# 4 Span and subspace

## 4.1 Linear combination

Let $\mathbf{x}_1 = [2, -1, 3]^T$ and let $\mathbf{x}_2 = [4, 2, 1]^T$, both vectors in the $\mathbf{R}^3$. We are interested in which other vectors in $\mathbf{R}^3$ we can get by just scaling these two vectors and adding the results. We can get, for instance,

$$3\mathbf{x}_1 + 4\mathbf{x}_2 = 3\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 5 \\ 13 \end{bmatrix}$$

and also

$$2\mathbf{x}_1 + (-3)\mathbf{x}_2 = 2\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + (-3)\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -8 \\ 3 \end{bmatrix}.$$

Each of these is an example of a "linear combination" of the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

> Linear combination.
>
> Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s$ be vectors in $\mathbf{R}^n$. A **linear combination of** $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s$ is an expression of the form
>
> $$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_s\mathbf{x}_s,$$
>
> with $\alpha_1, \alpha_2, \dots, \alpha_s \in \mathbf{R}$.

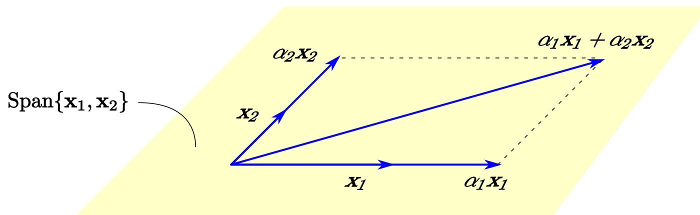
Note that a linear combination is a single vector; it is the result of scaling the given vectors and adding them together. For instance, the linear combination $3\mathbf{x}_1 + 4\mathbf{x}_2$ is the single vector $[22, 5, 13]^T$.

## 4.2 Span

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two vectors in $\mathbf{R}^3$. The "span" of the set $\{\mathbf{x}_1, \mathbf{x}_2\}$ (denoted $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$) is the set of all possible linear combinations of $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \{\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 \,|\, \alpha_1, \alpha_2 \in \mathbf{R}\}.$$

If $\mathbf{x}_1$ and $\mathbf{x}_2$ are not parallel, then one can show that $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ is the plane determined by $\mathbf{x}_1$ and $\mathbf{x}_2$. This seems reasonable, since every vector in the this plane appears to be expressible in the form $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$ for suitable scalars $\alpha_1$ and $\alpha_2$:




**4.2.1 Example** Let $\mathbf{x}_1 = [2, -1, 3]^T$ and $\mathbf{x}_2 = [4, 2, 1]^T$.

(a) Determine whether $[2, -5, 8]^T$ is in $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

(b) Determine whether $[-2, 9, 0]^T$ is in $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

*Solution*

(a) We are wondering whether there exist numbers $\alpha_1$ and $\alpha_2$ such that $[2, -5, 8]^T = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$, that is,

$$\begin{bmatrix} 2 \\ -5 \\ 8 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 + 4\alpha_2 \\ -\alpha_1 + 2\alpha_2 \\ 3\alpha_1 + \alpha_2 \end{bmatrix}.$$

Equating components leads to a system of equations with augmented matrix

$$\begin{array}{c} \left[\begin{array}{cc|c} 2 & 4 & 2 \\ -1 & 2 & -5 \\ 3 & 1 & 8 \end{array}\right] \end{array} \begin{array}{cc} 1 & -3 \\ 2 & \\ & 2 \end{array} \sim \left[\begin{array}{cc|c} 2 & 4 & 2 \\ 0 & 8 & -8 \\ 0 & -10 & 10 \end{array}\right] \begin{array}{c} \\ \frac{1}{8} \\ \frac{1}{10} \end{array}$$

$$\sim \left[\begin{array}{cc|c} 2 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{array}\right] \begin{array}{c} \\ 1 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 2 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right].$$

Since there is no pivot in the augmented column, we know that a solution exists. Therefore, $[2, -5, 8]^T$ is in $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

