Chapter 1

Vectors and Vector Spaces

1.1 Vector Spaces

Underlying every vector space (to be defined shortly) is a scalar field F. Examples of scalar fields are the real and the complex numbers

$$R := \text{real numbers}$$
$$\mathbb{C} := \text{complex numbers}.$$

These are the only fields we use here.

Definition 1.1.1. A *vector space* V is a collection of objects with a (vector) addition and scalar multiplication defined that closed under both operations and which in addition satisfies the following axioms:

(i) $(\alpha + \beta)x = \alpha x + \beta x$ for all $x \in V$ and $\alpha, \beta \in F$

(ii) $\alpha(\beta x) = (\alpha\beta)x$

(iii) $x + y = y + x$ for all $x, y \in V$

(iv) $x + (y + z) = (x + y) + z$ for all $x, y, z \in V$

(v) $\alpha(x + y) = \alpha x + \alpha y$

(vi) $\exists O \in V \ni 0 + x = x$; 0 is usually called the *origin*

(vii) $0x = 0$

(viii) $ex = x$ where e is the multiplicative unit in F.

The "closed" property mentioned above means that for all $\alpha, \beta \in F$ and $x, y \in V$

$$\alpha x + \beta y \in V$$

(i.e. you can't leave V using vector addition and scalar multiplication). Also, when we write for $\alpha, \beta \in F$ and $x \in V$

$$(\alpha + \beta)x$$

the '+' is in the field, whereas when we write $x + y$ for $x, y \in V$, the '+' is in the vector space. There is a multiple usage of this symbol.

Examples.

(1) $R_2 = \{(a_1, a_2) \mid a_1, a_2 \in R\}$ two dimensional space.

(2) $R_n = \{(a_1, a_2, \dots, a_n) \mid a_1, a_2, \dots, a_n \in R\}$, n dimensional space. $(a_1, a_2, \dots, a_n)$ is called an n-tuple.

(3) $\mathbb{C}_2$ and $\mathbb{C}_n$ respectively to R_2 and R_n where the underlying field is $\mathbb{C}$, the complex numbers.

(4) $P_n = \left\{ \sum\limits_{j=0}^{n} a_j x^j \mid a_0, a_1, \dots, a_n \in R \right\}$ is called the *polynomial space* of all polynomials of degree n. Note this includes not just the polynomials of exactly degree n but also those of lesser degree.

(5) $\ell_p = \{(a_i, \dots) \mid a_i \in R, \Sigma |a_i|^p < \infty\}$. This space is comprised of vectors in the form of infinite-tuples of numbers. Properly we would write

$$\ell_p(R) \quad \text{or} \quad \ell_p(\mathbb{C})$$

to designate the field.

(6) $T_N = \left\{ \sum\limits_{n=1}^{N} a_n \sin n\pi x \mid a_1, \dots, a_n \in R \right\}$, trigonometric polynomials.

Standard vectors in R^n

$$
\begin{aligned}
e_1 &= (1, 0, \dots, 0) \\
e_2 &= (0, 1, 0, \dots, 0) \\
e_3 &= (0, 0, 1, 0, \dots, 0) \\
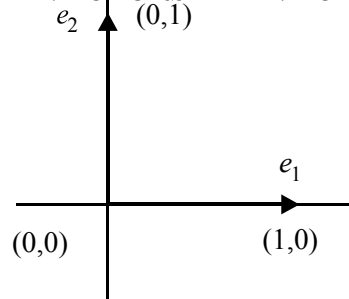&\vdots \\
e_n &= (0, 0, \dots, 0, 1)
\end{aligned}
$$

These are the *unit** vectors which point in the n *orthogonal** directions.

*Precise definitions will be given later.
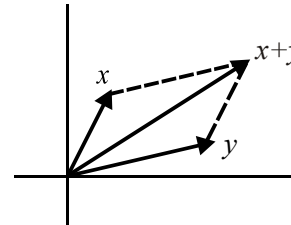
For R^2, the standard vectors are

$e_1 = (1,0)$

$e_2 = (0,1)$

Graphical representation of e_1 and e_2 in the usual two dimensional plane.

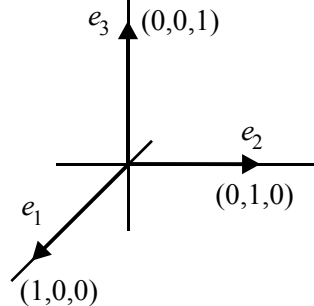Recall the usual vector addition in the plane uses the parallelogram rule

For R^3, the standard vectors are

$e_1 = (1,0,0)$

$e_2 = (0,1,0)$

$e_3 = (0,0,1)$

Graphical representation of e_1, e_2, and e_3 in the usual

Linear algebra is the mathematics of vector spaces and their subspaces. We will see that many questions about vector spaces can be reformulated as questions about arrays of numbers.

1.1.1   Subspaces

Let V be a vector space and $U \subset V$. We will call U a *subspace* of V if U is closed under vector addition, scalar multiplication and satisfies all of the vector space axioms. We also use the term *linear subspace* synonymously.

Examples. Proofs will be given later

$$\text{let} \quad V = R^3 = \{(a,b,c) \mid a,b,c \in R\} \tag{1.1}$$
$$U = \{(a,b,0) \mid a,b \in R\}.$$

Clearly $U \subset V$ and also U is a subspace of V.

$$\text{let} \quad v_1, v_2 \in R^3 \tag{1.2}$$
$$W = \{av_1 + bv_2 \mid a,b \in R\}$$
$$W \text{ is a subspace of } R^3.$$

In this case we say W is "spanned" by $\{v_1, v_2\}$. In general, let $S \subset V$, a vector space, have the form

$$S = \{v_1, v_2, \ldots, v_k\}.$$

The *span* of S is the set

$$U = \left\{ \sum_{j=1}^{k} a_j v_j \mid a_1, \ldots, a_k \in R \right\}.$$

We will use the notion

$$\mathfrak{S}(v_1, v_2, \ldots, v_k)$$

for the *span* of a set of vectors.

Definition 1.1.2. We say that

$$u = a_1 v_1 + \cdots + a_k v_k$$

is a *linear combination* of the vectors $v_1, v_2, \ldots, v_k$.

Theorem 1.1.1. *Let V be a vector space and $U \subset V$. If U is closed under vector addition and scalar multiplication, then U is a subspace of V.*

Proof. We remark that this result provides a "short cut" to proving that a particular subset of a vector space is in fact a subspace. The actual proof of this result is simple. To show (i), note that if $x \in U$ then $x \in V$ and so

$$(ab)x = ax + bx.$$

Now $ax, bx, ax + bx$ and $(a+b)x$ are *all* in U by the closure hypothesis. The equality is due to vector space properties of V. Thus (i) holds for U. Each of the other axioms is proved similarly. $\square$

A very important corollary follows about spans.

