4. Correlation Coefficients

(1) Ordinary correlation coefficient

The correlation between two random variabl
X_1 and X_2 is

$$\rho = \rho_{x_1,x_2} = \frac{cov(X_1,X_2)}{\sqrt{var(X_1)var(X_2)}}$$

Let the mean vector, variance matrix and correlation matrix of $\underline{X}' = (x_1, x_2)$ be denoted by:

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and let $\sigma_{11} = \sigma_1^2$ $(\sigma_1 \geq$
, $\sigma_{22} = \sigma_2^2$ $(\sigma_2 \geq 0)$, then

$$\rho_{x_1,x_2} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

It is well known that

$$|\rho_{x_1,x_2}| \leq 1$$

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Partial variance covariance matrix

The elements of the v-covariance matrix of the conditional dist. $\Sigma_{11.2}$ are called the partial variance and covariance matrix for they measure the variation and dependence of the variates in the first set conditional upon the fixed values of those in the second set.

if we denot the ij elements of the matrix $\boxed{\Sigma_{11.2}}$ by

$\Sigma_{11.2} = \overline{\sigma_{ij \cdot q+1, q+2, \dots, p}}$ where $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$

$$\Sigma_{ii.j} = \begin{pmatrix} \overline{\sigma_{ii \cdot q+1, q+2, \dots, p}} & \overline{\sigma_{ij \cdot q+1, q+2, \dots, p}} \\ \overline{\sigma_{ji, q+1, q+2, \dots, p}} & \overline{\sigma_{jj, q+1, q+2, \dots, p}} \end{pmatrix}$$

Know. , if $i=1$, $j=2$ then

$$\Sigma_{11.2} = \begin{bmatrix} \overline{\sigma_{11, 3, 4, \dots, p}} & \overline{\sigma_{12, 3, 4, \dots, p}} \\ \overline{\sigma_{21, 3, 4, \dots, p}} & \overline{\sigma_{22, 3, 4, \dots, p}} \end{bmatrix}$$

where:

$\overline{\sigma_{ij, q+1, \dots, p}}$ be the partial covariance which is $\boxed{i,j}$ the elements of matrix $\boxed{\Sigma_{11.2}}$

$\sigma_{ii, q+1, \dots, p}$ be the partial variance which is the $\boxed{i,i}$ the elements of matrix $\boxed{\Sigma_{11.2}}$

Partial Correlation Coefficient:

The partial correlation of the ith and jth variates of first set, with all members of the second sets is:

$$\rho_{ij,q+1,q+2,\dots,p} = \frac{\sigma_{ij,q+1,q+2,\dots,p}}{\sqrt{\sigma_{ii,q+1,q+2,\dots,p}}\sqrt{\sigma_{jj,q+1,q+2,\dots,p}}}$$

where

$\rho_{ij,q+1,\dots,p}$; is the partial correlation cofficient between x_i and x_j of $X^{(1)}$ when $X_{q+1},\dots,X_p$ of $X^{(2)}$ are fixed.

also: if the number of fixed varintes is small the partial correlat may be computed as:

$$\rho_{ij,h} = \frac{\rho_{ij} - \rho_{ih}\rho_{jh}}{\sqrt{(1-\rho_{ih}^2)(1-\rho_{jh}^2)}}$$

and $\rho_{ij.h} = \rho_{ij,q+1,q+2,\dots,p}$

$i,j = 1,2,\dots,p$

$h = q+1,\dots,p$

(2) Partial correlation coefficient

Let:

$\sigma_{ij.q+1,\dots,p}$: be the i,jth element of $\Sigma_{11.2}$
we call these partial covariance

$\sigma_{ii.q+1,\dots,p}$: is a partial variances

The partial correlation between X_1 and X_2 of the vector $\underline{X}_1$ holding $X_3,\dots,X_p$ the elements of the second vector $\underline{X}_2$ fixed is

$$\rho_{12.3\dots p} = \frac{\sigma_{12.3\dots p}}{\sqrt{\sigma_{11.3\dots p}}\sqrt{\sigma_{22.3\dots p}}}$$

The partial correlation coefficient allows to measure the linear dependence of any two variables of the first set by removing the linear association of the variables in the second set. We denote

$$\Sigma_{11.2} = \begin{pmatrix} \sigma_{11.3\dots p} & \sigma_{12.3\dots p} \\ \sigma_{21.3\dots p} & \sigma_{22.3\dots p} \end{pmatrix}$$

For example, when $p=3$

$$\rho_{12.3} = \frac{\sigma_{12.3}}{\sqrt{\sigma_{11.3}}\sqrt{\sigma_{22.3}}} = \frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{(1-\rho_{13}^2)(1-\rho_{23}^2)}}$$