(b) Arguing as above, we are led to a system with augmented matrix

$$\begin{array}{c}\left[\begin{array}{cc|c} 2 & 4 & -2 \\ -1 & 2 & 9 \\ 3 & 1 & 0 \end{array}\right]\begin{array}{cc} 1 & -3 \\ 2 & \\ & 2 \end{array}\end{array} \sim \left[\begin{array}{cc|c} 2 & 4 & 1 \\ 0 & 8 & 16 \\ 0 & -10 & 6 \end{array}\right]\begin{array}{c} \\ \frac{1}{8} \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 2 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & -10 & 6 \end{array}\right]\begin{array}{c} \\ 10 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 2 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 26 \end{array}\right].$$

Since there is a pivot in the last column, we conclude that $[-2, 9, 0]^T$ is not in $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

□

Here is the general definition of span:

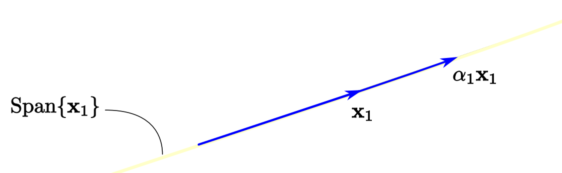
Span.

Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ be a set of vectors in $\mathbf{R}^n$. The **span of** $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ (denoted $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$) is the set of all linear combinations of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$. In symbols,

$$\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\} = \{\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_s\mathbf{x}_s \mid \alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbf{R}\}.$$

We sometimes say "the span of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$" to mean the span of the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$.

The span of a single nonzero vector $\mathbf{x}_1$ in $\mathbf{R}^3$ (or $\mathbf{R}^2$) is the line through the origin determined by $\mathbf{x}_1$. (Reason: $\mathrm{Span}\{\mathbf{x}_1\}$ is the set of all possible linear combinations of the vector $\mathbf{x}_1$, that is, all vectors of the form $\alpha_1\mathbf{x}_1$ where $\alpha_1 \in \mathbf{R}$. So $\mathrm{Span}\{\mathbf{x}_1\}$ is the set of all multiples of $\mathbf{x}_1$ and is therefore the line through the origin determined by $\mathbf{x}_1$):



**4.2.2 Example** Let $\mathbf{x}_1 = [2, 1]^T$ and $\mathbf{x}_2 = [1, 3]^T$. Show that $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \mathbf{R}^2$.

*Solution* The way to show that two sets are equal is to show that each is a subset of the other. It is automatic that $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\} \subseteq \mathbf{R}^2$ (since every linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ is a vector in $\mathbf{R}^2$). So we just need to show that $\mathbf{R}^2 \subseteq \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, that is, show that every vector in $\mathbf{R}^2$ can be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.

Let $\mathbf{b} = [b_1, b_2]^T$ be an arbitrary vector in $\mathbf{R}^2$. We are trying to show that there exist scalars $\alpha_1$ and $\alpha_2$ such that $\mathbf{b} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$, that is,

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 + \alpha_2 \\ \alpha_1 + 3\alpha_2 \end{bmatrix}.$$

This leads to a system with augmented matrix

$$\begin{array}{cc} \left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 1 & 3 & b_2 \end{array}\right] & \begin{array}{c} \\ -2 \end{array} \end{array} \sim \left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 0 & -5 & b_1 - 2b_2 \end{array}\right].$$

Since there is no pivot in the augmented column, a solution $\alpha_1$ and $\alpha_2$ exists. Therefore, $\mathbf{b}$ can be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. This finishes the proof that $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \mathbf{R}^2$. □

An argument similar to that just given shows that $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \mathbf{R}^2$ whenever $\mathbf{x}_1$ and $\mathbf{x}_2$ are *nonparallel* vectors in $\mathbf{R}^2$. (The nonparallel assumption means that $\mathbf{x}_2$ is not a multiple of $\mathbf{x}_1$ so the row echelon form of the corresponding augmented matrix will have a pivot in each row and hence no pivot in the augmented column.)

On the other hand, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are *parallel* and nonzero, then both vectors lie on the same line through the origin and $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ is this line. (In this case, $\mathbf{x}_2$ is a multiple of $\mathbf{x}_1$ so linear combinations of the two vectors amount to multiples of the single vector $\mathbf{x}_1$.)

## 4.3 Subspace

When a subset of $\mathbf{R}^n$ contains the origin and is closed under addition and scalar multiplication, we call it a "subspace."

