3.3 Stability Criteria

- Modern computer programs such as *Matlab* directly give the poles of a particular transfer function.
  - Helps us determine the stability of that system right away!
- Sometimes, the control engineers wish to test for stability for an *entire class* of systems:
  - Controller parameters ("gains") are usually embedded into the coefficients of the characteristic polynomial.
  - The range of control parameters ("design region") that yield a stable system are sought.
- In such cases, stability tests are utilized:
  - Jury Test
  - *w transforms* along with the Routh-Hurwitz Criterion

3.3.1 Routh's Criterion for Discrete-time Systems

To apply the well-known *Routh Criterion*, a continuous-time equivalent for the discrete-time system must be obtained through Tustin's transformation:

$$s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)\Rightarrow \quad z=\frac{1+s\frac{T}{2}}{1-s\frac{T}{2}}$$

Some engineers replace s by "w" and call the resulting operation as *"w transform"* which states the fact that the equation above do not directly take us back to the original s-plane! Anyhow,

$$\left.\frac{Y(z)}{R(z)}\right|_{z=\frac{1+s\frac{T}{2}}{1-s\frac{T}{2}}}=F\left(z=\frac{1+s\frac{T}{2}}{1-s\frac{T}{2}}\right)\Rightarrow \quad F'(s)=\frac{B'(s)}{A'(s)}$$

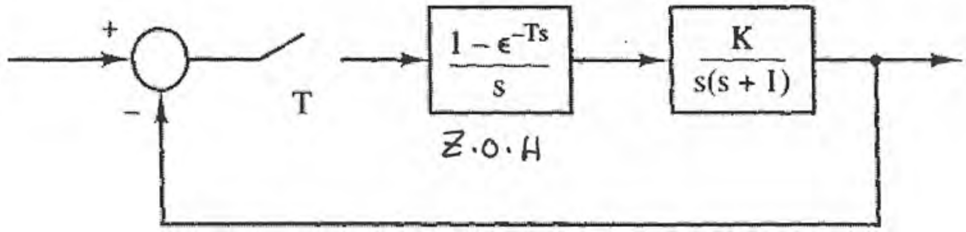
where F'(z) is the closed-loop transfer function. The coefficients of A'(s) is used to form Routh's array.

This can **be applied**, when solved the problem [by using bilinear Transformation].

Or **using** z=(w+1)/(w-1) [by using modified Routh stability]

Example:

Consider the system shown in Figure



Find the range of (K) for stable System.

with $T = 0.1$ s. The open-loop function is

using P.F.M

$$G(s) = \frac{1 - \epsilon^{-Ts}}{s}\left[\frac{1}{s(s+1)}\right] \quad \text{or} = (1 - z^{-1}) Z\left\{ \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1} \right\}$$

Hence, from the z-transform tables we obtain

$$G(z) = \frac{z-1}{z}\left[\frac{(\epsilon^{-T} + T - 1)z^2 + (1 - \epsilon^{-T} - T\epsilon^{-T})z}{(z-1)^2(z - \epsilon^{-T})}\right]$$

O.L.T.F

$$= \frac{0.00484z + 0.00468}{(z-1)(z-0.905)}$$

Then $G(w)$ is given by [using bilinear Transformation]

$$G(w) = G(z)\big|_{z = [1 + (T/2)w]/[1 - (T/2)w]} = G(z)\big|_{z = (1 + 0.05w)/(1 - 0.05w)}$$

or

$$G(w) = \frac{-0.00016w^2 - 0.1872w + 3.81}{3.81w^2 + 3.80w}$$

1. Given a characteristic equation of the form

$$F(w) = b_n w^n + b_{n-1} w^{n-1} + \cdots + b_1 w + b_0 = 0$$

form the Routh array as

$$\begin{array}{c|cccc} w^n & b_n & b_{n-2} & b_{n-4} & \cdots \\ w^{n-1} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ w^{n-2} & c_1 & c_2 & c_3 & \cdots \\ \vdots & d_1 & d_2 & d_3 & \cdots \\ w^1 & j_1 & & & \\ w^0 & k_1 & & & \end{array}$$

