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## Partitioned matrices

Suppose we partition $A$ as follows.

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

where $A_{11}$ and $A_{22}$ are both square and non singular.

and Then the elements of the inverse are given by:

$$A^{-1} = \left[\begin{array}{c|c} Q^{-1} & -Q^{-1} A_{12} A_{22}^{-1} \\ \hline -A_{22}^{-1} A_{21} Q^{-1} & A_{22}^{-1} + A_{22}^{-1} A_{21} Q^{-1} A_{12} A_{22}^{-1} \end{array}\right]$$

Where

$$Q = A_{11} - (A_{12} A_{22}^{-1} A_{21})$$

it is necessary to compute $|A|$ So:

(i) if $A_{11}$ is non singular then $|A| = |A_{11}| \cdot |A_{22} - A_{21} A_{11}^{-1} A_{12}|$

(ii) if $A_{22}$ is non singular then $|A| = |A_{22}| \cdot |A_{11} - A_{12} A_{22}^{-1} A_{21}|$

H.W. 3

ex) let $A = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ \hline 3 & -1 & 4 & 0 & 0 \\ 3 & -1 & 0 & 4 & 0 \\ 3 & -1 & 0 & 0 & 4 \end{array}\right]$ find ① $A^{-1}$ ② $|A|$

sol

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$A_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$A_{11}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_{22}^{-1} = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

$$Q = A_{11} - A_{12} A_{22}^{-1} A_{21}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 3 & -1 \\ 3 & -1 \end{pmatrix}$$

$$Q = \begin{pmatrix} -5/4 & 3/4 \\ -9/2 & 5/2 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} 10 & -3 \\ 18 & -5 \end{pmatrix}$$

$$A_{12} A_{22}^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad A_{22}^{-1} A_{21} = \begin{bmatrix} 3/4 & -\frac{1}{4} \\ 3/4 & -\frac{1}{4} \\ 3/4 & -\frac{1}{4} \end{bmatrix}$$

① $A^{-1} = \begin{bmatrix} 10 & -3 & -1 & -1 & -1 \\ 18 & -5 & -2 & -2 & -2 \\ -3 & 1 & 1/2 & 1/4 & 1/4 \\ -3 & 1 & 1/4 & 1/2 & 1/4 \\ -3 & 1 & 1/4 & 1/4 & 1/2 \end{bmatrix}$

② $|A| = |A_{11}| \cdot |A_{22} - A_{21} A_{11}^{-1} A_{12}|$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \left| \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 3 & -1 \\ 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \right|$$

$|A| = \boxed{16}$

iii - If $\lambda_i > 0$ then $A$ is positive definite and if $\lambda_i \geq 0$ then $A$ is positive semidefinite.

v - If $\lambda_i \neq \lambda_j$ for a symmetric matrix $A$, their corresponding vectors $\underline{X}_i$ and $\underline{X}_j$ are called orthogonal $\longrightarrow \underline{X}_i' \underline{X}_j = 0$

H.W Find the normalized eigenvectors of the matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

2.4 vector Derivatives

Consider $f: R^p \rightarrow R$ and a vector $\underline{X}$ of order $(p \times 1)$. Let $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_p}$ be the partial derivatives, then

where $f(x)$ be continuous and of the elements of the vector $\underline{X}$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$

① $$\frac{\partial f(\underline{x})}{\partial \underline{x}} = \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \vdots \\ \partial f/\partial x_p \end{pmatrix} \rightarrow \text{Column vector}$$

and $$\frac{\partial f(\underline{x})}{\partial \underline{x}'} = \begin{pmatrix} \partial f/\partial x_1 & \frac{\partial f}{\partial x_2} & \cdots & \partial f/\partial x_p \end{pmatrix}$$

$\rightarrow$ row vector

② If $f(\underline{x}) = \underline{a}' \underline{x}$, where $\underline{a}' = (a_1 \cdots a_p)$ is constant vector, then

Hessian matrix

$$H = \frac{\partial^2 f(x)}{\partial \underline{x}' \partial \underline{x}} = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_p \partial x_1} & & \frac{\partial^2 f(x)}{\partial x_p^2} \end{pmatrix}$$

Hessian

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$$f(x) = x' A x$$

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$$= \boxed{2A}$$

$$\frac{\partial h(\underline{x})}{\partial \underline{x}} = \frac{\partial (\underline{a}'\underline{x})}{\partial \underline{x}} = \underline{a}$$

