62 (B) I

eorem 14

if h_1 and h_2 are Borel measurable function from Ω to $\overline{\mathbb{R}}$ so are h_1+h_2, h_1-h_2, $h_1 \cdot h_2$, h_1/h_2 assuming these are will defined, prove

اذا h_1, h_2 هي Borel measurable
فانه لأي دالة ناتجة من العمليات عليها
عليها هي ايضا Borel measurable

mma 6

A composition of measurable function is measurable (in the sens of Borel measurable).

i.e

if $g:(\Omega_1, F_1) \to (\Omega_2, F_2)$

and $h:(\Omega_2, F_2) \to (\Omega_3, F_3)$

$$\begin{tikzcd}
(\Omega_1, F_1) \arrow[r, "g"] \arrow[dr, "h\circ g"'] & (\Omega_2, F_2) \arrow[d, "h"] \\
 & (\Omega_3, F_3)
\end{tikzcd}$$

$h \circ g$

اي انها تشبه عبارة
عن تركيبة دالات.

proof

let $B \in F_3 \Rightarrow h^{-1}(B) \in F_2$

hence. $g^{-1}[h^{-1}(B)] \in F_1$

hence $g^{-1} \circ h^{-1}(B) \in F_1$ عكس التركيبة

$\Rightarrow (h \circ g)^{-1}(B) \in F_1$

$(f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1}$ ملاحظة

$\Rightarrow$ Borel measurable.

The

تعريف

<u>Def</u>: if $h: R \to R$, h is said to be <u>lebsgue measurable</u>
iff. the inverse image of any Borel set is lebsgue measurable

(Ω, F) $\quad B \subset R$ $\qquad$ توضيح الفرق بين Borel و lebsgue
(Ω, F) $\Omega \to R$ $B(R)$ $\qquad$ حيث ان lebsgue اوسع من Borel

$(R, B(R)) \to R \quad (B(R)$

Borel $\mu[a,b] = b - a$

Lebesgue $\mu(a,b] < \infty$
$\qquad f(b) - f(a)$

<u>Remark</u>
every Borel measurable fun. is Lebsgue measurable
but not conversely.

مفهوم L.M اكثر عمومية من مفهوم B.M

<u>i.e</u>
The concept of lebsgue measurable it is more gene
than that of Borel measurable

تركيب

<u>Remark</u> The Composition of two lebsgue measurable fun.
need not be Lebsgue measurable fun.

Th

Def: if $h: R \to R$, h is said to be <u>lebsgue measurable</u> تعريف
iff. the inverse image of any Borel set is lebsgue measurable.

$(\Omega, \mathcal{F})$
(Ω, f) $\Omega \to R$ $B(R)$ — $B \in R$

توضيح الفرق ما بين Borel و lebsgue

[illegible] ان lebsgue [illegible] Borel

$(R, B(R)) \to R$ $(B(R))$

Borel $\mu(a,b] = b - a$

lebcsgue $\mu(a,b] < \infty$

$f(b) - f(a)$

Remark

every Borel measurable fun. is Lebsgue measurable but not conversely.

مفهوم L.M أكثر عمومية من مفهوم B.M

i.e

The concept of lebsgue measurable it is more gene[ral] than that of Borel measurable

Remark The Composition of two lebsgue measurable fun. need not be Lebsgue measurable fun. تركيب

Theorem is:

let $h : \Omega \longrightarrow \bar{R}^n$: if P_i is the projection map of $\bar{R}^n$ onto $\bar{R}$ taking $(x_1, \dots, x_n)$ to x_i :

set $h_i = P_i \circ h$ for $i = 1, 2, \dots, n$

then h is Borel measurable iff h_i is Borel Measurable for all $i = 1, 2, \dots, n$

$$\begin{tikzcd} \Omega \arrow[r, "h"] \arrow[dr, "h_i = P_i \circ h"'] & \bar{R}^n \arrow[d, "P_i"] \\ & \bar{R} \end{tikzcd}$$

theorem 16

عندما نقول التكامل موجود exists
يجب ان لا يظهر غير معرف.

