Headway Models

1. Relation between Headway and Capacity

The headway distribution describe which headways can be observed with which probability. One might argue that this is related to the flow, since the average headway is the flow. The flow is determined by the demand. However, in the bottleneck the flow is determined by the minimum headway at which drivers follow each other. Or in other words, the headway distribution determines the capacity in the bottleneck. Suppose the headway distribution is given by P(h).
Then, the average headway can be determined by:

$$\langle h \rangle = \int_0^{\inf} hP(h)dh$$

This is a mathematical way to describe the average headway once the distribution is given. The flow in the active bottleneck, hence the capacity is inverse of the mean headway,

$$C = 1/ \left\langle h_{\text{bottleneck}} \right\rangle$$

The number of vehicles arriving in a certain period could be a useful measure. This holds for instance for traffic lights, where the number of arrivals per red period is relevant. As illustrated in figure below, there could be different lanes for different directions at a traffic light. The idea is that the traffic towards one direction will not block the traffic to other directions, hence, the length should be long enough to allow the number of vehicles in the red period. The average number of vehicles in a red period can be determined from the flow. However, mostly requirements are that in p% of the red times (under a constant demand) the queue should not exceed the dedicated lane. In that case, the headway distribution can form the basis for the calculations.

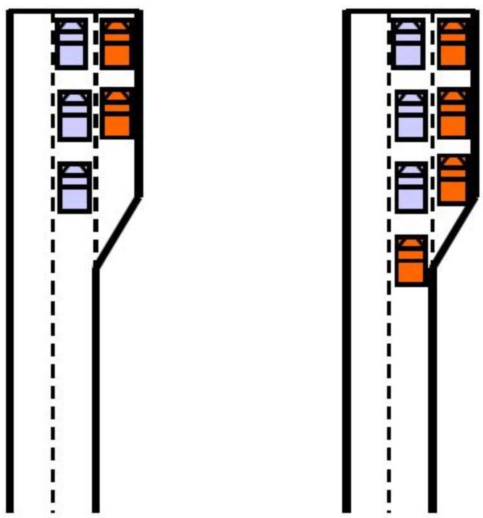


Figure : A queuing area. The orange coloured vehicles turn right, whereas the blue ones continue straight.

2. Arrivals per Interval

This section describes the number of arrivals per time interval. For different conditions, this distribution is different. Table below gives an overview of the distributions described in this section, and gives some characteristics. One should differentiate between the probability of number of arrivals (a macroscopic characteristic, based on aggregating over a certain duration of time), described in this section, and the probability of a headway (a microscopic characteristic).

Table : Overview of the means and variances of the different distributions. In this table, q is the flow in the observation period, p is the probability of including the observation in the period.

Distribution	Mean	Variance
Poisson	q	q=mean
Binomial	np	$np(1-p)$ <mean
Negative binomial	$n(1p)/p$	$n(1p)/p^2$ >mean

2.1 Poisson

The first distribution function described here is the Poisson distribution. One will observe this distribution function once the arrivals are independent . The resulting probability is described by a so-called Poisson distribution function. Mathematically, this function is described by:

$$P(X=k)=\frac{q^k}{k!}e^{-q}$$

This equation gives the probability that k vehicles arrive if the average arrival rate per period is q. Hence note that one needs to rescale q to units of number of vehicles per aggregation period!

Figure below shows examples of the Poisson distribution. Note that for low values of the flow (expected value smaller than 1), the probability is decreasing. If the flow is higher, there is a maximum at the number of arrivals which is at a higher value than 1.