2. Only the first two rows of the array are obtained from the characteristic equation. The remaining rows are calculated as follows.

$$c_1 = \frac{b_{n-1} b_{n-2} - b_n b_{n-3}}{b_{n-1}} \qquad d_1 = \frac{c_1 b_{n-3} - b_{n-1} c_2}{c_1}$$

$$c_2 = \frac{b_{n-1} b_{n-4} - b_n b_{n-5}}{b_{n-1}} \qquad d_2 = \frac{c_1 b_{n-5} - b_{n-1} c_3}{c_1}$$

$$c_3 = \frac{b_{n-1} b_{n-6} - b_n b_{n-7}}{b_{n-1}} \qquad \vdots$$

3. Once the array has been formed, the Routh–Hurwitz criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of sign changes of the coefficients in the first column of the array.

Then the characteristic equation is given by

$$1 + KG(w) = (3.81 - 0.00016K)w^2 + (3.80 - 0.1872K)w + 3.81\,K = 0$$

The Routh array derived from this equation is

$$\begin{array}{c|lll} w^2 & 3.81 - 0.00016K \quad 3.81K & \Rightarrow & K < 23{,}813 \\ w^1 & 3.80 - 0.1872K & \Rightarrow & K < 20.3 \\ w^0 & 3.81K & \Rightarrow & K > 0 \end{array}$$

Hence, for no sign changes to occur in the first column, it is necessary that K be in the range $\underline{\underline{0 < K < 20.3}}$, and this is the range of K for stability.

Consider again the system

with $T = 1$ s

$$1 + KG(w) = 1 + KG(z)|_{z = (1+0.5w)/(1-0.5w)}$$
$$= 1 + \frac{K\left[0.368\left[\frac{1+0.5w}{1-0.5w}\right] + 0.264\right]}{\left[\frac{1+0.5w}{1-0.5w}\right]^2 - 1.368\left[\frac{1+0.5w}{1-0.5w}\right] + 0.368}$$

or

$$1 + KG(w) = 1 + \frac{-0.0381K(w-2)(w+12.14)}{w(w+0.924)}$$
$$= \frac{(1-0.0381K)w^2 + (0.924 - 0.386K)w + 0.924K}{w(w+0.924)}$$

Thus the characteristic equation may be expressed as

$$(1 - 0.0381K)w^2 + (0.924 - 0.386K)w + 0.924K = 0$$

The Routh array is then

$$\begin{array}{l|lll} w^2 & 1 - 0.0381K & 0.924K & \Rightarrow \quad K < 26.2 \\ w^1 & 0.924 - 0.386K & & \Rightarrow \quad K < 2.39 \\ w^0 & 0.924K & & \Rightarrow \quad K > 0 \end{array}$$

Hence the system is stable for $\underline{\underline{0 < K < 2.39.}}$

In a manner similar to that employed in continuous-time systems, the frequency of oscillation at $K = 2.39$ can be found from the w^2 row of the array. Recalling that ω_w is the imaginary part of w, we obtain the auxiliary equation

$$(1 - 0.0381K)w^2 + 0.924K|_{K=2.39} = 0.9089w^2 + 2.181 = 0$$

or

$$w = \pm j\sqrt{\frac{2.181}{0.9089}} = \pm j1.549$$

Then $\omega_w = 1.549$

and this expression gives the relationship between frequencies in the s-plane and frequencies in the w-plane.

$$\boxed{\omega_w = \frac{2}{T}\tan\frac{\omega T}{2}}$$ ⇒ Frequency of oscillation at w-plane Calculated from auxiliary Eqn.

$$\boxed{\omega = \frac{2}{T}\tan^{-1}\frac{\omega_w T}{2}} = \frac{2}{1}\tan^{-1}\left[\frac{(1.549)(1)}{2}\right] = 1.32 \text{ rad/s}$$

← Calculated by Formula ↑ Freqeny of oscillation at s-plane

and is the s-plane (real) frequency at which this system will oscillate with $K = 2.39$.