Ex Let $\underline{X} \sim N_3(\underline{\mu}, \Sigma)$ with $\underline{\mu} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

and $\Sigma = \begin{pmatrix} 4 & -1 & 0 \\ & 2 & 1 \\ & & 6 \end{pmatrix}$

Find $\rho_{13\cdot 2}$

Sol $\rho_{13\cdot 2} = \dfrac{\sigma_{13\cdot 2}}{\sqrt{\sigma_{11\cdot 2}}\sqrt{\sigma_{33\cdot 2}}}$

$$\Sigma = \begin{array}{c} \begin{array}{ccc} X_1 & X_3 & X_2 \end{array} \\ \left(\begin{array}{cc|c} 4 & 0 & -1 \\ 0 & 6 & 1 \\ \hline -1 & 1 & 2 \end{array}\right) \end{array} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

$\therefore \Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 7/2 & 1/2 \\ 1/2 & 11/2 \end{pmatrix}$$

$$\rho_{13\cdot 2} = \frac{1/2}{\sqrt{7/2}\sqrt{11/2}} = 0.114$$

Remark: The partial correlation matrix between subvector X_1 holding the elements of the second subvector X_2 is defined as follows.

$$\rho_{11\cdot2} = D^{-\frac{1}{2}} \Sigma_{11\cdot2} D^{-\frac{1}{2}} \qquad \boxed{\sigma_i^{-\frac{1}{2}} = \frac{1}{\sqrt{\sigma_i}}}$$

where $D = \mathrm{diag}(\Sigma_{11\cdot2}) = \mathrm{diag}(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

space of $\Sigma_{11\cdot2}$ containing the reciprocal square root of $\Sigma_{11\cdot2}$

Ex For the last example, find $\rho_{11\cdot2}$

Sol

$$D^{\frac{1}{2}} = \begin{pmatrix} \sqrt{7/2} & 0 \\ 0 & \sqrt{11/2} \end{pmatrix}$$

$$D^{-\frac{1}{2}} = \begin{pmatrix} \sqrt{2/7} & 0 \\ 0 & \sqrt{2/11} \end{pmatrix}$$

$$\rho_{11\cdot2} = D^{-\frac{1}{2}} \Sigma_{11\cdot2} D^{-\frac{1}{2}}$$

$$= \begin{pmatrix} \sqrt{2/7} & 0 \\ 0 & \sqrt{2/11} \end{pmatrix} \begin{pmatrix} 7/2 & 1/2 \\ 1/2 & 11/2 \end{pmatrix} \begin{pmatrix} \sqrt{2/7} & 0 \\ 0 & \sqrt{2/11} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0.114 \\ 0.114 & 1 \end{pmatrix}$$

note

$\rho_{11\cdot2} = D^{-\frac{1}{2}} \Sigma_{11\cdot2} D^{-\frac{1}{2}}$

$\rho_{11\cdot2} = D^{-1/2} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) D^{-\frac{1}{2}}$

ملاحظة.

① الارتباطات الجزئية تحسب من مصفوفة التباين والتباين المشترك الشرطية $(V-C)$

② عناصر مصفوفة $(V-C)$ الشرطية التي على القطر تسمى التباينات الجزئية partial variance اما العناصر التي حول القطر فتسمى (التباينات المشتركة) (partial covariance)

مثلاً لدينا $X_1 X_2 / X_3$

$$\Sigma_{11.2} = \begin{pmatrix} \sigma_{11.3} & \sigma_{12.3} \\ \sigma_{21.3} & \sigma_{22.3} \end{pmatrix}$$

$$\begin{pmatrix} \text{تباين} & \text{تباين مشترك} \\ \text{تباين مشترك} & \text{تباين} \end{pmatrix}.$$

③ يحسب معامل الارتباط الجزئي بنفس القانون المحسوب فيه معامل الارتباط البسيط اي ان

$$\text{معامل الارتباط الجزئي} = \frac{\text{التباين المشترك}}{\sqrt{\text{التباين للأول} \cdot \text{التباين للثاني}}}$$

④ اختصاراً في الكتابة نرمز لمجموعة متغيرات الكيف (الثوابت) fixed. خاصة اذا كانت تتضمن اكثر من عنصر فنرمز لها مثلاً $\boxed{C}$.
مثلاً اذا كانت مجموعة الكيف $\boxed{C}$ هي $\boxed{p, \dots, q+1}$ فنرمز لها بالرمز $\boxed{C}$ لتبسيط كتابة القانون

⑤ مصفوفة الارتباطات الجزئية يمكن حسابها بالصيغة التالية

$$\rho_{11.2} = D^{-\frac{1}{2}} \left(\Sigma_{11.2}\right) D^{-\frac{1}{2}}$$

where: D : diagonal of $(\Sigma_{11.2})$

اي ان D مصفوفة قطرية عناصر القطر فيها هي نفس عناصر القطر للمصفوفة $\Sigma_{11.2}$
اما $D^{-\frac{1}{2}}$ فهي " " " " " تمثل مقلوب الجذر التربيعي لعناصر المصفوفة D