(a) if $\int_\Omega h\,d\mu$ exists and $c \in R$

then we have

$\int_\Omega ch\,d\mu$ and equal $c\int_\Omega h\,d\mu$

(b) if $g(w) \geq h(w) \quad \forall w$ then

$$\int_\Omega g(w)\,d\mu \geq \int_\Omega h(w)\,d\mu$$

in the sence that

if $\int_\Omega h\,d\mu$ exists and greater than $-\infty$

then

$\int_\Omega g\,d\mu$ exists

and $\int_\Omega g\,d\mu \geq \int_\Omega h\,d\mu$, if $\int_\Omega g\,d\mu$ e and is less than $+\infty$

then $\int h \, d\mu$ exists

and $\int_{\Omega} g \, d\mu \geq \int_{\Omega} h \, d\mu$

(c) if $\int_{\Omega} h \, d\mu$ exists

then $\left| \int_{\Omega} h \, d\mu \right| \leq \int_{\Omega} |h| \, d\mu$

(d) if $h \geq 0$ and $B \in \mathcal{F}$ then

$$\int_B h \, d\mu = \sup \left\{ \int_B s \, d\mu , \ 0 \leq s \leq h , \ s \text{ simple} \right\}$$

$B \subset E$

(e) if $\int_\Omega h\,d\mu$ exists, so does $\int_A h\,d\mu$, $\forall A \in \mathcal{F}$

if $\int_\Omega h\,d\mu$ finite then $\int_A h\,d\mu$ is also finite $\forall A \in \mathcal{F}$

ندما يقول if ثم then
ان العكس غير صحيح
اذا ذكر iff
ند ذلك يصبح العكس صحيح.

proof

(a) if h simple then proof is easy (check)

if not let $h = h^+ - h^-$ and $c > 0$,

$$(ch)^+ = ch^+$$

$$(ch)^- = ch^-$$

$$\Rightarrow \int_\Omega ch\,d\mu = \int_\Omega (ch^+ - ch^-)\,d\mu$$

$$= \int_\Omega ch^+\,d\mu - \int_\Omega ch^-\,d\mu$$

$$= c\int_\Omega h^+\,d\mu - c\int_\Omega h^-\,d\mu$$

$$= c\left[\int_\Omega (h^+ - h^-)\,d\mu\right]$$

$$= c\int_\Omega h\,d\mu$$

let $h = h^+ - h^-$ and $c < 0$

$$(ch)^+ = -ch^-$$

$$(ch)^- = -ch^+$$

$$\Rightarrow \int_\Omega ch\,d\mu = \int_\Omega (ch)^+ d\mu - \int_\Omega (ch)^- d\mu$$

$$= -\int_\Omega ch^- d\mu + \int_\Omega ch^+ d\mu$$

$$= c\int_\Omega (h^+ - h^-)\,d\mu$$

$$= c\int_\Omega h\,d\mu$$

prove $\boxed{(b, c, d, e)}$

Basic Integration theorems

مرجع كتاب Ash

Measure space (Ω, F, μ)

all fun. to be considered map Ω to $\bar{R}$

Theorem (17)

let h be a Borel measurable fun. such that

$\int_\Omega h\, d\mu$ exists define.

$$\lambda(B) = \int_B h\, d\mu \quad , B \in F$$

then λ is countably additive on F

thus

if $h \geq 0$, λ is a measure nonnegativity

تحقق الشرط

فتشكل تجمع Measure

اذا h هي Borel measurable fun.

$\lambda(B) = \int_B h\, d\mu$, $B \in F$

λ countably additive on F

if $h \geq 0 \Rightarrow \lambda$ is a measure

تحقق شرط عدم السالبية.

Theorem 18 Monotone convergens theorem

Let $h_1, h_2, \ldots$ form an increasing sequence of non-negative Borel measurable functions and Let

$$h(\omega) = \lim_{n \to \infty} h_n(\omega) \quad , \omega \in \Omega \quad , \text{ then}$$

$$\int_\Omega h_n \, d\mu \Longrightarrow \int_\Omega h \, d\mu$$

{Note that $\int_\Omega h_n \, d\mu$ increases with n }

i.e

$$\underset{\text{non-negative}}{0 \leq h_n \uparrow h} \quad \text{implies} \quad \int_\Omega h_n \, d\mu \uparrow \int_\Omega h \, d\mu$$

$h_1, h_2, \ldots$ increasing seq. اذا

$$\int_\Omega h_n \, d\mu = \int_\Omega h \, d\mu$$

Theorem 19 Additivity theorem

let f and g be Borel measurable and assume $f+g$ is well define then.

$$\int_\Omega (f+g)\, d\mu = \int_\Omega f\, d\mu + \int_\Omega g\, d\mu$$

provided that all terms are well defined.

corollories

a) if $h_1, h_2, \dots$ are B.M. then $\int_\Omega \sum_{n=1}^{\infty} h_n \, d\mu = \sum_{n=1}^{\infty} \int_\Omega h_n \, d\mu$

(b) if h is B.M. integrable iff $|h|$ is integrable
$h^+ \, h^-$

(c) if g, h are B.M. with $|g| \le h$, h is integrable then g is integrable.