The same system was used in both examples in this section, but with different sample periods. For $T = 0.1$ s, the system is stable for $0 < K < 20.3$. For $T = 1$ s, the system is stable for $0 < K < 2.39$. Hence we can see the dependency of system stability on the sample period. The degradation of stability with increasing T (decreasing sampling frequency) is due to the delay (phase lag) introduced by the

Sampler and hold unit.

Return to P94 {Using Jury Test}

$$G(z) = \frac{(0.00484z + 0.00468)K}{(z-1)(z-0.905)} \quad , \text{ IF } T = 0.1 \text{ sec.}$$

c/cs

$$\therefore Q(z) = z^2 + (0.00484K - 1.905)z + (0.905 + 0.00468K) = 0$$

apply the Jury Test Conditions :-

$0 < K < 20.3 \Rightarrow$ Same reslut see P95

IF $K = 20.3 \Rightarrow$ Sub in $Q(z)$

$$Q(z) = z^2 - 1.8067z + 1.000004 = 0$$

$$\Downarrow \quad z_{1,2} = 0.9034 \pm j0.429 = 1\angle \pm 25.4^\circ = 1\angle \pm 0.4434 \text{ rad}$$
$$= 1\angle \pm \omega T$$

$$\therefore \omega = \frac{0.4434}{0.1} = 4.434 \text{ rad/sec}$$ [This is the s-plane oscillation Frequency]

To calculate ω_w

$$\omega_w = \frac{2}{0.1}\tan\frac{0.4434 \text{ rad}}{2} = 4.508 \text{ rad/sec}$$ [This is the w-plane oscillation Frequency]

Example

Determine the stability of the following system (with T = 1 sec) using Routh's criterion:

C.L.T.F $\Rightarrow$ $$\frac{Y(z)}{R(z)} = F(z) = 0.0484\frac{z+0.9672}{(z-1)(z-0.9048)}$$

Solution: [using bilinear Transformatio] , Here $s \Rightarrow jw \Rightarrow w$

Tustin's (bilinear) transformation leads to

$$F(z)\Big|_{z=\frac{1+\frac{s}{2}}{1-\frac{s}{2}}} \cong F'(s) = 0.0484\frac{\frac{1+\frac{s}{2}}{1-\frac{s}{2}}+0.9672}{\left(\frac{1+\frac{s}{2}}{1-\frac{s}{2}}-1\right)\left(\frac{1+\frac{s}{2}}{1-\frac{s}{2}}-0.9048\right)}$$

$$z = \frac{1+(\frac{T}{2})w}{1-(\frac{T}{2})w}$$

Thus, C.L.T.F $\Rightarrow$ $$F'(s) = -\frac{\left(\frac{s}{120}+1\right)\left(\frac{s}{2}-1\right)}{s\left(\frac{s}{0.0999}+1\right)}$$

The characteristic polynomial becomes

$$A'(s) = 10.01s^2 + s + 0$$

or $= 10.01\,w^2 + w + 0$

Routh Array:

s^2:	10.01	0
s^1:	1	0
s^0:	$0 \approx \varepsilon$	

All first-column coefficients are bigger than zero $\Rightarrow$ system is *(marginally) stable.*

Example:

[by using modified Routh stability]

Let the c/cs equ. Is:-

$$1+G(z)=1+\frac{0.632Kz}{z^2-1.368z+0.368}=0$$

Determine K for the stable system

Sub. $Z=\frac{W+1}{W-1}$

$$Q(w)=0.632K\,w^2+1.264W+(2.736-0.632K)=0$$

In terms of the Routh criterion:

$$\begin{array}{ll} 0.632K & 2.736-0.632K \\ 1.264 & \\ 2.736-0.632K & \end{array}$$