Cofficient of Determination: R^2

The cofficient of determination of the ith variates of the first set with all members of the second set is:

$$R^2_{1.q+1,q+2,\dots,p} = 1 - \frac{\sigma_{ii,q+1,\dots,p}}{\sigma_{ii}}$$

or $$= \frac{\sigma_{(i)} \Sigma_{22}^{-1} (\sigma_{i})'}{\sigma_{ii}}$$

where $$\sigma_{ii,q+1,\dots,p} = \sigma_{ii} - \sigma_i \Sigma_{22}^{-1} \sigma_i' = Var(x_i/\underline{x}_2)$$

Since $\sigma_{(i)} = E(x_i)(\underline{x}^{(2)})$ is the ith row of the matrix Σ_{12}

$$\Sigma_{22} = E(\underline{x}^{(2)})(\underline{x}_i^{(2)})'$$

$$\sigma_{(i)}' = E(x^{(2)})(x_i)$$

note

$$\Sigma^{*} = \begin{pmatrix} \sigma_{ii} & \sigma_{(i)}' \\ \sigma_{(i)} & \Sigma_{22} \end{pmatrix} \quad \text{if } \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$$

(3) Multiple Correlation coefficient

Consider $\underline{X}$ partitioned into:

$$\underline{X} = \begin{pmatrix} x_1 \\ \hline x_2 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} x_1 \\ \underline{X}_2 \end{pmatrix}$$

The multiple correlation coefficient between the variable X_1 and the elements of the subvector $\underline{X}_2' = (x_2, \ldots, x_p)$ is given as follows.

$$R_{i.q+1,\ldots,p} = \sqrt{\frac{\sigma_{(i)} \Sigma_{22}^{-1} \sigma_{(i)}'}{\sigma_{ii}}}$$

or

$$R^2_{i.q+1,\ldots,p} = \frac{\sigma_{(i)} \Sigma_{22}^{-1} \sigma_{(i)}'}{\sigma_{ii}}$$

where:

$\sigma_{(i)}$: is the ith row of Σ_{12}.

σ_{ii} : is the variance of the ith variabl

In particular, when $p=3$

$$R_{1.23} = \sqrt{\frac{\sigma_{(1)} \Sigma_{22}^{-1} \sigma_{(1)}'}{\sigma_{11}}} = \sqrt{\frac{\rho_{12}^2 + \rho_{13}^2 - 2\rho_{12}\rho_{13}\rho_{23}}{1 - \rho_{23}^2}}$$

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also:

The multiple Correlation Coefficient

يقيس علاقة ارتباط متغير
مع مجموعة المتغيرات

is the maximum Correlation between X_i and the linear Combination $\beta \underline{X}^{(2)}$ is called the multiple Correlation Coefficient between X_i and $\beta \underline{X}^{(2)}$ which is given by:

$$R_{i,q+1,\dots,p} = \sqrt{\frac{\sigma_{(i)} \Sigma_{22}^{-1} \sigma_{(i)}'}{\sigma_{ii}}} = \frac{E\, \beta \underline{X}^{(2)} X_i}{\sqrt{\sigma_{ii}\, E \beta \underline{X}^{(2)} \underline{X}^{(2)\prime} \beta'}}$$

proof

since $\beta = \sigma_{(i)} \Sigma_{22}^{-1}$

Then

$$R_{i,q+1,\dots,p} = \frac{Cov(X_i, \beta \underline{X}^{(2)})}{\sqrt{V(X_i)\, V(\beta \underline{X}^{(2)})}}$$

$Var(\beta X^{(2)})$

$$= \frac{E X_i \beta \underline{X}^{(2)} - E X_i\, E \beta X^{(2)}}{\sqrt{[E X_i^2 - (E X_i)^2]\, Var(\beta X^{(2)})}}$$

بنفرض انه مساوي
للصفر بيطلع
لنا

$$= \frac{E X_i \beta \underline{X}^{(2)} - E X_i\, E \beta \underline{X}^{(2)}}{\sqrt{[E X_i^2 - (E X_i)^2][V(\beta X^{(2)})]}}$$

بنفرض انه مساوي
للصفر بيطلع لنا

$$= \frac{\beta E\, \underline{X}^{(2)} X_i}{\sqrt{\sigma_{ii}\, \beta E(X^{(2)} X^{(2)\prime}) \beta'}} = \frac{\beta \sigma_{(i)}'}{\sqrt{\sigma_{ii}\, \beta \Sigma_{22} \beta'}}$$

$$= \frac{\sigma_{(i)} \Sigma_{22}^{-1} \sigma_{(i)}'}{\sqrt{\sigma_{ii}\, \sigma_{(i)} \Sigma_{22}^{-1} \Sigma_{22} \Sigma_{22}^{-1} \sigma_{(i)}'}} = \frac{\sigma_{(i)} \Sigma_{22}^{-1} \sigma_{(i)}'}{\sqrt{\sigma_{ii}\, \sigma_{(i)} \Sigma_{22}^{-1} \sigma_{(i)}'}}$$

$$R_{i,q+1,\dots,p} = \sqrt{\frac{\sigma_{(i)} \Sigma_{22}^{-1} \sigma_{(i)}'}{\sigma_{(ii)}}}$$