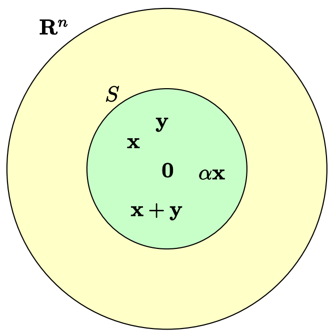
SUBSPACE.

A subset $S$ of $\mathbf{R}^n$ is called a **subspace** if the following hold:

(a) $\mathbf{0} \in S$,

(b) $\mathbf{x}, \mathbf{y} \in S$ implies $\mathbf{x} + \mathbf{y} \in S$,

(c) $\mathbf{x} \in S, \alpha \in \mathbf{R}$ implies $\alpha\mathbf{x} \in S$.




In other words, a subset $S$ of $\mathbf{R}^n$ is a subspace if it satisfies the following:

(a) $S$ contains the origin $\mathbf{0}$,

(b) $S$ is closed under addition (meaning, if $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $S$, then their sum $\mathbf{x} + \mathbf{y}$ is also in $S$),

(c) $S$ is closed under scalar multiplication (meaning, if $\mathbf{x}$ is a vector in $S$ and $\alpha$ is any scalar, then the product $\alpha\mathbf{x}$ is also in $S$).

**4.3.1 Example** Let $S$ be the subset of $\mathbf{R}^2$ given by

$$S = \{ \begin{bmatrix} 2t \\ -t \end{bmatrix} \mid t \in \mathbf{R} \}.$$

Show that $S$ is a subspace of $\mathbf{R}^2$.

*Solution* We check that $S$ satisfies the three properties in the definition of subspace.

(a) We have

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(0) \\ -(0) \end{bmatrix} \in S$$

(the vector $[2(0), -(0)]^T$ is in $S$ because it is of the form $[2t, -t]^T$).

(b) Let $\mathbf{x}, \mathbf{y} \in S$. (Must show that $\mathbf{x}+\mathbf{y} \in S$.) Since $\mathbf{x}$ is in $S$, it can be written $\mathbf{x} = [2t, -t]^T$ for some $t \in \mathbf{R}$. Similarly, $\mathbf{y} = [2s, -s]^T$ for some $s \in \mathbf{R}$ (if we had used $t$ instead of $s$ here, then we would have been assuming that $\mathbf{y}$ was the same as $\mathbf{x}$, which might not be the case). Therefore,

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} 2t \\ -t \end{bmatrix} + \begin{bmatrix} 2s \\ -s \end{bmatrix} = \begin{bmatrix} 2t + 2s \\ -t - s \end{bmatrix} = \begin{bmatrix} 2(t+s) \\ -(t+s) \end{bmatrix} \in S.$$

(c) Let $\mathbf{x} \in S$ and $\alpha \in \mathbf{R}$. (Must show that $\alpha\mathbf{x} \in S$.) Since $\mathbf{x}$ is in $S$, it can be written $\mathbf{x} = [2t, -t]^T$ for some $t \in \mathbf{R}$. Therefore,

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} 2t \\ -t \end{bmatrix} = \begin{bmatrix} 2(\alpha t) \\ -(\alpha t) \end{bmatrix} \in S.$$

Therefore, $S$ is a subspace of $\mathbf{R}^2$. □

**4.3.2 Example** Let $S$ be the subset of $\mathbf{R}^3$ given by

$$S = \{ \begin{bmatrix} t+s \\ 2t \\ 3t-s \end{bmatrix} \mid t, s \in \mathbf{R} \}.$$

Show that $S$ is a subspace of $\mathbf{R}^3$.

*Solution* We check that $S$ satisfies the three properties in the definition of subspace.

