16	17	18	19	20	21
A_E	z	J_1	J_2	q_{ss}	Q_{ss}	Q_s	G_s	G_s
[-]	[-]	[-]	[-]	[m³/s/m]	[m³/s]	[m³/s]	[kg/s]	[N/s]
2.03E−03	9.222	2.62E−02	−1.59E−01	0.00E+00	0.00E+00	*0.00E+00*	0	0
1.30E−03	5.832	4.46E−02	−2.87E−01	1.54E−08	1.38E−06	*8.54E−06*	0	0
8.87E−04	4.124	6.90E−02	−4.63E−01	6.36E−07	5.73E−05	*2.37E−04*	1	6
7.32E−04	3.367	9.11E−02	−6.19E−01	3.00E−06	2.70E−04	*8.92E−04*	2	23
6.48E−04	2.916	1.13E−01	−7.67E−01	7.77E−06	6.99E−04	*1.97E−03*	5	51
5.32E−04	2.062	2.02E−01	−1.34E+00	5.35E−05	4.81E−03	*9.31E−03*	25	242
4.49E−04	1.684	3.10E−01	−1.96E+00	1.52E−04	1.37E−02	*2.14E−02*	57	557
4.39E−04	1.458	4.43E−01	−2.62E+00	3.27E−04	2.94E−02	*4.03E−02*	107	1047
4.12E−04	1.304	6.10E−01	−3.37E+00	6.14E−04	5.52E−02	*6.91E−02*	183	1798
3.69E−04	1.166	8.74E−01	−4.47E+00	1.20E−03	1.08E−01	*1.26E−01*	333	3268
3.29E−04	1.065	1.21E+00	−5.73E+00	2.13E−03	1.92E−01	*2.13E−01*	566	5548
2.61E−04	0.922	2.14E+00	−8.90E+00	5.62E−03	5.05E−01	*5.34E−01*	1416	13888
2.14E−04	0.825	3.47E+00	−1.29E+01	1.23E−02	1.10E+00	*1.14E+00*	3019	29619
1.82E−04	0.753	5.25E+00	−1.77E+01	2.35E−02	2.12E+00	*2.16E+00*	5727	56183

col.	symbol	explanations	expression
13	A_E	dimensionless height, eq. 6.52a,	$A_E = \frac{z_{sb}}{h} = \frac{2d_{35}}{h}$
14	z	Rouse exponent, eq. 6.50a, v_{ss} : settling velocity (see Fig. 6.10)	$z = \frac{v_{ss}}{\kappa\, u_*'}$
15	J_1	Einstein's first integral (see Fig. 6.12)	
16	J_2	Einstein's second integral (see Fig. 6.12)	
17	q_{ss}	solid discharge, as suspended load, by volume and by unit width, eq. 6.58,	$q_{ss} = q_{sb}(P_e I_1 + I_2)$
18	Q_{ss}	solid discharge, as suspended load, by volume	$Q_{ss} = q_{ss}\, b$
19	Q_s	solid discharge, as total load, by volume	$Q_s = Q_{sb} + Q_{ss}$
20	G_s	solid discharge, as total load, by mass	$G_s = Q_s\, \rho_s$
21	G_s	solid discharge, as total load, by weight	$G_s = Q_s\, \rho_s\, g$

2° The formula of *Graf* et *Acaroglu* (1968), which allows the calculation of t total load transported by the flow, is given by :

$$\Phi_A = f(\Psi_A) \tag{6.(}$$

with the parameter of transport :

$$\Phi_A = \frac{C_s\ UR_h}{\sqrt{(s_s-1)gd_{50}^{\ 3}}} \tag{6.(}$$

and the parameter of shear intensity :

$$\Psi_A = \frac{(s_s-1)\ d_{50}}{S_e\ R_h} \tag{6.(}$$

where the equivalent diameter is taken as (see Table 6.3) :

$d \equiv d_{50} = 0.00032$ [m].

It is to be noted, that R_h is the total hydraulic radius and $C_s = q_s/q$ is t average concentration by volume. The functional relationship is evaluat according to eq. 6.63.