Suppose that in the joint dist. specified by the partitioned mean vector and covariance matrix the first set contains a single variate $\boxed{X_1}$ and the second set contains q variates. It is designed to find that Linear componed. $Y = BX^{(2)}$ $BX^{(2)}$ of the second set having the greatest correlation with X_1 that correlation is

$$r_{xy} = \frac{Cov(X_1, y)}{\sqrt{V(X_1)\ V(y)}} = \frac{\beta' \delta_{12}}{\sigma_{11}\sqrt{\beta' \Sigma_{22} \beta}}$$

$$Y = BX^{(2)'}$$
$$EY = B\,E X^{(2)'} = B\mu_2$$

where

$$\begin{aligned} Cov(X_1, y) &= E(X_1 - EX_1)(y - Ey) = E(X_1 - \mu_1)(Y - B\mu_2)' \\ &= E(X_1 - \mu_1)(BX^{(2)'} - B\mu^{(2)'}) \\ &= E(X_1 - \mu_1)(X^{(2)} - \mu^{(2)})B' \\ &= \delta_{12} B' \end{aligned}$$

where σ_1 is the single element, of Σ_{11} and $\delta_{11} = \Sigma_{12}'$

to find β which maximized the correlation we must put some constant and then find the value of β which make the correlation maximized and that is.

$$\underline{\beta} = \Sigma_{22}^{-1} \delta_{12}$$

$$\underline{\beta} = \Sigma_{21}^{-1} \Sigma_{21}$$

$$\begin{pmatrix} Y \\ X_1 \\ X_2 \end{pmatrix} = V\text{-}Cov = \left(\begin{array}{c|cc} v(y) & cov(X_1, y) & cov(X_2, y) \\ \hline & V(X_1) & Cov(X_1, X_2) \\ & & V(X_2) \end{array}\right)$$

$$\Sigma_{22}$$

and the vector is called the vector of rgression cofficients of variate X_1 upon the elemats of X_2
it is will be necessary to consider all regression cofficients of one set of p variate or asecond set of q and we shall write that set of parameters as the $(p \times q)$ matrix

$$\boxed{B = \Sigma_{12}\Sigma_{22}^{-1}}.$$

Then the maximum Correlation between X_1 and the linear Compond $B' X^{(2)}$ is

$$\rho_{1.2\ldots q+1} = \frac{B'\Sigma_{21}}{\sigma_1 \sqrt{B'\Sigma_{22} B}}$$

$$= \frac{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}{\sigma_1 \sqrt{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}\Sigma_{22}^{-1}\Sigma_{21}}}$$

where $B = \Sigma_{22}^{-1}\Sigma_{21}$

$$\Rightarrow \rho_{1.2\ldots q+1} = \frac{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}{\sigma_1\sqrt{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}} = \frac{\sqrt{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}}{\sigma_1}$$

Ex If $\underline{X} \sim N_3\left(\underline{0}, \begin{pmatrix} 4 & -1 & 0 \\ & 2 & 1 \\ & & 6 \end{pmatrix}\right)$

Find (a) $R^2_{1.23}$

(b) $R^2_{2.13}$

Sol $\underline{X} = \begin{pmatrix} x_1 \\ \hline x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ \underline{x}_2 \end{pmatrix}$

$$\Sigma = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{c|cc} 4 & -1 & 0 \\ \hline -1 & 2 & 1 \\ 0 & 1 & 6 \end{array}\right) \end{array} = \begin{pmatrix} \Sigma_{11} & \sigma_{(1)} \\ \sigma'_{(1)} & \Sigma_{22} \end{pmatrix}$$

$$\Sigma_{22}^{-1} = \frac{\operatorname{adj}\Sigma_{22}}{|\Sigma_{22}|} = \begin{pmatrix} 6/11 & -1/11 \\ -1/11 & 2/11 \end{pmatrix}$$

$$R^2_{1.23} = \frac{\sigma_{(1)} \Sigma_{22}^{-1} \sigma'_{(1)}}{\sigma_{11}}$$

$$= \frac{(-1 \quad 0)\begin{pmatrix} 6/11 & -1/11 \\ -1/11 & 2/11 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix}}{4}$$

$$= \frac{6}{44}$$

properties of multiple correlation

(1) if $\underline{X}_1 = X_1$, $\underline{X}_2 = X_2, \dots X_{q+1}$ then the partial covariance matrix is