We have: $0 < K < 4.33$

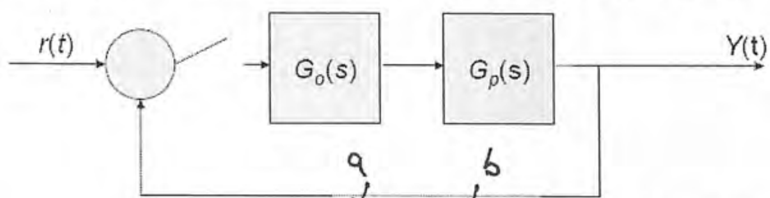
$K > 0$

$(2.736-0.632K) > 0$

$K < 4.329$

To study the effect of gain(K) on the stability

Example: Stability of a closed-loop system



$$G_p(s)=\frac{K}{s(s+1)};\,G(z)=\frac{K(\overset{a}{0.3678}z+\overset{b}{0.2644})}{z^2-1.3678z+0.3678}=\frac{K(az+b)}{z^2-(1+a)z+a}$$

The poles of the closed - loop transfer function t(z) are the roots of the equation

$$[1+G(z)]=0:z^2-(1+a)z+a+Kaz+Kb=0$$

$$K=1:\quad z^2-z+0.6322=(z-0.5+j0.6182)(z-0.5-j0.6182)=0$$

The system is stable because the roots lie within the unit circle. When $K=10$

$$z^2+2.310z+3.012=(z+1.115+j1.295)\,(z+1.115-j1.295)\ \text{(unstable)}$$

This system is stable for : $0 < K < 2.39$ ⇒ See P 105

Second - order sampled system is unstable for increased gain where the continuous is stable for all values of gain.

$$Q(s)=s^2+s+K$$

$$\begin{array}{c|cc} s^2 & 1 & K \\ s^1 & 1 & 0 \\ s^0 & K & \end{array}$$

∴ stable for all values of $K>0$

Example:- Find the range of (K) for stable system

$$G(s) = \frac{K}{s+3}$$

and with digital-to-analog converter (DAC) and analog-to-digital converter (ADC) if the sampling period is 0.02 s.

Solution

The transfer function for analog subsystem ADC and DAC is

$$G_{ZAS}(z) = (1 - z^{-1})\mathscr{Z}\left\{\mathscr{L}^{-1}\left[\frac{G(s)}{s}\right]\right\}$$

$$= (1 - z^{-1})\mathscr{Z}\left\{\mathscr{L}^{-1}\left[\frac{K}{s(s+3)}\right]\right\}$$

Using the partial fraction expansion

$$\frac{K}{s(s+3)} = \frac{K}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

we obtain the transfer function

$$G_{ZAS}(z) = \frac{1.9412 \times 10^{-2} K}{z - 0.9418}$$

For unity feedback, the closed-loop characteristic equation is

$$1 + G_{ZAS}(z) = 0$$

which can be simplified to

$$z - 0.9418 + 1.9412 \times 10^{-2} \quad K = 0$$

$$\Downarrow$$

The stability conditions are $z + a$, where $a = -.9418 + 1.9412 \times 10^{-2} K$

$$0.9418 - 1.9412 \times 10^{-2} \quad K < 1$$

$$-0.9418 + 1.9412 \times 10^{-2} \quad K < 1$$

Thus, the stable range of K is

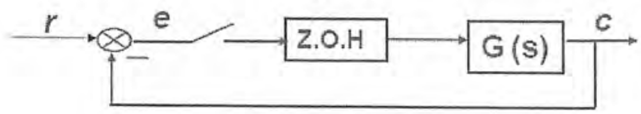
$$-3 < K < 100.03$$

$|a| < 1 \quad \therefore$

or

$-1 < a < 1$

Example



$$T = 1s \qquad G(s) = \frac{K}{s(s+5)}$$

1) Determine K for the stable system.