(a) We have

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (0)+(0) \\ 2(0) \\ 3(0)-(0) \end{bmatrix} \in S.$$

(b) Let $\mathbf{x}, \mathbf{y} \in S$. (Must show that $\mathbf{x}+\mathbf{y} \in S$.) Since $\mathbf{x}$ is in $S$, it can be written $\mathbf{x} = [t+s, 2t, 3t-s]^T$ for some $t, s \in \mathbf{R}$. Similarly, $\mathbf{y} = [r+q, 2r, 3r-q]^T$ for some $r, q \in \mathbf{R}$. Therefore,

$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} t+s \\ 2t \\ 3t-s \end{bmatrix} + \begin{bmatrix} r+q \\ 2r \\ 3r-q \end{bmatrix} = \begin{bmatrix} (t+s)+(r+q) \\ (2t)+(2r) \\ (3t-s)+(3r-q) \end{bmatrix} = \begin{bmatrix} (t+r)+(s+q) \\ 2(t+r) \\ 3(t+r)-(s+q) \end{bmatrix} \in S$$

(this last vector is in $S$ since it is in the right form with $t+r$ playing the role of the $t$ and $s+q$ playing the role of the $s$).

(c) Let $\mathbf{x} \in S$ and $\alpha \in \mathbf{R}$. (Must show that $\alpha\mathbf{x} \in S$.) Since $\mathbf{x}$ is in $S$, it can be written $\mathbf{x} = [t+s, 2t, 3t-s]^T$ for some $t, s \in \mathbf{R}$. Therefore,

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} t+s \\ 2t \\ 3t-s \end{bmatrix} = \begin{bmatrix} \alpha(t+s) \\ \alpha(2t) \\ \alpha(3t-s) \end{bmatrix} = \begin{bmatrix} (\alpha t)+(\alpha s) \\ 2(\alpha t) \\ 3(\alpha t)-(\alpha s) \end{bmatrix} \in S.$$

Therefore, $S$ is a subspace of $\mathbf{R}^3$. $\square$

**4.3.3 Example** Let $S$ be the subset of $\mathbf{R}^2$ given by

$$S = \{ \begin{bmatrix} t \\ t^2 \end{bmatrix} \mid t \in \mathbf{R}\}.$$

Is $S$ a subspace of $\mathbf{R}^2$? Explain.

*Solution* We check to see whether $S$ satisfies the three properties of subspace.

(a) We have

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (0) \\ (0)^2 \end{bmatrix} \in S.$$

(b) Let $\mathbf{x}, \mathbf{y} \in S$. (Must show that $\mathbf{x} + \mathbf{y} \in S$.) Since $\mathbf{x}$ is in $S$, it can be written $\mathbf{x} = [t, t^2]^T$ for some $t \in \mathbf{R}$. Similarly, $\mathbf{y} = [s, s^2]^T$ for some $s \in \mathbf{R}$. Therefore,

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} t \\ t^2 \end{bmatrix} + \begin{bmatrix} s \\ s^2 \end{bmatrix} = \begin{bmatrix} t + s \\ t^2 + s^2 \end{bmatrix}.$$

If this last vector is to be in $S$, then the second component must be the square of the first component. But since $(t + s)^2 = t^2 + 2ts + s^2$, we see that this need not be the case. In fact, it is not the case when, say, $t = 1$ and $s = 1$.

Everything we have done up to this point can be considered scratch work. It was done just to come up with an idea for a counterexample. To solve the problem, all we really need to write is this:

Let $\mathbf{x} = [1, 1]^T$ and $\mathbf{y} = [1, 1]^T$. Then $\mathbf{x}$ is in $S$, since $\mathbf{x} = [1, 1]^T = [(1), (1)^2]^T$ so $\mathbf{x}$ is of the right form. Similarly, $\mathbf{y}$ is in $S$. However,

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin S$$

(this last vector is not in $S$ because its second component is not the square of its first component). Therefore, $S$ is not a subspace of $\mathbf{R}^2$. $\square$

**4.3.4 Example** Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two vectors in $\mathbf{R}^n$ and let $S = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$. Show that $S$ is a subspace of $\mathbf{R}^n$.

*Solution* We check that $S$ satisfies the three properties of subspace.