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$$

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$\Sigma_{11.2} = \sigma_1^2 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$\Sigma_{11.2} = \sigma_1^2\left(1 - \frac{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}{\sigma_1^2}\right)$$

$$\Sigma_{11.2} = \sigma_1^2\left(1 - \rho_{11.2}^2\right)$$

$$\Rightarrow \rho_{11.2}^2 = 1 - \frac{\Sigma_{11.2}}{\sigma_1^2}$$

(2) for the special case bivariate normal density $q=1$ then the regression coefficient is:

$$B' = \Sigma_{12}\Sigma_{22}^{-1} = \frac{\sigma_{12}}{\sigma_{22}} = \frac{\rho_{12}\sigma_1\cancel{\sigma_2}}{\sigma_2\cancel{\sigma_2}} = \boxed{\rho_{12}\frac{\sigma_1}{\sigma_2}}$$

$$\rho_{1.2} = \sqrt{\frac{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}{\sigma_1^{2}}} = \sqrt{\frac{\rho_{12}\cancel{\sigma_1}\cancel{\sigma_2}\frac{1}{\cancel{\sigma_2^2}}\rho_{12}\cancel{\sigma_1}\cancel{\sigma_2}}{\cancel{\sigma_1^2}}} = \sqrt{\rho_{12}^2} = \rho_{12}$$

(3) if $\underline{X}_1 = X_2$, $\underline{X}_2 = X_1 \Rightarrow \rho_{12} = \rho_{21} = \rho_{2.1}$

(4) The multiple correlation is invariate under nonsingular transformation original variates:

let $y_1 = aX_1 + b$

$y_2 = cX_2 + d$

where a, b scaler, c matrix of $q \times q$ has full rank and d vector. then

$$\underset{X}{\rho_{1.2\dots q+1}} = \underset{Y}{\rho^2_{1.2\dots q+1}}$$

$$\underset{Y}{\rho^2_{1.2\dots(q+1)}} = \frac{\underset{Y}{\Sigma_{12}}\,\underset{Y}{\Sigma_{22}^{-1}}\,\underset{Y}{\Sigma_{21}}}{\underset{Y}{\sigma_1^2}}.$$

where $\Sigma_Y = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$

and $\underset{Y}{\Sigma_{11}} = V(y_1) = a^2 V(x_1) = \boxed{a^2 \sigma_1^{\,2}} = a \, \Sigma_{11} \, a'$

$\underset{Y}{\Sigma_{22}} = E(y_2 - Ey_2)^2$ — $y_2 = cx_2 - d$

$\underset{Y}{\Sigma_{22}} = E(cx_2 + \cancel{d} - cEx_2 - \cancel{d})(cx_2 + \cancel{d} - cEx_2 - \cancel{d})'$

$\boxed{\underset{Y}{\Sigma_{22}} = c \underset{X}{\Sigma_{22}} c'}$

$\underset{Y}{\Sigma_{12}} = E(y_1 - Ey_1)(y_2 - Ey_2)'$

$= E(aX_1 + \cancel{b} - aEX_1 - \cancel{b})(cX_2 + \cancel{d} - cEX_2 - \cancel{d})'$

$= a\, E(X_1 - \mu)(X_2 - \mu)' c'$

$\boxed{\underset{Y}{\Sigma_{12}} = a \underset{X}{\Sigma_{12}} c'}$

$$\rho_{1.2\cdots q+1} = \frac{\cancel{a} \underset{X}{\Sigma_{12}} c' \, (c \underset{X}{\Sigma_{22}} c')^{-1} \, c \underset{X}{\Sigma_{21}} \cancel{a}}{\cancel{a^2} \cdot \sigma_1^{\,2}}$$

$$= \frac{\Sigma_{12} \, c' \, c'^{-1} \, \Sigma_{22}^{-1} \, c^{-1} \, c \, \Sigma_{21}}{\sigma_1^{\,2}}$$

$$\underset{X}{\rho_{1.2\cdots q+1}} = \frac{\Sigma_{12} \, \Sigma_{22}^{-1} \, \Sigma_{21}}{\sigma_1^{\,2}}$$

Note

if $\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}$ is partitioned as $\underline{X} = \begin{pmatrix} X_1 \\ \hline X_2 \\ \vdots \\ X_p \end{pmatrix} \Rightarrow X = \begin{pmatrix} X_1 \\ \hline X_2 \end{pmatrix}$

Then.

