Chapter Two
Steady Heat Conduction

• INTRODUCTION

in heat transfer analysis, we are often interested in the rate of heat transfer through a medium under steady conditions and surface temperatures. Such problems can be solved easily without involving any differential equations by the introduction of thermal resistance concepts in an analogous manner to electrical circuit problems. In this case, the thermal resistance corresponds to electrical resistance, temperature difference corresponds to voltage, and the heat transfer rate corresponds to electric current.

• ONE-DIMENSIONAL STEADY HEAT CONDUCTION IN A PLANE WALL, A CYLINDER, AND A SPHERE

• A PLANE WALL

Consider steady heat conduction through the walls of a house during a winter day. We know that heat is continuously lost to the outdoors through the wall. We intuitively feel that heat transfer through the wall is in the normal direction to the wall surface, and no significant heat transfer takes place in the wall in other directions (Fig. 2–1).

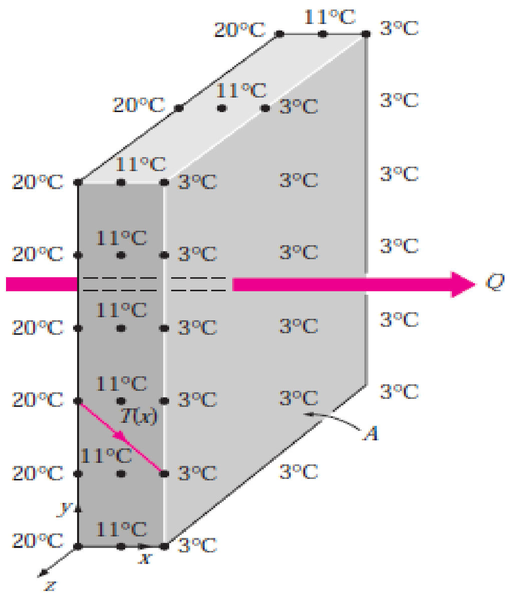


Figure 2-1 Heat flow through a wall is one dimensional when the temperature of the wall varies in one direction only

Recall that heat transfer in a certain direction is driven by the temperature gradient in that direction. The small thickness of the wall causes the temperature gradient in that direction to be large. Further, if the air temperatures in and outside the house remain constant, then heat transfer through the wall of a house can be modeled as ***steady and one-dimensional***. The temperature of the wall in this case will depend on one direction only (say the x-direction) and can be expressed as T(x).

The ***energy balance*** for the wall can be expressed as

$$(Rate\ of\ heat\ transfer\ into\ the\ wall) - (Rate\ of\ heat\ transfer\ out\ of\ the\ wall) = (Rate\ of\ change\ of\ the\ energy\ of\ the\ wall)$$

Or

$$Q_{in} - Q_{out} = \frac{dE_{wall}}{dt} \qquad 2\text{-}1$$

But ${dE_{wall}}/{dt}$ = 0 for *steady* operation, since there is no change in the temperature of the wall with time at any point. Hence, the rate of heat transfer into the wall must be equal to the rate of heat transfer out of it. In other words, *the rate of heat transfer through the wall must be constant,* $\boldsymbol{Q_{cond,wall}}$**= constant.**

Consider a plane wall of thickness L and average thermal conductivity *k*. The two surfaces of the wall are maintained at constant temperatures of T_1 and T_2. For one-dimensional steady heat conduction through the wall, we have *T*(*x*). Then Fourier's law of heat conduction for the wall can be expressed as

$$Q_{cond,wall} = -kA\frac{dT}{dx} \qquad 2\text{-}2$$

where the rate of conduction heat transfer $Q_{cond,wall}$ and the wall area A are constant. Thus we have dT/dx = constant, which means that the temperature through the wall varies linearly with x. That is, the temperature distribution in the wall under steady conditions is a straight line (Fig. 2–2).