$dx$ -- $x'a$

③ If $h(\underline{x}) = \underline{X}' A \underline{X}$, where $A$ is symmetric matrix of constant, then

$$\frac{\partial h(\underline{x})}{\partial \underline{x}} = \frac{\partial (\underline{X}' A \underline{X})}{\partial \underline{x}} = 2A\underline{X}$$

A may be identity matrix then $\frac{\partial f(\underline{x})}{\partial \underline{x}} = 2\underline{x}$

④ we can also find the second derivatives

$\frac{\partial^2 h(\underline{x})}{\partial \underline{x} \partial \underline{x}'}$ is the $(p \times p)$ matrix of elements of

$$\frac{\partial^2 h(\underline{x})}{\partial x_i \partial x_j}, \quad i = 1, 2, \dots, p \text{ and } j = 1, 2, \dots, p$$

$\frac{\partial^2 h(\underline{x})}{\partial \underline{x} \partial \underline{x}'}$ is called the Hessian of $h$.

The Hessian of the quadratic form $\underline{X}' A \underline{X}$ is

$$\frac{\partial^2 \underline{X}' A \underline{X}}{\partial \underline{x} \partial \underline{x}'} = 2A.$$

$$, H = \frac{\partial^2 f(x)}{\partial \underline{x}' \partial \underline{x}} = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_p \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_p^2} \end{pmatrix}$$

Hessian matrix

**Ex** Consider $\underline{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\underline{a}' = (3 \quad 4)$

and $A = \begin{pmatrix} 3 & 5 \\ 5 & 4 \end{pmatrix}$

(a) If $h(\underline{x}) = \underline{a}'\underline{x} = 3x_1 + 4x_2$

$$\frac{\partial h(\underline{x})}{\partial \underline{x}} = \begin{pmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \underline{a}$$

(b) If $h(\underline{x}) = \underline{X}' A \underline{X} = (x_1 \;\; x_2)\begin{pmatrix} 3 & 5 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$= 3x_1^2 + 4x_2^2 + 10x_1x_2$$

$$\frac{\partial h(\underline{x})}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial h(\underline{x})}{\partial x_1} \\ \frac{\partial h(\underline{x})}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 6x_1 + 10x_2 \\ 8x_2 + 10x_1 \end{bmatrix}$$

$$= \begin{bmatrix} 6x_1 + 10x_2 \\ 10x_1 + 8x_2 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 10 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2\begin{bmatrix} 3 & 5 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2A\underline{X}$$

(4) If $h(\underline{x}) = A\underline{X} \longrightarrow \frac{\partial h(\underline{x})}{\partial \underline{X}} = \frac{\partial (A\underline{X})}{\partial \underline{X}} = A'$

(5) If $h(\underline{x}) = A\underline{X} \longrightarrow \frac{\partial h(\underline{x})}{\partial \underline{x}'} = \frac{\partial (A\underline{X})}{\partial \underline{X}'} = A$



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(7) if $h(x) = (a - cx)'(a - cx)$

General Q. f. then

$$\frac{\partial h(x)}{\partial \underline{x}} = -2c'K(a - cx)$$

since $K$ be an $n \times n$ symatrix matrix

and $c$ be a $n \times p$ matrix of constant

and $a$ be an $n \times 1$ vectors.

(8) let $A$ is $n \times n$ matrix so.

$$\frac{\partial |A|}{\partial a_{ij}} = \begin{cases} 2A_{ij} & , i \neq j \\ A_{ij} & , i = j \end{cases}$$

(9) $$\frac{\partial (x'Ax)}{\partial A} = \underline{x}x' \quad , \quad \frac{\partial x'Ax}{\partial \underline{x}'} = 2x'A$$

(10) $$\frac{\partial x'A^{-1}x}{\partial A} = -A^{-1}\underline{x}x'A^{-1} , \cancel{\frac{\partial x'A^{-1}x}{\partial \underline{x}'} = 2\underline{x}'A}$$

(11) $$\frac{\partial (x'\Sigma^{-1}x)}{\partial \Sigma} = -\Sigma^{-1}\underline{x}\underline{x}'\Sigma^{-1}$$

(12) $$\frac{\partial \log |\Sigma|}{\partial \Sigma} = \frac{1}{|\Sigma|} \operatorname{adj}(\Sigma) = \Sigma^{-1}$$

(13) $$\frac{\partial |\Sigma|}{\partial \Sigma} = \operatorname{adj} \Sigma .$$

maximization Q.F. subject to constraints.

