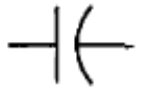# Electrical Network Transfer Functions

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \dfrac{1}{C} \displaystyle\int_0^1 i(\tau)d\tau$	$i(t) = C\dfrac{dv(t)}{dt}$	$v(t) = \dfrac{1}{C}q(t)$	$\dfrac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \dfrac{1}{R}v(t)$	$v(t) = R\dfrac{dq(t)}{dt}$	R	$\dfrac{1}{R} = G$
Inductor	$v(t) = L\dfrac{di(t)}{dt}$	$i(t) = \dfrac{1}{L}\displaystyle\int_0^1 v(\tau)d\tau$	$v(t) = L\dfrac{d^2q(t)}{dt^2}$	Ls	$\dfrac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t) - $ V (volts), $i(t) - $ A (amps), $q(t) - $ Q (coulombs), $C - $ F (farads), $R - \Omega$ (ohms), $G - \Omega$ (mhos), $L - $ H (henries).
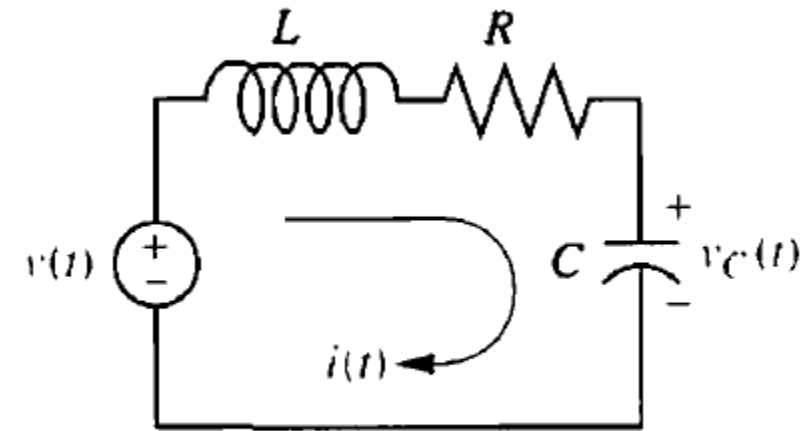
Simple Circuits via Mesh Analysis

FIGURE 2.3   *RLC* network

We are to treat the capacitor voltage as the output and the applied voltage as the input.

Summing the voltages around the loop, assuming zero initial conditions,

yields the integro-differential equation for this network as

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

Changing variables from current to charge using $i(t) = dq(t)/dt$ yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t)$$

From the voltage-charge relationship for a capacitor in the Table from previous slide, $q(t) = Cv_C(t)$ Substituting this in the previous Eq.

$$LC\frac{d^2 v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t)$$

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V(s)$$

Solving for the transfer function, $V_C(s)/V(s)$, we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Let us now develop a technique for simplifying the solution for future problems. First, take the Laplace transform of the equations in the voltage-current column of the Table in the first slide assuming zero initial conditions.

For the capacitor,

$$V(s) = \frac{1}{Cs} I(s)$$

For the resistor,

$$V(s) = RI(s)$$

For the inductor,

$$V(s) = LsI(s)$$

Now define the following transfer function: We call this particular transfer function *impedance.*

$$\frac{V(s)}{I(s)} = Z(s)$$

The Laplace transform of Eq

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

assuming zero initial conditions, is

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s)$$

FIGURE 2.5 **Laplace-transformed network**

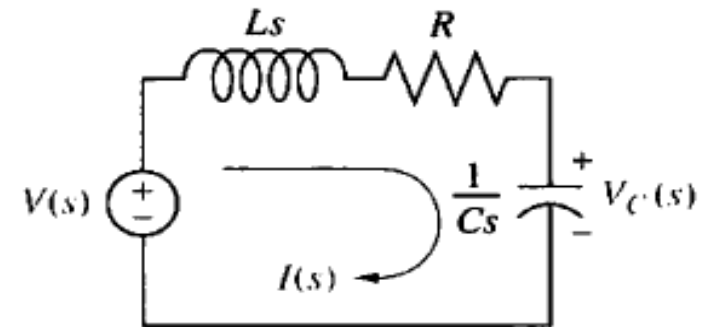Notice that the above Eq. , which is in the form

[Sum of impedances] $I(s)$ = [Sum of applied voltages]

1) Redraw the original network showing all time variables, such as $v(t)$, $i(t)$ and $v_C(t)$ as Laplace transforms $V(s)$, $I(s)$, and $V_C(s)$ respectively.