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

$$\Rightarrow \quad \Sigma^* = \begin{pmatrix} \sigma_{ii} & \sigma'_{(i)} \\ \sigma_{(i)} & \Sigma_{22} \end{pmatrix}$$

$\sigma'_{(i)}$ is the ith row of Σ_{12}

هو صف وللتوضيح

$$\Sigma^* = \left(\begin{array}{c|cc} \overset{\sigma_{ii}}{\sigma_{11}} & \overbrace{\sigma_{12} \quad \sigma_{13}}^{\sigma'_{(i)}} & \\ \hline \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \underbrace{\sigma_{31}}_{\sigma_{(i)}} & \sigma_{32} & \underbrace{\sigma_{33}}_{\Sigma_{22}} \end{array}\right)$$

لذلك فان σ_{ii} تكون عنصر واحد.

$$V(X_i \,/\!/\, \underline{X}_2 = \underline{x}_2) = \sigma_{ii \cdot q+1, q+2, \dots, p} = \sigma_{ii} - \sigma'_{(i)} \Sigma_{22}^{-1} \sigma_{(i)}$$

توضيح

Assume that $\mu = 0$.

Then $x - \mu$ is replaced by $\underline{X}$ choose a component x_i of $\underline{X}_1$ then.

$$E(\underline{X}_1 / X_2) = \underline{\mu}_1 + \Sigma_{12} \Sigma_{22}^{-1} (X_2 - \mu_2)$$

$$= 0 + \Sigma_{12} \Sigma_{22}^{-1} (X_2 - 0) \qquad (\mu$$

$$\boxed{E(\underline{X}_1 / X_2) = \Sigma_{12} \Sigma_{22}^{-1} \underline{X}_2}$$

$$\Rightarrow E(x_i / \underline{X}_2) = \sigma_{(i)}' \Sigma_{22}^{-1} \underline{X}_2$$

$$= \beta' \underline{X}_2$$

because $\boxed{\beta' = \sigma_{(i)}' \Sigma_{22}^{-1}}$

and $\sigma_{(i)}'$ is the ith row of Σ_{12}

توضيح 2

$$Y = aX_1 + bX_2 + cX_3$$

$$Y = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$\boxed{Y = \beta' \underline{X}_2}$$

linear Combination.

توضيح 3.

$\Sigma_{12} \Sigma_{22}^{-1}$ is called the matrix of regression cofficient of $\underline{X}_1$ on X_2

and the i, j elements of $\Sigma_{12} \Sigma_{22}^{-1}$ is denoted by $\beta_{ij \cdot 2, \dots, j-1, j+1, \dots, p}$

توضيح 4.

$\beta = \Sigma_{22}^{-1} \underline{\sigma}(i)$ is a column vector of ~~areg~~ a regression cofficient of X_1 on the elements of $\underline{X}_2$

ملاحظة

توضيح ؟

Note

$$\rho_{1.2\ldots q+1} = \sqrt{\frac{\sigma_{(i)}' \Sigma_{22}^{-1} \sigma_{(i)}}{\sigma_{ii}}}$$

$$\rho^2_{1.2\ldots q+1} = \frac{\sigma_{(i)}' \Sigma_{22}^{-1} \sigma_{(i)}}{\sigma_{ii}}$$

$1 + \bigcirc = 1 + \bigcirc$

$1 - \bigcirc = 1 - \bigcirc$

نطرح واحد
من الطرفين

فينتج لنا .

$$1 - \rho^2_{1.2\ldots q+1} = 1 - \frac{\sigma_{(i)}' \Sigma_{22}^{-1} \sigma_{(i)}}{\sigma_{ii}}$$

نوحد المقام

$$1 - \rho^2_{1.2\ldots q+1} = \frac{\sigma_{ii} - \sigma_{(i)}' \Sigma_{22}^{-1} \sigma_{(i)}}{\sigma_{ii}}$$ ... العلاقة

$$\Rightarrow \boxed{1 - \rho^2_{1.2\ldots q+1} = \frac{|\Sigma^*|}{\sigma_{ii}\, |\Sigma_{22}|}}$$

نتيج

where $|\Sigma^*| = |\Sigma_{22}| \cdot |\Sigma_{11.2}|$

if $\Sigma^* = \begin{pmatrix} \sigma_{ii} & \sigma'_{(i)} \\ \sigma_{(i)} & \Sigma_{22} \end{pmatrix}$

but $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

then $\quad = \sigma_{ii} - \underline{\sigma}'_{(i)}\Sigma_{22}^{-1}\underline{\sigma}_{(i)}$

[illegible]

$\Rightarrow \quad |\Sigma^*| = |\Sigma_{22}|\,|\sigma_{ii} - \sigma'_{(i)}\Sigma_{22}^{-1}\sigma_{(i)}|$

[illegible] - [illegible]

$\Rightarrow \quad \dfrac{|\Sigma^*|}{|\Sigma_{22}|} = (\sigma_{ii} - \sigma'_{(i)}\Sigma_{22}^{-1}\sigma_{(i)})$

نعوض في العلاقة فينتج لنا .