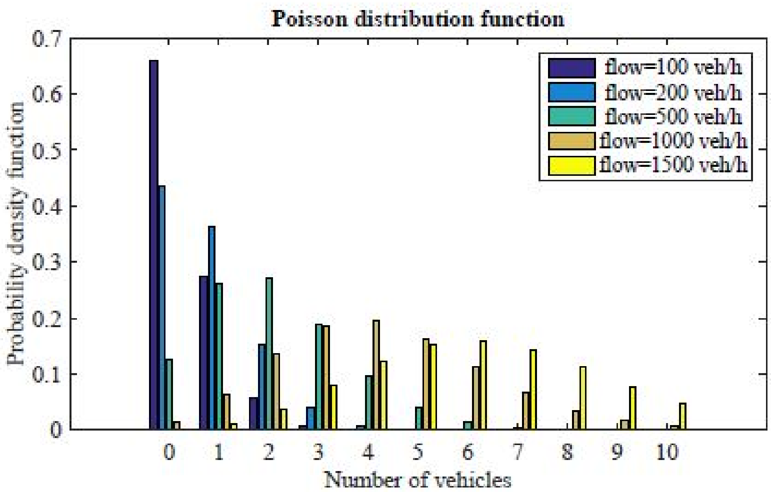


Figure: Example of the Poisson distribution for different flow values; the flow is indicated in veh/h, and these are the probabilities for arrivals in 15 seconds.

Figure below shows the best fits of this distribution on real life data. This distribution is accurate if the flow is low, and is not so good if the flow increases. This is because once the flow is low, the assumption of independent arrivals does not hold any more. Once vehicles are bound by the minimum headway, the arrivals are not independent any more. This restriction come more into play once the flow is high.

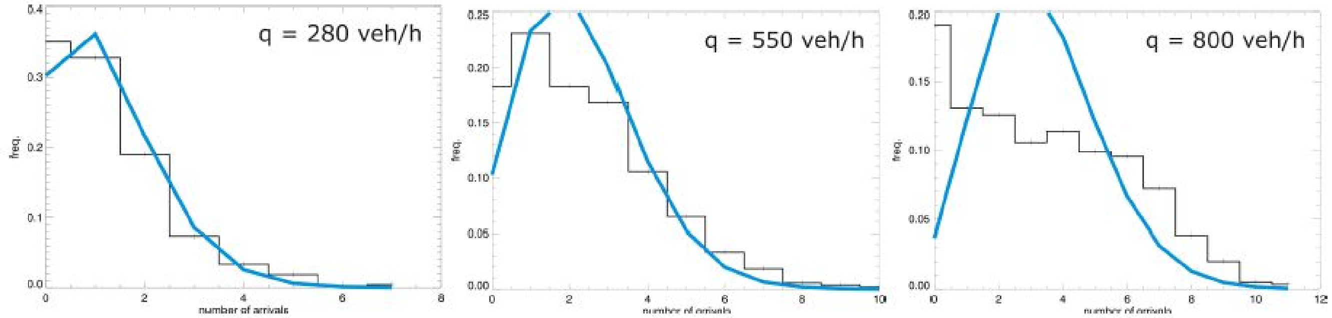


Figure : Illustration of the number of arrivals from real world data.

2.3 Binomial

The binomial distribution can be used if there are correlations between the arrivals. For instance on busy roads, one can expect that more vehicles drive at a minimum headway. Whereas in case of the Poisson distribution, the variance of the distribution was equal to the mean, in this case the variance is smaller (since more drivers drive at a certain headway). The mathematical equation describing the function is:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

The idea behind the distribution is that one does n tries, each with an independent success rate of p. The number of successes is k. The mean of the distribution is np, as shown in previous Table. A certain flow specifies the mean of the arrivals, which hence determines np. This gives a freedom to choose n or p, by which one can match the spread of the function. The number of observations in the distribution can never exceed n, so an reasonable choice of n would be the interval time divided by the minimum headway. Figure below shows examples of the binomial distribution function. Note that the variance of the binomial function is smaller than the Poisson distribution for the same flow. This can be a reason to choose this function.