2) Replace the component values with their impedance values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values.

We now redo the previous Example.

$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$$

Solving for $I(s)/V(s)$,

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R + \dfrac{1}{Cs}}$$

But the voltage across the capacitor, *VQ (S)*, is the product of the current and the impedance of the capacitor. Thus,

$$V_C(s) = I(s)\frac{1}{Cs}$$

Solving the Eq. for $I(s)$,

$$I(s) = V_c(s) * Cs$$

substituting *I(s)* into Eq. in the previous slide.

$$\frac{V_c(s) * Cs}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}$$

$$\frac{V_c(s)}{V(s)} = \frac{1}{Cs(Ls + R + \frac{1}{Cs})} = \frac{1}{LCs^2 + CRs + \frac{Cs}{Cs}}$$

$$= \frac{1}{LCs^2 + CRs + 1} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Simple Circuits via Nodal Analysis

We will Repeat the previous Example using nodal analysis and without writing a differential equation.

The transfer function can be obtained by summing currents flowing out of the node whose voltage is *Vc(s)* in Figure below We assume that currents leaving the node are positive and currents entering the node are negative. The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor.
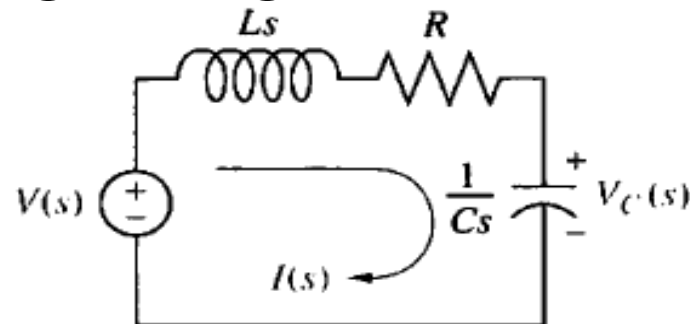

FIGURE 2.5 Laplace-transformed network

From the impedance function we know that $I(s) = V(s)/Z(s)$, hence

$$\frac{V_c(s)}{1/Cs} + \frac{V_c(s) - V(s)}{R + Ls} = 0$$

Where $\frac{V_c(s)}{1/Cs}$ is the current flowing out of the node through the capacitor, and $\frac{V_c(s) - V(s)}{R + Ls}$ is the current flowing out of the node through the series resistor and inductor.

Solving the Eq. for the transfer function $V_C(s)/V(s)$

Use Matlab to solve the Eq.

syms Vc V R L C s

eq = (((Vc)/(1/(C*s)))+((Vc-V)/(R+(L*s))))==0

sol = solve(eq,Vc)

nsol = sol/V

Simple Circuits via Voltage Division

Repeat the previous Example using voltage division and the transformed circuit.

The voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances. Thus,

$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \dfrac{1}{Cs}\right)} V(s)$$

Solving for the transfer function $V_C(s)/V(s)$ yields the same result

Complex Circuits via Mesh Analysis

To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

1) Replace passive element values with their impedances.

2) Replace all sources and time variables with their Laplace transform.

3) Assume a transform current and a current direction in each mesh.

4) Write Kirchhoffs voltage law around each mesh.

5) Solve the simultaneous equations for the output.

6) Form the transfer function.