Corollary 1.1.1. *Let V be a vector space and $S = \{v_1, v_2, \dots, v_k\} \subset V$. Then $\mathfrak{S}(v_1, \dots, v_k)$ is a linear subspace of V.*

Proof. We merely observe that

$$\mathfrak{S}(v_1, \dots, v_k) = \left\{ \sum_1^k a_j v_j \mid a_1, \dots, a_k \in R \text{ or } C \right\}.$$

This means that the closure is built right into the definition of span. Thus, if

$$v = a_1 v_1 + \cdots + a_k v_k$$
$$w = b_1 v_1 + \cdots + b_k v_k$$

then both

$$v + w = (a_1 + b_1)v_1 + \cdots + (a_k + b_k)v_k$$

and

$$cv = ca_1 v + ca_2 v + \cdots + ca_k v$$

are in U. Thus U is closed under both operations; therefore U is a subspace of V. $\qquad\square$

Example 1.1.1. (Product spaces.) Let V and W be vector spaces defined over the same field. We define the new vector space $Z = V \times W$ by

$$Z = \{(v, w) \mid \quad u \in V, \ w \in W\}$$

We define vector addition as $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and scalar multiplication by $\alpha(v, w) = (\alpha v, \alpha w)$. With these operations, Z is a vector space, sometimes called the *product* of V and W.

Example 1.1.2. Using set-builder notation, define $V_{13} = \{(a, 0, b) \mid a, b, \in R\}$. Then U is a subspace of R_3. It can also be realized as the subspace of the standard vectors $e_1 = (1, 0, 0)$ and $e_3 = (0, 0, 1)$, that is to say $V_{13} = \mathfrak{S}(e_1, \ e_3)$.

Example 1.1.3. More subspaces of R_3. There are two other important methods to construct subspaces of R_3. Besides the set builder notation used above, we have just considered the method of spanning sets. For example, let $S = \{v_1, v_2\} \subset R_3$. Then $\mathfrak{S}(S)$ is a subspace of R_3. Similarly, if $T = \{v_1\} \subset R_3$. Then $\mathfrak{S}(T)$ is a subspace of R_3. A third way to construct subspaces is by using inner products. Let $x, w \in R_3$. Expressed in coordinates $x = (x_1, x_2, x_3)$ and $w = (w_1, w_2, w_3)$. Define the inrner product of x and w by $x \cdot w = x_1 w_1 + x_2 w_2 + x_3 w_3$. Then $U_w = \{x \in R_3 \mid x \cdot w = 0\}$ is a subpace of R_3. To prove this it is necessary to prove closure under vector addition and scalar multiplication. The latter is easy to see because the inner product is homogeneous in α, that is, $(\alpha x) \cdot w = \alpha x_1 w_1 + \alpha x_2 w_2 + \alpha x_3 w_3 = \alpha (x \cdot w)$. Therefore if $x \cdot w = 0$ so also is $(\alpha x) \cdot w$. The additivity is also straightforward. Let $x, y \in U$. Then the sum

$$
\begin{aligned}
(x + y) \cdot w &= (x_1 + y_1) w_1 + (x_2 + y_2) w_2 + (x_3 + y_3) w_3 \\
&= (x_1 w_1 + x_2 w_2 + x_3 w_3) + (y_1 w_1 + y_2 w_2 + y_3 w_3) \\
&= 0 + 0 = 0
\end{aligned}
$$

However, by choosing two vectors $v, w, \in R_3$ we can define $U_{v,w} = \{x \in R_3 \mid x \cdot y = 0 \text{ and } x \cdot w = 0\}$. Establishing $U_{v,w}$ is a subspace of R_3 is proved similarly. In fact, what is that both these sets of subspaces, those formed by spanning sets and those formed from the inner products are the same set of subspaces. For example, referring to the previous example, it follows that $V_{13} = \mathfrak{S}(e_1, \ e_3) = U_{e_2}$. Can you see how to correspond the others?

1.2 Linear independence and linear dependence

One of the most important problems in vector spaces is to determine if a given subspace is the span of a collection of vectors and if so, to determine a spanning set. Given the importance of spanning sets, we intend to examine the notion in more detail. In particular, we consider the concept of uniqueness of representation.

Let $S = \{v_1, \dots, v_k\} \subset V$, a vector space, and let $U = \mathfrak{S}(v_1, \dots, v_k)$ (or $\mathfrak{S}(S)$ for simpler notation). Certainly we know that any vector $v \in U$ has the representation

$$
v = a_1 v_1 + \cdots + a_k v_k
$$

for some set of scalars $a_1, \dots, a_k$. Is this representation unique? **Or**, can we

find another set of scalars $b_1, \ldots, b_k$ not all the same as $a_1, \ldots, a_k$ respectively for which

$$v = b_1 v_1 + \cdots + b_k v_k.$$

We need more information about S to answer this question either way.

Definition 1.2.1. Let $S = \{v_1, \ldots, v_k\} \subset V$, a vector space. We say that S is *linearly dependent* (l.d.) if there are scalars $a_1, \ldots, a_k$ not all zero for which

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0. \qquad (\star)$$

Otherwise we say S is *linearly independent* (l.i.).

Note. If we allow all the scalars to be zero we can always arrange for $(\star)$ to hold, making the concept vacuous.

Proposition 1.2.1. *If $S = \{v_1, \ldots, v_k\} \subset V$, a vector space, is linearly dependent, then one member of this set can be expressed as a linear combination of the others.*

Proof. We know that there are scalars $a_1, \ldots, a_k$ such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$$

Since not all of the coefficients are zero, we can solve for one of the vectors as a linear combination of the other vectors. $\qquad \square$

Remark 1.2.1. Actually we have shown that there is *no* vector with a unique representation in $\mathfrak{S}(S)$.

Corollary 1.2.1. *If $0 \in S = \{v_1, \ldots, v_k\}$, then S is linearly dependent.*

Proof. Trivial. $\qquad \square$

Corollary 1.2.2. *If $S = \{v_1, \ldots, v_k\}$ is linearly independent then every subset of S is linearly independent.*

1.3 Bases

The idea of a basis is that of finding a minimal generating set for a vector space. Through basis, unicity of representation and a number of other useful properties, both theoretical and computational, can be concluded. Thinking of the concept in operations research ideas, a basis will be a redundancy free and complete generating set for a vector space

Definition 1.3.1. Let V be a vector space and $S = \{v_1, \ldots, v_k\} \subset V$. We call S a *spanning set* for the subspace $U = \mathfrak{S}(S)$.