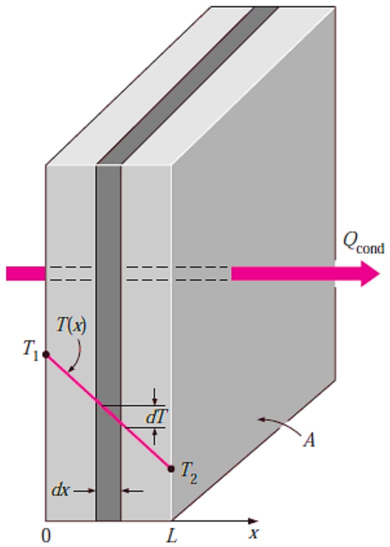


Figure 2-2 Under steady conditions, the temperature distribution in a plane wall is a straight line.

Separating the variables in the preceding equation and integrating from $x = 0$, where $T(0) = T_1$, to $x = L$, where $T(L) = T_2$, we get

$$\int_{x=0}^{L} Q_{cond,wall} dx = \int_{T=T_1}^{T_2} kAdT \qquad 2\text{-}3$$

Performing the integrations and rearranging gives

$$Q_{cond,wall} = kA\frac{T_1 - T_2}{L} \qquad \text{(W)} \qquad 2\text{-}4$$

<u>Note:-</u> The rate of heat conduction is available, the temperature $T(x)$ at any location x can be determined by replacing $T2$ in Eq. 2–4 by T, and L by x.

• The Thermal Resistance Concept

• For heat conduction

Equation 2–4 for heat conduction through a plane wall can be rearranged as

$$Q_{cond,wall} = \frac{T_1 - T_2}{R_{wall}} \qquad \text{(W)} \qquad 2\text{-}5$$

Where

$$R_{wall} = \frac{L}{kA} \qquad (°\text{C/W}) \qquad 2\text{-}6$$

is the ***thermal resistance*** of the wall against heat conduction or simply the **conduction resistance** of the wall. Note that the thermal resistance of a medium depends on the ***geometry* and the *thermal properties* of the medium**.

Equation 2-6 for heat flow is analogous to the relation for ***electric current flow*** *I*, expressed as

$$I = \frac{V_1 - V_2}{R_{elc}} \qquad \text{2-7}$$

where $R_{elc} = L/\sigma_{elc}\, A$ is the *electric resistance* and $V_1 - V_2$ is the ***voltage difference*** across the resistance (σ_{elc} is the electrical conductivity). Thus, the ***rate of heat transfer*** through a layer corresponds to the ***electric current***, the ***thermal resistance*** corresponds to ***electrical resistance***, and the ***temperature difference*** corresponds to ***voltage difference*** across the layer (Fig. 2–3).

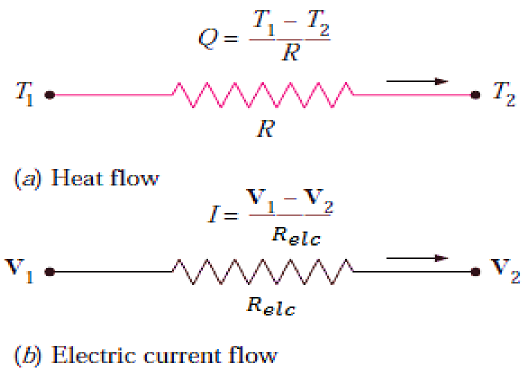


Figure 2-3 Analogy between thermal and electrical resistance concepts.

- ## For heat convection

Consider convection heat transfer from a solid surface of area A_s and temperature T_s to a fluid whose temperature sufficiently far from the surface is T_∞, with a convection heat transfer coefficient *h*. Newton's law of cooling for convection heat transfer rate $Q_{conv} = hA_s(T_s - T_\infty)$ can be rearranged as

$$Q_{conv} = \frac{(T_s - T_\infty)}{R_{conv}} \quad \text{(W)} \qquad \text{2-8}$$

Where

$$R_{conv} = \frac{1}{hA_s} \quad \text{(°C/W)} \qquad \text{2-9}$$

is the ***thermal resistance*** of the surface against heat convection, or simply the **convection resistance** of the surface (Fig. 2–4).

