Lec. 9   **Digital Filter Design**

9.1 Introduction:

A discrete time filter takes a discrete time input sequence x(n) and produces a discrete time output sequence y(n).

A special class of a discrete time shift-invariant system can be characterized by a unit sample response h(n), a system function H(Z), or difference equation.

$$\sum_{k=0}^{N} a_k \, y(n-k) = \sum_{k=0}^{M} b_k \, x(n-k) \tag{9.1}$$

$$H(Z) = \frac{\sum_{k=0}^{M} b_k \, Z^{-k}}{\sum_{k=0}^{N} a_k \, Z^{-k}} \tag{9.2}$$

$$H(e^{jW}) = \frac{\sum_{k=0}^{M} b_k \, e^{-jWk}}{\sum_{k=0}^{N} a_k \, e^{-jWk}} \quad , \quad Z = e^{jW} \tag{9.3}$$

A filter may be required to have a given frequency response, or specific response to an impulse, step, or ramp, or simulate a continuous analog system. The simulation of analog filter is shown in Fig. (9.1).
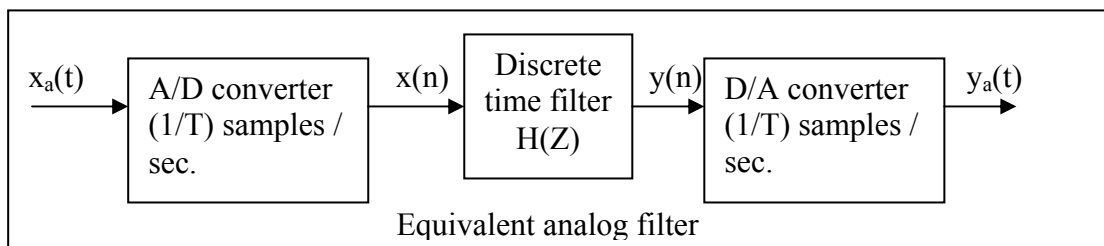

Fig. (9.1) Equivalent analog filter

A/D converter consists of sampler, quantizer, and coder.

D/A converter consists of decoder, sample and hold, and low-pass filter.

9.1.1 Definitions

1. If unit sample response h(n) is of finite duration, the system is said to be a finite impulse response (FIR) system. Eq. (9.1) represents FIR system if $a_0 \neq 0$ and $a_k = 0$ for k=1, 2,..N.

2. If unit sample response h(n) is of infinite duration, the system is said to be an infinite impulse response (IIR) system.

3. IIR filter is usually implemented by recursive realization (is one in which the present value of the output depends on both the input present and or past values), i.e., *with feedback.*

4. FIR filter is usually implemented by either a nonrecursive realization (*without feedback*) or an FFT realization.

9.1.2 A comparison between FIR and IIR filters:

FIR	IIR
1- Finite impulse response h(n) $n_1 \leq n \leq n_2$	1- Infinite impulse response h(n) $n_1 \leq n \leq \infty$
2-Complex requires large number of computations	2- Simple, does not require large number of computations
3- Due to large number of computations, it requires large memory	3- Dose not require large memory
4- Always stable because its poles lie at the origin	4- Stable only if its poles lie inside the unit circle of the Z-plane
5- Linear phase characteristics	5- nonlinear phase characteristics

9.2 Infinite Impulse Response (IIR) filter format

An IIR filter is described using the difference equation (9.1) as:

$$y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$$
$$- a_1 y(n-1) - \cdots - a_N y(n-N).$$

(9.4)

The IIR filter transfer function given in eq.(9.2) as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}},$$

(9.5)

Example (1): Given the following IIR filter:

y(n) = 0.2 x(n) + 0.4 x(n − 1) + 0.5 y(n − 1),

Determine the transfer function, nonzero coefficients, and impulse response.

Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.2 + 0.4z^{-1}}{1 - 0.5z^{-1}}.$$

$b_0 = 0.2$, $b_1 = 0.4$, and $a_1 = -0.5$.

Using the inverse z-transform and shift theorem, we obtain the impulse response as

$$h(n) = 0.2(0.5)^n u(n) + 0.4(0.5)^{n-1} u(n-1).$$

9.3 Techniques for designing H(Z) for IIR filter:

9.3.1 Design by using numerical solutions of differential equations:

A continuous time linear filter is specified by the following difference equation:

$$\sum_{k=0}^{N} c_k \frac{d^k y_a(t)}{dt^k} = \sum_{k=0}^{M} d_k \frac{d^k x_a(t)}{dt^k} \tag{9.6}$$

$$H_a(S) = \frac{\sum_{k=0}^{M} d_k S^k}{\sum_{k=0}^{N} c_k S^k} \tag{9.7}$$

Approximate the derivates using *first backward* differences:

$$\nabla^{(1)} [y(n)] = [y(n) - y(n-1)] / T \tag{9.8}$$

Higher order backward differences are found by applying the *first backward* difference repeatedly, as follows:

$$\nabla^{(k)} [y(n)] = \nabla^{(1)} [\nabla^{(k-1)} [y(n)]] \tag{9.9}$$

Using the kth order differences as approximations to the derivatives given in eq. (9.6), we have:

$$\sum_{k=0}^{N} c_k \nabla^{(k)} [y_a(nT)] = \sum_{k=0}^{M} d_k \nabla^{(k)} [x_a(nT)] \tag{9.10}$$

