Lecture 2
Stability

Bounded-Input Bounded-Output (BIBO) Stability
Asymptotic Stability
Lyapunov Stability
Linear Approximation of a Nonlinear System

Bounded-Input Bounded-Output (BIBO) stablility

Definition: For any constant *N*, *M* >0

Any bounded input yields bounded output, i.e.

$$|u(t)| \le N < \infty \to |y(t)| \le M < \infty$$

For linear systems: $T(s) = \dfrac{p(s)}{q(s)} = C(sI - A)^{-1}B$

BIBO Stability ⇔ All the poles of the transfer function lie in the LHP.

$q(s) = 0$ ⇒ Solve for poles of the transfer function $T(s)$

⇓

Characteristic Equation

Asymptotic stablility

When $u(t)=0$, i. e. the system $\dot{x}=Ax$

$x(t)\rightarrow 0$ as $t\rightarrow\infty$

For linear systems:

$$\dot{x}=Ax+Bu$$
$$y=Cx$$

Asymptotically stable $\Leftrightarrow$ All the eigenvalues of the *A* matrix have negative real parts (i.e. in the LHP)

$$T(s)=\frac{p(s)}{q(s)}=C(sI-A)^{-1}B=\frac{C\,adj[sI-A]B}{|sI-A|}$$

$|sI-A|=0$ ➡ Solve for the eigenvalues for A matrix

Note: Asy. Stability is indepedent of B and C Matrix

Asy. Stability from Model Decomposition

Suppose that all the eigenvalues of A are distinct. $A\in R^{n\times n}$

Let v_i the eigenvector of matrix *A* with respect to eigenvalue λ_i

i.e. λ_i, satisfying $Av_i=\lambda_i v_i,\quad i=1,\cdots,n$

Coordinate Matrix $T=[v_1,\ v_2,\ \cdots,v_n]$

$$\Rightarrow\quad \dot{\xi}=T^{-1}AT\xi$$

$$\Rightarrow\quad \begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \vdots\\ \dot{\xi}_n\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0 & 0\cdots & 0\\ 0 & \lambda_2 & 0 & 0\\ \vdots & 0 & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_n\end{bmatrix}$$

$$\overline{A}=T^{-1}AT$$
$$\overline{B}=T^{-1}B$$
$$\overline{C}=CT$$

$$\dot{z}=T^{-1}ATz+T^{-1}Bu$$
$$y=CTz+Du$$

$$x(t)=T\zeta(t)=v_1e^{\lambda_1 t}\xi_1(0)+v_2e^{\lambda_2 t}\xi_2(0)+\cdots+v_ne^{\lambda_n t}\xi_n(0),\quad \xi(0)=T^{-1}x(0)$$

Hence, system Asy. Stable $\Leftrightarrow$ all the eigenvales of A at lie in the LHP

Asymptotic Stablility versus BIBO Stability

In the absence of pole-zero cancellations, transfer function poles are identical to the system eigenvalues. Hence BIBO stability is equivalent to asymptotical stability.

Conclusion: If the system is both controllable and observable, then

BIBO Stability $\Leftrightarrow$ Asymptotical Stability

Methods for Testing Stability

- Asymptotically stable
  - All the eigenvalues of *A* lie in the LHP
- BIBO stable
  - Routh-Hurwitz criterion
  - Root locus method
  - Nyquist criterion
  - ....etc.

Lyapunov Stablility

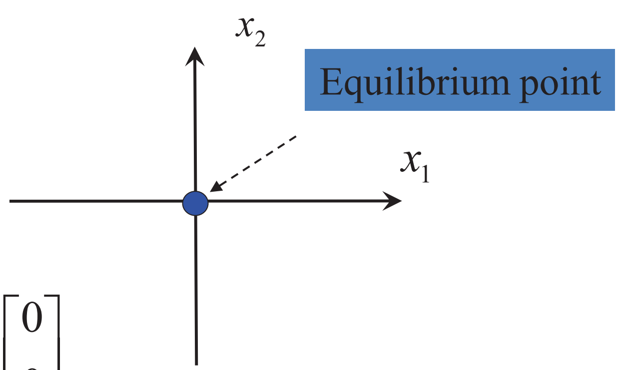
A state x_e of an autonomous system is called an equilibrium state, if starting at that state the system will not move from it in the absence of the forcing input.