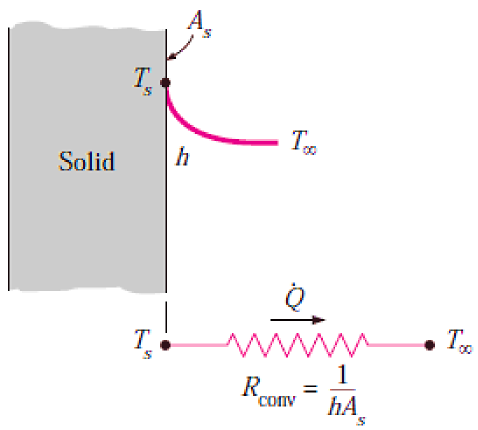


Figure 2-4 Schematic for convection resistance at a surface.

Note:- Equation 2-9 for convection resistance is valid for surfaces of any shape

• Thermal Resistance Network

Now consider steady one-dimensional heat flow through a plane wall of thickness L, area A, and thermal conductivity k that is exposed to convection on both sides to fluids at temperatures $T_{\infty 1}$ and $T_{\infty 2}$ with heat transfer coefficients h_1 and h_2, respectively, as shown in Fig. 2–5. Assuming $T_{\infty 1} > T_{\infty 2}$, the variation of temperature will be as shown in the figure. Note that the temperature varies linearly in the wall, and asymptotically approaches $T_{\infty 1}$ and $T_{\infty 2}$ in the fluids as we move away from the wall.

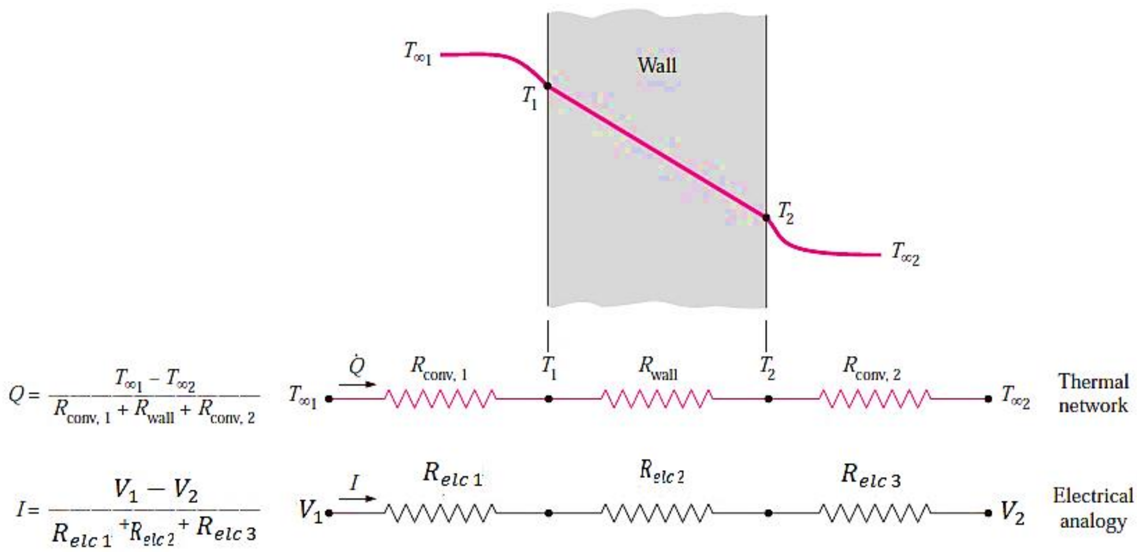


Figure 2-5 The thermal resistance network for heat transfer through a plane wall subjected to convection on both sides, and the electrical analogy.