As in the previous case, the calculations are programmed on a microcompu using a spreadsheet program. The computation sheet prepared in this way presented in Table Ex. 6.F.3.

3° The formula of *Ackers* et *White* (1973), which allows the calculation average concentration, C_s , by volume is given by :

$$C_s = G_{gr}\ \frac{d_{35}}{h_m}\ \left(\frac{U}{u_*}\right)^{n_w} \tag{6.6}$$

where the equivalent diameter is taken as (see Table 6.3) :

$d \equiv d_{35} = 0.00029$ [m]

The sediment-transport parameter is calculated as :

$$G_{gr} = C_w\ \left(\frac{F_{gr}}{A_w} - 1\right)^{m_w} \tag{6.6}$$

with the mobility parameter defined as :

$$F_{gr} = \frac{u_*^{\ n_w}}{\sqrt{(s_s-1)\ g\ d_{35}}}\ \left[\frac{U}{\sqrt{32}\ \log\ (10h_m/d_{35})}\right]^{(1-n_w)} \tag{6.6}$$

The dimensionless diameter for $d \equiv d_{35}$ is determined using :

Table Ex.6.F.3

Computation sheet for determining the *stage*—**solid**-discharge curve, using the method of *Graf* et *Acaroglu* (1968)

$S_f = 0.0005$ [-] $d_{50} = 0.00032$ [m]
$\rho = 999.1$ [kg/m³] $\rho_s = 2650$ [kg/m³]

1	2	3	4	5	6	7	8	9	10
h	R_h	U	Q	Ψ_A	Φ_A	C_s	Q_s	G_s	G_s
[m]	[m]	[m/s]	[m³/s]	[-]	[-]	[-]	[m³/s]	[kg/s]	[N/s]
0.29	0.28	0.16	4.2	3.718	0.380	1.91E-04	*7.95E-04*	2	21
0.44	0.44	0.29	11.7	2.401	1.143	2.06E-04	*2.41E-03*	6	63
0.65	0.65	0.45	26.5	1.639	2.990	2.39E-04	*6.33E-03*	17	165
0.79	0.78	0.58	41.4	1.356	4.821	2.48E-04	*1.03E-02*	27	266
0.89	0.88	0.69	55.8	1.203	6.522	2.49E-04	*1.39E-02*	37	362
1.09	1.07	1.05	103.7	0.992	10.615	2.20E-04	*2.28E-02*	60	592
1.29	1.26	1.33	157.4	0.839	16.163	2.22E-04	*3.49E-02*	92	907
1.32	1.29	1.58	191.3	0.821	17.091	1.93E-04	*3.69E-02*	98	960
1.41	1.37	1.81	232.4	0.773	19.864	1.85E-04	*4.30E-02*	114	1118
1.57	1.52	2.06	297.3	0.694	26.121	1.91E-04	*5.69E-02*	151	1478
1.76	1.70	2.30	371.6	0.622	34.445	2.03E-04	*7.54E-02*	200	1960
2.23	2.13	2.72	559.3	0.496	60.791	2.41E-04	*1.35E-01*	358	3507
2.71	2.57	3.11	780.9	0.411	97.757	2.82E-04	*2.20E-01*	583	5721
3.19	3.00	3.46	1026.7	0.353	143.802	3.20E-04	*3.28E-01*	870	8531

col.	symbol	explanations	expression
1	h	flow depth (see Table Ex.6.F.1)	
2	R_h	total hydraulic radius (see Table Ex.6.F.1)	
3	U	average velocity (see Table Ex.6.F.1)	
4	Q	liquid discharge (voir Tableau Ex.6.F.1)	
5	Ψ_A	shear-stress intensity parameter, eq. 6.61,	$\Psi_A = \dfrac{(s_s - 1)\, d_{50}}{S_f\, R_h}$
6	Φ_A	transport parameter, eq. 6.63,	$\Phi_A = 10.39\, \Psi_A^{-2.52}$
7	C_s	concentration by volume in the section, eq. 6.62,	$C_s = \Phi_A \dfrac{\sqrt{(s_s - 1)\, g\, d_{50}^3}}{U\, R_h}$
8	Q_s	solid discharge, as total load, by volume	$Q_s = C_s\, Q$
9	G_s	solid discharge, as total load, by mass	$G_s = Q_s\, \rho_s$
10	G_s	solid discharge, as total load, by weight	$G_s = Q_s\, g\, \rho_s$