(Lagrange multipliers)

To maximize or minimize one fun. $\boxed{f(\underline{x})}$ subject to constraints $\boxed{g(\underline{x}) = c}$ on the value of $\boxed{\underline{x}}$, a more general method is that of Lagrange multipliers

For new. fun.

$$\phi = h(x, \lambda) = f(x) - \lambda\,(g(x) - c)$$

two conditions must be met:

$$\frac{\partial h(x,\lambda)}{\partial \lambda} = \frac{\partial f(x)}{\partial \underline{x}} - \lambda \frac{\partial g(x)}{\partial \underline{x}} \quad \cdots (1)$$

$= 0$ ثم نشتق بالرمز

$$\frac{\partial h(x,\lambda)}{\partial \lambda} = -g(x) + c \quad \cdots (2)$$

$= 0$

Solve (1) for ($\underline{x}$) after aliminating $\lambda$ to find the extrem value

<u>Note</u>

if the Q.F. is to be maximiz then $\lambda$ must be the greatest chr. root. of $A$ and $\underline{x}$ is associat vector. similarly if the Q.F. is to be minimize the $\lambda$ must be the minimum chr. roots. of $A$.

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ex. Find the extreme values for
$z = 2x_1^2 + 2x_1x_2 + 3x_2^2$
subject to the constraint $x_1^2 + x_2^2 = 1$.

Sol

$$Q = 2x_1^2 + 2x_1x_2 + 3x_2^2 - \lambda(x_1^2 + x_2^2 - 1)$$

$$\frac{\partial Q}{\partial x_1} = 4x_1 + 2x_2 - 2\lambda x_1 = 0 \quad \text{-----} (1)$$

$$\frac{\partial Q}{\partial x_2} = 2x_1 + 2x_2 - 2\lambda x_2 = 0 \quad \text{-----} (2)$$

Dividing. equ (1) by $x_1$ we have

$$4 + 2\frac{x_2}{x_1} - 2\lambda = 0$$

and dividing eq. (2) by $x_2$ we have.

$$\frac{x_2}{x_1} = \frac{1}{\lambda - 3}$$

$$\therefore \quad \lambda - 2 = \frac{1}{\lambda - 3}$$

$$\Rightarrow (\lambda - 2)(\lambda - 3) = 1$$

$$\lambda^2 - 5\lambda + 5 = 0$$

الدستور $\Rightarrow \lambda = \frac{5 \mp \sqrt{25 - 20}}{2}$

$$\lambda = \frac{5 \mp \sqrt{5}}{2}$$

$$\Rightarrow \lambda_1 = \frac{5 + \sqrt{5}}{2} = 3.618034$$

عند التعويض يكون اكبر
ما يمكن لانه اكبر

$$\lambda_2 = \frac{5 - \sqrt{5}}{2} = 1.381966$$

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ما يمكن لانه اصغر

now write eq(1) & eq(2) as

$$2x_1 + x_2 = \lambda x_1$$
$$x_1 + 3x_2 = \lambda x_2$$

multiple the first eq. by $x_1$
the second equation by $x_2$
and sum the two resulting equation

$$\underbrace{2x_1^2 + 2x_1x_2 + 3x_2^2}_{Q.F.} = \lambda \qquad (\text{using } x_1^2+x_2^2=1)$$

Q.F. it will be max when the Large root $\lambda_1$ replace $\lambda$
and min when the smallest root $\lambda$ replace $\lambda$.

$$4x_1 + 2x_2 - 2\lambda x_1 = 0 \quad \text{--}① \qquad \text{من } ①$$
$$x_1(2x_1 + x_2 = \lambda x_1)$$

