Manipulation of Algebraic Functions

Before Adding some properties that are usefull in simplifying algebraic expressions, it is helpful to introduce some terminology that will make the discussion simpler.

<u>A literal</u> is the apperance of a variable or its complement. It is used as a measure of the complexity of an expression. For example, the expression $a\bar{b} + b\bar{c}d + \bar{a}d + \bar{e}$ contains <u>eight</u> literals.

<u>A Product term</u> is one or more literals connected by AND operators. In the above example, there are <u>four</u> product terms, $a\bar{b}$, $b\bar{c}d$, $\bar{a}d$, and $\bar{e}$.

<u>A Standrd Product term</u>, also called a minterm is a product term that includes each variable of the problem. For a function of four variables, w, x, y, and z,

$$\left.\begin{matrix}\bar{w}xy\bar{z}\\ wxyz\end{matrix}\right\}\text{standard Product terms}\qquad w\bar{y}z\ \}\ \text{not Standard Product term}$$

<u>A Sum of Products</u> expression (oftem abbreviated SOP) is one or more product terms connected by OR operators

$\bar{w}xy\bar{z} + w\bar{x}\bar{y}\bar{z} + w\bar{x}yz + wxyz$ (4 Product terms)

$x + \bar{w}y + wx\bar{y}z$ (3 Product terms)

$\bar{x} + \bar{y} + z$ (3 Product terms)

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A Canonical Sum or Sum of Standard Product terms is just a sum of products expression where all of the terms are standard product terms. The first example above is the only canonical sum.

Ex: Reduce the following Expression

$$\bar{x}y\bar{z} + \bar{x}yz + x\bar{y}\bar{z} + x\bar{y}z + xyz$$

sol^n

$$= (\bar{x}y\bar{z} + \bar{x}yz) + (x\bar{y}\bar{z} + x\bar{y}z) + xyz \qquad \text{P2a}$$

$$= \bar{x}y(\bar{z} + z) + x\bar{y}(\bar{z} + z) + xyz \qquad \text{P8a}$$

$$= \bar{x}y \cdot 1 + x\bar{y} \cdot 1 + xyz \qquad \text{P5a}$$

$$= \bar{x}y + x\bar{y} + xyz \qquad \text{P3b}$$

The last three steps can be combined into a single step. We can add a property

P9a. $ab + a\bar{b} = a$ P9b. $(a+b)(a+\bar{b}) = a$

Ex: Prove the above example equal to $\bar{x}y + x\bar{y} + xz$

$$\bar{x}y\bar{z} + \bar{x}yz + x\bar{y}\bar{z} + x\bar{y}z + xyz + x\bar{y}z$$

$$= (\bar{x}y\bar{z} + \bar{x}yz) + (x\bar{y}\bar{z} + x\bar{y}z) + (xyz + x\bar{y}z)$$

$$= \bar{x}y(\bar{z} + z) + x\bar{y}(\bar{z} + z) + xz(y + \bar{y})$$

$$= \bar{x}y \cdot 1 + x\bar{y} \cdot 1 + xz \cdot 1$$

$$= \bar{x}y + x\bar{y} + xz$$

Another Property

P10 a. $a + \bar{a}b = a + b$ P10 b. $a(\bar{a}+b) = ab$

Ex: Prove $\bar{a}\bar{b}\bar{c} + \bar{a}b\bar{c} + \bar{a}bc + a\bar{b}\bar{c} = \bar{b}\bar{c} + \bar{a}b$

Sol:

$\bar{a}\bar{c} + \bar{a}bc + a\bar{b}\bar{c}$ P9a

$\bar{a}\bar{c} + \bar{a}b + a\bar{b}\bar{c}$ P10a

$\bar{a}\bar{c} + \bar{a}b + \bar{b}\bar{c}$

A Sum term is one or more literals connected by OR operators.

A Standard Sum Term, also called a maxterm, is a sum term that includes each variable of the problem, either uncomplemented or complemented. For a function of four variables, w, x, y, and z:

$\left.\begin{array}{l} \bar{w}+x+y+\bar{z} \\ w+x+y+z \end{array}\right\}$ Standard Sum terms

$w+\bar{y}+z$ is not a standard Sum term

A Product of Sums expression (POS) is one or more sum terms connected by AND operators.