Table Ex.6.F.4

Computation sheet for determining the *stage*—**solid**-discharge curve, using the method of *Ackers* et *White* (1973)

$\rho = 999.1$ [kg/m³] $\quad \nu = 1.186 \times 10^{-6}$ [m²/s]

$\rho_s = 2650$ [kg/m³] $\quad d_{35} = 0.00029$ [m] $\Rightarrow d_* = (g\,(s_s - 1)/\nu^2)^{1/3}\, d_{35} = 6.536$

$n_w = 1 - 0.56 \log d_* = 0.5434 \quad m_w = 9.66 / d_* + 1.34 = 2.8180$

$A_w = 0.23/\sqrt{d_*} + 0.14 = 0.2300 \quad C_w = 10^{(2.86 \log d_* - (\log d_*)^2 - 3.53)} = 0.0137$

1	2	3	4	5	6	7	8	9	10
h	u_*	U	Q	F_{gr}	G_{gr}	C_s	Q_s	G_s	G_s
[m]	[m/s]	[m/s]	[m³/s]	[-]	[-]	[-]	[m³/s]	[kg/s]	[N/s]
0.29	0.04	0.16	4.2	0.256	2.979E−05	6.69E−08	*2.79E−07*	0	0
0.44	0.05	0.29	11.7	0.369	3.316E−03	5.85E−06	*6.83E−05*	0	2
0.65	0.06	0.45	26.5	0.490	1.937E−02	2.65E−05	*7.04E−04*	2	18
0.79	0.06	0.58	41.4	0.573	4.242E−02	5.22E−05	*2.16E−03*	6	56
0.89	0.07	0.69	55.8	0.638	6.923E−02	8.03E−05	*4.48E−03*	12	117
1.09	0.07	1.05	103.7	0.808	1.845E−01	2.10E−04	*2.17E−02*	58	565
1.29	0.08	1.33	157.4	0.939	3.268E−01	3.41E−04	*5.37E−02*	142	1397
1.32	0.08	1.58	191.3	1.020	4.446E−01	4.95E−04	*9.48E−02*	251	2463
1.41	0.08	1.81	232.4	1.099	5.808E−01	6.44E−04	*1.50E−01*	397	3891
1.57	0.09	2.06	297.3	1.197	7.845E−01	8.11E−04	*2.41E−01*	639	6269
1.76	0.09	2.30	371.6	1.289	1.014E+00	9.64E−04	*3.58E−01*	949	9314
2.23	0.10	2.72	559.3	1.467	1.570E+00	1.22E−03	*6.81E−01*	1805	17707
2.71	0.11	3.11	780.9	1.626	2.210E+00	1.44E−03	*1.12E+00*	2970	29133
3.19	0.12	3.46	1026.7	1.769	2.906E+00	1.63E−03	*1.68E+00*	4440	43559

col.	symbol	explanations	expression
1	h	flow depth (see Table Ex.6.F.1)	$h \cong h_m = S/B$
2	u_*	total shear velocity (see Table Ex.6.F.1)	
3	U	average velocity (see Table Ex.6.F.1)	
4	Q	liquid discharge (see Table Ex.6.F.1)	
5	F_{gr}	parameter of mobility, eq. 6.64,	$F_{gr} = \dfrac{u_*^{\,n_w}}{\sqrt{(s_s-1)\,g d_{35}}} \left[\dfrac{U}{\sqrt{32}\,\log(10h/d_{35})}\right]^{(1-n_w)}$
6	G_{gr}	transport parameter, eq. 6.65,	$G_{gr} = C_w \left(\dfrac{F_{gr}}{A_w} - 1\right)^{m_w}$
7	C_s	concentration by volume, eq. 6.66,	$C_s = G_{gr} \dfrac{d_{35}}{h} \left(\dfrac{U}{u_*}\right)^{n_w}$
8	Q_s	solid discharge, as total load, by volume	$Q_s = C_s\, Q$
9	G_s	solid discharge, as total load, by mass	$G_s = Q_s\, \rho_s$
10	G_s	solid discharge, as total load, by weight	$G_s = Q_s\, g\, \rho_s$