Under steady conditions we have

$$\begin{pmatrix} Rate\ of\ heat \\ convection \\ into\ the\ wall \end{pmatrix} = \begin{pmatrix} Rate\ of\ heat \\ conductio \\ through\ the\ wall \end{pmatrix} = \begin{pmatrix} Rate\ of\ heat \\ convection \\ from\ the\ wall \end{pmatrix}$$

Or

$$Q = h_1 A(T_{\infty 1} - T_1) = kA\frac{T_1 - T_2}{L} = h_2 A(T_2 - T_{\infty 2}) \qquad 2\text{-}10$$

which can be rearranged as

$$Q = \frac{(T_{\infty 1} - T_1)}{{}^{1}\!/_{h_1 A}} = \frac{T_1 - T_2}{{}^{L}\!/_{kA}} = \frac{(T_2 - T_{\infty 2})}{{}^{1}\!/_{h_2 A}}$$

$$= \frac{(T_{\infty 1} - T_1)}{R_{conv1}} = \frac{T_1 - T_2}{R_{wall}} = \frac{(T_2 - T_{\infty 2})}{R_{conv2}} \qquad 2\text{-}11$$

Adding the numerators and denominators yields

$$Q = \frac{(T_{\infty 1} - T_{\infty 2})}{R_{total}} \quad \text{(W)} \qquad 2\text{-}12$$

Where

$$R_{total} = R_{conv1} + R_{wall} + R_{conv2} = {}^{1}\!/_{h_1 A} + {}^{L}\!/_{kA} + {}^{1}\!/_{h_2 A} \quad (°C/W) \qquad 2\text{-}13$$

Note that the heat transfer area ***A*** is constant for a plane wall, and the rate of heat transfer through a wall separating two mediums is equal to the temperature difference divided by the total thermal resistance between the mediums.

We can rewrite equation (2-12) as below:

$$\Delta T = QR \quad (°C) \qquad 2\text{-}14$$

which indicates that the ***temperature drop*** **across** any layer is equal to the ***rate of heat transfer*** **times the *thermal resistance*** across that layer**.**

• Overall Heat Transfer Coefficient

It is sometimes convenient to express heat transfer through a medium in an analogous manner to Newton's law of cooling as

$$Q = UA\Delta T \ \text{(W)} \qquad 2\text{-}15$$

where U is the **overall heat transfer coefficient**

A comparison of Eqs. (2–12) and (2–15) reveals that

$$UA = \frac{1}{R_{total}} \qquad 2\text{-}16$$

Therefore, for a unit area, the overall heat transfer coefficient is equal to the inverse of the total thermal resistance.

• Multilayer Plane Walls

In practice we often encounter plane walls that consist of several layers of different materials. The thermal resistance concept can still be used to determine the rate of steady heat transfer through such *composite* walls. As you may have already guessed, this is done by simply noting that the conduction resistance of each wall is L/kA connected in series, and using the electrical analogy.
That is, by dividing the ***temperature difference*** between two surfaces at known temperatures by the ***total thermal resistance*** between them.
Consider a plane wall that consists of two layers (such as a brick wall with a layer of insulation). The rate of steady heat transfer through this two-layer composite wall can be expressed as Fig. 2–6.