Suppose that V is a vector space, and $S = \{v_1, \ldots, v_k\}$ is a linearly independent spanning set for V. Then S is called a *basis* of V. Modify this definition correspondingly for subspaces.

Proposition 1.3.1. *If S is a basis of V, then every vector has a unique representation.*

Proof. Let $S = \{v_1, \ldots, v_k\}$ and $v \in V$. Then

$$v = a_1 v_1 + \cdots + a_k v_k$$

for some choice of scalars. If there is a second choice of scalars $b_1, \ldots, b_k$ not all the same, respectively, as $a_1, \ldots, a_k$, we have

$$v = b_1 v_1 + \cdots + b_k v_k$$

and

$$0 = (a_1 - b_1)v_1 + \cdots + (a_k - b_k)v_k.$$

Since not all of the differences $a_1 - b_1, \ldots, a_k - b_k$ are zero we must have that S is linearly dependent. This is a contradiction to our hypothesis, and the result is proved. $\square$

Example. Let $V = R^3$ and $S = \{e_1, e_2, e_3\}$. Then S is a basis for V.

Proof. Clearly V is spanned by S. Now suppose that

$$0 = a_1 e_1 + a_2 e_2 + a_3 e_3$$

or

$$(0, 0, 0) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1)$$
$$= (a_1, a_2, a_3).$$

Hence $a_1 = a_2 = a_3 = 0$. Thus the set $\{e_1, e_2, e_3\}$ is linearly independent. $\square$

Remark 1.3.1. Note how we resolved the linearly dependent/linearly independent issue by converting a vector problem to a numbers problem. This is at the heart of linear algebra.

Exercise. Let $S = \{v_1, v_2\} = \{(1,0,1), (1,-1,0)\} \subset R^3$. Show that S is linearly independent and therefore a basis of $\mathfrak{S}(S)$.

1.4   Extension to a basis

In this section, we show that given a linearly independent set of vectors from a vector space with a finite spanning set, it is possible add to this set more vectors until it becomes a basis. Thus any set of linearly independent vectors can be a part (subset) of a basis.

Theorem 1.4.1 (Extension to a basis). *Assume that the given vector space V has a finite spanning set S_1, i.e. $V = \mathfrak{S}(S_1)$. Let $S_0 = \{x_1, \ldots, x_\ell\}$ be a linearly independent subset of V so that $\mathfrak{S}(S_0) \subsetneqq V$. Then, there is a subset S_1' of S_1, such that $S_0 \cup S'$ is a basis for V.*

Proof. Our intention is to add vectors to S_0 keeping it linearly independent and eventually becoming a basis. There are a couple of steps.
Steps.

1. Since $\mathfrak{S}(S_1) \supsetneqq \mathfrak{S}(S_0)$, there is a vector $y_1 \in S_1$ such that $S_{0,1} = \{S_0, y_1\}$ is linearly independent and thus $\mathfrak{S}(S_{0,1}) \supsetneqq \mathfrak{S}(S_0)$.

2. Continue this process generating sets

$$S_{0,1} = \{S_0, y_1\}$$
$$S_{0,2} = \{S_{0,1}, y_2\}$$
$$\vdots$$
$$S_{0,j} = \{S_{0,j-1}, y_{j-1}\}$$
$$\vdots$$

   At each step $S_{0,1}, S_{0,2}, \ldots$ are linearly independent sets. Since S_1 is finite we must eventually have that

$$\mathfrak{S}(S_{0,m}) = \mathfrak{S}(S_1) = V.$$

3. Since $S_{0,m}$ is linearly independent and spans V, it must be a basis.

$\square$

Remark 1.4.1. In the proof it was important to begin with *any* spanning set for V and to extract vectors from it as we did. Assuming merely that there exists a finite spanning set and extracting vectors directly from V leads to a problem of *terminus*. That is, when can we say that the new linearly independent set being generated in Step 2 above is a spanning set for V? What we would need is a theorem that says something to the effect that if V has a finite basis, then every linearly independent set having the same number of vectors is also a basis. This result is the content of the next section. However, to prove it we need the Extension theorem.

Corollary 1.4.1. *If $S = \{v_1, \dots, v_k\}$ is linearly dependent then the representation of vectors in $\mathfrak{S}(S)$ is not unique.*

Proof. We know there are scalars $a_1, \dots, a_k$ not all zero, for which

$$a_1 v_1 + \cdots + a_k v_k = 0$$

let $v \in \mathfrak{S}(S)$ have the representation

$$v = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k.$$

Then we also have the representation

$$v = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \cdots + (a_k + b_k)v_k$$

establishing the result. $\square$

Remark 1.4.2. The upshot of this construction is that we can *always* construct a basis from a spanning set. In actual practice this process may be quite difficult to carry out. In fact, we will spend some time achieving this goal. The main tool will be matrix theory.

1.5 Dimension

One of the most remarkable features of vector spaces is the notion of dimension. We need one simple result that makes this happen, the basis theorem.

Theorem 1.5.1 (Basis Theorem). *Let $S = \{v_1, \dots, v_k\} \subset V$ be a basis for V. Then every basis of V has k elements.*

Proof. We proceed by induction. Suppose $S = \{v_1\}$ and $T = \{w_1, w_2\}$ are both bases of V. Then since S is a basis

$$w_1 = \alpha_1 v_1 \quad w_2 = \alpha_2 v_1$$

and therefore

$$\frac{1}{\alpha_1} w_1 - \frac{1}{\alpha_2} w_2 = 0$$

which implies that T is linearly dependent (we tacitly assumed that both α_1 and α_2 were nonzero. Why can we do this?)

The next step is to assume the result holds for bases having up to k elements. Suppose that $S = \{v_1, \ldots, v_{k+1}\}$ and $T = \{w_1, \ldots, w_{k+2}\}$ are both bases of V. Now consider $S' = \{v_1, \ldots, v_k\}$. We know that $\mathfrak{S}(S') \subsetneq \mathfrak{S}(S) = \mathfrak{S}(T) = V$. By our extension of bases result, there is a vector $w_{\ell_1} \in T$ such that

$$S'_1 = \{v_1, \ldots, v_k, w_{\ell_1}\}$$

is linearly independent and $\mathfrak{S}(S'_1) \subset \mathfrak{S}(S) = V$. If $\mathfrak{S}(S'_1) \subsetneq V$, our extension result applies again to give a vector v_{ℓ_1} such that

$$S'_{11} = \{v_1, \ldots, v_k, w_{\ell_1}, v_{\ell_1}\}$$

is linearly independent The only possible selection is $v_{\ell_1} = v_{k+1}$. But in this case w_{ℓ_i} will depend on $v_1, \ldots, v_k, v_{k+1}$, and that is a contradiction. Hence $\mathfrak{S}(v_1, \ldots, v_k, w_{\ell_1}) = V$.