The Z. Transform of the 1st and k^{th} order differences are given below:

$$Z\{\nabla^{(1)} [y(n)] = Y(Z) \{1 - Z^{-1}\}] / T \tag{9.11}$$

$$Z\{\nabla^{(k)} [y(n)] = Y(Z) [\{1 - Z^{-1}\}] / T]^k \tag{9.12}$$

Letting $x(n) = x_a(nT)$, and $y(n) = y_a(nT)$. Taking the Z. Transform of eq. (9.10):

$$\sum_{k=0}^{N} c_k Y(Z) [\{1 - Z^{-1}\} / T]^k = \sum_{k=0}^{M} d_k X(Z) [\{1 - Z^{-1}\} / T]^k$$

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{\sum_{k=0}^{M} d_k [\{1 - Z^{-1}\} / T]^k}{\sum_{k=0}^{N} c_k [\{1 - Z^{-1}\} / T]^k} \tag{9.13}$$

Comparing eq. (9.7) and eq. (9.13), we find:

$$H(Z) = H_a(S) \Big|_{S \to \frac{(1-Z^{-1})}{T}} \tag{9.14}$$

H.W: If $H(S) = \dfrac{1}{(S+1)\,(S+2)}$ use the numerical solutions of differential equations to obtain

H(Z) for, a) T = 1 sec., and b) f_s = 100 Hz.

9.3.2 Bilinear transformation (BLT) Design method:

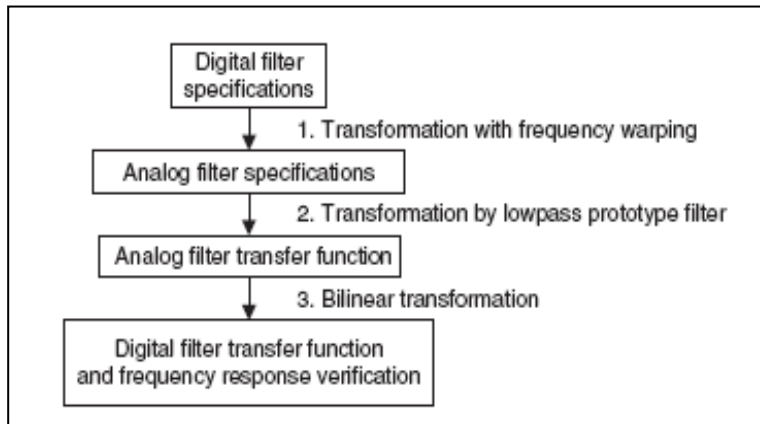Figure (9.2) illustrates a flow chart of the BLT design used

Fig. 9.2 General procedure for IIR filter design using bilinear transformation.

$$H(Z) = H_a(S) \Big|_{S \to \frac{2}{T}\frac{(1-Z^{-1})}{(1+Z^{-1})}} \tag{9.15}$$

$$\because S = \frac{2}{T}\frac{(1-Z^{-1})}{(1+Z^{-1})}$$

$$j\,\Omega = \frac{2}{T}\frac{(1-e^{-jW})}{(1+e^{-jW})} \tag{9.16}$$

$$\Omega = -j\,\frac{2}{T}\,\frac{e^{-jW/2}}{e^{-jW/2}}\,\frac{e^{jW/2}-e^{-jW/2}}{e^{jW/2}+e^{-jW/2}}$$

$$\Omega = \frac{2}{T}\tan\left(\frac{W}{2}\right) \quad , rad/\sec \tag{9.17}$$

$$W = 2\tan^{-1}\left(\frac{\Omega\,T}{2}\right) \quad , rad \tag{9.18}$$

As (W/2) becomes smaller, we get more linear characteristics [(W/2) << 1]. If the bilinear transformation is applied to an H_a(S) with critical frequency Ω_c, the digital filter will have critical frequency W_c.

$$W_c = 2 \tan^{-1} \left(\frac{\Omega_c T}{2} \right) \tag{9.19}$$

If the resulting H(Z) is used in an A/D-H(Z)-D/A structure, the equivalent critical frequency becomes:

$$W_c = \Omega_{ceq} \, T \tag{9.20}$$

$$\Omega_{ceq} = \frac{2}{T} \tan^{-1} \left(\frac{\Omega_c T}{2} \right) \tag{9.21}$$

Which will give Ω_c only if $\Omega_c T / 2$ is so small, that $\tan^{-1}(\Omega T / 2) \approx \Omega T / 2$.

In bilinear transformation, the design of digital filter does not depend on the sampling rate (T =1, *prewarp case*). For a low-pass filter, with $S \rightarrow S / \Omega_c$, and applying eq. (9.17), then:

$$S = \frac{2}{T} \frac{(1 - Z^{-1})}{(1 + Z^{-1}) \, \Omega_c} = \frac{(1 - Z^{-1})}{(1 + Z^{-1}) \, \tan(\frac{W_c}{2})}$$

Example (2): Design and realize a digital low-pass filter using bilinear transformation method to satisfy the following c/cs:

1. $- 3.01$ dB cutoff frequency of $0.5 \, \pi$ rad
2. Magnitude down at least 15 dB at $0.75 \, \pi$ rad.