- Example: Given the network of the Figure below find the transfer function $\dfrac{I_2(s)}{V(s)}$
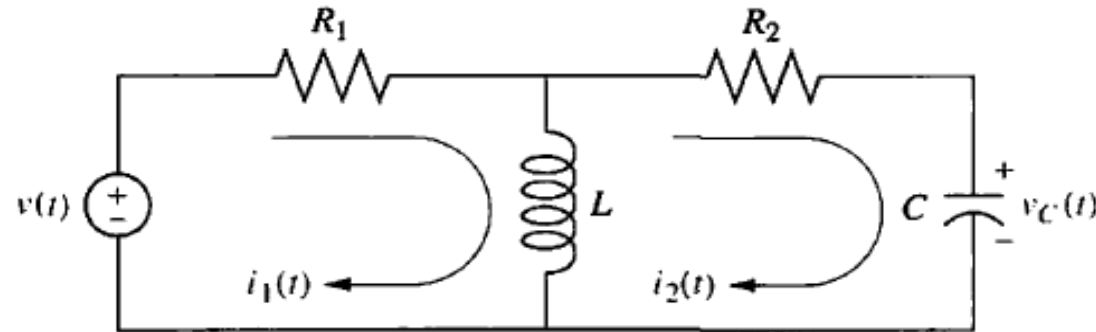
The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions. The result is shown in the Figure below
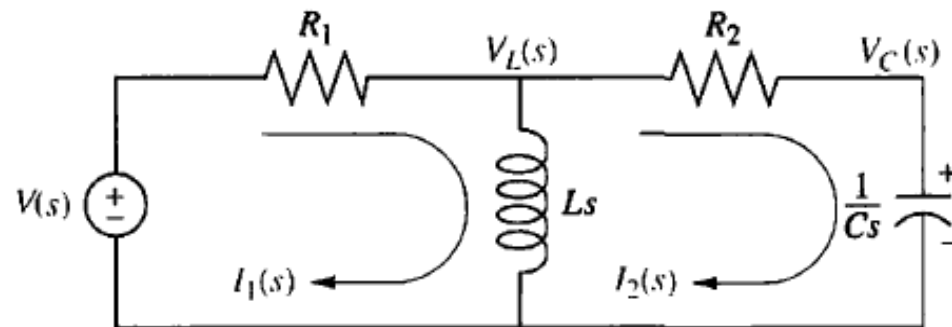
The circuit with which we are dealing requires two simultaneous equations to solve for the transfer function. These equations can be found by summing voltages around each mesh through which the assumed currents, $I_1(s)$ and $I_2(s)$, flow
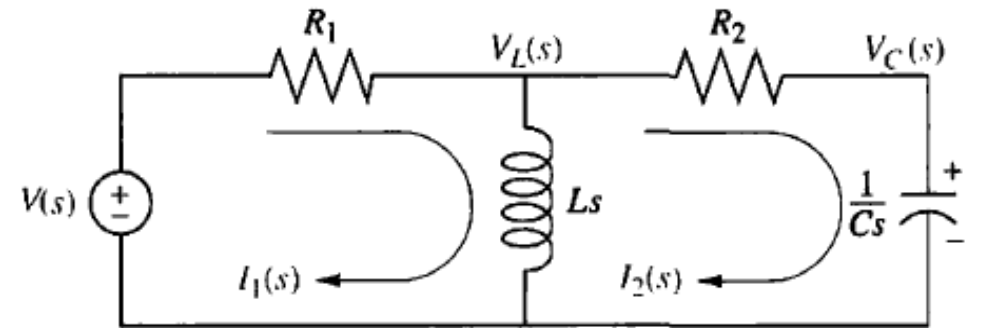
Around Mesh 1, where $I_1(s)$ flows,

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

Around Mesh 2, where $I_2(s)$ flows,

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0$$

Combining terms

$$(R_1 + Ls)\, I_1(s) - LsI_2(s) = V(s)$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs} \right) I_2(s) = 0$$

We can use Cramer's rule

$$I_2(s) = \frac{\begin{vmatrix} R_1 + Ls & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{Ls\ V(s)}{\Delta}$$

Where $\Delta = \begin{vmatrix} R_1 + Ls & -Ls \\ -Ls & Ls + R_2 + \dfrac{1}{Cs} \end{vmatrix}$