$$1 - \rho^2_{1.2\ldots q+1} = \frac{|\Sigma^*|}{\sigma_{ii}\,|\Sigma_{22}|}$$

ex: let $X \sim N(\mu, \Sigma)$ find $\rho_{1.23}$

where
$$\Sigma = \left(\begin{array}{c|cc} 6 & 1 & -1 \\ \hline 1 & 4 & 0 \\ -1 & 0 & 2 \end{array}\right)$$

By using $\quad 1-\rho^2_{1.23} = \dfrac{|\Sigma^*|}{\sigma_{11}\,|\Sigma_{22}|}$

where $\Sigma = \Sigma^* = \begin{pmatrix} \sigma_{11} & \sigma_{12}' \\ \sigma_{12} & \Sigma_{22} \end{pmatrix}$

then $|\Sigma^*| = |\Sigma|$

$$= 6\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} - (1)\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix}$$

$$= (6)(8) - 2 - 4$$

$|\Sigma^*| = 42$

and $|\Sigma_{22}| = 8$

then $\quad 1-\rho^2_{1.23} = \dfrac{42}{6(8)} = \dfrac{42}{48} = 0.88$

$\therefore \rho^2_{1.23} = 1 - 0.88 = 0.12$

$\rho_{1.23} = \sqrt{0.12}$

$\rho_{1.23} = 0.35$

ملاحظة
توضيح 5 .

Note 5

The multiple Correlation is invariate under non-singular transformation

هذه خاصية من خصائص الارتباط المتعدد
وتعني بان قيمة معامل الارتباط المتعدد
لا تتغير بوجود التحويلات غير الاعتيادية.

for $y_1 = a x_1 + b$

$y_2 = C x_2 + d$ where a, b scalars.
d is avector of q constant
c is amatrix of full rank of order $q \times q$

where a, b scalars.
d is avector of q constant.
c is amatrix of full rank of order $q * q$

ملاحظة
full rank : تعني ان المصفوفة كل الاعمدة لها مستقلة .
وان عدد الصفوف = عدد الاعمدة.
وكذلك ان المحددة موجودة
وبذلك فانها لها رتبة

$|c| \neq 0$

$r(c) = q$

ولذلك فاذا حسبنا معامل الارتباط المتعدد ال X فان قيمة هذا المعامل
لا تتغير عن قيمة Y_1 بشرط ان تكون قيمة التحويل اعتيادية

و ذلك يعني بان قيمة معامل الارتباط المتعدد للمتجه $\underline{X}$
لا تختلف عن قيمة معامل " " المحسوب للمتجه $\underline{Y}$
بشرط ان يكون التحويل من X ال Y يتم من خلال المصفوفة c
التي تكون مصفوفة غير اعتيادية $|c| \neq 0$ وهي ترمز
بمصفوفة التحويل

<u>example</u> : let $X \sim N_3(\underline{\mu}, \Sigma)$

$$\Sigma = \begin{pmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

and Suppose the linear transformation

$$Y_1 = 5X_1 + 2$$
$$Y_2 = 3X_2 + X_3 - 1$$
$$Y_3 = X_2 + X_3 + 2$$

Show that the matrix correlation cofficient $\rho_{1.23}$ is invariate for $\underline{Y}$ under this transformation.

Sol To find $\rho_{1.23}$ for $\underline{X}$

for X $\quad \Sigma = \left(\begin{array}{c|cc} 6 & 1 & -1 \\ \hline 1 & 4 & 0 \\ -1 & 0 & 2 \end{array}\right) = \begin{pmatrix} \sigma_{(11)} & \sigma'_{(1)} \\ \sigma_{(1)} & \Sigma_{22} \end{pmatrix}$

$$\rho_{1.23} = \sqrt{\frac{(1 \;\; -1)\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{6}}$$

$$= \sqrt{\frac{3/4}{6}}$$

$$\boxed{\rho_{1.23} = 0.35}$$

for Y

$$Y_2 = C X_2 + d$$

$$\begin{pmatrix} Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} X_2 \\ X_3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$|C| = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 3-1 = \boxed{2}$$ exist $\neq 0$

موجود

$\therefore C$ is a non singular matrix

and $\boxed{r(C) = 2}$

$$\underline{Y} = A\underline{X}$$

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$\Sigma_Y = A \Sigma_X A'$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Sigma_Y = \left(\begin{array}{c|cc} 150 & 10 & 0 \\ \hline 10 & 38 & 14 \\ 0 & 14 & 6 \end{array}\right) = \left(\begin{array}{c|c} \sigma_{11} & \sigma_{(1)}' \\ \hline \sigma_{(1)} & \Sigma_{22} \end{array}\right)$$

$$\therefore \quad \rho_{1\cdot 23} = \sqrt{\frac{(10 \;\; 0)\begin{pmatrix} 38 & 14 \\ 14 & 6 \end{pmatrix}^{-1}\begin{pmatrix} 10 \\ 0 \end{pmatrix}}{150}}$$

$$\rho_{1.23} = 0.35$$

توضيح او ملاحظة

Note. 6.

if Σ and Σ_{22} both have the same Rank q, then X_1 can be expressed exactly as the linear compon~~ent~~ of the q. variates of the second set then the multiple Correlation is exactly unity.