In other words, consider the system $\dot{x} = f(x(t), u(t))$

equilibrium state x_e must satisfy $f(x_e, 0) = 0, \quad \forall t \geq t_0$

Example:

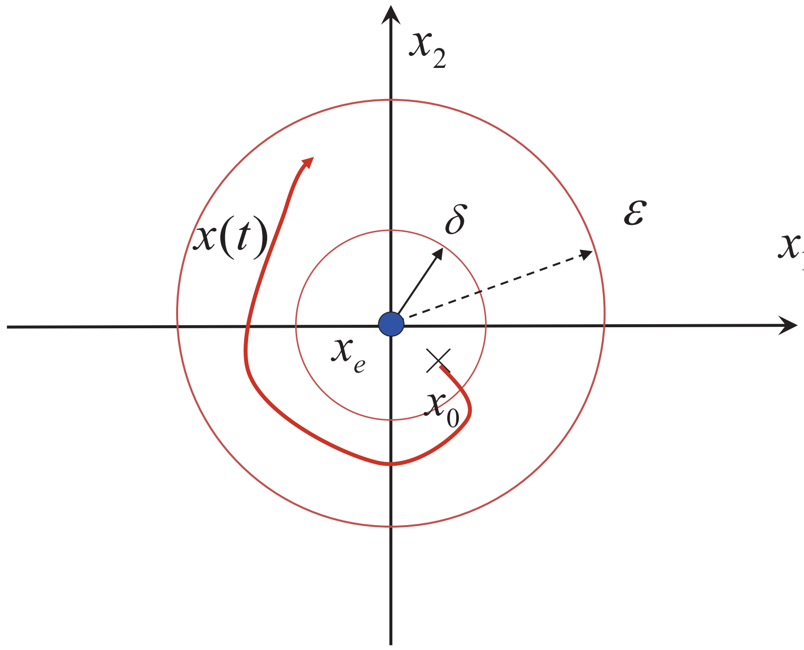
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$



Set $u(t) = 0$,

we get $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Definition: *An equilibrium state* x_e *of an autonomous system is* ***stable in the sense of Lyapunov*** *if for every* $\varepsilon > 0$ *, exist a* $\delta(\varepsilon) > 0$ *such that* $\|x_0 - x_e\| < \delta \Rightarrow \|x(t, x_0) - x_e\| < \varepsilon$ *for* $\forall t \geq t_0$

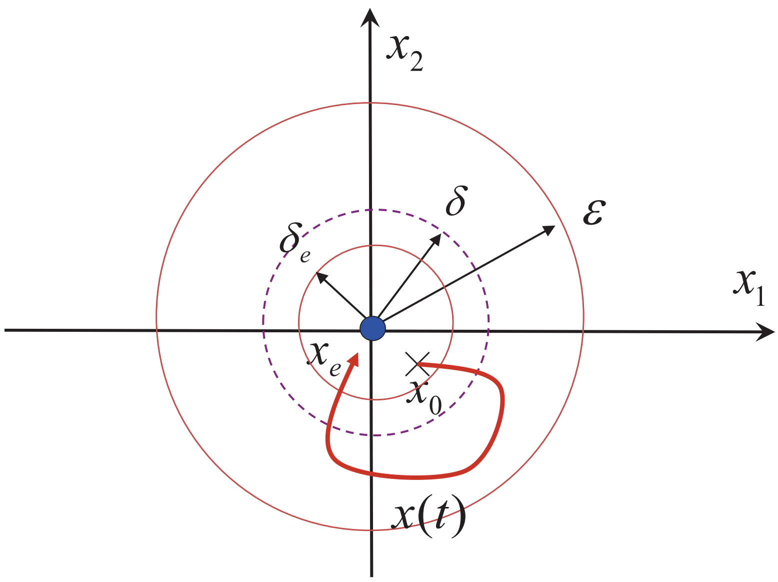


Definition: *An equilibrium state* x_e *of an autonomous system is* ***asymptotically stable*** *if*