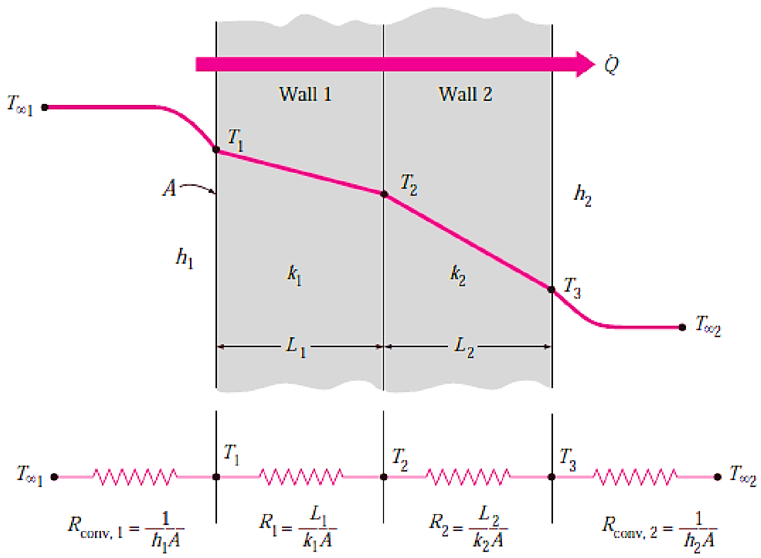


Figure 2-6 The thermal resistance network for heat transfer through a two-layer plane wall subjected to convection on both sides

$$Q = \frac{(T_{\infty 1} - T_{\infty 2})}{R_{total}}$$

where R_{total}l is the ***total thermal resistance***, expressed as

$$R_{total} = R_{con,1} + R_{wall,1} + R_{wall,2} + R_{con,2}$$

$$= {}^{1}\!/_{h_1 A} + {}^{L_1}\!/_{k_1 A} + {}^{L_2}\!/_{k_2 A} + {}^{1}\!/_{h_2 A} \qquad 2\text{-}17$$

The $R_{wall,1}$ and $R_{wall,2}$ in relations above indicate the first and the second layers, respectively.

This result for the two-layer case is analogous to the single-layer case, except that an additional resistance is added for the additional layer. This result can be extended to plane walls that consist of three or more layers by adding an additional resistance for each additional layer.

Note:- The thermal resistance concept is widely used in practice because it is intuitively easy to understand and it has proven to be a powerful tool in the solution of a wide range of heat transfer problems. But its use is limited to systems through which the rate of heat transfer remains constant; that is, to systems involving steady heat transfer with no heat generation (such as resistance heating or chemical reactions) within the medium.

Example 1/

Consider a 0.8-m-high and 1.5-m-wide double-pane window consisting of two 4-mm-thick layers of glass (k = 0.78 W/m · °C) separated by a 10-mm-wide stagnant air space (k = 0.026 W/m · °C). Determine the steady rate of heat transfer through this double-pane window and the temperature of its inner surface for a day during which the room is maintained at 20°C while the temperature of the outdoors is −10°C. Take the convection heat transfer coefficients on the inner and outer surfaces of the window to be h_1 = 10 W/m^2 · °C and h_2 = 40 W/m^2 · °C, which includes the effects of radiation.

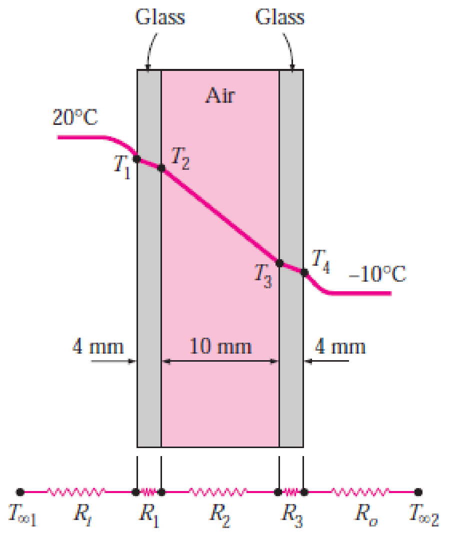


Figure 2-7 Schematic for Example 1

Solution:

A= 0.8 m x 1.5 m = 1.2 m^2

$$R_i = R_{\text{conv},1} = \frac{1}{h_1 A} = \frac{1}{(10\ \text{W/m}^2\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.08333^\circ\text{C/W}$$

$$R_1 = R_3 = R_{\text{glass}} = \frac{L_1}{k_1 A} = \frac{0.004\ \text{m}}{(0.78\ \text{W/m}\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.00427^\circ\text{C/W}$$

$$R_2 = R_{\text{air}} = \frac{L_2}{k_2 A} = \frac{0.01\ \text{m}}{(0.026\ \text{W/m}\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.3205^\circ\text{C/W}$$

$$R_o = R_{\text{conv},2} = \frac{1}{h_2 A} = \frac{1}{(40\ \text{W/m}^2\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.02083^\circ\text{C/W}$$