The next step is to remove the vector v_k from S'_1 and apply the extension to conclude that the span of the set

$$S'_2 = \{v_1, \ldots, v_{k-1}, w_{\ell_1}, w_{\ell_2}\}$$

is V. We continue in this way eventually concluding that

$$S'_{k+1} = \{w_{\ell_1}, w_{\ell_2}, \ldots, w_{\ell_{k+1}}\}$$

has span V. But $S'_{k+1} \subsetneq T$, whence T is linearly dependent. $\qquad\square$

Proposition 1.5.1 (Reduced spanning sets).   *(a) Suppose that $S = \{v_1, \ldots, v_k\}$ spans V and v_j depends (linearly) on*

$$S_j = \{v_1, \ldots, v_{j-1}, v_{j+1} \ldots v_k\}.$$

Then S_j also spans V.

(b) *If at least one vector in S is nonzero (that is $V \neq \{0\}$, the smallest vector space), then there is a subset $S_0 \subset S$ that is linearly independent and spans V.*

Proof. (Left to reader.) □

Definition 1.5.1. The *dimension* of a vector space V is the (unique) number of vectors in a basis of V. We write $\dim(V)$ for the dimension.

Remark 1.5.1. This definition make sense possible only because of our basis theorem from which we are assured all every linearly independent spanning sets of V, that is all bases, have the same number of elements.

Examples.

(1) $\dim(R_n) = n$,

(2) $\dim(P_n) = n + 1$.

Exercise. Let $M = $ all rectangular arrays of two rows and three columns with real entries. Find a basis for M, and find the dimension of M. Note

$$M = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \,\middle|\, a, b, c, d, e, f \in R \right\}$$

Example 1.5.1. $P_n = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0\}$ is the vector space of polynomials of degree n. We claim that the powers, $x^0 = 1$, $x, x^2, \ldots, x^n$ are linearly independent, and since

$$P_n = \mathfrak{S}(1, x, \ldots, x^n)$$

they form a basis of P_n.

Proof. There are several ways we can prove this fact. Here is the most direct and it requires essentially no machinery. Suppose they are linearly dependent, which means that there are coefficients $a_0, a_1, \ldots, a_n$ so that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \qquad (\star)$$

the function. (This functional view is critically important because every polynomial has roots.) There must be a coefficient *which is nonzero* and which corresponds to the highest power. Let us assume that $a_n \neq 0$, for convenience, and with no loss in generality.

Solve for x^n to get

$$x^n = -\frac{a_{n-1}}{a_n}x^{n-1} + \cdots + -\frac{a_1}{a_n}x - \frac{a_0}{a_n} \qquad (\star\star)$$

Now compute the ratio of this expression divided by x^n on both sides, and let $x \to \infty$. The left side of course will be 1. Again for convenience we take $n = 2$. So, condensing terms we will have

$$\frac{b_1 x + b_0}{x^2} = b_1\left(\frac{1}{x}\right) + b_0\left(\frac{1}{x^2}\right) = 1$$

where $b_j = -a_j/a_2$. But as $x \to \infty$ the expression $b_1\left(\frac{1}{x}\right) + b_0\left(\frac{1}{x^2}\right) \to 0$. This is a contradiction. It cannot be that the functions 1, x, and x^2 are linearly dependent.

In the general case for n we have

$$b_{n-1}\left(\frac{1}{x}\right) + b_{n-2}\left(\frac{1}{x^2}\right) + \cdots + b_0\left(\frac{1}{x^n}\right) = 1,$$

where $b_j = -a_j/a_n$. Apply the same limiting argument to obtain the contradiction. Thus

$$T = \{1, x, \dots, x^n\}$$

is a basis of P_n.

A calculus proof is available. It is also based on the fact that if the powers are linearly independent and $(\star)$ holds, then we can assume that the same relation $(\star\star)$ is true. Now take the nth derivative of both sides. We obtain

$$n! = 0$$

a contraction, and the result if proved

Finally, one more technique used to prove this result is by using the Fundamental Theorem of Algebra.

Theorem 1.5.2. *Every polynomial $(\star)$ of exactly n^{th} degree (i.e. with $a_n \neq 0$) has exactly n roots counted with multiplicity (i.e. if $q(x) = q_n x^n + q_{n-1}x^{n-1} + \cdots + q_1 x + q_0 \in P_n(\mathbb{C})$, $q_n \neq 0$ then the number of solutions of $q(x) = 0$ is exactly n).*

From $(\star)$ above we have an nth degree polynomial that is zero for *every* x. Thus the polynomial is zero, and this means *all* the coefficients are zero. This is a contradiction to the hypothesis, and therefore the theorem is proved. $\qquad\qquad\square$

Remark 1.5.2.

$$P_0 \subsetneqq P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n \subsetneqq \cdots .$$

On the other hand this is not true for the Euclidean spaces $R_1, R_2, \ldots$. However, we may say that there is a subspace of R_3 which is "like" R_2 in every possible way. Do you see this? We have

$$R_2 = \{(a, b) \mid a, b \in R\}$$
$$R_3 = \{(a, b, c) \mid a, b, c \in R\}.$$

No element in R_2, an ordered *pair*, can be in R_3, a set of ordered triples. However

$$U = \{(a, b, 0) \mid a, b \in R\}$$

is "like" R_2 is just about every way. Later on we will give a precise mathematical meaning to this comparison.

Example 1.5.2. Find a basis for the subspace V_0 of R_3 of all solutions to

$$x_1 + x_2 + x_3 = 0 \qquad (\star)$$

where $x = (x_1, x_2, x_3) \in R^3$.