$$d_* = d_{35}\left((s_s-1)\,\frac{g}{\nu^2}\right)^{1/3} \cong 6.5 \tag{6.26}$$

which allows calculation of the coefficients, n_w, m_w, A_w and C_w (see point 6.5.2.4°).

Again, the calculations are programmed on a microcomputer using a spreadsheet program. The computation sheet is presented in Table Ex. 6.F.4.

The Fig. Ex. 6.F.2 gives the *stage*—**solid**-discharge curves — supplemented by the *stage*—**liquid**-discharge curve — for the **total load** calculated using the total-load relations of (1) *Einstein*, (2) *Graf* et *Acaroglu* et 3) *Ackers* et *White* as well as for the **bed load** calculated using the bed-load relation of (1) *Einstein*. (*Einstein*'s method, 1950, is an indirect method, thus it allows the evaluation of the bed load transport automatically.).

The Fig. Ex. 6.F.2 shows that the three methods used for the calculations do not give the same values for the solid discharge, Q_s. It is important however to remind that the formulae for the sediment transport can only give the engineer an idea about the order of magnitude of the solid discharge that one should reasonably expect in a particular flow situation. It should also be clear that the sediment-transport capacity (voir sect. 6.1.4) has been calculated.

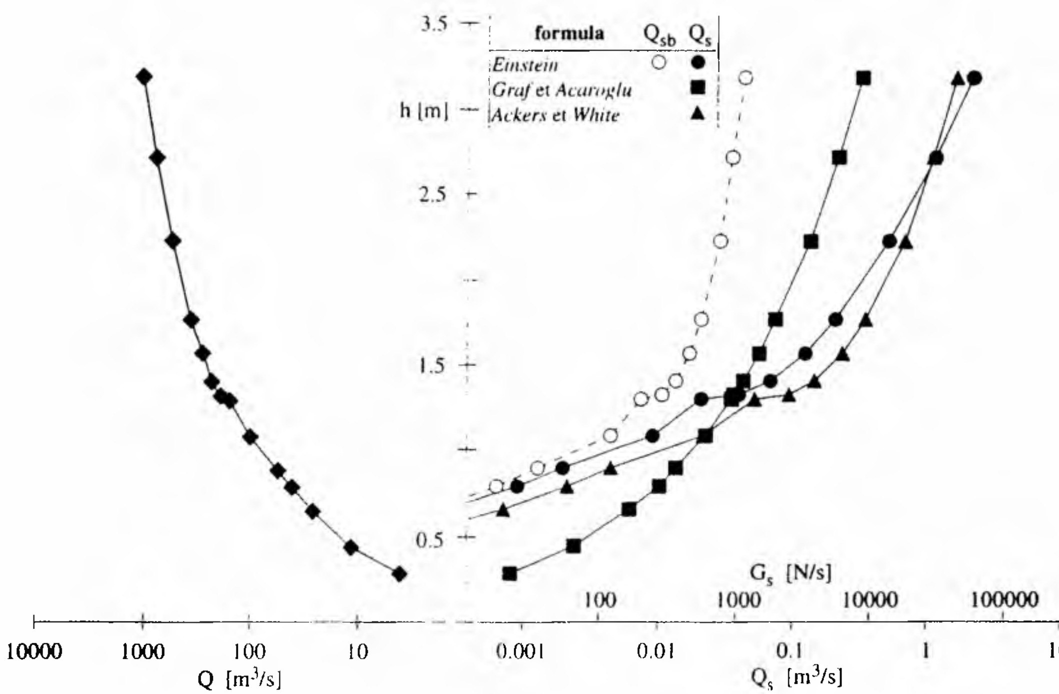


Fig. Ex.6.F.2 *Stage—liquid*-discharge and *stage—solid*-discharge curves.