Noting that all three resistances are in series, the total resistance is

$$\begin{aligned} R_{\text{total}} &= R_{\text{conv},1} + R_{\text{glass},1} + R_{\text{air}} + R_{\text{glass},2} + R_{\text{conv},2} \\ &= 0.08333 + 0.00427 + 0.3205 + 0.00427 + 0.02083 \\ &= 0.4332^\circ\text{C/W} \end{aligned}$$

Then the steady rate of heat transfer through the window becomes

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} = \frac{[20 - (-10)]^\circ\text{C}}{0.4332^\circ\text{C/W}} = 69.2\ \text{W}$$

The inner surface temperature of the window in this case will be

$$T_1 = T_{\infty 1} - \dot{Q}R_{\text{conv},1} = 20^\circ\text{C} - (69.2\ \text{W})(0.08333^\circ\text{C/W}) = 14.2^\circ\text{C}$$

• GENERALIZED THERMAL RESISTANCE NETWORKS

The thermal resistance concept or the electrical analogy can also be used to solve steady heat transfer problems that involve parallel layers or combined series-parallel arrangements. Consider the composite wall shown in figure 2–8, which consists of two parallel layers.

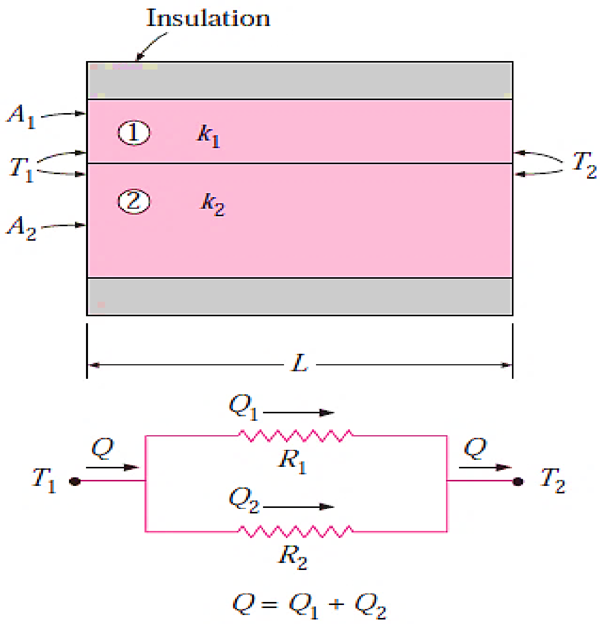


Figure 2-8 Thermal resistance network for two parallel layers

The thermal resistance network, which consists of two parallel resistances, can be represented as shown in the figure. Noting that the total heat transfer is the sum of the heat transfers through each layer, we have

$$Q = Q_1 + Q_2 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad 2\text{-}18$$

Utilizing electrical analogy, we get

$$Q = \frac{T_1 - T_2}{R_{total}} \qquad 2\text{-}19$$

Where

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} \rightarrow R_{total} = \frac{R_1 R_2}{R_1 + R_2} \qquad 2\text{-}20$$

since the resistances are in parallel.

Now consider the combined series-parallel arrangement shown in figure 2–9.