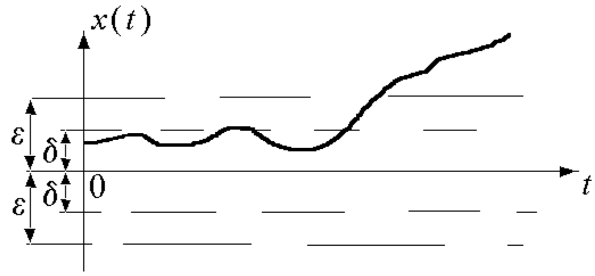
(i) it is stable

(ii) there exist a $\delta_e > 0$ *such that*

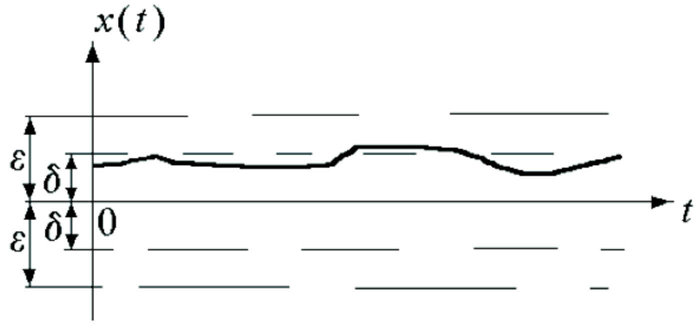
$$\|x_0 - x_e\| < \delta_e \Rightarrow \|x(t) - x_e\| \to 0, \text{ as } \quad t \to \infty$$



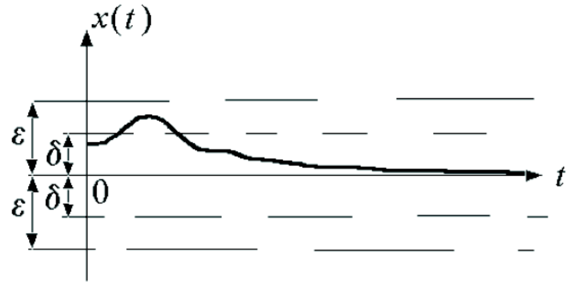
Instability in 1D



Stability Concept in 1D



Asymptotic Stability in 1D



Lyapunov Theorem

Consider the system $\dot{x} = f(x)$ (6.1)

Eq. State: $x_e = 0 \quad (\because f(0) = 0)$

A function $V(x)$ is called a Lapunov fuction $V(x)$ if

(1) $V(x) > 0, \forall x \neq 0$

(2) $V(0) = 0$ for $x = 0$

(3) $\dfrac{dV(x)}{dt} = \dfrac{dV(x)}{dx} f(x) \leq 0$

Then eq. state of the system (6.1) is stable.

Moreover, if the Lyapunov function satisfies

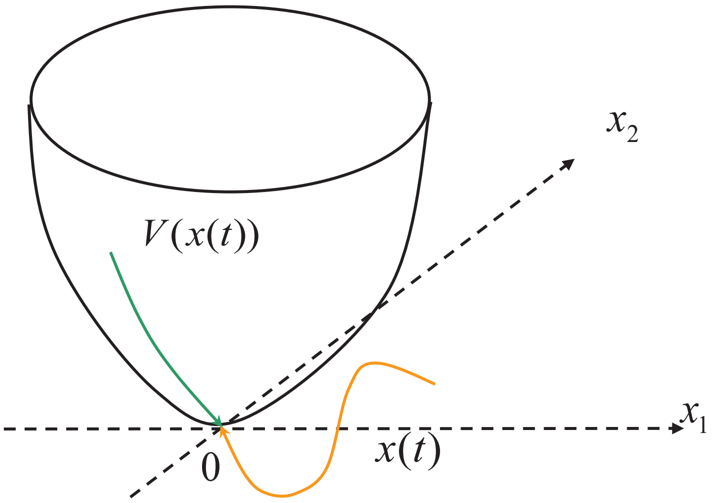
$$\frac{dV(x)}{dt} < 0, \forall x \neq 0 \quad \text{and} \quad \frac{dV(x)}{dt} = 0 \iff x = 0$$

Then eq. state of the system (6.1) is asy. stable.

Explanation of the Lyapunov Stability Theorem

1. The derivative of the Lyapunov function along the trajectory is negative.

2. The Lyapunov function may be consider as an energy function of the system.



Lyapunov's method for Linear system: $\dot{x} = Ax$ where $|A| \neq 0$

The eq. state $x = 0$ *is asymptotically stable.*

$\Leftrightarrow$

For any p.d. matrix Q *, there exists a p.d. solution of the Lyapunov equation*

$$A^T P + PA = -Q$$

Proof: Choose $V(x) = x^T P x$

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + PA) x \qquad \because A^T P + PA = -Q \\ &= -x^T Q x < 0, \text{ for } x \neq 0 \end{aligned}$$

Hence, the eq. state x=0 is asy. stable by Lapunov theorem.