$(w+x)(w+y)$ 2 terms

$w(x+y)$ 2 terms

$w+x$ 1 term

w 1 term

P11a. $\overline{(a+b)} = \bar{a}.\bar{b}$

P11b. $\overline{(ab)} = \bar{a} + \bar{b}$

<u>CAUTION</u>: $\boxed{\begin{array}{l} \overline{ab} \neq \bar{a}\bar{b} \\ \overline{(a+b)} \neq \bar{a}+\bar{b} \end{array}}$

P12a. $a + ab = a$

P12b. $a(a+b) = a$

One more tool is useful in the algebraic simplification of switching functions. The operator consensus (indicated by the symbol ¢) is defined as follows:

For any two product terms where exactly one variable appears uncomplemented in one and complemented in the other the consensus is defined as the product of the remaining literals. If no such variable exists or if more than one such variable exists, then the consensus is undefined.

$a t_1 \text{ ¢ } \bar{a} t_2 = t_1 t_2$ (where t_1 and t_2 represent product terms)

examples:

$a\bar{b}c \text{ ¢ } \bar{a}d = \bar{b}cd$

$a\bar{b}c \text{ ¢ } \bar{a}cd = \bar{b}cd$

$ab\bar{c} \text{ ¢ } bc\bar{d} = ab\bar{d}$

$ab\bar{c} \text{ ¢ } b\bar{c}d =$ undefined − no such variable.

P13a. $a t_1 + \bar{a} t_2 + t_1 t_2 = a t_1 + \bar{a} t_2$

P13b. $(a+t_1)(\bar{a}+t_2)(t_1+t_2) = (a+t_1)(\bar{a}+t_2)$

P13a states that the consensus term is redundant and can be removed a sum of product expression.

P14a. $ab + \bar{a}c = (a+c)(\bar{a}+b)$

<u>A Canonical Product or Product of Standard Sum Terms</u> is just a product of sums expression where all of the terms are standard sum terms.

<u>Ex:</u> Simplify the expression in Product of sum form.

$$g = (\bar{w}+\bar{x}+y+\bar{z})(\bar{w}+\bar{x}+y+z)(w+\bar{x}+y+\bar{z})$$

<u>sol'n</u>

$$g = (\bar{w}+\bar{x}+y)(w+\bar{x}+y+\bar{z}) \qquad \text{P9b}$$

$$g = \bar{x}+y+\bar{w}(w+\bar{z}) \qquad \text{P8b}$$

$$= \bar{x}+y+\bar{w}\bar{z} \qquad \text{P10b}$$

$$= (\bar{x}+y+\bar{w})(\bar{x}+y+\bar{z}) \qquad \text{P8b}$$

<u>Canonical and Standard Forms</u>

If we have n variables, we can combine it to form 2^n minterms. The 2^n different minterms may be determined by a method similar to the one shown below for 3 variables.

			Minterm		Maxterms	
X	Y	Z	Term	Designation	Term	Designation
0	0	0	$\bar{x}\bar{y}\bar{z}$	m_0	$x+y+z$	M_0
0	0	1	$\bar{x}\bar{y}z$	m_1	$x+y+\bar{z}$	M_1
0	1	0	$\bar{x}y\bar{z}$	m_2	$x+\bar{y}+z$	M_2
0	1	1	$\bar{x}yz$	m_3	$x+\bar{y}+\bar{z}$	M_3
1	0	0	$x\bar{y}\bar{z}$	m_4	$\bar{x}+y+z$	M_4
1	0	1	$x\bar{y}z$	m_5	$\bar{x}+y+\bar{z}$	M_5
1	1	0	$xy\bar{z}$	m_6	$\bar{x}+\bar{y}+z$	M_6
1	1	1	xyz	m_7	$\bar{x}+\bar{y}+\bar{z}$	M_7

The binary numbers from 0 to 2^n-1 are listed under the n variables. Each minterm is obtained from an AND term of the n variables, with each variable being primed if the corresponding bit of the binary number is a 0 and unprimed if a 1. m_j is the symbol for each minterm, where j denotes the decimal equivalent of the binary number of the minterm designated.