Solution:

Step (1): applying eq. (9.17), where T=1 (*prewarp case*)

$$\Omega_u = 2 \tan \left(\frac{W_1}{2} \right) = 2 \tan(0.5 \, \pi / 2) = 2$$

$$\Omega'_r = 2 \tan \left(\frac{W_2}{2} \right) = 2 \tan(0.75 \, \pi / 2) = 4.8282$$

Step (2) : applying eq. (8.4) and (8.5a):

$$n = \left\lceil \frac{\log_{10} \{ (10^{-0.1 \, k_1} - 1) / (10^{-0.1 \, k_2} - 1) \}}{2 \log_{10} (1 / \Omega_r)} \right\rceil$$

$$n = \left\lceil \frac{\log_{10} \{ (10^{3.01/10} - 1) / (10^{15/10} - 1) \}}{2 \log_{10} (2 / 4.8282)} \right\rceil = \lceil 1.9412 \rceil = 2$$

$$\Omega_c = 2 / (10^{3.01/10} - 1)^{1/4} = 2 \quad rad \, / \sec$$

Referring to lecture 8, Table (1) to write the normalized Butterworth LPF equation, and then using LP $\rightarrow$ LP transformation:

$$H_a(S) = \frac{1}{S^2 + \sqrt{2}\, S + 1} \Big|_{S \rightarrow S/2} = \frac{4}{S^2 + 2\sqrt{2}\, S + 4}$$

Step (3): Applying bilinear transformation, eq.(9.15), T = 1

$$H(Z) = \frac{4}{[\frac{2(1-Z^{-1})}{(1+Z^{-1})}]^2 + 2\sqrt{2}\,[\frac{2(1-Z^{-1})}{(1+Z^{-1})}] + 4} = \frac{1 + 2Z^{-1} + Z^{-2}}{3.4142135 + 0.5857865\, Z^{-2}}$$

y(n) = 0.2928932 { x(n) + 2 x(n−1) + x(n−2) } − 0.1715729 y(n−2)

9.3.3 Digital-to digital transformation design method

1. Use digital specifications to calculate the order of *digital unit bandwidth low-pass Butterworth prototype* and corresponding critical frequency *Wp*. The order of the digital filter can be obtained by using eq. (9.17) of the prewarped digital frequencies Ω_u, and Ω_r' in the standard formula for the analog Butterworth filter { eq. (8.4) }, as:

2. $$n = \left| \frac{\log_{10} \{ (10^{-0.1 k_1} - 1) / (10^{-0.1 k_2} - 1) \}}{2 \log_{10} \{ \tan(W_1/2) / \tan(W_2/2) \}} \right| \qquad (9.22)$$

$$W_P = 2 \tan^{-1} \{ (10^{-k_1/10} - 1)^{-1/2n} \tan(W_1/2) \} \qquad (9.23)$$

Note: Refer to lecture 8, Table (2) to substitute for Ω_r in eq. (9.22) in terms of eq.(9.17).

Table (1) Digital-to digital transformation

3- From Table (1), calculate $\alpha = \dfrac{\sin\{(\theta_P - W_1)/2\}}{\sin\{(\theta_P + W_1)/2\}}$ (9.24)

4- Table (2) gives H_{Bn} (Z) for *normalized low-pass Butterworth* digital filter. Calculate

$$H(Z) = H_{Bn}(Z) \Big|_{Z^{-1} \to \frac{(Z^{-1}-\alpha)}{(1+\alpha Z^{-1})}}$$ (9.25)

Example (3): Use Digital-to digital transformation method. Find H(Z) for LP digital filter that satisfies the following requirements:

1- A − 3.0102 dB cutoff digital frequency of 0.5 π rad.

2- Attenuation at and past 0.75 π rad is at least 15 dB

Solution:

$$n = \left\lceil \frac{\log_{10}\{(10^{0.30102}-1)/(10^{1.5}-1)\}}{2\log_{10}\{\tan(0.5\pi/2)/\tan(0.75\pi/2)\}} \right\rceil = \lceil 1.9412 \rceil = 2$$

$$W_P = 2\tan^{-1}\{(10^{0.30102}-1)^{-1/4}\tan(0.5\pi/2)\} = 0.5\pi \ , \ \theta_P = 1 \text{ (normalized)}$$

$$\alpha = \frac{\sin\{(1-0.5\pi)/2\}}{\sin\{(1+0.5\pi)/2\}} = -0.293401993$$

Using Table (2) that gives H_{Bn} (Z) for *normalized low-pass Butterworth* digital filter

$$H_{B2}(Z) = \frac{0.144106\,(1+Z^{-1})^2}{1 - 0.677496\,Z^{-1} + 0.253921\,Z^{-2}}$$

Applying eq.(9.25) , then:

$$H(Z) = \frac{(1+Z^{-1})^2}{3.4142 + 0.5858\,Z^{-2}}$$

9.3.4 Impulse invariant design method

If $h_a(t)$ represents the response of an analog filter to a unit impulse $\delta(t)$, then the unit sample response of a discrete-time filter used in an A/D-H(Z)-D/A structure is selected to be the sampled version of h(n).

$$H(Z) = Z\{h(n)\} = Z\{h_a(t)\big|_{t=nT}\}$$ (9.26)

If an analog filter with system function $H_a(S)$ is given, the corresponding impulse invariant design filter has

$$H(Z) = Z\{L^{-1}\,H_a(S)\big|_{t=nT}\}$$ (9.27)

Example (4): Find H(Z) corresponding to the impulse invariant design using sampling rate of (1/T) samples / sec. for an analog filter $H_a(S)$ specified as: $H_a(S) = A/(S+\alpha)$

Solution:

$h_a(t) = L^{-1} H_a(S) = A e^{-\alpha t} u(t)$

$h(n) = h_a(t)_{t=nT} = A e^{-\alpha nT} u(nT)$

$$H(Z) = Z\{h(n)\} = \frac{A Z}{Z - e^{-\alpha T}}$$

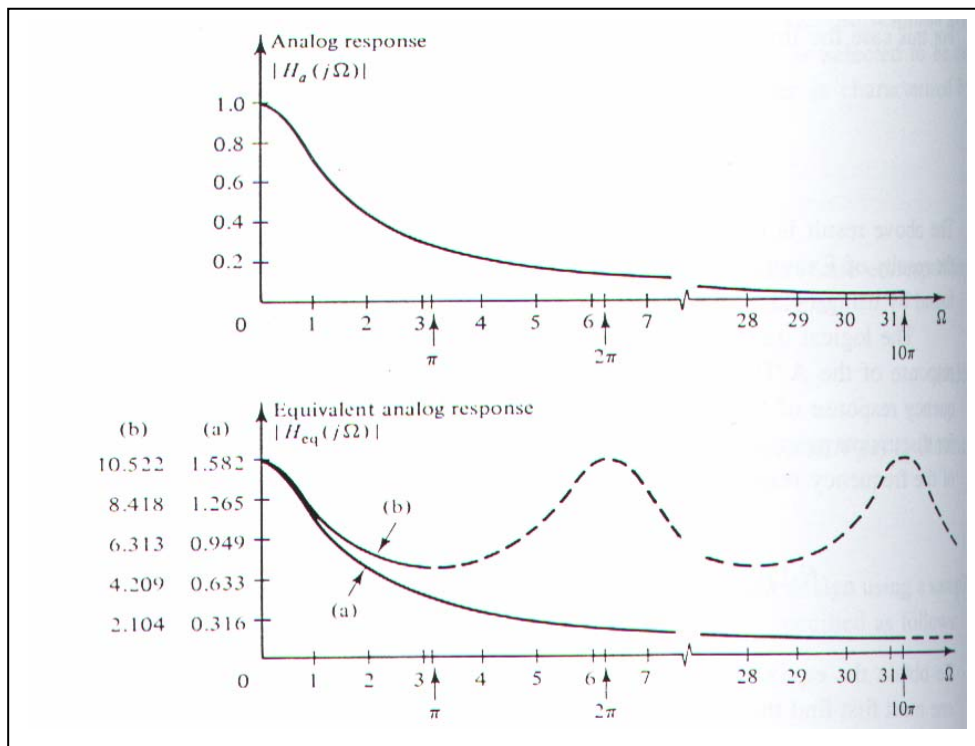$$\left| H_a(j\Omega) \right| = \frac{A}{(\alpha^2 + \Omega^2)^{1/2}} \quad , S = j\Omega$$

$$H(e^{jW}) = \frac{A e^{jW}}{e^{jW} - e^{-\alpha T}} = \frac{A}{1 - e^{-jW} e^{-\alpha T}} \quad , \quad Z = e^{jW}$$

$$H_{eq}(j\Omega) = H(e^{jW})\Big|_{W \to \Omega T} = \frac{A e^{j\Omega T}}{e^{j\Omega T} - e^{-\alpha T}} \quad , \quad \Omega < \pi/T$$

$$H_{eq}(j\Omega) = \frac{A}{\{1 - e^{-\alpha T}\cos(\Omega T)\} + j e^{-\alpha T}\sin(\Omega T)} \quad , \quad \Omega < \pi/T$$

$$\left| H_{eq}(j\Omega) \right| = \frac{A}{\sqrt{1 + e^{-2\alpha T} - 2 e^{-\alpha T}\cos(\Omega T)}} \quad , \quad \Omega < \pi/T$$

(a) Represents $\alpha = 1$, $T = 0.1$, $\left| H_a(j\Omega) \right|$ and $\left| H_{eq}(j\Omega) \right|$ are very close.

(b) Represents $\alpha = 1$, $T = 1$, $\left| H_a(j\Omega) \right|$ and $\left| H_{eq}(j\Omega) \right|$ are different.

Good results using impulse invariant design are obtained when the time between samples is selected small.

9.4 Pole-Zero Placement Method for Simple Infinite Impulse Response Filters Design

This section introduces a pole-zero placement method for a simple IIR filter design. Let us first examine effects of the pole-zero placement on the magnitude response in the z-plane shown in Fig .(9.3).

In the z-plane, when we place a pair of complex conjugate zeros at a given point on the unit circle with an angle θ, we will have a numerator factor of $(z - e^{j\theta})(z - e^{-j\theta})$ in the transfer function. Its magnitude contribution to the frequency response at $z = e^{jW}$ is $(e^{jW} - e^{j\theta})(e^{jW} - e^{-j\theta})$. When W = θ, the magnitude will reach zero.

When a pair of complex conjugate poles are placed at a given point within the unit circle, we have a denominator factor of $(z - r e^{j\theta})(z - r e^{-j\theta})$, where r is the radius chosen to be less than and close to 1 to place the poles inside the unit circle. The magnitude contribution to the frequency response at W = θ will rise to a large magnitude, since the first factor $(e^{j\theta} - r e^{j\theta}) = (1 - r) e^{+j\theta}$ gives a small magnitude of $1 - r$, which is the length between the pole and the unit circle at the angle W = θ. Note that the magnitude of $e^{+j\theta}$ is 1.