من ②

$$2x_1 + 6x_2 - 2\lambda x_2 = 0$$
$$x_2(x_1 + 3x_2 = \lambda x_2)$$

$$2x_1^2 + x_2x_1 = \lambda x_1^2$$
$$x_1x_2 + 3x_2^2 = \lambda x_2^2$$

$$2x_1^2 + x_1x_2 + x_2x_1 + 3x_2^2 = \lambda x_1^2 + x_2^2$$
$$2x_1^2 + 2x_1x_2 + 3x_2^2 = \lambda(\underbrace{x_1^2 + x_2^2}_{=1})$$

$$\underbrace{2x_1^2 + 2x_1x_2 + 3x_2^2}_{Q.f} = \lambda$$

انتهى الفصل الاول
الذي تضمن المقدمة

# Chapter Two

## Multivariate Normal Distribution

### 1. Moments of Multidimensional

we assume that aset of $p$ random variables $X_1, X_2, \dots X_p$, we denoted by a vector:

$$\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$$

which is called arandom vector.

The mean or expectation of $\underline{X}$ is defined to be the vector of expectation:

$$E(\underline{X}) = \begin{pmatrix} E(X_1) \\ \vdots \\ E(X_p) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix} = \underline{\mu}$$

The expected value of the random vector.

where $\underline{\mu}$ is called the mean vector of the random vector $\underline{X}$ and $\mu_i = E(X_i)$ is the mean of each univariate random variable $x_i$, $i = 1, 2, \dots, p$.

Let $\sigma_{ij}$ denote the covariance between $x_i$ and $x_j$, then,

$$\sigma_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)] = Cov(x_i, x_j)$$

if $i=j$ the covariance of $x_i, x_j$ is the variance of $x_i$

and $\sigma_{ii} = E[(x_i - \mu_i)^2] = var(x_i) = \sigma_i^2$

be the variance of $x_i$

The variance of the random vector $\underline{X}$ which called var-cov or (dispersion) matrix is de as hollows:

$$\Sigma = var(\underline{X}) = E[(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})']$$

$$\Sigma = E\begin{bmatrix} X_1-\mu_1 \\ \vdots \\ X_p-\mu_p \end{bmatrix}\begin{bmatrix}(X_1-\mu_1) & \cdots & (X_p-\mu_p)\end{bmatrix}$$

$$= E\begin{bmatrix} (X_1-\mu_1)^2 & (X_1-\mu_1)(X_2-\mu_2) & \cdots & (X_1-\mu_1)(X_p-\mu \\ \vdots & (X_2-\mu)^2 & \cdots & \\ (X_p-\mu_p)(X_1-\mu_1) & & \cdots & (X_p-\mu_p \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & & \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} = \Sigma$$

where $\Sigma$ is a symmetric matrix called v-c matrix.

Let $A$ be $(q*p)$ constant matrix and $\underline{a}$ $(p*1$ constant vector. Three properties of variance of vector $\underline{X}$ is written as hollows:

(1) $var(\underline{X}+\underline{a}) = var(\underline{X})$

(2) $var(A\underline{X}) = A\Sigma A'$

(3) $var(\underline{a}'\underline{X}) = \underline{a}'\Sigma\underline{a}$

The variance of the random vector $\underline{X}$ which it is called var-cov or (dispersion) matrix is defined as follows:

$$\Sigma = \text{var}(\underline{X}) = E\left[(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})'\right]$$

$$\Sigma = E\begin{bmatrix} X_1-\mu_1 \\ \vdots \\ X_p-\mu_p \end{bmatrix}\begin{bmatrix}(X_1-\mu_1) & \cdots & (X_p-\mu_p)\end{bmatrix}$$

$$= E\begin{bmatrix} (X_1-\mu_1)^2 & (X_1-\mu_1)(X_2-\mu_2) & \cdots & (X_1-\mu_1)(X_p-\mu_p) \\ \vdots & (X_2-\mu_2)^2 & \cdots & \\ (X_p-\mu_p)(X_1-\mu_1) & & \cdots & (X_p-\mu_p)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & & \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} = \Sigma$$

where $\Sigma$ is a symmetric matrix called v-c matrix

Let $A$ be $(q*p)$ constant matrix and $\underline{a}$ $(p*1)$ constant vector. Three properties of variance of vector $\underline{X}$ is written as follows:

(1) $\text{var}(\underline{X}+\underline{a}) = \text{var}(\underline{X})$

(2) $\text{var}(A\underline{X}) = A\Sigma A'$

(3) $\text{var}(\underline{a}'\underline{X}) = \underline{a}'\Sigma\underline{a}$

proof(2)

$$\begin{aligned}
var(A\underline{X}) &= E\left[(A\underline{X}-E(A\underline{X}))(A\underline{X}-E(A\underline{X}))'\right] \\
&= E\left[(A\underline{X}-AE(\underline{X}))(A\underline{X}-AE(\underline{X}))'\right] \\
&= E\left[(A\underline{X}-A\underline{M})(A\underline{X}-A\underline{M})'\right] \\
&= E\left[A(\underline{X}-\underline{M})(\underline{X}-\underline{M})'A'\right] \\
&= AE(\underline{X}-\underline{M})(\underline{X}-\underline{M})'A' \\
&= A\Sigma A'
\end{aligned}$$

The correlation between the ith and jth variables of the random vector $\underline{X}$ is given as:

$$\rho_{ij} = \frac{cov(x_i, x_j)}{\sqrt{var(x_i)}\sqrt{var(x_j)}} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

The correlation matrix of the random vector $\underline{X}$ is anologous to the covariance which is defined as

$$R = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & & & \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{pmatrix} = [\rho_{ij}]$$

of course, the matrix $R$ is symmetric, since $\rho_{ij} = \rho_{ji}$.

The correlation matrix can be obtained from the covariance matrix and vice versa.

If $D^{\frac{1}{2}} = diag(\sqrt{\sigma_{11}}, \sqrt{\sigma_{22}}, \ldots, \sqrt{\sigma_{pp}})$

$$= \begin{pmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & 0 \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{pp}} \end{pmatrix}$$

Then:

(i) $D^{\frac{1}{2}} R D^{\frac{1}{2}} = \Sigma$

(ii) $R = D^{-\frac{1}{2}} \Sigma D^{-\frac{1}{2}}$

Ex Let $\Sigma = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{pmatrix}$

Find $R$

sol $D^{\frac{1}{2}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$D^{-\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

$$R = (D^{-\frac{1}{2}}) \Sigma (D^{-\frac{1}{2}}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$$

The correlation matrix can be obtained from the covariance matrix and vice versa.

If $D^{\frac{1}{2}} = \text{diag}(\sqrt{\sigma_{11}}, \sqrt{\sigma_{22}}, \cdots, \sqrt{\sigma_{pp}})$

$$= \begin{pmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & 0 \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{pp}} \end{pmatrix}$$

Then:

(i) $D^{\frac{1}{2}} R D^{\frac{1}{2}} = \Sigma$

(ii) $R = D^{-\frac{1}{2}} \Sigma D^{-\frac{1}{2}}$

<u>Ex</u> Let $\Sigma = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{pmatrix}$

Find $R$

<u>sol</u> $D^{\frac{1}{2}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$$D^{-\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$$

$$R = (D^{-\frac{1}{2}}) \Sigma (D^{-\frac{1}{2}}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$$

$$\therefore R = \begin{pmatrix} 1 & \frac{1}{6} & \frac{1}{5} \\ \frac{1}{6} & 1 & -\frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & 1 \end{pmatrix}$$

Remark

$$\begin{aligned}
\operatorname{var}(\underline{X}) &= \operatorname{cov}(\underline{X}, \underline{X}) \\
&= E[(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})'] \\
&= E(\underline{X}\underline{X}' - \underline{\mu}\underline{X}' - \underline{X}\underline{\mu}' + \underline{\mu}\underline{\mu}') \\
&= E(\underline{X}\underline{X}') - \underline{\mu}E(\underline{X}') - \underline{\mu}E(\underline{X}) + \underline{\mu}\underline{\mu}' \\
&= E(\underline{X}\underline{X}') - \underline{\mu}\underline{\mu}'
\end{aligned}$$

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## Conditional dist.

is the dist. of one set of a random variables given that the variables of the second group have been set equal to specified constant values.

The density fun. of the conditional dist. of $x_1, \dots, x_r$ given that $X_{r+1} = x_{r+1}, \dots, X_p = x_p$ is:

$$h(\underbrace{\overset{\text{مجهول}}{x_1, \dots, x_r}}_{\text{عدد } r} \mid \underbrace{\overset{\text{معلوم}}{x_{r+1}, \dots, x_p}}_{\text{عدد } p-r}) = \frac{f(x_1, \dots, x_r, x_{r+1}, \dots, x_p)}{\underset{\text{معلوم.}}{g(x_{r+1}, \dots, x_p)}}$$

where $f(x_1, \dots, x_r, x_{r+1}, \dots, x_p)$
is the joint density of the complete set p variables

and $g(x_{r+1}, \dots, x_p)$ is the joint density of the $(p-r)$ fixed variables. (معلوم)

التوزيع الشرطي
هو التوزيع لمجموعة معينة من المتغيرات باعتبار ان المجموعة الاخرى تأخذ قيم محددة معطاة:

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let :

$X' = [X_1, \ldots, X_p]$ with density fun.