(a) We have

$$\mathbf{0} = 0\mathbf{x}_1 + 0\mathbf{x}_2 \in S.$$

(b) Let $\mathbf{x}, \mathbf{y} \in S$. (Must show that $\mathbf{x}+\mathbf{y} \in S$.) Since $\mathbf{x}$ is in $S$, it can be written $\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$ for some $\alpha_1, \alpha_2 \in \mathbf{R}$. Similarly, $\mathbf{y} = \beta_1\mathbf{x}_1 + \beta_2\mathbf{x}_2$ for some $\beta_1, \beta_2 \in \mathbf{R}$. Therefore,

$$\mathbf{x}+\mathbf{y} = (\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2) + (\beta_1\mathbf{x}_1 + \beta_2\mathbf{x}_2) = (\alpha_1+\beta_1)\mathbf{x}_1 + (\alpha_2+\beta_2)\mathbf{x}_2 \in S$$

(this last vector is in $S$ because it is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ and $S$ is the set of all such linear combinations).

(c) Let $\mathbf{x} \in S$ and $\alpha \in \mathbf{R}$. (Must show that $\alpha\mathbf{x} \in S$.) Since $\mathbf{x}$ is in $S$, it can be written $\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$ for some $\alpha_1, \alpha_2 \in \mathbf{R}$. Therefore,

$$\alpha\mathbf{x} = \alpha(\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2) = (\alpha\alpha_1)\mathbf{x}_1 + (\alpha\alpha_2)\mathbf{x}_2 \in S.$$

Therefore, $S$ is a subspace of $\mathbf{R}^n$. □

## 4.4 Examples of subspaces

THEOREM.

(a) $\{\mathbf{0}\}$ *is a subspace of* $\mathbf{R}^n$.

(b) $\mathbf{R}^n$ *is a subspace of* $\mathbf{R}^n$.

*Proof.* (a) We need to show that $S = \{\mathbf{0}\}$ satisfies the three properties of a subspace. First, $\mathbf{0}$ is in $S$ so property (a) is satisfied. Next, if $\mathbf{x}, \mathbf{y} \in S$, then $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} = \mathbf{0}$, so $\mathbf{x}+\mathbf{y} = \mathbf{0}+\mathbf{0} = \mathbf{0}$, implying $\mathbf{x}+\mathbf{y} \in S$ and (b) is satisfied. Finally, if $\mathbf{x} \in S$ and $\alpha \in \mathbf{R}$, then $\mathbf{x} = \mathbf{0}$, so $\alpha\mathbf{x} = \alpha\mathbf{0} = \mathbf{0}$, implying $\alpha\mathbf{x} \in S$ and (c) is satisfied.

(b) $\mathbf{R}^n$ is a subspace of itself since it contains $\mathbf{0}$ and it is closed under addition and scalar multiplication and therefore satisfies the three properties. □

THEOREM.

(a) *The subspaces of* $\mathbf{R}^2$ *are*

$$\{\mathbf{0}\}, \quad \textit{lines through origin}, \quad \mathbf{R}^2.$$

(b) *The subspaces of* $\mathbf{R}^3$ *are*

$$\{\mathbf{0}\}, \quad \textit{lines through origin}, \quad \textit{planes through origin}, \quad \mathbf{R}^3.$$

*Proof.* Here is informal reasoning for why (a) is true. First, $\{\mathbf{0}\}$ and $\mathbf{R}^2$ are subspaces by the previous theorem. Also, an argument just like that in Example 4.3.1 shows that any line through the origin is a subspace.

We also need to argue that this list covers all possible subspaces of $\mathbf{R}^2$. Let $S$ be an arbitrary subspace of $\mathbf{R}^2$. Since $S$ is a subspace, it contains the zero vector. If it contains only the zero vector, then it is $\{\mathbf{0}\}$, which is one in the list. Otherwise, it must contain a nonzero vector $\mathbf{x}$. Since $S$ is closed under scalar multiplication, it must contain all multiples of $\mathbf{x}$ and hence the line through the origin containing this vector. If this is all it contains then $S$ is again one in the list. Otherwise, it contains a vector $\mathbf{y}$ not in the line determined by $\mathbf{x}$. Then, by the closure properties of $S$ it must contain all linear combinations of $\mathbf{x}$ and $\mathbf{y}$ so that $S$ is $\mathbf{R}^2$ (using an argument similar to that in 4.2.2), which is again in the list. This handles all of the possibilities.