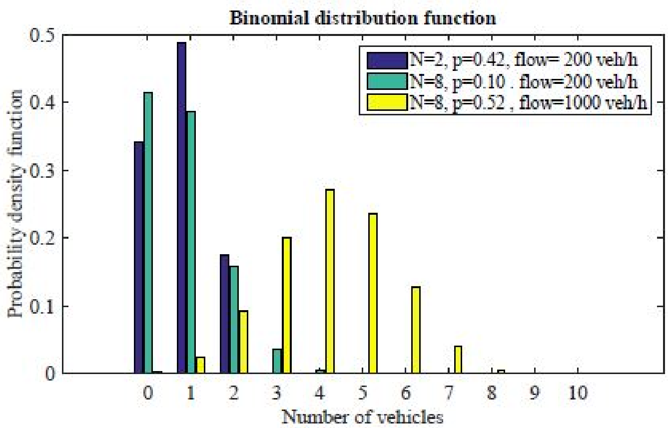


Figure: Examples of the binomial distribution function for the number of arrivals in an aggregation period.

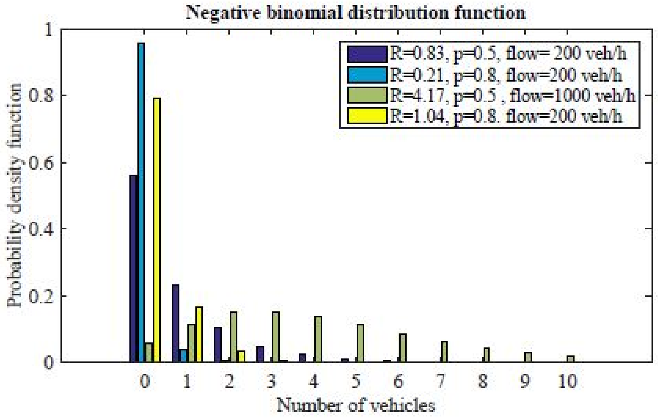
2.4 <u>Negative binomial</u>

The negative binomial distribution can be used if there are negative correlations between the arrivals. In traffic this happens for instance downstream of a signalised intersection. If one observes several vehicles at a short headway, one gets a larger probability that net next headway will be large because the traffic light will switch to red. The probability distribution function for the number of observed vehicles in an aggregation interval is:

$$P(X = k) = \binom{k + r - 1}{k} p^k (1 - p)^r$$

This distribution describes when one observes individual and independent process with a success rate of p. One observes so until r failures are observed. X is the stochastic indicating how many successes are observed.

Figure below shows the value of this function for different parameter sets. Note that the variance (and hence standard deviation) be set independently from the mean, like in the binomial distribution function. for this distribution (see previous table), the mean is given by n(1p)/p, and the variance is given by n(1p)/p^2, which is the mean divided by p.
Since p is a probability and has a value between 0 and 1, we can derive that the variance is larger than the mean. A larger variance is what one would intuitively expect downstream of a signalised intersection. This characteristic can be used to have an idea of the distribution to use.



<u>Figure</u> : Example of the negative binomial function for the number of arrivals in an aggregation period.

3. Headway distributions

3.1 Exponential

The first distribution is the exponential distribution, shown in figure below. This is defined by:

$$P(h|q) = q \exp -qh$$

Note that this equation has a single parameter, being the flow q. That is the inverse of the average headway. The underlying assumption of this distribution is that all drivers can choose their moment of arrival independently. Consider that each (infinitesimal small) time step a driver considers to leave with a fixed probability. One then gets a exponential distribution function for the headway.
This is a very good assumption on quiet roads, when there are no interactions between the vehicles. The interactions occur once vehicles are limited in choosing their headway, mostly indicated by the minimum headway.
The only good way to test whether the exponential data describes the data one observes is to do a proper statistical test (e.g., a KolmogorovSmirnov test). There are also rules of thump. A characteristic for this distribution is that the standard deviation is equal to the mean. If one has data of which one thinks the arrivals are independent and this criterion is satisfied, the arrivals are very likely to be exponentially distributed.
This distribution function for the headways matches the Poisson distribution function for the number of arrivals.