Asymptotically stable in the large
(globally asymptotically stable)
(1) The system is asymptotically stable for all the initial states $x(t_0)$.
(2) The system has only one equilibrium state.
(3) For an LTI system, asymptotically stable and globally asymptotically stable are equivalent.

Lyapunov Theorem (Asy. Stability in the large)

If the Lyapunov function $V(x)$ further satisfies

$$\text{(i)}\ \forall \|x\| < \infty, V(x) < \infty$$

$$\text{(ii)}\ \|x\| \to \infty, V(x) \to \infty$$

Then, the (asy.) stability is global.

Sylvester's criterion

A symmetric $n \times n$ matrix Q is p.d. if and only if all its n leading principle minors are positive.

Definition

The i-th leading principle minor $|Q_i|$ $i = 1,2,3,\cdots,n$ of an $n \times n$ matrix Q is the determinant of the $i \times i$ matrix extracted from the upper left-hand corner of Q.

Example 6.1:

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \qquad |Q_1| = |q_{11}|$$

$$|Q_2| = \begin{vmatrix} q_{11} & q_{21} \\ q_{21} & q_{22} \end{vmatrix} \qquad |Q_3| = |Q|$$

Remark:

(1) $|Q_1|, |Q_2|, \cdots |Q_n|$ *are all negative* $\not\Leftrightarrow$ *Q is n.d.*

(2) All leading principle minors of –Q are positive $\Leftrightarrow$ *Q is n.d.*

Example:

$$V(x) = 2x_1^2 + 4x_1x_3 + 3x_1^2 + 6x_2x_3 + x_3^2$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$|Q_1| = 2 > 0$

$|Q_2| = 6 > 0$

$|Q_3| = -24 < 0$

Q is not p.d.

Example:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$$

Let $Q = I$, Assume $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$

Solve for $A^T P + PA = -I$

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$|p_{11}| = 3 > 0$ $\quad |P| = 5 > 0$ $\Rightarrow$ P is p.d.

System is asymptotically stable

The Lyapunov function is: $V(x) = x^T P x = \frac{1}{2}(3x_1^2 + 2x_1x_2 + 2x_2^2)$

$$\dot{V}(x) = -(x_1^2 + x_2^2)$$

Linear approximation of a function around an operating point x_e

Let $f(x)$ be a differentiable function.

Expanding the nonlinear equation into a *Taylor series* about the operation point x_e , we have

$$f(x) = f(x_e) + \frac{df(x)}{dx}\bigg|_{x=x_e} \frac{(x-x_e)}{1!} + \frac{d^2 f(x)}{dx^2}\bigg|_{x=x_e} \frac{(x-x_e)^2}{2!} + \cdots$$

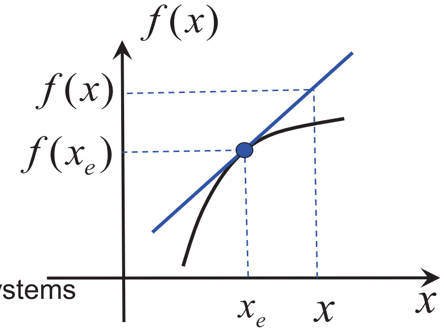
Neglecting all the high order terms, to yield

$$f(x) \approx f(x_e) + \frac{df(x)}{dx}\bigg|_{x=x_e} \frac{(x-x_e)}{1!} = f(x_e) + m\cdot(x-x_e)$$

$$f(x) - f(x_e) \approx m\cdot(x-x_e)$$

where

$$m = \frac{df(x)}{dx}\bigg|_{x=x_e}$$



Multi-dimensional Case:

Let x be a *n*-dimensional vector, i.e. $x \in R^n$

$$f(x_1,\cdots,x_n)$$
$$= f(x_{1e},\cdots,x_{ne}) + \frac{\partial f}{\partial x_1}\bigg|_{x=x_e}(x_1 - x_{1e}) + \frac{\partial f}{\partial x_2}\bigg|_{x=x_e}(x_2 - x_{2e}) + \cdots + \frac{\partial f}{\partial x_n}\bigg|_{x=x_e}(x_n - x_{ne})$$
$$= f(x_{1e},\cdots,x_{ne}) + \frac{\partial f}{\partial x}\bigg|_{x=x_e}(x - x_e), \quad \text{where } \frac{\partial f}{\partial x}\bigg|_{x=x_e} = \left[\frac{\partial f}{\partial x_1}\bigg|_{x=x_e}, \cdots, \frac{\partial f}{\partial x_n}\bigg|_{x=x_e}\right]$$