$f_1(x_1), \ldots, f_p(x_p)$

and the c.d.f. for each $X_i$

$F_1(x_1), \ldots, F_p(x_p)$

if those variable are independent then

$$f(x_1, \ldots, x_p) = f_1(x_1) \ldots f_p(x_p)$$

$$F(x_1, \ldots, x_p) = F_1(x_1) \ldots F_p(x_p)$$

such that Factorization of the density and dis. fun. imply that the $x_i$ are independent variable

Also if $F(x_1, \ldots, x_p)$ is absolutely continuous

the density fun. is:

$$\boxed{f(x_1, \ldots, x_p) = \frac{\partial^p F(x_1, \ldots, x_p)}{\partial x_1 \ldots \partial x_p}}$$

The properties for the Covariance.-

① $Cov(cX_i, dX_j) = cd\, Cov(X_i, X_j)$

② $Cov(X_i + a, X_j + b) = Cov(X_i, X_j)$ , since a & b are constant.

③ if $a'X = a_1X_1 + a_2X_2 + \dots + a_pX_p$

then $Var(\underline{a}'\underline{X}) = \sum_{i=1}^{p}\sum_{j=1}^{p} a_i a_j \sigma_{ij}$

$\boxed{Var(\underline{a}'\underline{X}) = \underline{a}' \Sigma \underline{a}}$

is the variance for the linear componend $\boxed{\underline{a}'\underline{X}}$

④ The Covariance of the two linear componend $\underline{a}'\underline{X}$ and $\underline{b}'\underline{X}$ in the same variable is:

$$Cov(\underline{a}'\underline{X}, \underline{b}'\underline{X}) = \sum_{i=1}^{p}\sum_{j=1}^{p} a_i b_j \sigma_{ij}$$

$$= \boxed{\underline{a}' \Sigma \underline{b}}$$

⑤ $\Sigma = \sigma_{ij} = E(X_i - \mu_i)(X_j - \mu_j)$ , $\forall i,j = 1,2,\dots,p$

$$\Sigma = \begin{cases} \sigma_{ij} = Cov(X_i, X_j) & i \neq j \\ \sigma_{ii} = Cov(X_i, X_i) = Var(X_i) = E(X_i - \mu)^2 & i = j \end{cases}$$

⑥ if the random ~~vectors~~ variates $X_1, X_2, \dots, X_p$ are independent then:

$$\Sigma = \begin{pmatrix} \sigma_{11} & & 0 \\ & \sigma_{22} & \\ 0 & & \ddots \; \sigma_{pp} \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \; \sigma_p^2 \end{pmatrix}.$$

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Where $Cov(X_i, X_j) = 0$ &

$Var(X_i) = \sigma_{ii} = \sigma_i^2$ اكتب اختصار.

7) more generally if A and B are $(r \times p)$ and $(s \times p)$ dimensions of the form variates:

$$\underline{Y} = A\underline{X} \quad \text{and} \quad \underline{Z} = B\underline{X}$$

will be given by the matrices:

$$Cov(\underline{Y}, \underline{Y}) = A \Sigma A'$$

$$Cov(\underline{Z}, \underline{Z}) = B \Sigma B'$$

$$Cov(Y, Z) = A \Sigma B'$$

The correlation coefficient of $x_i$ and $x_j$ is defined as:

8) $$\rho_{ij} = \frac{Cov(x_i, x_j)}{\sqrt{V(x_i)}\,\sqrt{V(x_j)}}$$

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$$= \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\,\sqrt{\sigma_{jj}}}.$$

$$\boxed{\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}}$$ where $-1 \le \rho_{ij} \le +1$

9) if $x_i$ and $x_j$ are independent then the $Cov(x_i, x_j)$ and their correlation $\rho_{ij}$ equal to zero

$$Cov(x_i, x_j) = 0$$

$$\Rightarrow \rho_{ij} = 0$$

But ingeneral the Covariance is not true.