Therefore, we can reduce the magnitude response using zero placement, while we increase the magnitude response using pole placement. Placing a combination of poles and zeros will result in different frequency responses. such as lowpass, highpass, bandpass, and bandstop. It is easy to compute filter coefficients for simple IIR filters. Practically, the pole-zero placement method has good performance when the bandpass and bandstop filters have very narrow bandwidth requirements and the lowpass and highpass filters have either very low cutoff frequencies close to the DC or very high cutoff frequencies close to the folding frequency (the Nyquist limit).
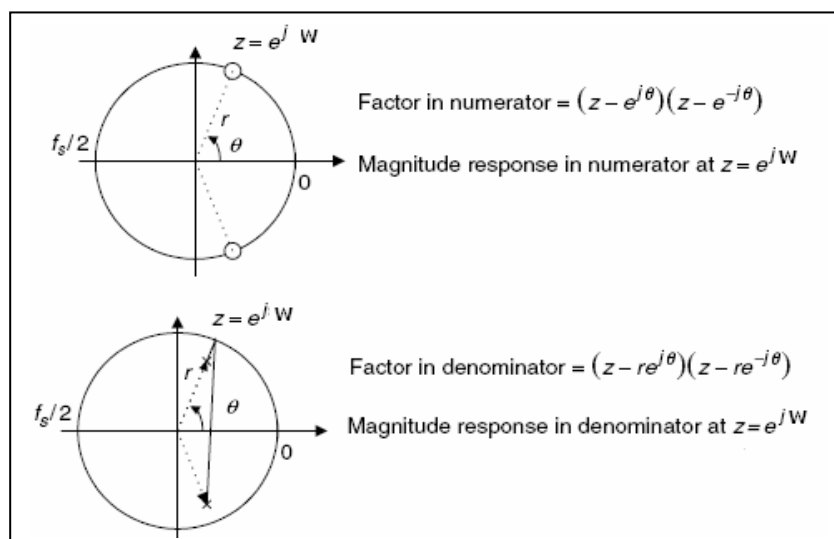

Fig. (9.3) Effects of pole-zero placement on the magnitude response.

9.4.1 Second-Order Bandpass Filter Design

Poles in a band-pass filter are complex conjugate, with the magnitude r controlling the bandwidth and the angle θ controlling the center frequency. The zeros are placed at z = 1, corresponding to DC, and z = -1, corresponding to the folding frequency.

The poles will raise the magnitude response at the center frequency while the zeros will cause zero gains at DC (zero frequency) and at the folding frequency. The following equations give the band-pass filter design formulas using pole-zero placement:
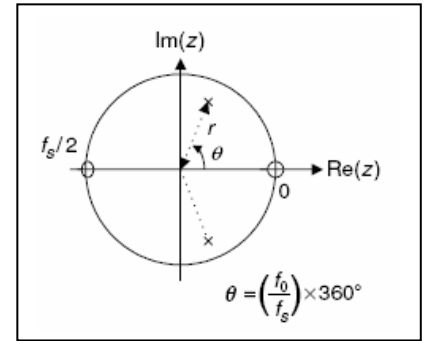
$$r \approx 1 - (BW_{3dB}/f_s) \times \pi, \text{ good for } 0.9 \le r < 1$$

$$\theta = \left(\frac{f_0}{f_s}\right) \times 360^0$$

$$H(z) = \frac{K(z-1)(z+1)}{(z - re^{j\theta})(z - re^{-j\theta})} = \frac{K(z^2 - 1)}{(z^2 - 2rz\cos\theta + r^2)},$$

$$K = \frac{(1-r)\sqrt{1 - 2r\cos 2\theta + r^2}}{2|\sin\theta|}.$$

$$\theta = \left(\frac{f_0}{f_s}\right) \times 360°$$

(9.28)

Where, K is a scale factor to adjust the band-pass filter to have a unit pass-band gain

Example (5): A second-order bandpass filter is required to satisfy the following specifications:

1. Sampling rate = 8,000 Hz

2. A 3 dB bandwidth: BW = 200 Hz

3. Narrow passband centered at f_0 = 1,000 Hz

4. Zero gain at 0 Hz and 4,000 Hz.

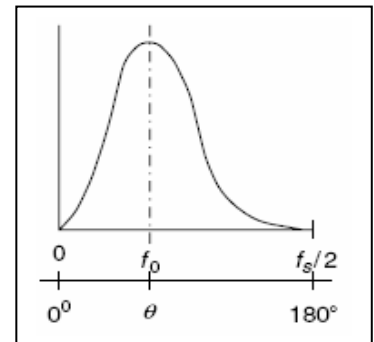Find the transfer function using the pole-zero placement method.