2) If Continuous determine stability

Solution

1)

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{K}{s(s+5)}\right]$$

$$= (1-e^{-Ts})Z\left[\frac{K}{s^2(s+5)}\right]$$

$$= (1-e^{-Ts})Z\left[\frac{K/5}{s^2} + \frac{-K/5}{s} + \frac{K/25}{s+5}\right]$$

$$= (1-z^{-1})\left(\frac{KTz/5}{(z-1)^2} - \frac{Kz/5}{z-1} + \frac{Kz/25}{z-e^{-5T}}\right)\Bigg|_{T=1}$$

$$\approx -\frac{K}{5}\cdot\frac{z^2 - 2.2067z + 0.2135}{(z-1)(z-0.0067)}$$

The charecteristic equation of the system:

$$1 + G(z) = 1 - \frac{K}{5}\cdot\frac{z^2 - 2.2067z + 0.2135}{(z-1)(z-0.0067)} = 0$$

$$(5\text{-}K)z^2 + (2.2067K - 5.0335)z + (0.0335 - 0.2135K) = 0$$

$$z = \frac{w+1}{w-1} \Rightarrow 0.9932w^2 + (9.993 - 1.573K)w + (10.067 - 2.4202K) = 0$$

$$0 < K < 4.16$$

$K < 6.35$

$K < 4.159$

2)

$$Q(s) = s^2 + 5s + K$$

s^2	1	K
s^1	5	0
s^0	K	

$\therefore$ For stable System all values of $K > 0$

Note that in example above[using modified Routh stability]

3.3.2 Jury Test

For continuous-time systems, the Routh–Hurwitz criterion offers a simple and convenient technique for determining the stability of low-ordered systems. However, since the stability boundary in the z-plane is different from that in the s-plane, the Routh–Hurwitz criterion cannot be directly applied to discrete-time systems if the system characteristic equation is expressed as a function of z. A stability criterion for discrete-time systems that is similar to the Routh–Hurwitz criterion and can be applied to the characteristic equation written as a function of z is the Jury stability test [2].

Jury's test will now be presented. Let the characteristic equation of a discrete-time system be expressed as

$$Q(z) = A_0 z^n + A_1 z^{n-1} + \cdots \ldots\ldots + A_n$$

Then form the array as shown in Table

$A_0 \quad A_1$ --------- A_n

$A_n \quad A_{n-1}$ -------- A_0

-- $\alpha_n = \dfrac{A_n}{A0}$

$B_0 \quad B_1$ ---------------- B_{n-1} $\qquad B_0 = A_0 - \alpha_n A_n$

$B_{n-1} \quad B_{n-2}$ ------ $B0$ $\qquad B_1 = A_1 - \alpha_n A_{n-1}$

$\vdots$

-- $\alpha_{n-1} = \dfrac{B_{n-1}}{B_0}$

C_0 ---------------- C_{n-K} $\qquad C_0 = B_0 - \alpha_{n-1} B_{n-1}$

C_{n-K} ---------------- C_0 $\qquad C_1 = B_1 - \alpha_{n-1} B_{n-2}$

$\vdots$

-- $\alpha_{n-2} = \dfrac{C_{n-k}}{C_0}$

The elements of 1st and 2nd rows are the cofficients of Q(z) in F/F and reverse order respectively .The 3rd row is obtanied by multiplying the 2nd row by (α_n) and subtracting this from the 1st row.The 4th row is the 3rd row written in reverse dirtection and this procedure is repeated until the last row has only one term.Where (n) is order of Q(z).