) if $\Sigma$ is the Covariance matrix and $\rho$ is the Correlation matrix (p.d.), then $\Sigma$ and $\rho$ are related as:

$$\boxed{\begin{aligned} \rho &= D\left(\frac{1}{\sigma_i}\right) \Sigma \, D\left(\frac{1}{\sigma_i}\right) \\ \Sigma &= D(\sigma_i)\, \rho \, D(\sigma_i) \end{aligned}}$$

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where the matrix of $p \times p$ Correlation is,

Since

$$\rho = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{bmatrix}$$

where $D(\sigma_i) \longrightarrow$ is a diagonal matrix of the standard deviation as a diagonal elements

$$D(\sigma_i) = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & \\ 0 & & \ddots \; \sigma_p \end{pmatrix}$$

$D(\sigma_i)$ مصفوفة قطرية عناصر:
الاقطار = المعياري $\sigma_i$

Where $\sigma_i$ is the standard deventions of the variates

(11) if $\rho = 1$ ($p \times p$) then the p-variate normal density fun. becomes the Univariate density fun.

and if $p = 2$ the p-variate normal density fun. becomes the bivariate joint normal density fun.

(12) if we standarize each variate by:

$$Z_i = \frac{X_i - \mu_i}{\sqrt{\sigma_{ii}}}$$

$$f(\underline{z}) = \frac{1}{(2\pi)^{p/2} |\rho|^{1/2}} \exp\left[-\frac{1}{2} \underline{z}' \rho^{-1} \underline{z}\right]$$

where $\underline{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix}$

$\rho$ is the Correlation matrix (p.d.)

H.w.

**Ex. (1)** Suppose that $\underset{\sim}{L} = \acute{a}\underline{x}$ , find $Var(L)$

where:

$$x' = (x_1 \quad x_2 \quad x_3)$$

$$a' = (15 \quad 1 \quad 10)$$

$$\Sigma_x = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

**Ex. (2)** Suppose that

$$Y_1 = 2X_1 + X_2 + X_3$$

$$Y_2 = X_2 + X_3$$

and

$$Z_1 = X_1 - X_2 + X_3$$

$$Z_2 = X_2 - X_3$$

Find ① $\Sigma_Y$

② $\Sigma_Z$

③ $Cov(Y, Z)$

where $\Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

**Ex: (3)** let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ , where $p = 2$ الدرجة

Define the Correlation matrix $\rho$

and the V-C matrix $\Sigma$

as arelated matrix.

Ex. 1

<u>Sol</u>

$$V(\underline{L}) = V(\acute{a}\,\underline{x}) = \acute{\underline{a}}\ \Sigma\ \underline{a}$$

$$= \begin{bmatrix} 15 & 1 & 10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ 1 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 28 & 36 \end{bmatrix} \begin{bmatrix} 15 \\ 1 \\ 10 \end{bmatrix}$$

$$= \left[ 26(15) + 28(1) + 36(10) \right]$$

$$= 390 + 28 + 360$$

$$V(\underline{L}) = \boxed{778}$$

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(2)

Ex.

Sol Y.

(1) $\underline{Y} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\underset{2\times 1}{\underline{Y}} = \underset{2\times 3}{A}\ \underset{3\times 1}{\underline{X}}$$

$$\Sigma_y = A \Sigma_x A'$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 6 & 5 \\ 4 & 6 & 7 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \Sigma_y = \begin{pmatrix} 19 & 21 \\ 21 & 27 \end{pmatrix} \\ \Sigma_y = \begin{pmatrix} v(y_1) & cov(y_1, y_2) \\ & v(y_2) \end{pmatrix} \end{array}}$$

(2) $\underline{Z} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\underset{3\times 1}{\underline{Z}} = \underset{2\times 3}{B}\ \underset{3\times 1}{\underline{X}}$$

$$\Sigma_z = B \Sigma B'$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \Sigma_z = \begin{pmatrix} 4 & -3 \\ -3 & 3 \end{pmatrix} \\ \Sigma_z = \begin{pmatrix} v(z_1) & cov(z_1, z_2) \\ & var(z_2) \end{pmatrix} \end{array}}$$