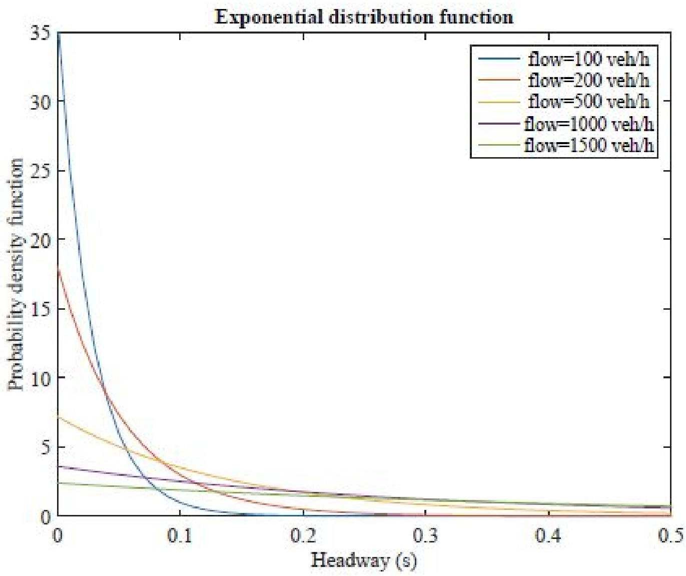


Figure: The probability density function for headways according to the exponential distribution for different flow values.

3.2 Composite headway models

Whereas the exponential distribution function works well for low flows, for higher flows the distribution function is not very good. For these situations, so called composite headway distributions are being used . The basic idea is that a fraction of the traffic Φ is driving freely following a headway distribution function $P_{free}(h)$. The other fraction of the traffic 1 - Φ is driving constraint, i.e., is following their leader, and have a headway distribution function $P_{constraint}(h)$.

In a composite headway distribution, these two distribution functions are combined. The combined distribution function can hence be expressed as:

$$P(h) = \Phi P_{\text{free}}(h) + (1 - \Phi) P_{\text{constraint}}(h)$$

A plot of the headway distribution function (in fact, a survival function of the headway, i.e. 1- the cumulative distribution function) is shown in figure below. The vertical axis is a logarithmic axis. Note that in these axis the exponential distribution function is a straight line. That is what is observed for the large headways. For the smaller headways (in the figure, for less then 7 seconds) this does not longer hold. This is due to the limitation of following distance. As the figure shows, this can be determined graphically.

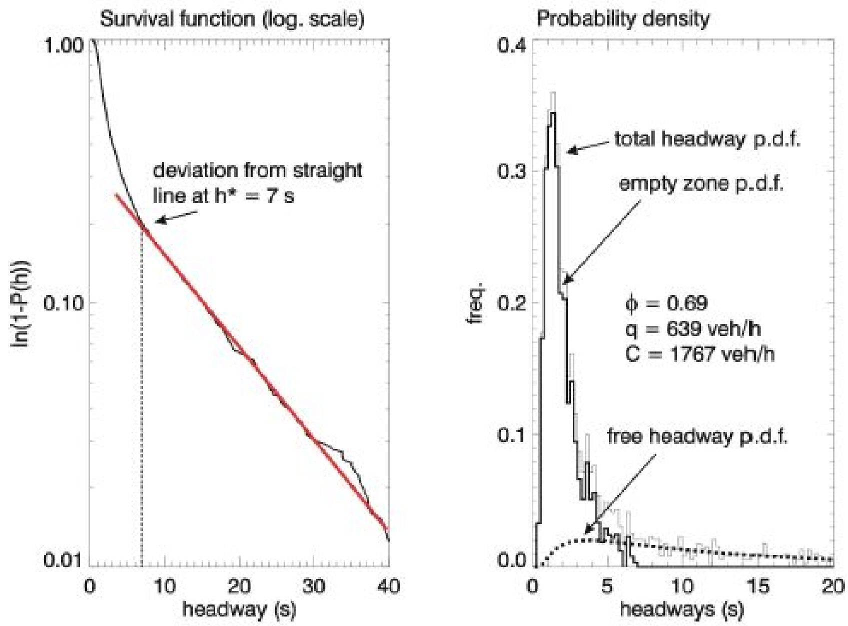


Figure : Example of the composite headway distribution and its estimation for real life data.

4. <u>Critical gap</u>