The necessary and sufficient conditions for the polynomial $Q(z)$ to have no roots outside or on the unit circle,

With A0 > 0 and

$$Q(1) > 0$$

$$(-1)^n Q(-1) > 0$$

A0 > and An

Example:

Consider Q(z)=$z^3 - 2z^2 + 1.5z - 0.4$

1 -2 1.5 -0.4

-0.4 1.5 -2 1

------------------------------------ $\propto 3 = \frac{-0.4}{1}$

3

α_n

$1 - (-0.4)(-0.4) = 0.84$

0.84 -1.4 0.7

0.7 -1.4 0.84

------------------------------ $\propto 2 = \frac{0.7}{0.84}$

0.2267 -.2333

-0.2333 0.2267

---------------------------- $\propto 1 = \frac{-0.2333}{0.2267}$

0.0445

Take the odd number of the 1st column (1, 0.84, 0.2267, 0.0445), since all these values are positive , Q(z) is stable , or because this coefficient's are all positive, it follows that all roots of Q(z) are inside the unit circle

1. Check the three conditions $Q(1) > 0$, $(-1)^n Q(-1) > 0$, and

requires no calculations. Stop if any of these conditions are not satisfied.

2. Construct the array, checking the conditions of

Each row is calculated stop if any condition is not satisfied.

3- $A0 > An$

Example:

Consider the c/cs equation is $T = 1$ sec.

$$Q(z) = z^2 + (0.368K - 1.368)z + (0.368 + 0.264K) = 0$$

The constraint $Q(1) > 0$ yields

$$1 + (0.368K - 1.368) + (0.368 + 0.264K) = 0.632K > 0 \Rightarrow K > 0$$

The constraint $(-1)^2 Q(-1) > 0$ yields

$$1 - 0.368K + 1.368 + 0.368 + 0.264K > 0 \Rightarrow K < \frac{2.736}{0.104} = 26.3$$

The constraint $A_n < A_0$ yields

$$0.368 + 0.264K < 1 \Rightarrow K < \frac{0.632}{0.264} = 2.39$$ see P96

Thus the system is stable for

$$0 < K < 2.39$$

The system is marginally stable for $K = 2.39$. For this value of K, the characteristic equation is

$$z^2 + (0.368K - 1.368)z + (0.368 + 0.264K)|_{K=2.39} = z^2 - 0.488z + 1 = 0$$

The roots of this equation are

$$z = 0.244 \pm j0.970 = 1\angle \pm 75.9° = 1\angle \pm 1.32 \text{ rad} = 1\angle \pm \omega T$$

Since $T = 1$ s, the system will oscillate at a frequency of 1.32 rad/s.

$\omega = 1.32$ rad/sec [This is the s-plane oscillation frequency]

$\omega_w = \frac{2}{T} \tan \frac{\omega T}{2} = 1.55$ {This is the w-plane oscillation freq.]

Example:

Find the range (K_p) for stable system [use Jury Test]

Consider the c/cs equation is

$$z^2 - (0.953 - 0.0952K_p)z + 0.905 - 0.0952K_p = 0$$

The constraint $Q(1) > 0$ yields

$$1 + 0.0952K_p - 0.953 + 0.905 - 0.0952K_p > 0$$

Thus this constraint is satisfied independent of K_p. The constraint $(-1)^2Q(-1) > 0$ yields

$$1 - 0.0952K_p + 0.953 + 0.905 - 0.0952K_p > 0$$

or since K_p is normally positive,

$$0 < K_p < 15.01$$

$K_p < 15$

The constraint $|A_n| < A_0$ yields

$$|0.905 - 0.0952K_p| < 1$$

or

$-1 < 0.905 - 0.0952K_p < 1$

$$K_p < 20.0$$

$K_p < 20.0$

$K_p > -0.997$

Homework

Use jury test for:-

1- $Q(z) = 2z^4 + 7z^3 + 10z^2 + 4z + 1$

2- $Q(z) = z^3 + z^2 + 0.5z + 0.25$

3- $Q(z) = z^3 - 1.8z^2 + 1.5z - 0.2$

4- $Q(z) = z^3 + 2z^2 + 1.9z + 0.8$

5- $Q(z) = z^3 + 5z^2 + 3z + 2$

6- $Q(z) = z^4 + 9z^3 + 3z^2 + 9z + 1$

7- $Q(z) = z^4 - 2z^3 + z^2 - 2z + 1$