Solution. First show that the set $V_0 = \{(x_1, x_2, x_3) \in R_3 \mid x_1 + x_2 + x_3 = 0\}$ is in fact a subspace of R_3. Clearly if $x = (x_1, x_2, x_3) \in V_0$ and $y = (y_1, y_2, y_3) \in V_0$ then $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in V_0$, proving closure under vector addition. Similarly V_0 is closed under scalar multiplication. Next, we seek a collection of vectors $v_1, v_2, \ldots, v_k \in V_0$ so that $\mathfrak{S}(v_1, \ldots, v_k) = V_0$. Let $x_3 = \alpha$ and $x_2 = \beta$ be free parameters. Then

$$x_1 = -(\alpha + \beta).$$

Hence all solutions of $(\star)$ have the form

$$x = (-(\alpha + \beta), \beta, \alpha)$$
$$x = \alpha(-1, 0, 1) + \beta(-1, 1, 0).$$

Obviously the vectors $v_1 = (-1, 0, 1)$ and $v_2 = (-1, 1, 0)$ are linearly independent, and x is expressed as being in the span of them. So, $V_0 = \mathfrak{S}(v_1, v_2)$. V_0 has dimension 2.

Theorem 1.5.3 (Uniqueness). *Let* $S = \{v_1, \dots, v_k\}$ *be a basis of* V. *Then each vector* $v \in V$ *has a unique representation with respect to* S.

Proof. Since $\mathfrak{S}(S) = V$ we have that

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

for some coefficients $a_1, a_2, \dots, a_k$ in the given field. (This is the representation of v with respect to S.) If it is *not* unique there is another

$$v = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k.$$

So, subtracting we have

$$(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_k - b_k)v_k = 0$$

where the differences $a_j - b_j$ are not all zero. This implies that S is a linearly dependent set. $\square$

Theorem 1.5.4. *Suppose that* $S = \{v_1, \dots, v_k\}$ *is a basis of the vector space* V. *Suppose that* $T = \{w_1, \dots, w_m\}$ *is a linearly independent subset of* V. *Then* $m \leq k$.

Proof. We know that S is a linearly independent spanning set. This means that every linearly independent set of k vectors is also a spanning set. Therefore, $m > k$ renders a contradiction as $T_0 = \{w_1, \dots, w_k\}$ is a spanning set and $w_{k+1} \in \mathfrak{S}(T_0)$. $\square$

Definition 1.5.2. If A is any set we define

$$|A| := \text{cardinality of } A,$$

that is to say $|A|$ is the number of elements of A.

Example 1.5.3. Let $T = \{1, x, x^2, x^3\}$. Then $|T| = 4$.

Theorem 1.5.5. *Both* R_k *and* $\mathbb{C}_k$ *are* k-*dimensional and* $S_k = \{e_1, e_2, \dots, e_k\}$ *is a basis of both.*

Proof. It is easy to see that $e_1, \dots, e_k$ are linearly independent, and any vector x in R_k has the form

$$x = a_1 e_1 + a_2 e_2 + \cdots + a_k e_k$$

for $a_1, \dots, a_k \in R$. Thus S_k is a linearly independent spanning set and hence a basis of R_k. $\square$

Question: What single change to the proof above gives the theorem for C_k?

The following results follow easily from previous results.

Theorem 1.5.6. *Let V be a k-dimensional vector space.*

(i) *Every set T with $|T| > k$ is linearly dependent.*

(ii) *If $D = \{v_1, \dots, v_j\}$ is linearly independent and $j < k$, then there are vectors $v_{\ell_1}, \dots, v_{\ell_{k-j}} \in V$ such that*

$$D \cup \{v_{\ell_1}, \dots, v_{\ell_{k-j}}\}$$

is a basis of V.

(iii) *If $D \subset V$, $|D| = k$, and D is either a spanning set for V or linearly independent, then D is a basis for V.*

1.6 Norms

Norms are a way of putting a measure of distance on vector spaces. The purpose is for the refined analysis of vector spaces from the viewpoint of many applications. It is also to all the comparison of various vectors on the basis of their length. Ultimately, we wish to discuss vector spaces as representatives of points. Naturally, we are all accustomed to the "shortest distance" distance from the Pythagorean theorem. This is an example of a norm, but we shall consider them as real valued functions with very special properties.

Definition 1.6.1. *Norms* on vector spaces over $\mathbb{C}$, or R. Let V be a vector space and suppose that $\| \cdot \|: V \to R^+$ is a function from V to the nonnegative reals for which

(i) $\|x\| \geq 0$ for all $x \in V$ and
     $\|x\| = 0$ if and only if $x = 0$
(ii) $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{C}, R$ and $x \in V$
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$ "The Triangle inequality".

Then $\| \cdot \|$ is called a *norm* on V. The second condition is often termed the (positive) homogeneity property.

Remark 1.6.1. The notation is a substitute function notation. The expression $\| \cdot \|$, without the vector, is just the way a norm is expressed.

Examples. Let $V = R_n$ (or $\mathbb{C}_n$). Define for $x = (x_1, \dots, x_n)$

(i) $\|x\|_2 = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{1/2}$ Euclidean norm

(ii) $\|x\|_1 = (|x_1| + |x_2| + \cdots + |x_n|)$ ℓ_1 norm

(iii) $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$ ℓ_∞ norm

Norm (ii) is read as: ell one norm. Norm (iii) is read as: ell infinity norm.

Proof that (ii) is a norm. Clearly (i) holds. Next

$$\|\alpha x\|_1 = (|\alpha x_1| + |\alpha x_2| + \cdots + |\alpha x_n|)$$
$$= (|\alpha||x_1| + |\alpha||x_2| + \cdots + |\alpha||x_n|)$$
$$= |\alpha|(|x_1| + |x_2| + \cdots + |x_n|) = |\alpha| \|x\|_1$$

which is what we needed to prove. Also,

$$\|x + y\|_1 = (|x_1 + y_1| + |x_2 + y_2| + \cdots + |x_n + y_n|)$$
$$\le (|x_1| + |y_1| + |x_2| + |y_2| + \cdots + |x_n| + |y_n|)$$
$$= (|x_1| + |x_2| + \cdots + |x_n|) + (|y_1| + |y_2| + \cdots + |y_n|)$$
$$= \|x\|_1 + \|y\|_1.$$

Here we used the fact that $|\alpha + \beta| \le |\alpha| + |\beta|$ for numbers.

To prove that (i) is a norm we need a very famous inequality.