③ $Cov(Y, Z) = Cov(A\underline{X}, B\underline{X}) = A \Sigma B'$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 6 & 5 \\ 4 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\underset{\Sigma_{YZ}}{Cov(Y, Z)} = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} Cov(Y_1, Z_1) & Cov(Y_2, Z_1) \\ Cov(Y_2, Z_2 & Cov(Y_2, Z_2) \end{bmatrix}$$

<u>Note</u>

$$\Sigma_X = E\{\underline{X} - E(\underline{X})\}\{\underline{X} - E(\underline{X})\}'$$

$\underline{Y} = A\underline{X}$

$$\Sigma_Y = E[\underline{Y} - E(\underline{Y})][\underline{Y} - E(\underline{Y})]'$$

$$= E[A\underline{X} - AE\underline{X}][A\underline{X} - AE(\underline{X})]'$$

$$= E\, A(X - E(X))\,(A(X - EX))'$$

$$= A\, \underline{E(X - EX)(X - EX)'}\, A'$$

$$\Sigma_Y = A \quad \Sigma_X\, A'$$

EX3

**Sol:** $\underline{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ , $\Sigma_x = \begin{pmatrix} V(X_1) & cov(X_1, X_2) \\ & V(X_2) \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$

V-C matrix of $x$

$\rho$ correlation matrix.

$$\rho = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{21} & 1 \end{pmatrix}$$

where $\rho_{11} = \dfrac{cov(X_1, X_1)}{\sqrt{V(X_1)\, V(X_1)}} = \dfrac{V(X_1)}{\sqrt{[V(X_1)]^2}} = \dfrac{\cancel{V(X_1)}}{\cancel{V(X_1)}} = 1$

$$\rho_{22} = \frac{cov(X_2, X_2)}{\sqrt{V(X_2)\, V(X_2)}} = \frac{V(\cancel{X_2})}{V(\cancel{X_2})} = 1$$

$$\rho_{12} = \frac{cov(X_1, X_2)}{\sqrt{V(X_1) \cdot V(X_2)}} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

لذلك لايجاد الارتباط نحتاج لمعرفة التباين والتباين المشترك
وباستخدام العلاقتين السابقتين حيث ان

$$\Sigma = D(\sigma_i)\, \rho\, D(\sigma_i) \qquad , i = 1, 2$$

$$= \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} 1 & \rho_{12} \\ \rho_{21} & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1 & \sigma_1 \rho_{12} \\ \sigma_2 \rho_{12} & \sigma_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1 \rho_{12} \sigma_2 \\ \sigma_2 \rho_{12} \sigma_1 & \sigma_2^2 \end{pmatrix}$$

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وبتعويض قيمة الارتباط حسب $\rho_{12}$

ينتج $\rho_{12} = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2}$

$$\sigma_1 \rho_{12} \sigma_2 = \cancel{\sigma_2} \left( \frac{\sigma_{12}}{\cancel{\sigma_1} \cancel{\sigma_2}} \right) \cancel{\sigma_1} = \sigma_{12}$$

$$\therefore \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

بعد ايجاد مصفوفة التباين والتباين المشترك $\Sigma$ .

من العلاقة نعرف معامل الارتباط .

بعد استخدام قانون الارتباط للعينات .

نجد الان مصفوفة الارتباط $\rho$ بدلالة مصفوفة التباين والتباين المشترك $\Sigma$

فالعلاقة ما بينهما حيث ان

$$\rho = D\left(\frac{1}{\sigma_i}\right) \Sigma \, D\left(\frac{1}{\sigma_i}\right)$$

$$= \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & \frac{1}{\sigma_2} \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & \frac{1}{\sigma_2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sigma_1^2}{\sigma_1} & \frac{\sigma_{12}}{\sigma_1} \\ \frac{\sigma_{12}}{\sigma_2} & \frac{\sigma_2^2}{\sigma_2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & \frac{1}{\sigma_2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\cancel{\sigma_1^2}}{\cancel{\sigma_1^2}} & \frac{\sigma_{12}}{\sigma_1 \sigma_2} \\ \frac{\sigma_{12}}{\sigma_2 \sigma_1} & \frac{\cancel{\sigma_2^2}}{\cancel{\sigma_2 \sigma_2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \rho_{12} \\ \rho_{21} & 1 \end{pmatrix}$$

$= \rho$ matrix.

the Correlation coefficient matrix