Part (b) is handled similarly. □

Consider $\mathbf{R}^3$. A plane through the origin is an analog of $\mathbf{R}^2$ (possibly tilted); we call it a "copy" of $\mathbf{R}^2$. Similarly, a line through the origin is a copy of $\mathbf{R}^1$. If we define $\mathbf{R}^0$ to be $\{\mathbf{0}\}$, then we can view the theorem as saying that the subspaces of $\mathbf{R}^3$ are copies of the spaces $\mathbf{R}^0, \mathbf{R}^1, \mathbf{R}^2, \mathbf{R}^3$. A similar statement about the subspaces of $\mathbf{R}^n$ holds for any $n$.

The next theorem says that the span of a set of vectors in $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$.

> THEOREM. *If* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$ *are vectors in* $\mathbf{R}^n$ *and* $S$ *is their span (i.e.,* $S = \operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$*), then* $S$ *is a subspace of* $\mathbf{R}^n$*.*

Example 4.3.4 handled the case of two vectors ($s = 2$). The proof of the general case is very similar so we omit it.

**4.4.1 Example** Give a geometrical interpretation of $S = \operatorname{Span}\{[1, 0, 4]^T, [2, -1, 3]^T\}$.

*Solution* A vector $\mathbf{x} = [x_1, x_2, x_3]^T$ is in $S$ if and only if there exist scalars $\alpha_1$ and $\alpha_2$ such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ -\alpha_2 \\ 4\alpha_1 + 3\alpha_2 \end{bmatrix}. \qquad (*)$$

This equation of vectors leads to a system with augmented matrix

$$\begin{array}{c}\left[\begin{array}{cc|c} 1 & 2 & x_1 \\ 0 & -1 & x_2 \\ 4 & 3 & x_3 \end{array}\right]\end{array}\begin{array}{l} -4 \\ \\ \\ \end{array} \sim \left[\begin{array}{cc|c} 1 & 2 & x_1 \\ 0 & -1 & x_2 \\ 0 & -5 & -4x_1 + x_3 \end{array}\right]\begin{array}{l} \\ -5 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & x_1 \\ 0 & -1 & x_2 \\ 0 & 0 & -4x_1 - 5x_2 + x_3 \end{array}\right].$$

The system has a solution if and only if there is no pivot in the augmented column. Therefore, there exist scalars $\alpha_1$ and $\alpha_2$ satisfying (*) if and only if $-4x_1 - 5x_2 + x_3 = 0$. We conclude that $S$ is the plane $-4x_1 - 5x_2 + x_3 = 0$. It is the plane through the origin determined by the two vectors $[1, 0, 4]^T$ and $[2, -1, 3]^T$. □

$\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s\}$ is the smallest subspace of $\mathbf{R}^n$ containing the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s$. The reason is that any subspace of $\mathbf{R}^n$ that contains the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s$ must contain all possible linear combinations of these vectors (by the closure properties of subspace) and must therefore contain their span.

The previous example illustrates this fact. The vectors $[1, 0, 4]^T$ and $[2, -1, 3]^T$ are not parallel (the second is not a multiple of the first), so the smallest subspace of $\mathbf{R}^3$ containing these vectors is the plane through the origin determined by the vectors. This is the plane found in the solution.

> THEOREM. *If $L : \mathbf{R}^n \to \mathbf{R}^m$ is a linear function, then*
>
> (a) $\text{im}\, L$ *is a subspace of* $\mathbf{R}^m$,
>
> (b) $\ker L$ *is a subspace of* $\mathbf{R}^n$.

*Proof.* Let $L : \mathbf{R}^n \to \mathbf{R}^m$ be a linear function.

(a) We check that $\text{im}\, L = \{L(\mathbf{x}) \,|\, \mathbf{x} \in \mathbf{R}^n\}$ satisfies the three properties of subspace.

  (a) Using the fact that a linear function sends $\mathbf{0}$ to $\mathbf{0}$, we have

$$\mathbf{0} = L(\mathbf{0}) \in \text{im}\, L.$$

  (b) Let $\mathbf{u}, \mathbf{v} \in \text{im}\, L$. (Must show that $\mathbf{u} + \mathbf{v} \in \text{im}\, L$.) Since $\mathbf{u}$ is in $\text{im}\, L$, it can be written $\mathbf{u} = L(\mathbf{x})$ for some $\mathbf{x} \in \mathbf{R}^n$. Similarly, $\mathbf{v} = L(\mathbf{y})$ for some $\mathbf{y} \in \mathbf{R}^n$. Therefore,

$$\mathbf{u} + \mathbf{v} = L(\mathbf{x}) + L(\mathbf{y}) = L(\mathbf{x} + \mathbf{y}) \in \text{im}\, L,$$

where we have used the first property in the definition of linear function.