In Similar fashion, Each maxterm is obtained from an OR term of the n variables, with each variable being unprimed if the corresponding bit is a 0 and primed if a 1. Note that each maxterm is the complement of its corresponding minterm, and vice versa.

A Boolean function can be expressed algebraically from a given truth table by forming a minterm for each combination of the variables that produces a 1 in the function, and then taking the OR of all those terms. For example, the function f_1 in the following table is determined by expressing the combinations 001, 100, and 111 as $\bar{X}\bar{Y}Z$, $X\bar{Y}\bar{Z}$, and XYZ, respectively. Since each one of these minterms results in $f_1 = 1$, we have

X	Y	Z	Function f_1	Function f_2
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

$f_1 = \bar{x}\bar{y}z + x\bar{y}\bar{z} + xyz = m_1 + m_4 + m_7$

Similarly, for f_2

$f_2 = \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz = m_3 + m_5 + m_6 + m_7$

The complement of f_1 is read as

$\bar{f}_1 = \bar{x}\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}yz + x\bar{y}z + xy\bar{z}$

If we take the complement of $\bar{f}_1$, we obtain the f_1

$f_1 = (x+y+z)(x+\bar{y}+z)(\bar{x}+y+\bar{z})(\bar{x}+\bar{y}+z)$

$(x+\bar{y}+\bar{z})$

$= M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$

Similarly, for the function f_2

$f_2 = (x+y+z)(x+y+\bar{z})(x+\bar{y}+z)(\bar{x}+y+z)$

$= M_0 M_1 M_2 M_4$

Ex: Express the Boolean function $F = A + \bar{B}C$ in a sum of minterms. ~~The~~

Sol^n: The first term A is missing two variables; therefore:

$A = A(B+\bar{B}) = AB + A\bar{B}$

This function is still missing one variable:

$A = AB(C+\bar{C}) + A\bar{B}(C+\bar{C})$

$= ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C}$

The second term $\bar{B}C$ is missing one variable:

$\bar{B}C = \bar{B}C(A+\bar{A}) = A\bar{B}C + \bar{A}\bar{B}C$

combining all terms, we have

$F = A + \bar{B}C$

$= ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C} + \bar{A}\bar{B}C + A\bar{B}C$

But $A\bar{B}C$ appears twice, and according to ~~theorem~~ P6a $(a+a=a)$, it is possible to remove one of them.

$F = \bar{A}\bar{B}C + A\bar{B}\bar{C} + A\bar{B}C + AB\bar{C} + ABC$

$= m_1 + m_4 + m_5 + m_6 + m_7$

It is sometimes convenient to express the Boolean function, when in its sum of minterms.

$F(A,B,C) = \sum(1,4,5,6,7)$

The summation symbol $\sum$ stands for the ORing of terms.
The numbers are the minterms of the function, the letters form a list of the variables.
The truth table below can be derived directly from the algebraic expression $(F = A + \bar{B}C)$.

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Ex: Express the Boolean function $F = xy + \bar{x}z$ in a product of maxterm form.

Sol^n: First convert the function into OR terms using the distributive law:

$$F = xy + \bar{x}z = (xy + \bar{x})(xy + z)$$
$$= (x + \bar{x})(y + \bar{x})(x + z)(y + z)$$
$$= (\bar{x} + y)(x + z)(y + z)$$

The function has three variables: x, y, and z. Each OR term is missing one variable; therefore:

$$\bar{x} + y = \bar{x} + y + z\bar{z} = (\bar{x} + y + z)(\bar{x} + y + \bar{z})$$
$$x + z = x + z + y\bar{y} = (x + y + z)(x + \bar{y} + z)$$
$$y + z = y + z + x\bar{x} = (x + y + z)(\bar{x} + y + z)$$

Combining all the terms and removing those that appear more than once, we finally obtain:

$$F = (x + y + z)(x + \bar{y} + z)(\bar{x} + y + z)(\bar{x} + y + \bar{z})$$
$$= M_0 M_2 M_4 M_5$$

A convenient way to express this function is as follows:

$$F(x, y, z) = \prod(0, 2, 4, 5)$$

The product symbol, $\prod$, denotes the ANDing of maxterms; the numbers are the maxterms of the function.