Forming the transfer function $G(s)$ yields,

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\begin{vmatrix} R_1 + Ls & -Ls \\ -LR_1 s & Ls + R_2 + \dfrac{1}{Cs} \end{vmatrix}}$$

$$= \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1 R_2 C + L)s + R_1}$$

Matlab Commands

Syms s R1 R2 L c V

A2=[(R1+L*s) V;-L*s 0]

A=[(R1+L*s) -L*s;-L*s (L*s+R2+(1/(c*s)))]

I2=det(A2)/det(A);
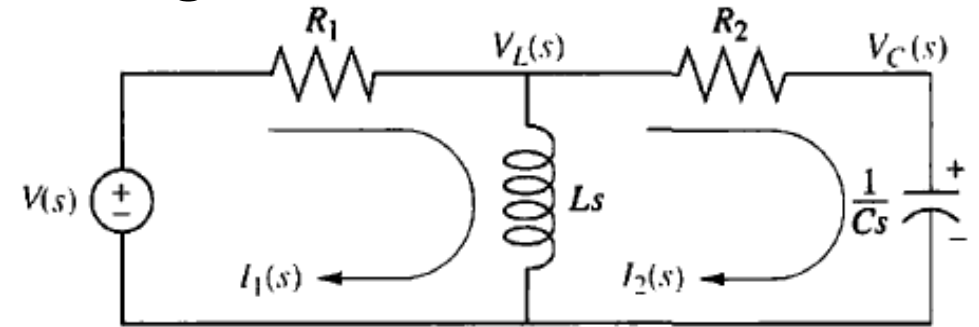
I2=simple(I2);

G=I2/V;

pretty (G)

Complex Circuits via Nodal Analysis

Find the transfer function, $\dfrac{V_C(s)}{V(s)}$, for the circuit in Figure *below* Use nodal analysis.

We sum currents at the nodes rather than sum voltages around the meshes. From the Figure the sum of currents flowing from the nodes marked $V_L(s)$ and $V_C(s)$ are, respectively,

$$\frac{V_L(s) - V(s)}{R_1} - \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0$$

$$CsV_C(s) + \frac{V_L(s) - V_C(s)}{R_2} = 0$$

Rearranging and expressing the resistances as conductance, $G_1 = 1/R_1$ and $G_2 = 1/R_2$ we obtain:

$$(G_1 + G_2 + \frac{1}{Ls})V_L(s) - G_2 V_C(s) = G_1 V(s)$$

$$-G_2 V_L(s) + (G_2 + Cs)V_C(s) = 0$$

Solving for the transfer function $\frac{V_C(s)}{V(s)}$ yields

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1 G_2}{C} s}{(G_1 + G_2)s^2 + \frac{G_1 G_2 L + C}{LC} s + \frac{G_2}{LC}}$$

Matlab Commands

syms C L s V Vc G1 G2

 A=[((G1)+(G2)+(1/(L*s))) G1*V;-G2 0]

A2= [((G1)+(G2)+(1/(L*s))) -G2;-G2 ((G2)+(C*s))]

Vc= det(A)/det(A2)
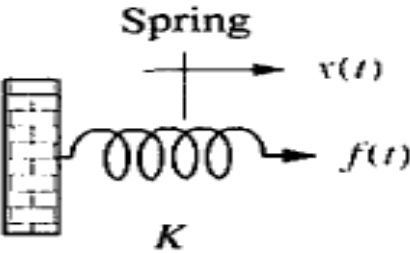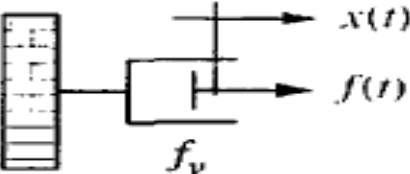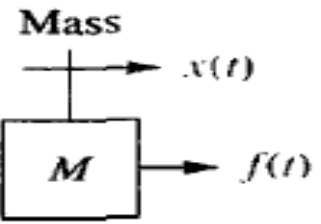
simplify(Vc)

G=Vc/V

pretty(G)

Translational Mechanical System Transfer Functions

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring $x(t)$ $f(t)$ K	$f(t) = K \int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper $x(t)$ $f(t)$ f_v	$f(t) = f_v v(t)$	$f(t) = f_v \dfrac{dx(t)}{dt}$	$f_v s$
Mass $x(t)$ M $f(t)$	$f(t) = M \dfrac{dv(t)}{dt}$	$f(t) = M \dfrac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t) = $ N (newtons), $x(t) = $ m (meters), $v(t) = $ m/s (meters/second), $K = $ N/m (newtons/meter), $f_v = $ N-s/m(newton-seconds/ meter), $M = $ kg (kilograms $= $ newton-seconds^2/meter).