Let f be a *m*-dimensional vector function, $f(x): R^n \to R^m$
i.e.

$$f(x_1,\cdots,x_n) = \begin{bmatrix} f_1(x_1,\cdots,x_n) \\ f_2(x_1,\cdots,x_n) \\ \vdots \\ f_m(x_1,\cdots,x_n) \end{bmatrix}$$

Linear approximation of a function around an operating point x_e

Special Case: $n=m=2$

$$f(x) - f(x_e) \approx \left.\frac{\partial f}{\partial x}\right|_{x=x_e} (x - x_e) = A(x - x_e)$$

where $x = [x_1, x_2]^T$ and

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_e} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{x=x_e} & \left.\frac{\partial f_1}{\partial x_2}\right|_{x=x_e} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{x=x_e} & \left.\frac{\partial f_2}{\partial x_2}\right|_{x=x_e} \end{bmatrix} = A$$

Linear approximation of an autonomous nonlinear systems $\dot{x}(t) = f(x(t))$

Let x_e be an equilibrium state, from

$$\dot{x} = f(x(t)) \approx A(x - x_e)$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_e} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{x=x_e} & \left.\frac{\partial f_1}{\partial x_2}\right|_{x=x_e} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{x=x_e} & \left.\frac{\partial f_2}{\partial x_2}\right|_{x=x_e} \end{bmatrix}$$

The linearization of $\dot{x}(t) = f(x(t))$ around the equilibrium state x_e is

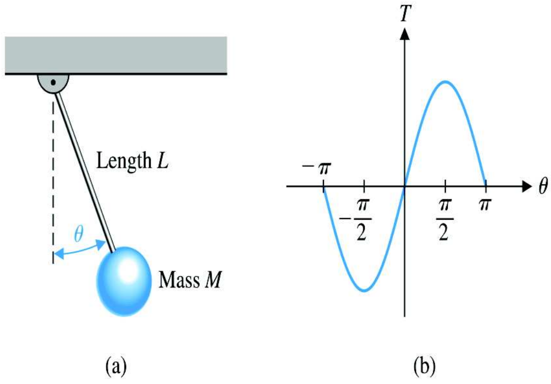
$$\dot{z} = Az \quad \text{where} \quad z = x - x_e \quad \text{and} \quad \dot{z} = \dot{x} - \dot{x}_e = \dot{x}$$

Example : Pendulum oscillator model

From Newton's Law we have

$$J\frac{d^2\theta}{dt^2} + MgL\sin\theta = 0$$

where J is the inertia.



(Reproduced from [1])

Define $x_1 = \theta, x_2 = \dot{\theta}$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{MgL}{J}\sin x_1 \end{bmatrix}$$

We can show that $x_e = 0$ is an equilibrium state.

Example (cont.):

Method 1: $f_1(x_2) = x_2 \Rightarrow f_1(x_2) - f(0) = (x_2 - 0) = z_2$

$$f_2(x_1) = \sin x_1$$

$$\Rightarrow f_2(x_1) - f_2(0) = \sin x_1 - \sin 0 \approx \left.\frac{d(\sin x_1)}{dx_1}\right|_{x_1=0}(x_1 - 0) = z_1$$

The linearization around the equilibrium state $x_e = 0$ is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{MgL}{J} z_1 \end{bmatrix}$$

where $z = x$ and $\dot{z} = \dot{x}$

Example (cont.):

Method 2: $f_1(x) = x_2$, $f_2(x) = -\dfrac{MgL}{J}\sin x_1$

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_e} = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial x_1}\right|_{x=x_e} & \left.\dfrac{\partial f_1}{\partial x_2}\right|_{x=x_e} \\ \left.\dfrac{\partial f_2}{\partial x_1}\right|_{x=x_e} & \left.\dfrac{\partial f_2}{\partial x_2}\right|_{x=x_e} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\dfrac{MgL}{J} & 0 \end{bmatrix} = A$$

The linearization around the equilibrium state $x_e = 0$ is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = Az = \begin{bmatrix} 0 & 1 \\ -\dfrac{MgL}{J} & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\dfrac{MgL}{J} z_1 \end{bmatrix}$$

where $z = x$ and $\dot{z} = \dot{x}$