(c) Let $\mathbf{u} \in \operatorname{im} L$ and $\alpha \in \mathbf{R}$. (Must show that $\alpha\mathbf{u} \in \operatorname{im} L$.) Since $\mathbf{u}$ is in $\operatorname{im} L$, it can be written $\mathbf{u} = L(\mathbf{x})$ for some $\mathbf{x} \in \mathbf{R}^n$. Therefore,

$$\alpha\mathbf{u} = \alpha L(\mathbf{x}) = L(\alpha\mathbf{x}) \in \operatorname{im} L,$$

where we have used the second property in the definition of linear function.

Therefore, $\operatorname{im} L$ is a subspace of $\mathbf{R}^m$.

(b) This proof is left as an exercise (see Exercise 4–9).

□

4 – Exercises

**4–1** Let $\mathbf{x}_1 = [1, -3, 2]^T$ and $\mathbf{x}_2 = [4, -7, -1]^T$.

(a) Determine whether $[2, -1, 6]^T$ is in $\operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

(b) Determine whether $[5, -5, -8]^T$ is in $\operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

**4–2** Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be vectors in $\mathbf{R}^n$ and let $S = \operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

(a) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are both in $S$.

(b) Show that if $\mathbf{x}$ is in $S$, then $-\mathbf{x}$ is in $S$.

**4–3** Let $\mathbf{x}_1 = [-1, 4]^T$, $\mathbf{x}_2 = [2, -8]^T$, and $\mathbf{x}_3 = [0, 6]^T$. Show that $\operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = \mathbf{R}^2$.

**4–4** Let $\mathbf{x}_1 = [1, 3, -5]^T$, $\mathbf{x}_2 = [-2, -1, 0]^T$, $\mathbf{x}_3 = [0, 5, -10]^T$ and let $S = \operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. Determine whether $S = \mathbf{R}^3$. If your answer is no, then find a vector in $\mathbf{R}^3$ that is not in $S$.

**4–5** Let $S$ be the subset of $\mathbf{R}^2$ given by

$$S = \{\begin{bmatrix} 4t \\ -3t \end{bmatrix} \mid t \in \mathbf{R}\}.$$

Show that $S$ is a subspace of $\mathbf{R}^2$.

**4–6** Let $S$ be the subset of $\mathbf{R}^3$ given by

$$S = \{\begin{bmatrix} 4t - 5s \\ 3t + s \\ s - t \end{bmatrix} \mid t, s \in \mathbf{R}\}.$$

Show that $S$ is a subspace of $\mathbf{R}^3$.

**4–7** Let $S$ be the upper half plane in $\mathbf{R}^2$. So $S$ consists of those vectors in $\mathbf{R}^2$ with second component $\geq 0$. In symbols, $S = \{[t, s]^T \mid t, s \in \mathbf{R},\ s \geq 0\}$. Using only the definition of subspace, determine whether $S$ is a subspace of $\mathbf{R}^2$.

**4–8** Let $S = \operatorname{Span}\{[2, -4, 1]^T\}$.

(a) Give a geometrical description of $S$.

(b) Find two planes having $S$ as their intersection. (Hint: Use the method for determining $S$ given in the solution to Example 4.4.1.)

**4–9** Let $L : \mathbf{R}^n \to \mathbf{R}^m$ be a linear function. Prove that $\ker L$ is a subspace of $\mathbf{R}^n$.

Hint: Verify the three properties in the definition of subspace with $\ker L$ playing the role of $S$. If you assume that $\mathbf{x}$ and $\mathbf{y}$ are elements of $\ker L$, then you know that $L(\mathbf{x}) = \mathbf{0}$ and $L(\mathbf{y}) = \mathbf{0}$. On the other hand, if you want to show that $\mathbf{x} + \mathbf{y}$ is an element of $\ker L$, then you need to show that $L(\mathbf{x} + \mathbf{y}) = \mathbf{0}$.