Solution: Applying eq.(9.28),

$$r = 1 - (200/8000)\pi = 0.9215$$

$$\theta = \left(\frac{1000}{8000}\right) \times 360 = 45^0.$$

$$K = \frac{(1 - 0.9215)\sqrt{1 - 2 \times 0.9215 \times \cos 2 \times 45^0 + 0.9215^2}}{2|\sin 45^0|} = 0.0755.$$

$$H(z) = \frac{0.0755(z^2 - 1)}{(z^2 - 2 \times 0.9215z\cos 45^0 + 0.9215^2)} = \frac{0.0755 - 0.0755z^{-2}}{1 - 1.3031z^{-1} + 0.8491z^{-2}}.$$

9.4.2 Second-Order Bandstop (Notch) Filter Design

For this type of filter, the pole placement is the same as the bandpass filter. The zeros are placed on the unit circle with the same angles with respect to the poles. This will improve passband performance. The magnitude and the angle of the complex conjugate poles determine the 3 dB bandwidth and the center frequency, respectively.
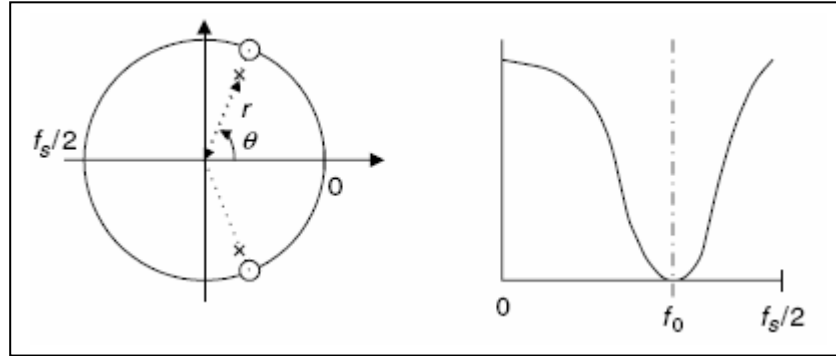
Design formulas for band-stop filters are given in the following equations:

$$r \approx 1 - (BW_{3dB}/f_s) \times \pi, \text{ good for } 0.9 \leq r < 1$$

$$\theta = \left(\frac{f_0}{f_s}\right) \times 360^0$$

$$H(z) = \frac{K(z - e^{j\theta})(z + e^{-j\theta})}{(z - re^{j\theta})(z - re^{-j\theta})} = \frac{K(z^2 - 2z\cos\theta + 1)}{(z^2 - 2rz\cos\theta + r^2)}.$$

$$K = \frac{(1 - 2r\cos\theta + r^2)}{(2 - 2\cos\theta)}.$$

$$(9.29)$$

Example (6): A second-order notch filter is required to satisfy the following specifications:

1. Sampling rate = 8,000 Hz

2. A 3 dB bandwidth: BW = 100 Hz

3. Narrow pass-band centered at f_0 = 1,500 Hz:

Find the transfer function using the pole-zero placement approach.

Solution:

$$r \approx 1 - (100/8000) \times \pi = 0.9607,$$

$$\theta = \left(\frac{1500}{8000}\right) \times 360^0 = 67.5^0.$$

$$K = \frac{(1 - 2 \times 0.9607\cos 67.5^0 + 0.9607^2)}{(2 - 2\cos 67.5^0)} = 0.9620.$$

$$H(z) = \frac{0.9620(z^2 - 2z \cos 67.5^0 + 1)}{(z^2 - 2 \times 0.9607z \cos 67.5^0 + 0.9607^2)}$$

$$= \frac{0.9620 - 0.7363z^{-1} + 0.9620z^{-2}}{1 - 0.7353z^{-1} + 0.9229}.$$

9.4.3 First-Order Low-pass Filter Design

The first-order pole-zero placement can be operated in two cases. The first situation is when the cutoff frequency is less than $f_s/4$. Then the pole-zero placement is shown in Fig. (9.4a).

As shown in Fig.(9.4a), the pole $z = \alpha$ is placed in the real axis. The zero is placed at $z = -1$ to ensure zero gain at the folding frequency (Nyquist limit). When the cutoff frequency is above $f_s / 4$, the pole-zero placement is adopted as shown in Fig.(9.4b).
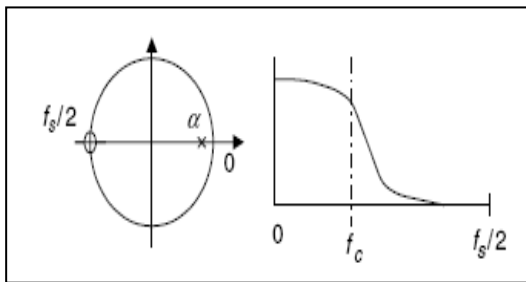

Fig. (9.4a) Pole-zero placement for the first-order lowpass filter with $f_c < f_s/4$.
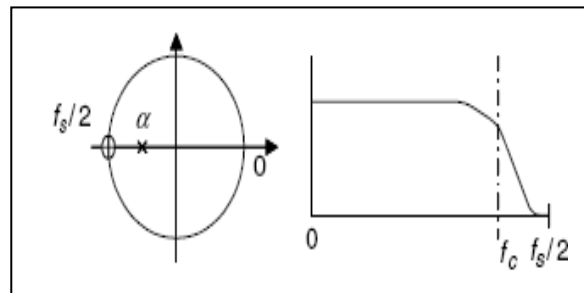
Fig.(9.4b) Pole-zero placement for the first-order lowpass filter with $f_c > f_s/4$.

Design formulas for lowpass filters using the pole-zero placement are given in the following equations:

$$H(z) = \frac{K(z + 1)}{(z - \alpha)},$$

$$K = \frac{(1 - \alpha)}{2}.$$

(9.30)

Example (7): A first-order lowpass filter is required to satisfy the following specifications:

1. Sampling rate = 8,000 Hz

2 A 3 dB cutoff frequency: f_c = 100 Hz

3. Zero gain at 4,000 Hz.

Find the transfer function using the pole-zero placement method.