6.6.2 Problems, unsolved

Ex. 6.1

A long channel of rectangular cross section has a bed slope of S_f = 0.0004 [-]. T channel bed is composed of a near-uniform granulate of d_{50} = 0.5 [mm] with a porosi of p = 0.3 [-]. The normal flow depth was measured as h = h_n = 2.10 [m]. This chanr enters a lake ; at the juncture the water levels of the channel and of the lake are the same

The water level in the lake is now *lowered* by Δh = 0.10 [m]. Determine, at what time t channel bed will be lowered by 90 % and by 50 % of Δh and this at two stations, situat at 1.5 [km] and at 20.0 [km] upstream of the juncture.

Ex. 6.2

In the channel, described in Ex. 6.1, the fixed point at the juncture will be ***raised*** l Δh = 0.20 [m]. Determine the temporal variation of the channel bed at a station, beii situated 20.0 [km] upstream of the juncture.

Ex. 6.3

The unit discharge in a river is q = 5 [m^2/s]. The bed has a slope of S_f = 0.0005 [-] ai the porosity is p = 0 [-] ; its granulate is given as d_{50} = 0.4 [mm].

Well downstream, this river enters a reservoir, where the water depth is kept constant at height of H = 5 [m]. Determine the aggradation one may expect in the reservoir after 2: [h] and 500 [h].

Ex. 6.4

The irrigation channel "Sivan" must be controlled for 81 days per year to deliver constant discharge of 10 [m^3/s] (during the remaining days of the year the discharge w be less). This rectangular channel, having a width of B = 5.0 [m], has a mobile bed with uniform granulate of d_{50} = 5 [mm] ; the bed slope is S_f = 0.0003 [-].

i) Calculate the solid discharge, which is annually transported.

ii) Study a proposal, which envisions that a spillway— blocking the flow at a wat depth being three times the normal flow depth — is installed at the downstream (the channel. What will the sediment deposition amount to and where will it tal place ?

Ex. 6.5

A reach of a river of length L = 38 [km] — being indicative of the Bas-Rhône — conve; a constant discharge of Q = 4000 [m^3/s]. In this reach the cross sections may t approximated by rectangular ones having a width of B = 250 [m] and having a consta bed slope of S_f = 0.0007 [-]. The roughness coefficient was estimated as being K_s = : [$m^{1/3}$/s] ; the diameter of the rather uniform granulate is d_{50} = 27.4 [mm].

It is envisioned to build a system of weirs, which would raise the water level by 10 [m], thus to H = 10 [m] + h_n. Study the evolution of bed level for a period of two years. Investigate the sediment-transport problem in the reach behind the weirs.

Ex. 6.6

The discharge in a channel, having a bed slope of $S_f = 0.00027$ [-] is $Q = 100$ [m^3/s] ; the width at the channel bed is b = 46 [m] and the side slopes are 2 / 1 . Flow in this channel is uniform and the temperature of the water is $T = 15$ [C°]. Samples of the bed material have been evaluated ; the granulometry is given in the following table and its density is $s_s = 2.65$ [-].

median diameter [mm]	granulometric fraction [%]
0.088	5
0.177	22
0.354	37
0.707	31
1.414	5

Calculate the total-load transport, Q_s, by making use of different available formulae. This is to be done :

i) for each individual granulometric fraction,
ii) for all fractions together, taking an equivalent diameter.

Ex. 6.7

For the channel studied in Ex. 6.6, verify if the transport of sediments is influenced by the water temperature ; the lowest and highest temperatures expected are respectively, $T = 10$ [C°] and $T = 20$ [C°]. Make the calculations using an equivalent diameter.

Ex. 6.8

For the channel studied in Ex. 6.6, determine the diameter of the armouring and the flow depth for which the armour layer turns unstable.