اذا كانت رتبة المصفوفة $\boxed{\Sigma}$ التي درجتها $\boxed{q+1}$ في $\boxed{q+1}$ اي $\boxed{(q+1)\times(q+1)}$
هي مصفوفة احادية رتبتها تساوي $\boxed{q}$ وهي عدد العناصر المتغيرات
للمجموعة الثانية ومن نفس الوقت هي تمثل رتبة Σ_{22}.
فان المتغير المعتمد X_1 يمكن التعبير عنه بعلاقة خطية تشمل
جميع متغيرات المجموعة الثانية بمعامل واحد لكل منها وفي هذه
الحالة نتوصل الى حالة الارتباط التام
عندما نحسب معامل الارتباط المتعدد

Note. 7 The Same multiple Correlation will obtained from the matrix of the correl as form as the (v-c) matrix.

Ex:

let $\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\underline{X} \sim N_3(\mu, \Sigma)$

find:

1. The conditional dist. of $(x_1, x_2 / x_3)$
2. The partial variance and partial covariance for $(x_1, x_2 / x_3)$
3. find the partial correlation of $x_1, x_2 / x_3$
4. find the partial correlation matrix of $x_1, x_2 / x_3$
5. find the correlation matrix.
6. Show that.

$$\rho_{12.3} = \frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{(1-\rho_{13}^2)(1-\rho_{23}^2)}}$$

الكراس
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الدفتر الابيض

Note: The conditional (v-c) matrix ($\Sigma_{11.2}$) also called The partial (v-c) matrix

توضيح مثلاً

$\Sigma_{11.2}$ من درجة (3×3)

هذا يعني

$$\rho = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ & 1 & \rho_{23} \\ & & 1 \end{pmatrix}$$

Ex: let $X \sim N_3(\mu, \Sigma)$, where $\mu = \begin{pmatrix} 5 \\ 0 \\ -3 \end{pmatrix}$, $\Sigma = \begin{bmatrix} 9 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

Find

ص 62 63 64 65 66

(1) The partial Correlation ~~Correlation~~ $\rho_{13.2}$

(2) " " " matrix for $X_2, X_3 / X_1$

(3) The Correlation matrix ρ.

(4) what is the value of $\sigma_{11.2}$ and $\sigma_{33.2}$

[illegible] الدفتر الأخير

ex: for a trivariate normal dist. : Show that:

(1) $R^2_{1.23} = 1 - \dfrac{\sigma_{11.23}}{\sigma_{11}}$

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(2) $\rho_{12.3} = \dfrac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{(1-\rho_{13}^2)(1-\rho_{23}^2)}}$

(3) $\rho_{13.2} = \dfrac{\rho_{13} - \rho_{12}\rho_{23}}{\sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)}}$

(4) $E(X_1 / X_2 X_3)$

ex: let $X \sim N_4(\underline{\mu}, \Sigma)$, where $\underline{\mu} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$ &

$$\Sigma = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

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Find (a) $\rho_{12.3}$, $\rho_{12.4}$, $\rho_{23.4}$

(b) $R^2_{1.23}$

(c) The conditional dist. of $X_1 X_2 / X_3$

(d) The regression matrix of $X_1 X_2 / X_3 X_4$

(e) The regression cofficient of $X_1 / X_2 X_3 X_4$

(f) " regression function of $X_1 / X_2 X_3 X_4$

دفتر ابيض ص ٧١

Ex. for the bivariate case let $X \sim N_2(\underline{\mu}, \Sigma)$

اكل ع ٧٢ ٧٣ دفتر ابيض

Find ① The regression coffizient β_{12} for X_1 ~~and~~ on X_2

② the " " β_{21} for X_2 on X_1

③ show that $\rho_{12} = |\rho|$

Ex: let $\underline{X} \sim N_4(\underline{\mu}, \Sigma)$, where $\underline{\mu} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$, $\Sigma = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1/2 \\ 0 & 0 & 2 & 1 \\ 0 & 1/2 & 1 & 1 \end{bmatrix}$

Find

① $f(X_1, X_2, X_3)$

② The conditional dist. for $X_1, X_2 / X_3$

③ $\rho_{12 \cdot 3}$, $\rho_{13 \cdot 4}$, $\rho_{12 \cdot 4}$, $\rho_{13 \cdot 4}$

④ Conditional dist for $X_1, X_2, X_3 / X_4$

⑤ $\rho_{1 \cdot 234}$

⑥ Conditional dist. for $X_1 / X_2, X_3$

اكل ص 74-75, 76, 77, 78, 71 الدفتر الابيض

⑦ $\rho_{2 \cdot 134}$, and $\rho_{4 \cdot 123}$