Example: Find the transfer function, $\dfrac{X(s)}{F(s)}$, for the system of Figure below
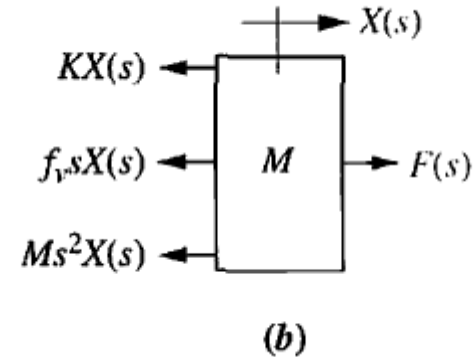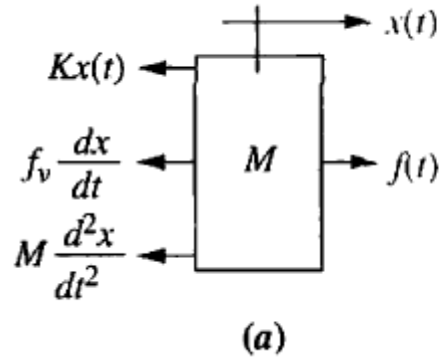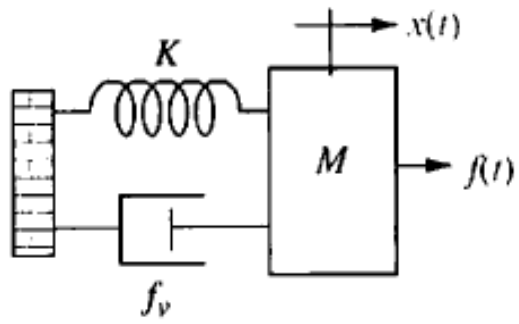

FIGURE 2.16   a. Free-body diagram of mass, spring, and damper system; b. transformed free-body diagram

Begin the solution by drawing the free-body diagram shown in Figure 2.16 above. Place on the mass all forces felt by the mass. We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left.

We now write the differential equation of motion using Newton's law to sum to zero all of the forces shown on the mass in Figure

$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s) \quad \text{or} \quad (Ms^2 + f_v s + K)X(s) = F(s)$$

Solving for the transfer function yields

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

Taking the Laplace transform of the force-displacement column in Table, we obtain for the spring,

$$F(s) = KX(s)$$
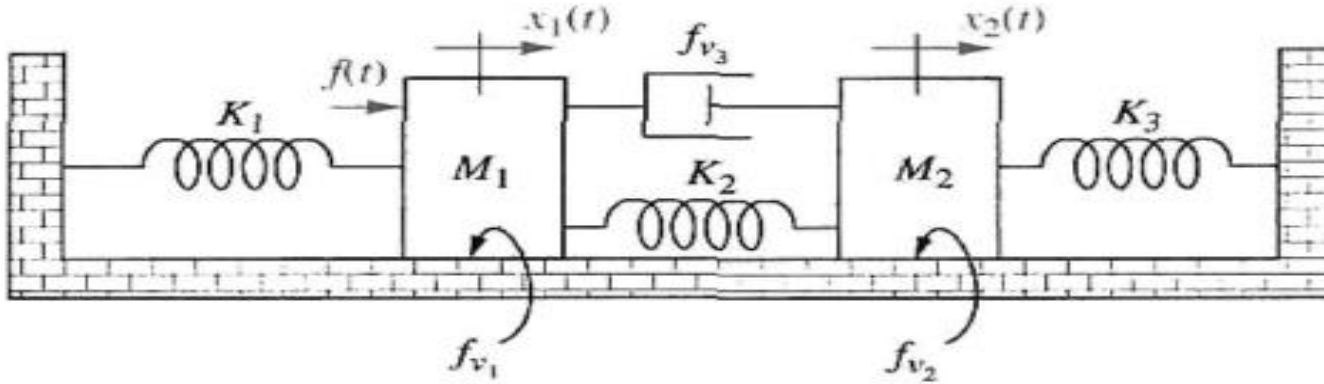
For the viscous damper,