Lemma 1.6.1 (Cauchy–Schwartz). *Given that* $a_1, \dots, a_n$ *and* $b_1, \dots, b_n$ *are in* $\mathbb{C}$*. Then*

$$\sum_1^n |a_i b_i| \le \left(\sum_1^n a_i^2 \right)^{1/2} \left(\sum_1^n b_i^2 \right)^{1/2}. \tag{$\star$}$$

Proof. We consider for the variable t

$$\Sigma(a_i + tb_i)^2 = \Sigma a_i^2 + 2t\Sigma a_i b_i + t^2 \Sigma b_i^2.$$

Note that $(\star)$ is obvious if $\sum_1^n a_i b_i = 0$. If not take

$$t = -\frac{\displaystyle\sum_1^n a_i^2}{\displaystyle\sum_1^n a_i b_i}.$$

Then

$$\Sigma(a_i + tb_i)^2 = \Sigma a_i^2 - 2\frac{\Sigma a_i^2}{\Sigma a_i b_i}\Sigma a_i b_i + \frac{\left(\Sigma a_i^2\right)^2}{\left(\Sigma a_i b_i\right)^2}\Sigma b_i^2$$

$$= -\Sigma a_i^2 + \frac{\left(\Sigma a_i^2\right)^2 \Sigma b_i^2}{\left(\Sigma a_i b_i\right)^2}$$

$$= \left(\Sigma a_i^2\right)\left(-1 + \frac{\Sigma a_i^2 \Sigma b_i^2}{\left(\Sigma a_i b_i\right)^2}\right).$$

Since the left side is ≥ 0 and since $\Sigma a_i^2 \geq 0$, we must have that

$$\left(-1 + \frac{\Sigma a_i^2 \Sigma b_i^2}{\left(\Sigma a_i b_i\right)^2}\right) \geq 0.$$

Solving this inequality we have

$$\left(\Sigma a_i b_i\right)^2 \leq \Sigma a_i^2 \Sigma b_i^2.$$

Now that square roots to get the result. $\qquad\square$

To prove that (i) is a norm, we note that conditions (i) and (ii) are straightforward. The truth of condition (iii) is a consequence of another famous result.

Theorem 1.6.1 (Minkowski). $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$.

Proof.

$$\Sigma(a_i + b_i)^2 = \Sigma a_i^2 + 2\Sigma a_i b_i + \Sigma b_i^2$$

$$\leq \Sigma a_i^2 + 2\left(\Sigma a_i^2\right)^{1/2}\left(\Sigma b_i^2\right)^{1/2} + \Sigma b_i^2$$

$$= \left(\left(\Sigma a_i^2\right)^{1/2} + \left(\Sigma b_i^2\right)^{1/2}\right)^2.$$

Taking square roots gives the result. $\qquad\square$

Continuity and Equivalence of Norms

Lemma 1.6.2. *Every vector norm on $\mathbb{C}_n$ is continuous in the vector components.*

Proof. Let $x \in \mathbb{C}_n$ and $\|\cdot\|$ some norm on $\mathbb{C}_n$. We need to show that if the vector $\delta \to 0$, in components, then $\|x + \delta\| \to \|x\|$. First, by the triangle inequality

$$
\begin{aligned}
\|x + \delta\| &\leq \|x\| + \|\delta\| \quad \text{or} \\
\|x + \delta\| - \|x\| &\leq \|\delta\|
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\|x\| &\leq \|x + \delta - \delta\| \\
&\leq \|x + \delta\| + \|\delta\| \quad \text{or} \\
-\|\delta\| &\leq \|x + \delta\| - \|x\|
\end{aligned}
$$

Therefore

$$
|\|x + \delta\| - \|x\|| \leq \|\delta\|
$$

Now expressing δ in components and standard bases vectors, we write $\delta = \delta_1 e_1 + \cdots + \delta_n e_n$ and

$$
\begin{aligned}
\|\delta\| &\leq |\delta_1| \, \|e_1\| + \cdots + |\delta_n| \, \|e_n\| \\
&\leq \max_{1 \leq i \leq n} |\delta_i| \; (\|e_1\| + \cdots + \|e_n\|) \\
&\leq M \max_{1 \leq i \leq n} |\delta_i|
\end{aligned}
$$

where $M = \|e_1\| + \cdots + \|e_n\|$. We know that if $\delta \to 0$ in components, then $\max_{1 \leq i \leq n} |\delta_i| \to 0$. Therefore $|\|x + \delta\| - \|x\|| \to 0$, as well. $\qquad\square$

Definition 1.6.2. Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two vector norms on $\mathbb{C}_n$. We say that these norms are *equivalent* if there are postive constants m, M such that for all $x \in \mathbb{C}_n$

$$
m \, \|x\|_a \leq \|x\|_b \leq M \, \|x\|_a
$$

The remarkable fact about vector norms on $\mathbb{C}_n$ is that they are *all* equivalent. The only tool we need to prove this is the following result: *Every continuous function on a compact set of $\mathbb{C}_n$ assumes its maximum (and minimum) on that set.* The term "compact" refers to a particular kind of set K, one which is both bounded and closed. Bounded means that for $\max_{x \in K} \|x\| \leq B < \infty$ and closed means that if $\lim_{n \to \infty} x_n = x$, then $x \in K$.

Theorem 1.6.2. *All norms on $\mathbb{C}_n$ are equivalent.*

Proof. Since equivalence of norms is an equivalence condition, we can take one of the norms to be the infinity norm $\|\cdot\|_\infty$. Denote the other norm by $\|\cdot\|$. Now define $K = \{\, x \mid \|x\|_\infty = 1 \,\}$. This set, called the unit ball in the infinity norm, is compact. Now we define

$$
\begin{aligned}
m &= \min_{x \in K} \|x\| \quad \text{and} \\
M &= \max_{x \in K} \|x\|
\end{aligned}
$$

Since $\|\cdot\|$ is a continuous function on K (from the lemma above) and since K is compact, we have that both the minimum and maximum are attained by specific vectors in K. Since these vectors are nonzero (they're in K) and because $\|x\|$ is positive for nonzero vectors, it must follow that $0 < m < M < \infty$. Hence, on K, the relation

$$
m \|x\| \leq \|x\|_\infty \leq M \|x\|
$$

holds true. For any vector $x \in \mathbb{C}_n$ we can write $x = \left(\dfrac{x}{\|x\|_\infty} \right) \|x\|_\infty$ and $\dfrac{x}{\|x\|_\infty} \in K$. Thus

$$
\begin{aligned}
m \left\| \frac{x}{\|x\|_\infty} \right\| &\leq \left\| \frac{x}{\|x\|_\infty} \right\|_\infty \leq M \left\| \frac{x}{\|x\|_\infty} \right\| \\
m \left\| \frac{x}{\|x\|_\infty} \right\| \|x\|_\infty &\leq \left\| \frac{x}{\|x\|_\infty} \right\|_\infty \|x\|_\infty \leq M \left\| \frac{x}{\|x\|_\infty} \right\| \|x\|_\infty \\
m \|x\| &\leq \|x\|_\infty \leq M \|x\|
\end{aligned}
$$

and the theorem is proved. $\qquad\square$

Example 1.6.1. Example. Find the estimates for the equivalence of $\|\cdot\|_2$ and $\|\cdot\|_\infty$