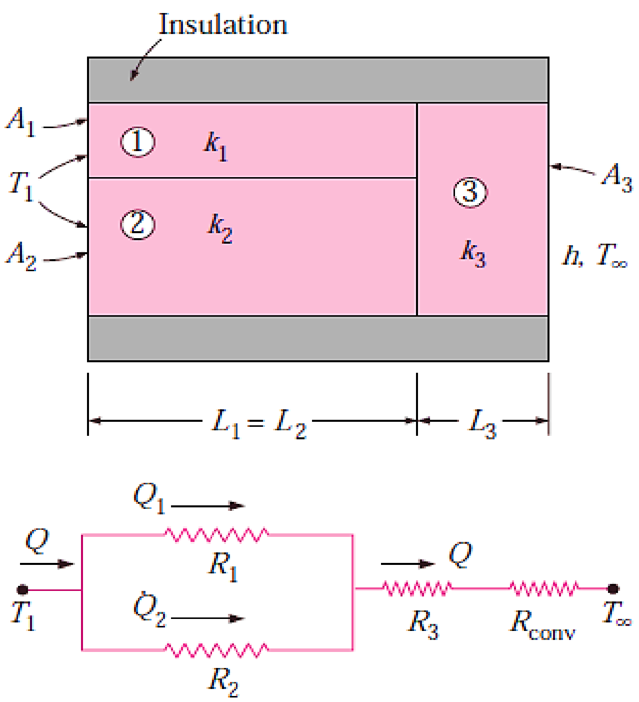


Figure 2-9 Thermal resistance network for combined series-parallel arrangement

The total rate of heat transfer through this composite system can again be expressed as

$$Q = \frac{T_1 - T_\infty}{R_{total}} \qquad 2\text{-}21$$

Where

$$R_{total} = R_{12} + R_3 + R_{conv} = \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_{conv} \qquad 2\text{-}22$$

And

$$R_1 = \frac{L_1}{k_1 A_1}, \quad R_2 = \frac{L_2}{k_2 A_2}, \quad R_3 = \frac{L_3}{k_3 A_3}, \quad R_{conv} = \frac{1}{h A_3} \qquad 2\text{-}23$$

Once the individual thermal resistances are evaluated, the total resistance and the total rate of heat transfer can easily be determined from the relations above.

Note:- there are two assumptions commonly used in solving complex multidimensional heat transfer problems by treating them as one-dimensional (say, in the x-direction)

1- Any plane wall normal to the x-axis is isothermal (i.e., to assume the temperature to vary in the x-direction only)

2- Any plane parallel to the x-axis is adiabatic (i.e., to assume heat transfer to occur in the x-direction only).

Example 2/

A 17-m-high and 5-m-wide wall consists of long 16-cm × 22-cm cross section horizontal bricks (k = 0.72 W/m · °C) separated by 17-cm-thick plaster layers (k = 0.22 W/m · °C). There are also 2-cm-thick plaster layers on each side of the brick and a 17-cm-thick rigid foam (k = 0.026 W/m · °C) on the inner side of the wall, as shown in Fig. 17–21. The indoor and the outdoor temperatures are 20°C and −10°C, respectively, and the convection heat transfer coefficients on the inner and the outer sides are h_1 = 10 W/m² · °C and h_2 = 25 W/m² · °C, respectively. Assuming one-dimensional heat transfer and disregarding radiation, determine the rate of heat transfer through the wall.

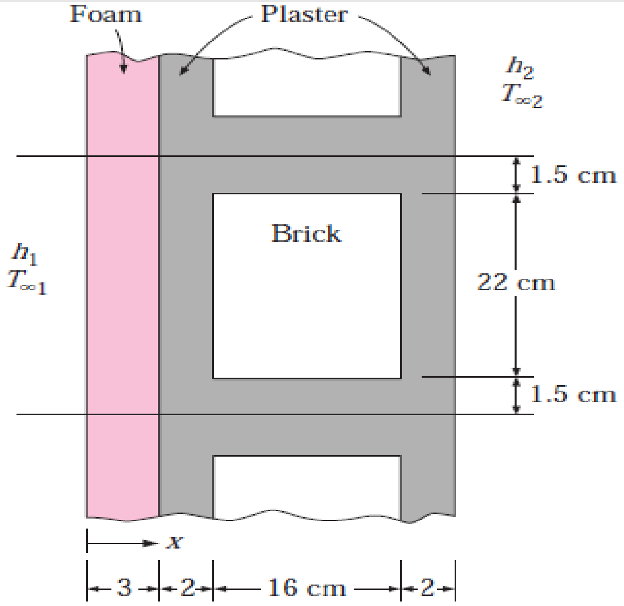
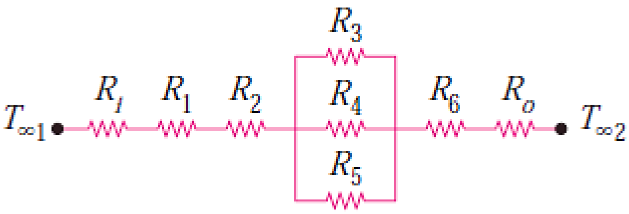