Solution: Since the cutoff frequency of 100 Hz is much less than $f_s / 4 = 2,000$ Hz, we determine the pole as:

$$\alpha \approx 1 - 2 \times (100/8000) \times \pi = 0.9215,$$

Which is above 0.9. Hence, we have a good approximation. The unit-gain scale factor is calculated by:

$$K = \frac{(1 - 0.9215)}{2} = 0.03925.$$

$$H(z) = \frac{0.03925(z + 1)}{(z - 0.9215)} = \frac{0.03925 + 0.03925z^{-1}}{1 - 0.9215z^{-1}}.$$

Note that we can also determine the unit-gain factor K by substituting $Z = e^{j0} = 1$ to the transfer function $H(Z) = (Z + 1) / (Z - \alpha)$, then find a DC gain. Set the scale factor to be a reciprocal of the DC gain. This can be easily done, that is,

$$DC\ gain = \frac{z + 1}{z - 0.9215}\bigg|_{z=1} = \frac{1 + 1}{1 - 0.9215} = 25.4777.$$

Hence, $K = 1/25.4777 = 0.03925.$

9.4.4 First-Order High-pass Filter Design

Similar to the low-pass filter design, the pole-zero placements for first-order high-pass filters in two cases are shown in Figures (9.5a) and (9.5b).
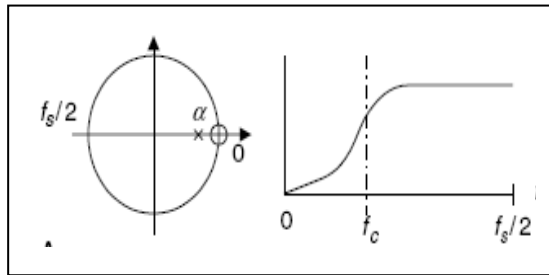

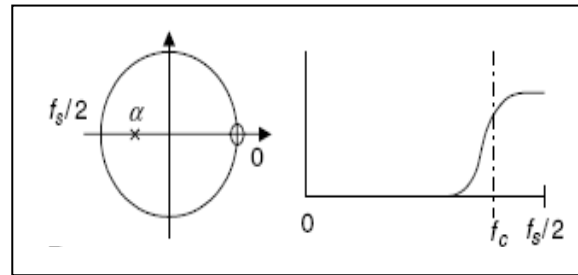**Fig.(9.5a) Pole-zero placement for the first-order highpass filter with $f_c < f_s/4$.**   **Fig.(9.5b) Pole-zero placement for the first-order highpass filter with $f_c > f_s/4$.**

Formulas for designing highpass filters using the pole-zero placement are listed in the following equations:

When $f_c < f_s/4$, $\alpha \approx 1 - 2 \times (f_c/f_s) \times \pi$, good for $0.9 \leq r < 1$.

When $f_c > f_s/4$, $\alpha \approx -(1 - \pi + 2 \times (f_c/f_s) \times \pi)$, good for $-1 < r \leq -0.9$

$$H(z) = \frac{K(z-1)}{(z-\alpha)}$$

$$K = \frac{(1+\alpha)}{2}.$$

(9.31)

Example (8): A first-order highpass filter is required to satisfy the following specifications:

1. Sampling rate = 8,000 Hz

2. A 3 dB cutoff frequency: fc = 3800 Hz

3. Zero gain at 0 Hz.

Find the transfer function using the pole-zero placement method.

Solution:

Since the cutoff frequency of 3,800 Hz is much larger than $f_s / 4 = 2,000$ Hz, we determine the pole as:

$$\alpha \approx -(1 - \pi + 2 \times (3800/8000) \times \pi) = -0.8429,$$

$$K = \frac{(1 - 0.8429)}{2} = 0.07854$$

$$H(z) = \frac{0.07854(z-1)}{(z+0.8429)} = \frac{0.07854 - 0.07854z^{-1}}{1 + 0.8429z^{-1}}.$$

Note that we can also determine the unit-gain scale factor K by substituting $Z = e^{j180} = -1$ into the transfer function $H(Z) = (Z - 1) / (Z - \alpha)$, finding a passband gain at the Nyquist limit $f_s/2 = 4,000$ Hz. We then set the scale factor to be a reciprocal of the passband gain. That is,

$$passband\ gain = \frac{z-1}{z+0.8429}\bigg|_{z=1} = \frac{-1-1}{-1+0.8429} = 12.7307.$$

Hence, $K = 1/12.7307 = 0.07854.$

9. 5 Application: Digital Audio Equalizer

For an audio application such as the CD player, the digital audio equalizer is used to make the sound as one desires by changing filter gains for different audio frequency bands. Other applications include adjusting the sound source to take room acoustics into account, removing undesired noise, and boosting the desired signal in the specified pass-band. The simulation is based on the consumer digital audio processor—such as a CD player—handling the 16-bit digital samples with a sampling rate of 44.1 kHz and an audio signal bandwidth at 22.05 kHz. A block diagram of the digital audio equalizer is depicted in Fig (9.6).