$$F(s) = f_v s X(s)$$

and for the mass,

$$F(s) = Ms^2 X(s)$$

If we define impedance for mechanical components as

$$Z_M(s) = \frac{F(s)}{X(s)}$$

Find the transfer function, , $\dfrac{X_2(s)}{F(s)}$, for the system of Figure below

The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still. Thus, two simultaneous equations of motion will be required to describe the system. The two equations come from free-body diagrams of each mass. Superposition is used to draw the free body diagrams. For example, the forces on M_1 are due to (1) its own motion and (2) the motion of M_2 transmitted to M_1 through the system. We will consider these two sources separately.

If we hold M_2 still and move M_1 to the right, we see the forces shown in Figure 2.18(a). If we hold M_1 still and move M_2 to the right, we see the forces shown in Figure 2.18(b). The total force on M_1 is the superposition, or sum, of the forces just discussed. This result is shown in Figure 2.18(c).
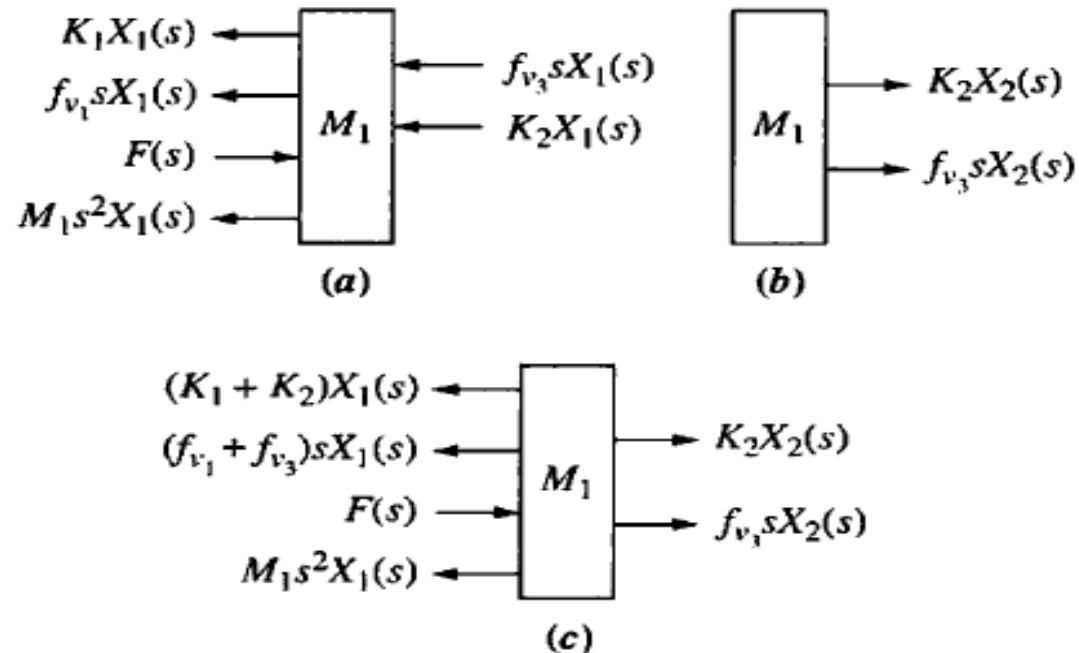

FIGURE 2.18   a. Forces on M_1 due only to motion of M_1; **b.** forces on M_1 due only to motion of M_2; **c.** all forces on M_1