Figure 2-10 Schematic for Example 2

Solution:



$$R_i = R_{\text{conv, 1}} = \frac{1}{h_1 A} = \frac{1}{(10\ \text{W/m}^2 \cdot {}^\circ\text{C})(0.25 \times 1\ \text{m}^2)} = 0.4^\circ\text{C/W}$$

$$R_1 = R_{\text{foam}} = \frac{L}{kA} = \frac{0.03\ \text{m}}{(0.026\ \text{W/m} \cdot {}^\circ\text{C})(0.25 \times 1\ \text{m}^2)} = 4.6^\circ\text{C/W}$$

$$R_2 = R_6 = R_{\text{plaster, side}} = \frac{L}{kA} = \frac{0.02\ \text{m}}{(0.22\ \text{W/m} \cdot {}^\circ\text{C})(0.25 \times 1\ \text{m}^2)}$$
$$= 0.36^\circ\text{C/W}$$

$$R_3 = R_5 = R_{\text{plaster, center}} = \frac{L}{kA} = \frac{0.16\ \text{m}}{(0.22\ \text{W/m} \cdot {}^\circ\text{C})(0.015 \times 1\ \text{m}^2)}$$
$$= 48.48^\circ\text{C/W}$$

$$R_4 = R_{\text{brick}} = \frac{L}{kA} = \frac{0.16\ \text{m}}{(0.72\ \text{W/m} \cdot {}^\circ\text{C})(0.22 \times 1\ \text{m}^2)} = 1.01^\circ\text{C/W}$$

$$R_o = R_{\text{conv, 2}} = \frac{1}{h_2 A} = \frac{1}{(25\ \text{W/m}^2 \cdot {}^\circ\text{C})(0.25 \times 1\ \text{m}^2)} = 0.16^\circ\text{C/W}$$

The three resistances R_3, R_4, and R_5 in the middle are parallel, and their equivalent resistance is determined from

$$\frac{1}{R_{\text{mid}}} = \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} = \frac{1}{48.48} + \frac{1}{1.01} + \frac{1}{48.48} = 1.03\ \text{W/}^\circ\text{C}$$

which gives

$$R_{\text{mid}} = 0.97^\circ\text{C/W}$$

Now all the resistances are in series, and the total resistance is

$$\begin{aligned} R_{\text{total}} &= R_i + R_1 + R_2 + R_{\text{mid}} + R_6 + R_o \\ &= 0.4 + 4.6 + 0.36 + 0.97 + 0.36 + 0.16 \\ &= 6.85^\circ\text{C/W} \end{aligned}$$

Then the steady rate of heat transfer through the wall becomes

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} = \frac{[20 - (-10)]^\circ\text{C}}{6.85^\circ\text{C/W}} = 4.38\ \text{W} \qquad \text{(per 0.25 m}^2\text{ surface area)}$$

or 4.38/0.25 = 17.5 W per m^2 area. The total area of the wall is A = 3 m × 5 m = 15 m^2. Then the rate of heat transfer through the entire wall becomes

$$\dot{Q}_{\text{total}} = (17.5\ \text{W/m}^2)(15\ \text{m}^2) = 263\ \text{W}$$