A seven-band audio equalizer is adopted for discussion. The center frequencies are listed in Table (2). The 3 dB bandwidth for each band-pass filter is chosen to be 50% of the center frequency. As shown in Fig (9.6), g_0 through g_6 are the digital gains for each band-pass filter output and can be adjusted to make sound effects, while $y_0(n)$ through $y_6(n)$ are the digital amplified bandpass filter outputs. Finally, the equalized signal is the sum of the amplified bandpass filter outputs and itself. By changing the digital gains of the equalizer, many sound effects can be produced. A IIR bandpass Butterworth filters are chosen for the audio equalizer. The coefficients are achieved using the BLT method.
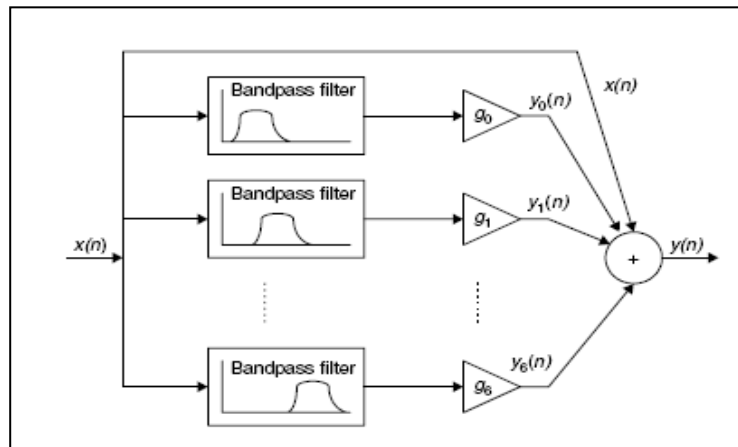

Fig. (9.6) Simplified block diagram of the audio equalizer.

Table (2) Specifications for an audio equalizer to be designed.

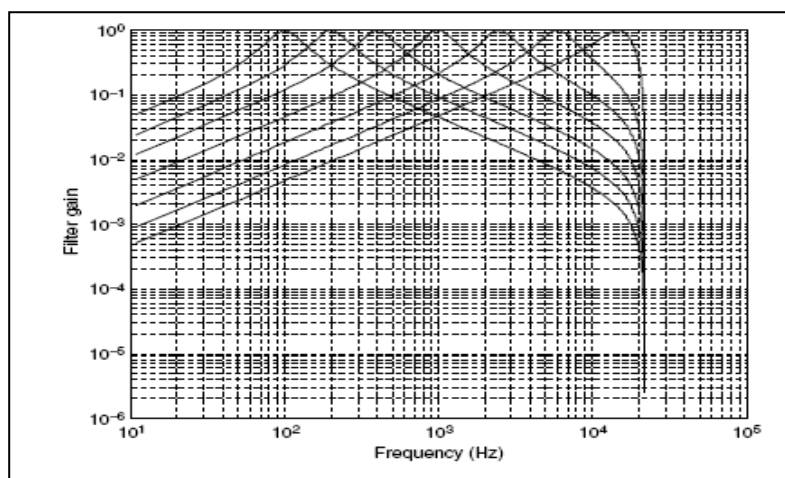| Center frequency (Hz)	100	200	400	1000	2500	6000	15000
Bandwidth (Hz)	50	100	200	500	1250	3000	7500

Fig. (9.7) Magnitude frequency responses for the audio equalizer.

The audio test signal having frequency components of 100 Hz, 200 Hz, 400 Hz, 1,000 Hz, 2,500 Hz, 6,000 Hz, and 15,000 Hz.

The gains set for the filter banks are: $g_0 = 10$; $g_1 = 10$; $g_2 = 0$; $g_3 = 0$; $g_4 = 0$; $g_5 = 10$; $g_6 = 10$. The frequency components at 100 Hz, 200 Hz, 6,000 Hz, and 15,000 Hz will be boosted by $20 \log_{10} 10 = 20$ dB. The top plot in Fig. (9.8), shows the spectrum for the audio test signal, while the bottom plot depicts the spectrum for the equalized audio test signal. Before audio digital equalization, the spectral peaks at all bands are at the same level; after audio digital equalization, the frequency components at bank 0, bank 1, bank 5, and bank 6 are amplified. The operation of the digital equalizer boosts the low frequency components and the high frequency components.
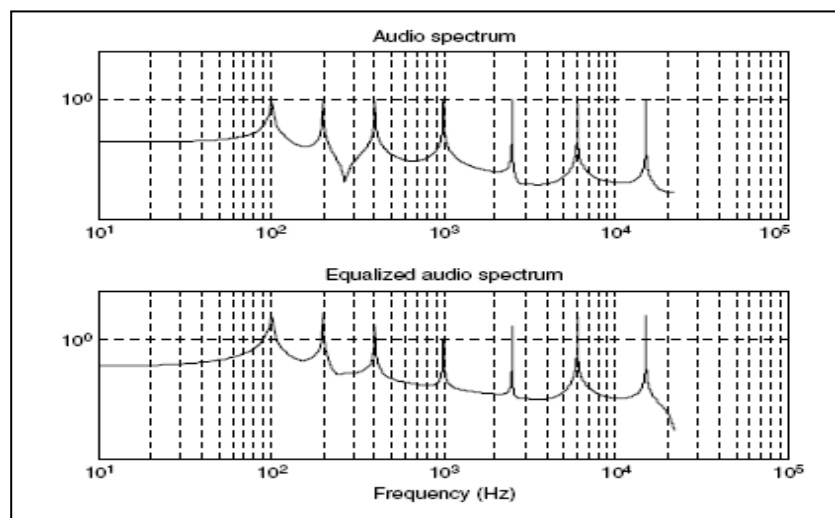

Fig. (9.8) Audio spectrum and equalized audio spectrum.