Solution. Let $x \in C_n$. Then, because we know for any finite sequences

that $\sum_{i=1}^{n} |a_i b_i| \leq \max_{1 \leq i \leq n} |a_i| \ \sum_{i=1}^{n} |b_i|$

$$
\begin{aligned}
\|x\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n} 1 \cdot |x_i|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\max_{1 \leq i \leq n} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} 1 \right)^{\frac{1}{2}} \\
&= n^{\frac{1}{2}} \|x\|_{\infty}
\end{aligned}
$$

On the other hand, by the Cauchy-Schwartz inequality

$$
\begin{aligned}
\|x\|_{\infty} &= \max_{1 \leq i \leq n} |x_i| \\
&\leq \sum_{i=1}^{n} |x_i| \\
&\leq \left(\sum_{i=1}^{n} 1 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \\
&= n^{\frac{1}{2}} \|x\|_2
\end{aligned}
$$

Putting these inequalities together we have

$$
n^{-\frac{1}{2}} \|x\|_2 \leq \|x\|_{\infty} \leq n^{\frac{1}{2}} \|x\|_2
$$

This makes $m = n^{-1/2}$ and $M = n^{\frac{1}{2}}$.

Remark 1.6.2. Note that the constants m and M depend on the dimension of the vector space. Though not the rule in all cases, it is mostly the situation.

Norms on polynomial spaces

Polynomial spaces, as we have considered earlier, can be given norms as well. Since they are function spaces, our norms usually need to consider all the values of the independent variable. In many, though not all, cases we need

to restrict the domains of the polynomials. With that in mind we introduce the notation

$$P_k(a, b) = P_k \text{ with domain restricted to the interval } [a, b]$$

We now define the function versions of the same three norms we have just studied. For functions $p(x)$ in $P_k(a, b)$ we define

1. $\|p(x)\|_2 = \left(\int_a^b | p(x) |^2 \ dx \right)^{\frac{1}{2}}$

2. $\|p(x)\|_1 = \int_a^b | p(x) | \ dx$

3. $\|p(x)\|_\infty = \max_{a \leq x \leq b} | p(x) |$

The positivity and homogeneity properties are fairly easy to prove. The triangle property is a little more involved. However, it has essentially been proved for the earlier norms. In the present case, one merely "integrates" over the inequality. Sometimes $\| \cdot \|_2$ is called the energy norm.

The integral norms are really the *norm* for polynomial spaces. Alternate norms use pointwise evaluation or even derivatives depending on the application. Here is a common type of norm that features the first derivative. For $p \in P_n(a, b)$ define

$$\|p\| = \max_{a \leq x \leq b} |p(x)| + \max_{a \leq x \leq b} |p'(x)|$$

As is evident this norm becomes large not only when the polynomial is large but also when its derivative is large. If we remove the term $\max_{a \leq x \leq b} |p(x)|$ from the norm above and define

$$N(p) = \max_{a \leq x \leq b} |p'(x)|$$

This function satisfies all the norm properties except one and thus is not a norm. (See the exercises.) Point evaluation-type norms take us too far afield of our goals partly because making point evaluations into norms requires some knowledge of interpolation and related topics. Leave it said that the obvious point evaluation functions such as $p(a)$ and the like will not provide us with norms.

1.7   Ordered Bases

Given a vector space V with a basis $S = \{v_1, v_2, \ldots, v_k\}$ we now know that every vector $v \in V$ has a representation with respect to the basis

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k.$$

But no order is implied. For example, for R_2 we have $S = \{e_1, e_2\} = \{e_2, e_1\}$ shows us that there is no particular order convey through the definition of a basis. When we place an order on a basis we will notice an underlying algebraic structure of all k-dimensional vector spaces.

Definition 1.7.1. Let V be a k-dimensional vector space with basis $S = \{v_1; v_2; \ldots; v_k\}$ is specified with a fixed and well defined order as indicated by their relevant positions. Then S is called an *ordered* basis. With ordered bases we obtain *coordinates*. Let V be a vector space of dimension k with ordered basis S, and suppose $v \in V$. For $1 \leq i \leq k$, we define the ith *coordinate* of v with respect to S to be the ith *coefficient* a_i in the representation

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_i v_i + \cdots + a_k v_k.$$

In this way we can associate each $v \in V$ with a k-tuple of numbers $(a_1, a_2, \ldots, a_k) \in R_k$ that are the coefficients of v with respect to S. The k-tuple is unique, owing to the fixed ordering of S. Conversely, for each ordered k-tuple $(a_1, a_2, \ldots, a_k) \in R_k$ there is associated a unique vector $v \in V$ given by $v = a_1 v_1 + a_2 v_2 + \cdots + a_i v_i + \cdots + a_k v_k$.

We will express this association as

$$v \sim (a_1, a_2, \ldots, a_k)$$

The following properties are each simple propositions:

- If $v \sim (a_1, a_2, \ldots, a_k)$ and $w \sim (b_1, b_2, \ldots, b_k)$ then

$$v + w \sim (a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k)$$

- If $v \sim (a_1, a_2, \ldots, a_k)$ and $\alpha \in R$ (or $\mathbb{C}$), then

$$\alpha v \sim \alpha(a_1, a_2, \ldots, a_k) = (\alpha a_1, \alpha a_2, \ldots, \alpha a_k)$$

- If $v \sim (a_1, a_2, \ldots, a_k) = (0, 0, \ldots 0)$, then $v = 0$.

We now define a special type of linear function from one linear space to another. The special condition is linearity of the map.

Definition 1.7.2. Let V and W be two vector spaces. We say that V and W are *homomorphic* if there is a mapping Φ between V and W for which

(1.) For v and w in V

$$\Phi(v + w) = \Phi(v) + \Phi(w)$$

(2.) For v and $\alpha \in R$ (or $\mathbb{C}$)

$$\Phi(\alpha v) = \alpha \Phi(v)$$

In this case we call Φ a *homomorphism* from V to W. Furthermore, we say that V and W are isomorphic if they are homomorphic and if

(3.) For each $w \in W$ there exists a unique $v \in V$ such that

$$\Phi(v) = w$$

In this case we call Φ a *isomorphism* from V to W.

We put all this together to show that finite dimensional vector spaces over the reals (resp. complex numers) and the standard Euclidean spaces R_k (resp. $\mathbb{C}_k$) are very, very closely related. Indeed from the point of view of isometry, they are identical.