- For M_2, we proceed in a similar fashion: First we move M_2 to the right while holding M_1 still; then we move M_1 to the right and hold M_2 still. For each case we evaluate the forces on M_2, The results appear in Figure 2.19.
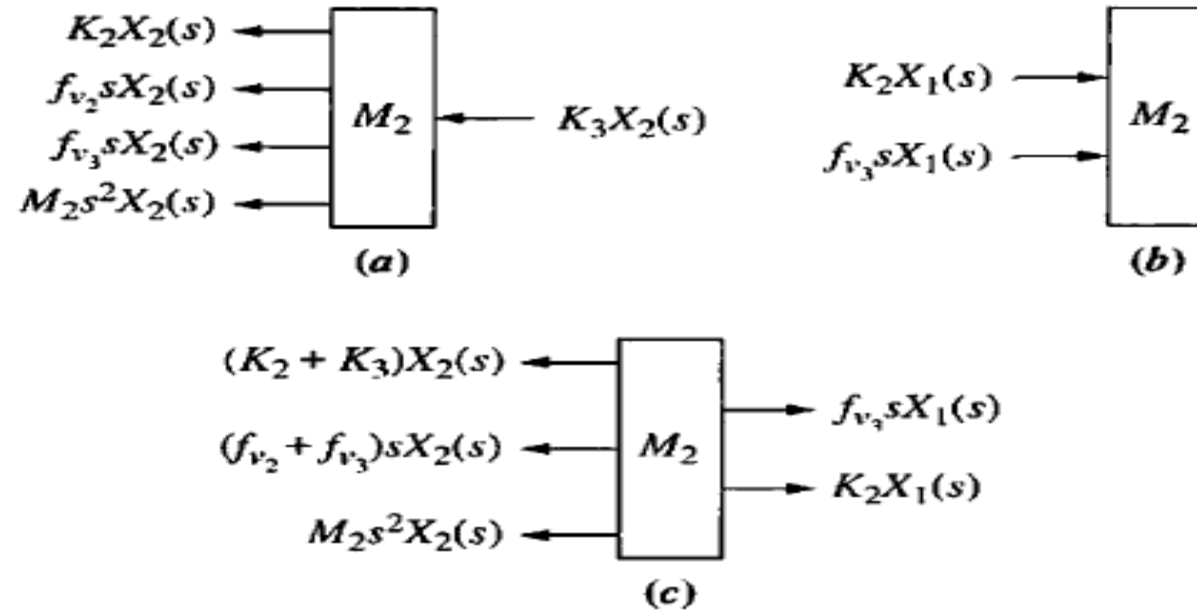

FIGURE 2.19 **a.** Forces on M_2 due only to motion of M_2; **b.** forces on M_2 due only to motion of M_1; **c.** all forces on M_2

The Laplace transform of the equations of motion can now be written from Figures 2.18(c) and 2.19(c) as

$$[M_1 s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - (f_{v3}s + K_2)X_2(s) = F(s)$$
$$-(f_{v3}s + K_2)X_1(s) + [M_2 s^2 + (f_{v2} + f_{v3})s + (K_2 + K_3)]X_2(s) = 0$$

From this, the transfer function $\dfrac{X_2(s)}{F(s)}$ is

$$\frac{X_2(s)}{F(s)} = G(s) = \frac{(f_{v_3}s + K_2)}{\Delta}$$

where

$$\Delta = \begin{vmatrix} [M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)] & -(f_{v_3}s + K_2) \\ -(f_{v_3}s + K_2) & [M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)] \end{vmatrix}$$

Matlab Commands

syms M1 M2 K1 K2 K3 fv1 fv2 fv3 s F

A=[((M1*s^2)+(fv1+fv3)*s)+(K1+K2) F; (-fv3*s)+K2 0]

A2=[((M1*s^2)+(fv1+fv3)*s)+(K1+K2) (-fv3*s)+K2;(-fv3*s)+K2 ((M2*s^2)+((fv2+fv3)*s)+(K2+K3))]

X2 = det(A)/det(A2)

G = X2/F

pretty(G)