Def. A condition is said to hold almost every where (a.e)

Def. A condition is said to hold almost every where (a.e) with respect to the measure μ (a.e $[\mu]$) iff there is a set $B \in F$ of μ-measure 0 ($\mu(B)=0$) such that the condition hold outside of B.

يتحقق

سؤال

Theorem 20

let f, g and h be B.M. fun.

ⓐ if $f=0$ a.e $[\mu]$, then $\int_\Omega f \, d\mu = 0$

ⓑ if $g=h$ a.e. $[\mu]$ and $\int_\Omega g \, d\mu$ exists then so does $\int_\Omega h \, d\mu$ and equal

تكملة - فرع ⓐ نظرية ٢٠

تكملة الفرع ⓐ

hence $\int_\Omega f \, d\mu = 0$

if $f = f^+ - f^-$, then $f^+ \leq |f|$ and $f^- \leq |f|$

f^+, f^- are a.e. $[\mu]$ since $f=0$ a.e $[\mu]$

then

$\int_\Omega f^+ d\mu = \int_\Omega f^- d\mu = 0$

$\Rightarrow \int_\Omega f \, d\mu = 0$

[there is integral theory fun. that difference only on a set of measure 0 may be identified.]

تطابق

corollories

a) if $h_1, h_2, \dots$ are B.M. then $\int_\Omega \sum_{n=1}^{\infty} h_n \, d\mu = \sum_{n=1}^{\infty} \int_\Omega h_n \, d\mu$

ⓑ if h is B.M. integrable iff $\underset{h^+ h^-}{|h|}$ is integrable

ⓒ if g, h are B.M. with $|g| \le h$, h is integrable then g is integrable.

Def. A condition is said to hold almost every where (a.e) with respect to the measure μ (a.e $[\mu]$) iff there is a set $B \in \mathcal{F}$ of μ-measure 0 ($\mu(B) = 0$) such that the condition hold outside of B.

تتحقق

تكملة - فرع ⓐ نظرية 20

Theorem 20

let f, g and h be B.M. fun.

ⓐ if $f = 0$ a.e $[\mu]$, then $\int_\Omega f \, d\mu = 0$

ⓑ if $g = h$ a.e. $[\mu]$ and $\int_\Omega g \, d\mu$ exists then so does $\int_\Omega h \, d\mu$ and equal

تكملة الفرع ⓐ

hence $\int_\Omega f \, d\mu = 0$

if $f = f^+ - f^-$ then $f^+ \le |f|$ and $f^- \le |f|$

f^+, f^- are a.e. $[\mu]$ since $f = 0$ a.e $[\mu]$

then

$\int_\Omega f^+ d\mu = \int_\Omega f^- d\mu = 0$

$\Rightarrow \int_\Omega f \, d\mu = 0$

[there is integral theory fun. that difference only on a set of measure 0 may be identified.]

تطابق

تماثل

تعتبر شيئاً واحداً

proof

(a) if f simple

$\Rightarrow f = \sum_{i=1}^{n} x_i I_{A_i}$

and $x_i \neq 0$ then we have $\mu(A_i) = 0$ by hypothesis

b/c $[f = 0 , a.e[\mu]]$

hence $\int_\Omega f \, d\mu = 0$, [either $f = 0$ or $\mu(A) = 0$]

if $f \geq 0$ and $0 \leq s \leq f$

$\therefore$ s simple then $s = 0$ a.e $[\mu]$

and hence.

$$\int_\Omega s \, d\mu = 0$$

$$\Rightarrow \int_\Omega f \, d\mu = 0$$

if $f = f^+ - f^-$, then

$f^+ \leq |f|$ and $f^- \leq |f|$

where f^+, f^- are a.e$[\mu]$ since $f = 0$ a.e $[\mu]$

Then

$$\int_\Omega f^+ d\mu = \int_\Omega f^- d\mu = 0$$

$$\Rightarrow \int_\Omega f \, d\mu = 0$$

let $A = \{\omega : g(\omega) = h(\omega)\}$

and

let $B = A^c$

then

$$g = g I_A + g I_B$$

and $h = h I_A + h I_B$

since

$$g I_B = h I_B = 0 \quad \text{except on sets of measure } 0$$

$[$ i.e $g I_B = h I_B = 0$ a.e $[\mu]]$

Then the result follows from (a) and theorem 19.

let $A = \{\omega : g(\omega) = h(\omega)\}$

and

let $B = A^c$

then

$$g = g I_A + g I_B$$

and $h = h I_A + h I_B$

since

$g I_B = h I_B = 0$ except on sets of measure 0

[i.e $g I_B = h I_B = 0$ a.e $[\mu]$]

Then the result follows from (a) and theorem 19.