Theorem 1.7.1. *If V is a k-dimensional vector space over R (resp $\mathbb{C}$), then V is isomorphic to R_k (resp. $\mathbb{C}_k$).*

This constitutes the beginning of the sufficiency of matrix theory as a tool to study finite dimentsional vector spaces.

Definition 1.7.3. The mapping $c_s : V \to R_k$ defined by

$$c_s(v) = (a_1, a_2, \ldots, a_k)$$

where $v \sim (a_1, a_2, \ldots, a_k)$ is the so-called *coordinate* map.

Example 1.7.1. We have shown that in R_3 the solutions to the equation $x_1 + x_2 + x_3 = 0$ for $x = (x_1, x_2, x_3) \in R_3$ is a subspace V_0 with basis $S = \{v_1, v_2\} = \{(-1, 0, 1), (-1, 1, 0)\}$. With respect to this basis $v_0 \in V_0$ if there are constants α_0, β_0 so that

$$v_0 = \alpha_0 (-1, 0, 1) + \beta_0 (-1, 1, 0)$$

With respect to this basis the coordinate map has the form

$$c_s(v_0) = (\alpha_0, \beta_0)$$

Therefore, we have established that V_0 is isomorphic to R_2.

1.8   Exercises.

1. Show that $\{(a, b, 0) \mid a, b \in R\}$ is a subspace of R^3 by proving that it is spanned by vectors in R^3. Find at least two sets of spanning sets.

2. Show that $\{(a, b, 1) \mid a, b \in R\}$ cannot be a subspace of R^3.

3. Show that $\{(a - b, 2b - a, a - b) \mid a, b \in R\}$ is a subspace of R^3 by proving that it is spanned by vectors in R^3.

4. For any $w \in R_3$, show that $U_w = \{x \in R_3 \mid x \cdot w = d \neq 0\}$ is **not** subspace of R_3.

5. Find a set of vectors in R_3 that spans the subspace $U_w = \{x \in R_3 \mid x \cdot w = 0\}$, where $w = (1, 1, 1)$.

6. Why can $\{(a - b, a^2, ab) \mid a, b \in R\}$ never be a subspace of R^3?

7. Let $Q = \{x_1, \ldots, x_k\}$ be a set of distinct points on the real line with $k < n$. Show that the subset P_Q of the polynomial space P_n of polynomials zero on the set Q is in fact a subspace of P_n. Characterize P_Q if $k > n$ and $k = n$.

8. In the product space defined above prove that definitions given the result is a vector space.

9. What is the product space $R_2 \times R_3$?

10. Find a basis for $Q = \{ax + bx^3 \mid a, b \in R\}$.

11. Let $T \subset P_n$ be those polynomials of exactly degree n. Show that T is not a subspace of P_n.

12. What is the dimension of

$$Q = \{ax + ax^2 + bx^3 \mid a, b \in Q\}.$$

What is a basis for Q?

13. Given that $S = \{x_1, \ x_2, \ \ldots, \ x_{2k}\}$ and $T = \{y_1, \ y_2, \ \ldots, \ y_{2k}\}$ are both bases of a vector space V. (Note, the space V has dimension $2k$.) Consider the set of any k integers $L = \{l_1, \ldots, l_k\} \subset \{1, 2, \ldots, 2k\}$. (i) Show that associated with $P = \{x_{l_1}, \ x_{l_2}, \ \ldots, \ x_{l_k}\}$ there are exactly k vectors from T, say $Q = \{y_{m_1}, \ y_{m_2}, \ \ldots, \ y_{m_k}\}$ so that $P \cup Q$ is also a basis for V. (ii) Is the set of vectors from T unique? Why or why not?

14. Given P_n. Define $Z_n = \{p'(x) \mid p(x) \in P_n\}$. (The notation $p'(x)$ is the standard notation for the derivative of the function $p(x)$ with respect to the variable x.) What is another way to express Z_n in terms of previously defined spaces?

15. Show that $\|x\|_\infty$ is a norm. (Hint. The condition (iii) should be the focal point of your effort.)

16. Let $S = \{x_1,\ x_2,\ \dots,\ x_n\} \subset R_n$. For each $j = 1, 2, \dots, n$ suppose $x_j \in S$ has the property that its first $j - 1$ entries equal zero and the jth entry is nonzero. Show that S is a basis of R_n.

17. Let $w = (w_1, w_2, w_3) \in R_3$, where all the components of w are strictly positive. Define $\|\cdot\|_w$ on R_3 by $\|x\|_w = \left(w_1 |x_1|^2 + w_2 |x_3|^2 + w_2 |x_3|^2\right)^{1/2}$. Show that $\|\cdot\|_w$ is a norm on R_3.

18. Show that equivalence of norms is an equivalence relation.

19. Define for $p \in P_n(a, b)$ the function $N(p) = \max_{a \le x \le b} |p'(x)|$. Show that this is not a norm on $P_n(a, b)$.

20. For $p \in P_n(a, b)$ define $\|p\| = \max_{a \le x \le b} |p(x)| + \max_{a \le x \le b} |p''(x)|$. Show this is a norm on $P_n(a, b)$.

21. Suppose that V is a vector space with dimension k. Find two (linearly independent) spanning sets $S = \{v_1, v_2, \dots, v_k\}$ and $W = \{w_1, w_2, \dots, w_k\}$ of V such that if any $m < k$ vectors are chosen from S and any $k - m$ vectors are chosen from T, the resulting set will be a basis for V.

22. For $p \in P_n(a, b)$ define $N(p) = \left|p\left(\frac{a+b}{2}\right)\right|$. Show this is not a norm on $P_n(a, b)$.

23. Find the estimates for the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

24. Find the estimates for the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_2$.

25. Show that P_n defined over the reals is isomorphic to R_{n+1}.

26. Show that T_n, the space of trigonometric polynomials, defined over the reals is isomorphic to R_n.

27. Show that the product space $\mathbb{C}_k \times \mathbb{C}_m$ is isomorphic to $\mathbb{C}_{k+m}$.

28. What is the relation between the product space $P_n \times P_n$ and P_{2n}? Find the polynomial space that is isomorphic to $P_n \times P_n$.

Terms.

Field
Vector space
scalar multiplication
Closed space
Origin
Polynomial space
Subspace, linear subspace
Span
Spanning set
Representation
Uniqueness of representation
linear dependence
linear independence
linear combination
Basis
Extension to a basis
Dimension
Norm
ℓ_2 norm; ℓ_1 norm; $\ell_2\infty$ norm
Cauchy-Schwartz inequality
Fundamental Theorem of Algebra
Cardinality
Triangle inquality