Theorem 21

let h be Borel measurable

(a) if h be integrable, then h is finite a.e

(b) if $h \geq 0$ and $\int_\Omega h \, d\mu = 0$ then $h = 0$ a.e.

Theorem 22 Extended monotone Convergence theorem

let $g_1, g_2, \ldots, g_\infty; h$ be Borel measurable

(a) if $g_n \geq h$ $\forall n$, where $\int_\Omega h \, d\mu > -\infty$

and $g_n \uparrow g$ then $\int_\Omega g_n \, d\mu \uparrow \int_\Omega g \, d\mu$

(b) if $g_n \leq h$ $\forall n$, where $\int_\Omega h \, d\mu < \infty$

and $g_n \uparrow g$ then $\int_\Omega g_n \, d\mu \downarrow \int_\Omega g \, d\mu$

مناسب

i.e the limit integral under propricte condition is equal to the integral of the limit.

fatou's lemma

let $f_1, f_2, \ldots, f$ be Borel measurable

ⓐ if $f_n \geq f \ \forall n$ where $\int_\Omega f \, d\mu > -\infty$

then $\liminf_{n\to\infty} \int_\Omega f_n \, d\mu \geq \int_\Omega \liminf_{n\to\infty} f_n \, d\mu$

ⓑ if $f_n \leq f, \forall n$ where $\int_\Omega f \, d\mu < \infty$

then $\limsup_{n\to\infty} \int_\Omega f_n \, d\mu \leq \int_\Omega \limsup_{n\to\infty} f_n \, d\mu$

مسيطر تقارب

Theorem 23 Dominated Convergence theorem

if $f_1, f_2, \dots, f$, g are Borel measurable

$|f_n| \leq g$, $\forall n$, where g is μ-integrable and

$f_n \to f$ a.e $[\mu]$, then f is μ-integrable and

$$\int_\Omega f_n \, d\mu \longrightarrow \int_\Omega f \, d\mu$$

<u>corallary</u> if $f_1, f_2, \dots, f$, g are Borel measurable

$|f_n| \leq g$, $\forall n$ where

$|g|^p$ is μ-integrable ($p > 0$ fixed)

and $f_n \to f$ a.e $[\mu]$

then

$|f|^p$ is μ-integrable and

$$\int_\Omega |f_n - f|^p \, d\mu \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

Theorem 24

if μ is σ- finite on F and g, h are Borel measurable

$$\int_\Omega g \, d\mu \text{ and } \int_\Omega h \, d\mu \text{ exist and}$$

$$\int_A g \, d\mu \leq \int_A h \, d\mu \, , \forall \, A \in F$$

then $g \leq h$ a.e $[\mu]$

Note

if μ is Lebesgue measure and $A = [a, b]$

and $\int_A f \, d\mu$ exists then we write

$$\int_A f \, d\mu = \int_a^b f(x) \, dx \quad \text{on } \mathbb{R}$$

and.

$$\int_A f \, d\mu = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \ldots dx_n \quad \text{on}$$

the end point may be deleted from the interval without changing the integral. since the Lebesgue measure of a single point is 0.

μ يساوي صفر

ASH ص 53

7 Comparison of Lebesgue and Rieman integral

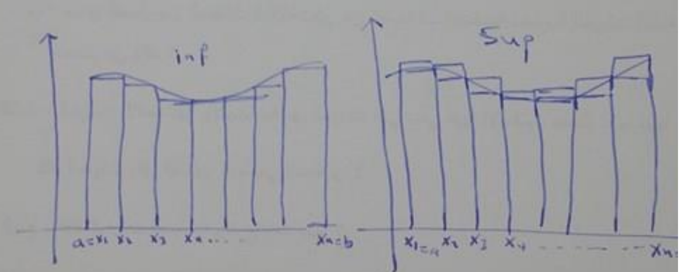
let [a,b] be abounded closed interval of reals and let f be bounded real valued fun on [a,b] fixed. Krough out the discussion
through ثابتة خلال شرح الموضوع.

if p apartition on the interval [a,b] such that

$p : a = x_1 < \dots < x_n = b$ جزء المساحة تحت المنحنى (الدالة)

نكون نبنيها.
we may be construct the upper and lower sum s o

relative to p as follows
النسبة



lower sum Upper sum

نستطيع جمع المستطيلات معنا تشكل المساحة تحت المنحني.

write smallest interval

let $M_i = \sup\{ f(y) : x_{i-1} < y \le x_i \}$, $i = 1, 2, \ldots, n$

and $m_i = \inf\{ f(y) : x_{i-1} < y \le x_i \}$, $i = 1, 2, \ldots, n$

and defin <u>step fun.</u> α and β called upper and lower fun. corresponding to p, by

$\Rightarrow \alpha(x) = M_i$ if $x_{i-1} < x \le x_i$, $i = 1, 2, \ldots, n$

$\Rightarrow \beta(x) = m_i$ if $x_{i-1} < x \le x_i$, $i = 1, 2, \ldots, n$

[$\alpha(a)$ and $\beta(a)$ may be chosen arbitrary]

The upper and lower sums given by

وزن كبير

$$U(p) = \sum_{i=1}^{n} u_i (x_i - x_{i-1})$$

وزن صغير

$$L(p) = \sum_{i=1}^{n} u_i (x_i - x_{i-1})$$

let $\Omega = [a,b]$,

$f = \beta[a,b]$, (the Lebesgue measurable subsets of [

u= Lebesgue measure since, α and β are simple functions we have.

$$U(p) = \int_a^b \alpha \, du$$

$$, L(p) = \int_a^b \beta \, du$$

now.

let $P_1, P_2, \cdots,$ be asequence of partitions on $[a,b]$

such that the P_{k+1} is refinement of P_k, $\forall k$
(تحسين دقة)

and such that $|P_k|$

(the length of Largest subintervual of P_k)

aproaches 0 as $k \to \infty$.

if α_k and β_k are upper and lower functions

Corresponding to P_k, then $\alpha_1 \geq \alpha_2 \geq \cdots \geq f \geq \cdots \geq \beta_k \geq \cdots \geq \beta_1$

thus α_k and β_k approach limit fun. α and β.

if $|f|$ is bounded by μ then all $|\alpha_k|$ and $|\beta_k|$ are bounded by μ also,

and the fun. that is constant at μ is integrable on $[a,b]$ with respect to μ.

since $\mu\,(a,b] = b-a < \infty$

by the demoniated convergence theorem

$$\lim_{k\to\infty} U(P_k) = \lim_{k\to\infty} \int_a^b \alpha_k \, d\mu = \int_a^b \alpha \, d\mu$$

$$\lim_{k\to\infty} L(P_k) = \lim_{k\to\infty} \int_a^b \beta_k \, d\mu = \int_a^b \beta \, d\mu.$$

if x is not an end point of any of the sub interval of the P_k, f is continuous at x

iff $\alpha(x) = f(x) = B(x)$

if $\lim U(P_k) = \lim L(P_k) =$ a finite number r independent of the particullar sequence of par[illegible]

is $r = r_{a,b}(f)$ is said to be the value of the Riemann integrable of f on $[a,b]$

if f is Riemann integrable then we have

$$r_{[a,b]} = \int_a^b \alpha \, d_\mu = \int_a^b B \, d_\mu$$

[illegible] [illegible]

Theorem 25   let f be abounded fun. defined a compact interv [a,b], let μ denote lebesgue measure on [a,b]

let f be abounded real-valued fun. on [a,b]

(a) f is R.I (Riemann integrable) on [a,b] iff f is continuous a.e (almost everywhere) [μ] on [a,b]

(with respect to) w.r.t. L.M. (μ Lebesgue Measure)

f is Riemann integrable iff f is continuous μ almost everyw on [a,b]

(b) if f is R. I on [a,b] then f is integrable (w.r.t.) L.M. on [a,b] and the two integrals are equal.

if f is Rimann integrable then f is lebesgue integrable

and $R-\int_a^b f(x)\,dx = \int_{[a,b]} f\,d\mu$

Remark:

theorem (25) holds in n dimension with [a,b] R^n (closed bounded).

مجموعة من الشروط اللازمة f $g_1, g_2, \ldots, g, h$ Borel measurable ترتيبي

① $g_n \geq h$	① $g_n < h$
② $\int_\Omega h\,d\mu > -\infty$	② $\int_\Omega h\,d\mu < \infty$
③ $g_n \uparrow g$	③ $g_n \downarrow g$
$\Rightarrow \int_\Omega g_n\,d\mu \uparrow \int_\Omega g\,d\mu$	$\Rightarrow \int_\Omega g_n\,d\mu \downarrow \int_\Omega g\,d\mu$

الشروط اللازمة هي bounded continuous

Remark

lebesgue integrable functions. under convenient condition exists and equal the integral of the limit.

The corresponding results for R.I. fun.s are more complicated because the limit of a sequence of R.I. fun.s. need not be R.I. fun. even if the entire sequence is uniformly bounded thus Riemann integrability of the limit fun. must be added~~tive~~ as a hypothesis and this is a serious limitation on the scope of the results

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